

# Explosion in a growth model with cooperative interaction on an infinite graph

Bruna de Oliveira Gonçalves<sup>1</sup>      Marina Vachkovskaia<sup>1</sup>

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<sup>1</sup>Departamento de Estatística, Instituto de Matemática, Estatística e Computação Científica, Universidade de Campinas, rua Sérgio Buarque de Holanda, 651, CEP 13083-859, Campinas SP, Brasil

e-mails: [ra162348@ime.unicamp.br](mailto:ra162348@ime.unicamp.br), [marinav@ime.unicamp.br](mailto:marinav@ime.unicamp.br),

## Abstract

In this paper we study explosion/non-explosion of a continuous time growth process with cooperative interaction on  $\mathbb{Z}_+$ . We consider symmetric neighborhood and different types of rate functions and prove that explosion occurs for exponential rates, but not for polynomial. We also present some simulations to illustrate the explosion types.

**Keywords:** growth process, adsorption, explosion

## Introduction

In this paper we study a probabilistic model for the asymptotic behavior of particles adsorption process on an infinite graph. Adsorption may be defined as the process of accumulation of a substance on a surface. The adsorption process can be classified into two types: Physical Adsorption or chemical adsorption. The physical adsorption happens when molecules or atoms adhere to the adsorbent surface by intermolecular interactions while in chemical adsorption there are chemical attraction forces or chemical bond between substance and surface [3].

The basic model of adsorption is the random sequential adsorption (RSA), this model refers to a process where particles are randomly introduced in a system without overlaps [4]. The RSA is not appropriate for many physical, chemical and biological processes, in these cases, the model of cooperative sequential adsorption (ASC) is more appropriate. In ASC the adsorption probability depends on a configuration formed by locations of previously adsorbed particles, which may contribute to the adsorption of the next particle or vice versa.

The adsorption is important in a number of situations, for example, the use of activated charcoal in refrigerators to retain odor from food, the use of activated charcoal to remove impurities in water, besides the adsorption presents decisive influence in the interactions between cells of the human body and implants (see [1], [5] and [7]).

Therefore, the adsorption process is of great importance to the research of materials that can, besides being used as adsorbents, be economically viable. Thus, the goal of this paper is to study the behavior of the adsorption process using a probabilistic model. Our model is motivated by multilayer adsorption models without restrictions on the numbers of adsorbed particles and considers the ASC model. We studied the asymptotic structure of the configuration formed by the particles since there is no limitation as to the number of particles adsorbed at each site.

The probability of adsorption at the site  $k$  is proportional to different rates, exponential  $e^{\lambda_k U_k}$  or polynomial  $(\lambda_i U_i(x))^\alpha$ , where  $\lambda_k > 0$ ,  $\alpha \in \mathbb{N}$  and  $U_k$  is the number of particles adsorbed in the neighborhood of site  $k$ . Thus the number of particle in the neighborhood of site  $k$  and the value of  $\lambda_k$  influence the adsorption probability of the next particle.

The study of the asymptotic structure of adsorption when time increases is a typical problem of the models of urns. In particular, the model is directly related to the model of Polya urns for the exponential case (see [2]). In the general model of Polya urn colored balls are added in an urn and a ball is randomly withdrawn from the urn, this ball is returned to the urn along with another additional ball of the same color as the drawn ball. In our model each site  $i$  correspond the different colors of the balls in the Polya urn and the probability of the site adsorbing the particle is proportional to  $\Gamma_i$ .

Our main result states that the explosion occurs in the exponential model for all lambdas, in the polynomial rate model degree at least 1 or any degree if  $\lambda_i$  grows fast enough in  $i$ , and no explosion occurs when  $\alpha \leq 1$  and lambdas are bounded. We also present some simulation to illustrate the types of

explosion.

## 1 The model and results

Define the growth process on  $\mathbb{N}$  in the following way. Let  $\xi_i(t) \in \{0, 1, 2, \dots\}$  be the number of particles at site  $i \in \mathbb{N}$  at time  $t \geq 0$ . We start with one particle at  $i = 1$ , so  $\xi_1(0) = 1$  and  $\xi_j(0) = 0$  for all  $j > 1$ .

Then, the process

$$X(t) = (\xi_1(t), \xi_2(t), \dots) \in \bigcup_{n=1}^{\infty} \{1, 2, \dots, n\}^{\mathbb{N}} =: \Omega$$

is a continuous time Markov chain with the following dynamics. Given  $X(t) = x = (\xi_1, \xi_2, \dots) \in \Omega$  the next particle is allocated to site  $i$  with rate

$$\Gamma_i(x) := \begin{cases} f(\lambda_i, U_i(x)), & U_i(x) \neq 0 \\ 0, & U_i(x) = 0, \end{cases} \quad (1)$$

where  $\lambda_i > 0$  for all  $i$  and  $U_i$  is given by

$$U_i = \begin{cases} \xi_{i-1} + \xi_i + \xi_{i+1}, & i > 1 \\ \xi_1 + \xi_2, & i = 1 \end{cases} \quad (2)$$

The quantity  $U_i(x)$  is called the potential of a site  $i$  given the configuration  $x$ . When  $\lambda_i > 0$  for all  $i$  and  $f$  is an increasing function, this dynamics characterizes a cooperative sequential adsorption.

Let  $\omega_1, \omega_2, \dots$  be transition moments:

$$\begin{aligned} \omega_1 &= \inf\{t, X(t) \neq X(0)\} \\ &\vdots \\ \omega_n &= \inf\{t > \omega_{n-1}, X(t) \neq X(\omega_{n-1})\} \\ &\vdots \end{aligned}$$

The transition matrix  $P$  of the embedded Markov chain is given by

$$P_{xy} = \begin{cases} \frac{\Gamma_i(x)}{\sum_{j=1}^{\infty} \Gamma_j(x)}, & \text{if } y = x + e_i \\ 0, & \text{otherwise} \end{cases}, \quad (3)$$

where  $e_i = (\xi_1, \xi_2, \dots)$  with  $\xi_i = 1$  e  $\xi_j = 0$  for all  $j \neq i$ . Note that  $\sum_{j=1}^{\infty} \Gamma_j(x) < \infty$ , since the number of particle is always finite if we start with a finite configuration.

Note that the sequence of random jump times  $\omega_n$  can be written as  $\omega_n := \sum_{j=1}^n \sigma_j$  and  $\omega_0 := 0$ , where  $\sigma_n$  is exponential time between adsorption of particles. Let  $\zeta = \lim_{n \rightarrow \infty} \omega_n$ , then we say that explosion occurs if  $\zeta < \infty$ . Checking whether or not the phenomenon of explosion occurs can be difficult, since it requires knowledge of the entire trajectory of the embedded chain, but there are some constructive criteria [6], that we are going to use. Define measurable functions in state space  $(\mathbb{X}, \mathcal{F}')$  by

$$m\mathcal{F}' := \{f : \mathbb{X} \rightarrow \mathbb{R} \mid f \text{ is } \mathcal{F}' \text{ - measurable}\}, \quad (4)$$

Generator's domain is then

$$\text{Dom}(\Gamma) := \{f \in m\mathcal{F}' : \sum_{y \in \mathbb{X} \setminus x} \Gamma_{x,y} |f(y)| < +\infty, \forall x \in \mathbb{X}\} \quad (5)$$

thus,  $\text{Dom}_+(\Gamma)$  is the set of non-negative functions on  $\mathbb{X}$  and the generator  $\Gamma f(x)$  is defined by

$$\Gamma f(x) := \sum_{y \in \mathbb{X}} \Gamma_{xy} f(y) \quad (6)$$

Now we are ready to state our main results.

**Theorem 1.1** *If*

$$\Gamma_i(x) := \begin{cases} e^{\lambda_i U_i(x)}, & U_i(x) \neq 0 \\ 0, & U_i(x) = 0, \end{cases} \quad (7)$$

where  $\lambda_i > 0$  for all  $i$  and  $U_i$  is defined in (2). Then for all  $x \in \mathbb{Z}_+^\infty$  we have  $\zeta < \infty$ , so the process explodes.

The next result addresses the occurrence or not of explosion in the polynomial model.

**Theorem 1.2** *If*

$$\Gamma_i(x) := \begin{cases} (\lambda_i U_i(x))^\alpha, & U_i(x) \neq 0 \\ 0, & U_i(x) = 0, \end{cases} \quad (8)$$

where  $\alpha > 0$ ,  $\lambda_i > 0$  for all  $i$  and  $U_i$  is defined in (2). We have

- (i) If  $\alpha > 1$ , then  $\zeta < \infty$  for all  $x \in \mathbb{Z}_+^{\mathbb{N}}$ .
- (ii) If  $\alpha \leq 1$  and  $\lambda_i \leq c$  for all  $i$  and  $c \in \mathbb{R}$ , then  $\mathbb{P}_x(\zeta = +\infty) = 1$  for all  $x \in \mathbb{Z}_+^{\mathbb{N}}$ .
- (iii) If  $\alpha \leq 1$ , but  $\lambda_i$  is increasing at a polynomial rate with a degree greater than or equal to  $\frac{1}{\alpha}$ , then  $\zeta < \infty$  for all  $x \in \mathbb{Z}_+^{\mathbb{N}}$ .

## 2 Proofs

We are going to use the following results from [6]. Theorem 1.9.:

**Theorem 2.1** *The following are equivalent:*

1. There exist  $f \in \text{Dom}_+(\Gamma)$  strictly positive and  $\varepsilon > 0$  such that  $\Gamma f(x) \leq -\varepsilon, \forall x$ .
2. The explosion time  $\zeta$  satisfies  $E_x \zeta < +\infty, \forall x \in \mathbb{X}$ .

The next theorem (Theorem 1.14 from [6]) guarantees the non-explosion:

**Theorem 2.2** *Let  $f \in \text{Dom}_+(\Gamma)$ . If*

1.  $f \rightarrow \infty$ ,
2. there exist an increasing (not necessarily strictly) function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  whose inverse is locally integrate but has non integrate tail (i.e.  $G(z) := \int_0^z \frac{dy}{g(y)} < +\infty$ , for all  $z \in \mathbb{R}_+$  but  $\lim_{z \rightarrow \infty} G(z) = \infty$ ),
3.  $\Gamma f(x) \leq g(f(x))$  for all  $x \in \mathbb{X}$ ,

then  $\mathbb{P}_x(\zeta = +\infty) = 1$  for all  $x \in \mathbb{X}$ .

## 2.1 Proof of theorem 1.1

Without loss of generality, assume that the configuration  $x \in \mathbb{Z}_+^\infty$  has at least one particle at site 1. Obviously, this will always be true if we start with one particle in 1. The function

$$f(x) = \frac{1}{e^{\lambda_1 U_1(x)}}, \quad (9)$$

where

$$U_i = \begin{cases} \xi_{i-1} + \xi_i + \xi_{i+1}, & i > 1 \\ \xi_1 + \xi_2, & i = 1 \end{cases}, \quad (10)$$

satisfies then the conditions of Theorem 2.1. Indeed,

$$\begin{aligned} \Gamma f(x) &= \sum_{y \in \mathbb{X}} \Gamma_{xy} f(y) \\ &= \sum_{y \in \mathbb{X} \setminus \{x\}} \Gamma_{xy} (f(y) - f(x)) \\ &= \sum_{i=1}^{\infty} \Gamma_{x, x+e_i} (f(x+e_i) - f(x)) \\ &= \sum_{i=1}^{\infty} e^{\lambda_i U_i(x)} \left( \frac{1}{e^{\lambda_1 U_1(x+e_i)}} - \frac{1}{e^{\lambda_1 U_1(x)}} \right) \end{aligned} \quad (11)$$

$$(12)$$

If the particle is adsorbed out of the neighborhood of site 1 then  $U_1\{x+e_i\} = U_1\{x\}$  and if the particle is adsorbed in the neighborhood of site 1 then  $U_1\{x+e_i\} = U_1\{x\} + 1$ . Thus,

$$\begin{aligned} \Gamma f(x) &= \sum_{i=1}^2 e^{\lambda_i U_i(x)} \left( \frac{1}{e^{\lambda_1 (U_1(x)+1)}} - \frac{1}{e^{\lambda_1 U_1(x)}} \right) \\ &= e^{\lambda_1 U_1(x)} e^{-\lambda_1 U_1(x)} (e^{-\lambda_1} - 1) + e^{\lambda_2 U_2(x)} e^{-\lambda_1 U_1(x)} (e^{-\lambda_1} - 1) \\ &= (e^{-\lambda_1} - 1) (1 + e^{(\lambda_2 U_2(x) - \lambda_1 U_1(x))}) \end{aligned} \quad (13)$$

Therefore, there exists  $\varepsilon > 0$  such that  $\Gamma f(x) < -\varepsilon$ , which implies  $\zeta < \infty$  a.s.  $\square$

## 2.2 Proof of theorem 1.2

Again, without loss of generality, assume that the configuration  $x \in \mathbb{Z}_+^\infty$  has at least one particle at site 1.

(i) Consider  $\alpha > 1$  and take

$$f(x) = \frac{1}{U_1(x)^\beta}, \quad (14)$$

where  $0 < \beta < \alpha - 1$ . Then,

$$\begin{aligned} \Gamma f(x) &= \sum_{y \in \mathbb{X}} \Gamma_{xy} f(y) \\ &= \sum_{i=1}^{\infty} \Gamma_{x, x+e_i} (f(x+e_i) - f(x)) \\ &= \sum_{i=1}^{\infty} (\lambda_i U_i(x))^\alpha \left( \frac{1}{U_1(x+e_i)^\beta} - \frac{1}{U_1(x)^\beta} \right) \end{aligned} \quad (15)$$

$$\begin{aligned} &= \sum_{i=1}^2 (\lambda_i U_i(x))^\alpha \left( \frac{1}{(U_1(x)+1)^\beta} - \frac{1}{(U_1(x))^\beta} \right) \\ &\leq -\lambda_1^\alpha \frac{U_1(x)^\alpha}{U_1(x)^\beta} \left( 1 - \left( 1 + \frac{1}{U_1(x)} \right)^{-\beta} \right) \end{aligned} \quad (16)$$

$$\leq -\frac{c\beta\lambda_1^\alpha U_1(x)^\alpha}{U_1(x)^{\beta+1}} < -\varepsilon, \quad (17)$$

as  $0 < \beta < \alpha - 1$ , so  $\zeta < \infty$  a.s. and by Theorem 2.1 explosion occurs.

(ii) Consider  $\alpha \leq 1$  and first suppose  $\lambda_i \leq 1$  for all  $i$  and consider

$$f(x) = \frac{\sum_1^\infty U_i(x)}{3}, \quad (18)$$

(note that the number of particles is always finite). The Item 1 of Theorem 2.2 is satisfied, since  $f \rightarrow \infty$ . Define  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by

$$g(y) := 3y + \sqrt{y}. \quad (19)$$

The function  $g(y)$  satisfies the conditions of Item 2 of Theorem 2.2. Indeed,

$$G(z) = \int_0^z \frac{dy}{g(y)} = \int_0^z \frac{dy}{3y + \sqrt{y}} = \frac{2}{3} \ln(3\sqrt{z} + 1) < +\infty, \quad (20)$$

for all  $z \in \mathbb{R}_+$  and moreover  $\lim_{z \rightarrow \infty} G(z) = \lim_{z \rightarrow \infty} \frac{2}{3} \ln(3\sqrt{z} + 1) = \infty$ .

Thus,

$$\begin{aligned} \Gamma f(x) &= \sum_{y \in \mathbb{X}} \Gamma_{xy} f(y) \\ &= \sum_{i=1}^{\infty} (\lambda_i U_i(x))^\alpha (f(x + e_i) - f(x)). \end{aligned} \quad (21)$$

As

$$f(x + e_i) = \begin{cases} f(x) + \frac{2}{3}, & i = 1, \\ f(x) + 1, & i \neq 1 \end{cases}, \quad (22)$$

we have

$$\begin{aligned} \Gamma f(x) &\leq \sum_{i=1}^{\infty} (\lambda_i U_i(x))^\alpha (f(x) + 1 - f(x)) \\ &= \sum_{i=1}^{\infty} (U_i(x))^\alpha \\ &\leq \sum_{i=1}^{\infty} U_i(x) + \sqrt{\frac{\sum_{i=1}^{\infty} U_i(x)}{3}}, \quad \text{as } \alpha \leq 1 \text{ and } U_i(x) \geq 1, \\ &= g(f(x)). \end{aligned} \quad (23)$$

As  $\Gamma f(x) \leq g(f(x))$  for all  $x \in \mathbb{X}$  where  $\alpha \leq 1$ , by Theorem 2.2 we have  $\mathbb{P}_x(\zeta = +\infty) = 1$  for all  $x \in \mathbb{X}$ , that is, no explosion a.s.

If  $\lambda_i \leq c$  with  $c > 1$ , for all  $i$  we can define  $f(x)$  as

$$f(x) = \frac{\sum_1^\infty \lambda_i U_i(x)}{3c}, \quad (24)$$

and proceed analogously.

(iii) Consider  $\alpha \leq 1$  and the following function:

$$f(x) = \frac{1}{i_{\max}^\beta}, \quad (25)$$

where  $i_{\max} = \max\{i : \xi_i \geq 1\}$ ,  $\beta > 0$ . Then,



$$\begin{aligned}
\Gamma f(x) &= \sum_{y \in \mathbb{X}} \Gamma_{xy} f(y) \\
&= \sum_{i=1}^{\infty} \Gamma_{x, x+e_i} (f(x+e_i) - f(x)) \\
&= \sum_{i=1}^{\infty} (\lambda_i U_i(x))^\alpha (f(x+e_i) - f(x)). \tag{26}
\end{aligned}$$

If the particle is placed in a place other than  $i_{\max} + 1$  then  $f(x+e_i) = f(x)$ , otherwise  $f(x+e_i) = \frac{1}{(i_{\max} + 1)^\beta}$ . Thus,

$$\begin{aligned}
\Gamma f(x) &= (\lambda_{i_{\max}+1} \xi_{i_{\max}})^\alpha \left( \frac{1}{(i_{\max} + 1)^\beta} - \frac{1}{i_{\max}^\beta} \right) \\
&\leq - \frac{\beta c (\lambda_{i_{\max}+1} \xi_{i_{\max}})^\alpha}{i_{\max}^{\beta+1}} \tag{27}
\end{aligned}$$

So, if the rate of growth of  $\lambda_i$  is polynomial with degree  $p \geq \frac{1}{\alpha}$  there exist  $\beta > 0$  and  $\varepsilon > 0$  such that  $\Gamma f(x) < -\varepsilon$  for all  $x \in \mathbb{X}$  and by Theorem 2.1 explosion occurs.  $\square$

### 3 Simulations

Simulations were performed considering a finite number of sites and different formulas for the parameters  $\lambda_i$ . The simulation of the process consisted of assigning pre-set values to  $\lambda$  and considering an initial configuration  $X(0)$ . The calculations were performed using Statistical Computational System R (R Core Team, 2017).

For  $i \mapsto \lambda_i$  we considered increasing functions, decreasing functions that have local maximums and constant functions. In addition, different initial configurations were considered. At each iteration a new site is chosen to receive a new particle with probability previously defined in (??). For the choice of site the inverse transform method was used.

An explosion can occur vertically or horizontally. We say that an explosion is vertical if a finite number of sites adsorbs all particles, and an explosion is horizontal otherwise (that is, infinite number of sites is occupied in finite time).

### 3.1 Exponential Adsorption Rate

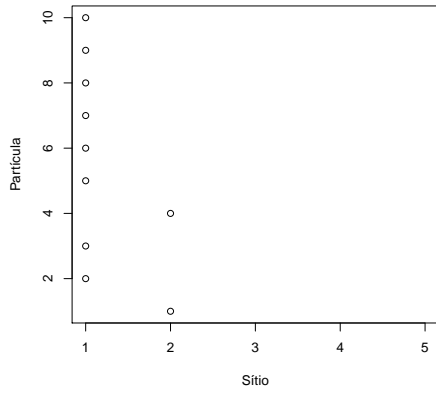
The next simulations illustrate the adsorption in the exponential model. That is

$$\Gamma_i(x) := \begin{cases} e^{\lambda_i U_i(x)}, & U_i(x) \neq 0 \\ 0, & U_i(x) = 0, \end{cases}$$

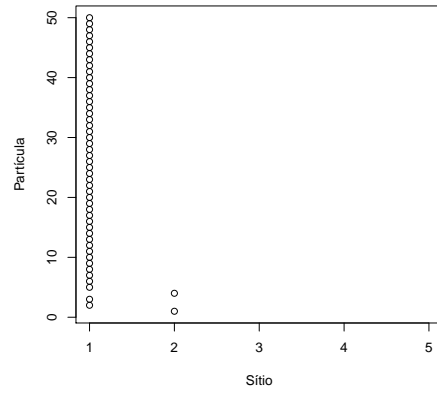
considering different formulas for the parameters  $\lambda_i$ .

#### 3.1.1 Decreasing Lambdas

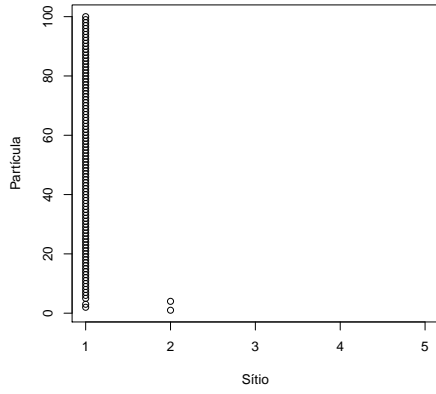
We can see that the process explodes vertically, at  $i = 1$  (maximum value of  $\lambda_i$ ). The figure 1 shows where each particle was adsorbed, each  $i$  site has probability of adsorption related to  $\lambda_i = 1/i$ . The initial configuration considered was  $x = (1, 0, \dots)$ . The Figures show the adsorption process of the first 10, 50, 100 and 1000 particles, respectively.



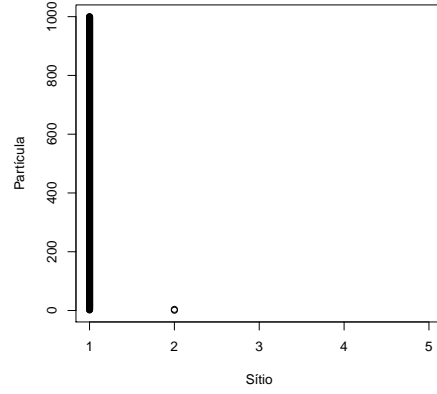
(a)

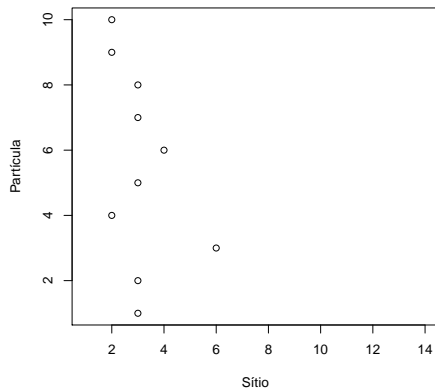


(b)

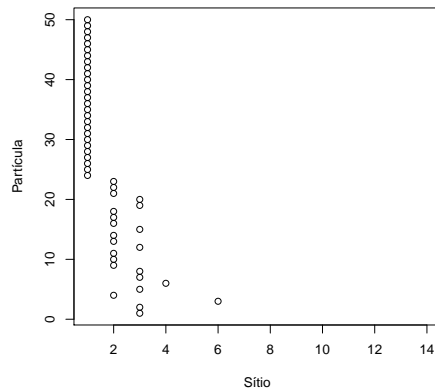


(c)

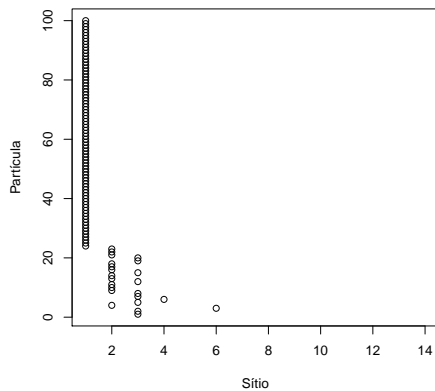




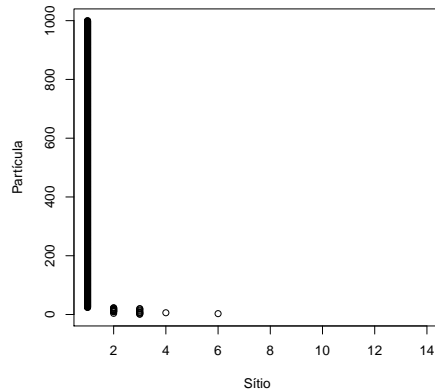
(a)



(b)



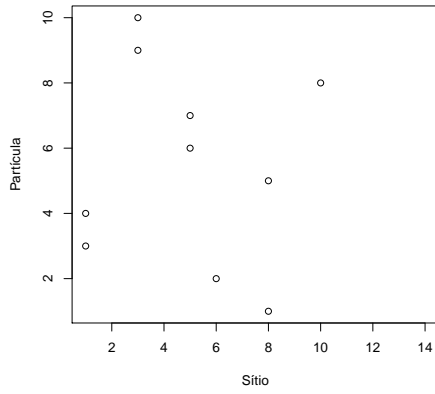
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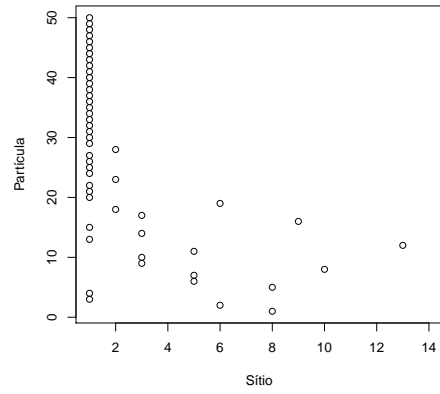
(d)

Figure 2: Particle Adsorption when  $\lambda_i = 1/i$ , initial configuration with particles in the first 12 sites, and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  (d)  $n=1000$ .

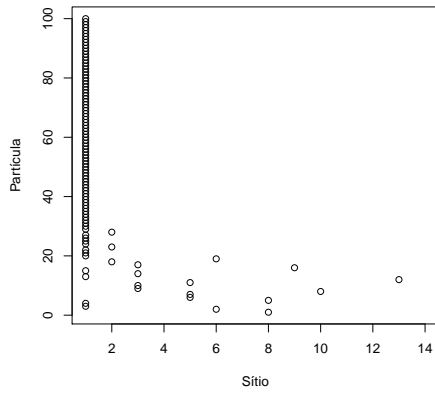
Figure 3 shows the site where each particle was adsorbed considering that each site  $i$  has probability of adsorption related to  $\lambda_i = 1/e^i$  and initial configuration  $x$  such that  $\xi_i = 0$  for all  $i > 12$ , ie the first 12 sites containing a random amount of particles.



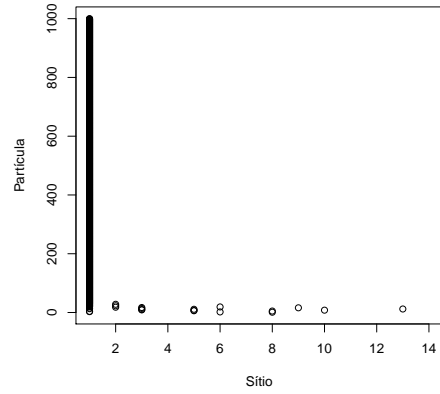
(a)



(b)



(c)



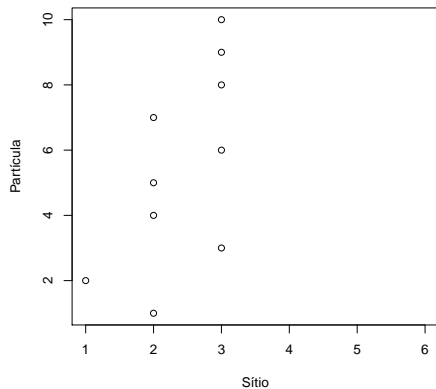
(d)

Figure 3: Particle Adsorption when  $\lambda_i = 1/e^i$ , initial configuration with particles in the first 12 sites, and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

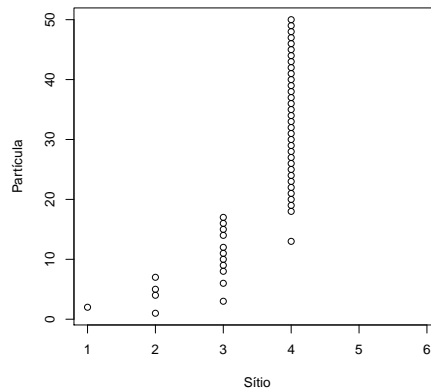
### 3.1.2 Increasing Lambdas

We know that the process explodes and we can see that the explosion is horizontal, although the number of particles at each site can be quite large. But, when  $\xi_i$  gets large, the probability that the next particle will fall at  $i + 1$  becomes close to 1, so its impossible for  $\xi_i$  to go to infinity. Figure 4 shows

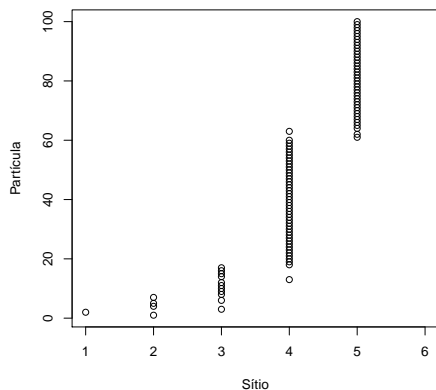
the site at which each particle was adsorbed considering that each  $i$  site has probability of adsorption related to  $\lambda_i = i$  and the initial configuration with one particle at site 1.



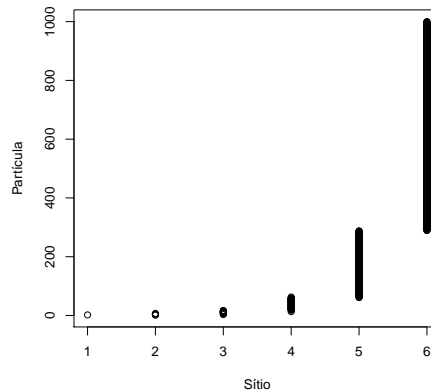
(a)



(b)



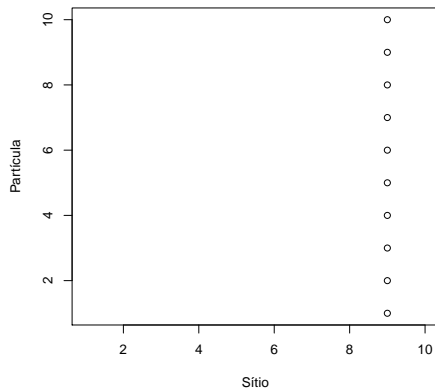
(c)



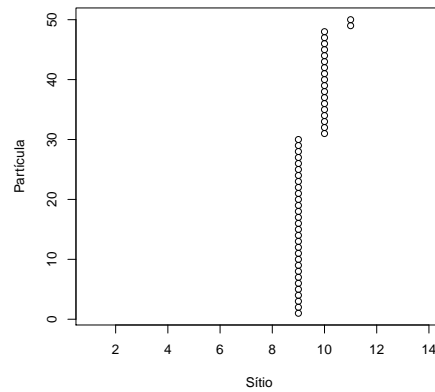
(d)

Figure 4: Particle Adsorption when  $\lambda_i = i$ , initial configuration  $x = (1, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

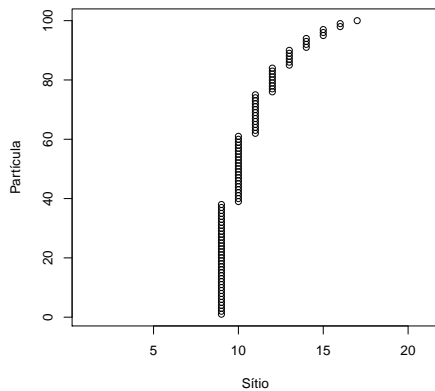
Figure 5 shows where each particle is adsorbed, considering an initial configuration with a random number of particles in the first 8 sites.



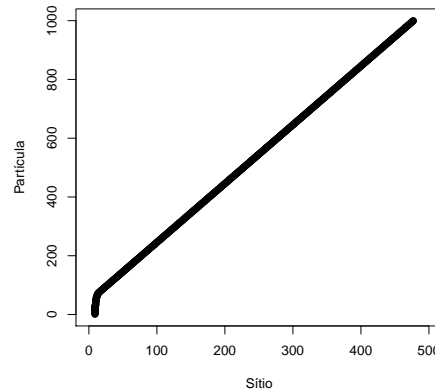
(a)



(b)



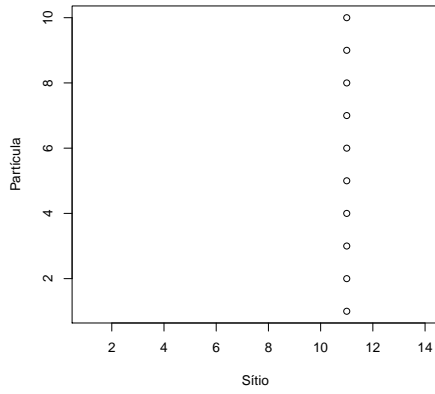
(c)



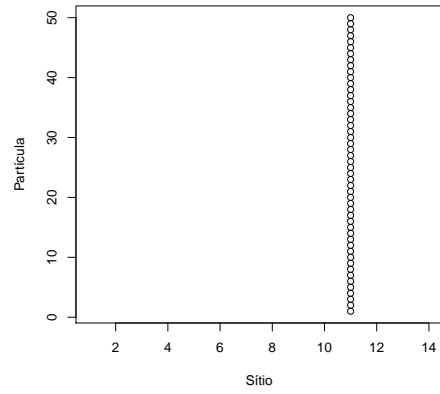
(d)

Figure 5: Particle Adsorption when  $\lambda_i = e^i$ , initial configuration  $x = (20, 9, 37, 44, 44, 9, 40, 50, 0, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

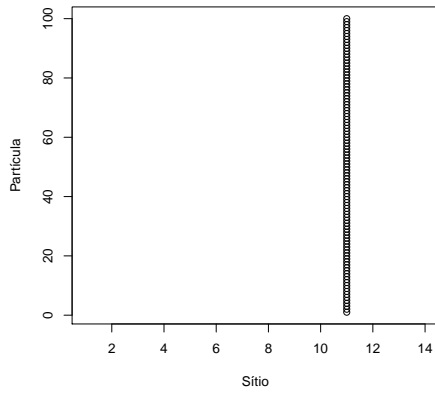
Figure 6 shows where each particle is adsorbed, considering an initial configuration with a random number of particles in the first 12 sites.



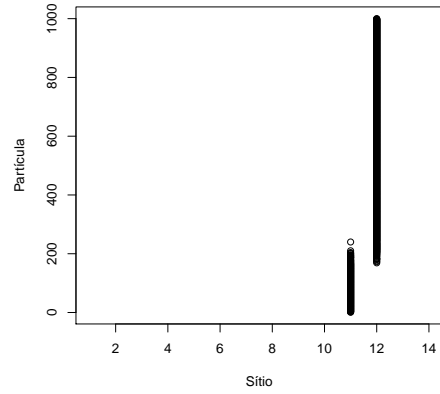
(a)



(b)



(c)



(d)

Figure 6: Particle Adsorption when  $\lambda_i = \ln(i)$ , initial configuration  $x = (34, 51, 41, 64, 50, 16, 21, 6, 29, 9, 37, 67, 0, 0 \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

### 3.1.3 General case

Simulations were performed with different initial configurations to verify the adsorption process of the particles when the function  $i \mapsto \lambda_i$  has local maxima. The explosion is vertical and always occurs at a local maximum.

Figure 7 shows the site at which each particle was adsorbed considering



that each  $i$  site has probability of adsorption related to  $\lambda_i = |\sin(i)|$  and the initial configuration with one particle at site 1.

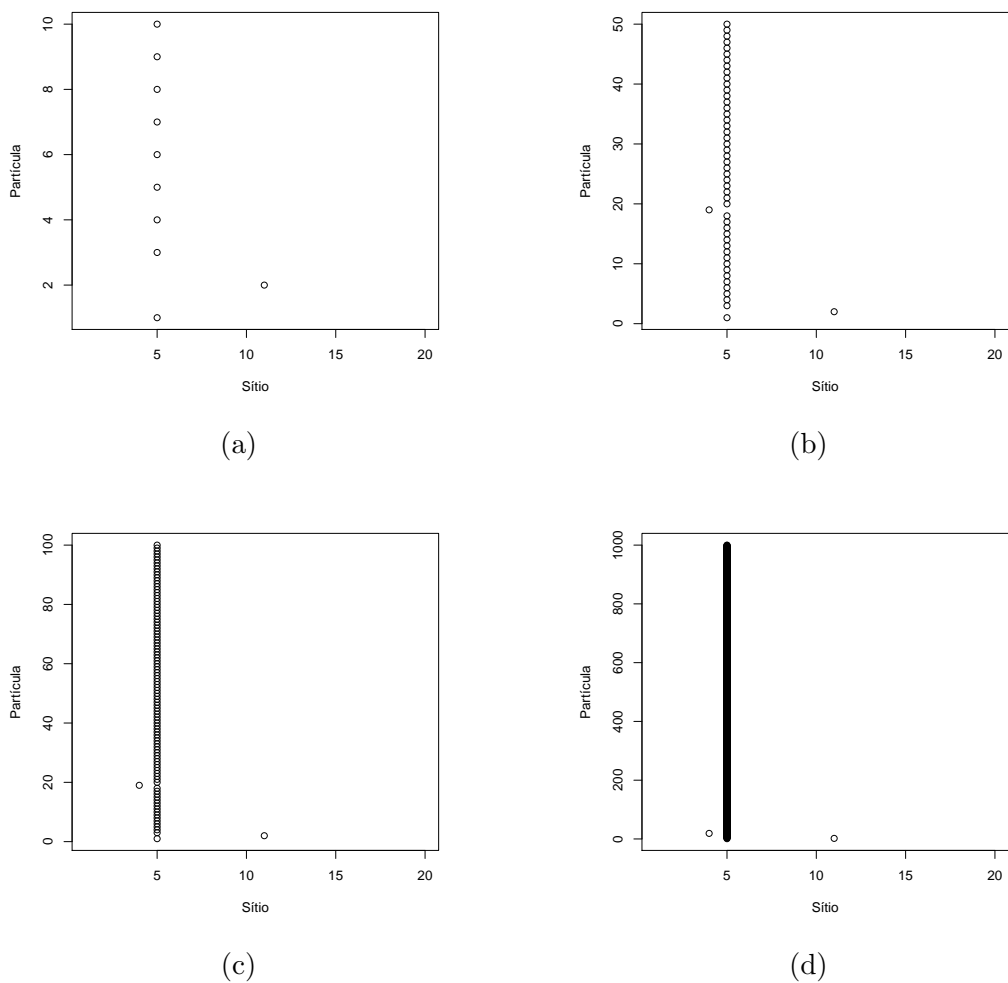
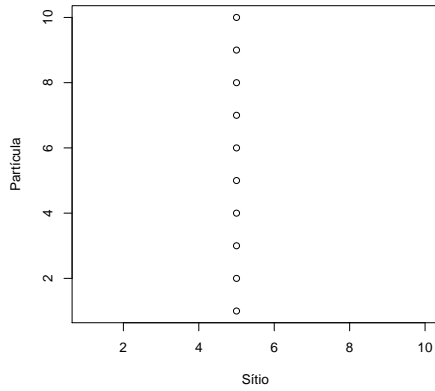


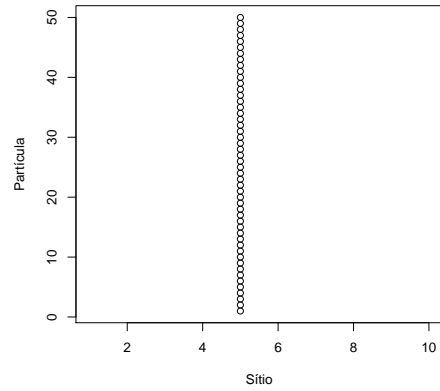
Figure 7: Particle Adsorption when  $\lambda_i = |\sin(i)|$ , initial configuration  $x = (1, 0, 0 \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

We can see in the figure 7 that the first particles are adsorbed by sites 1 and 2 and after some particles have been adsorbed the site 2 will adsorb all the next particles.

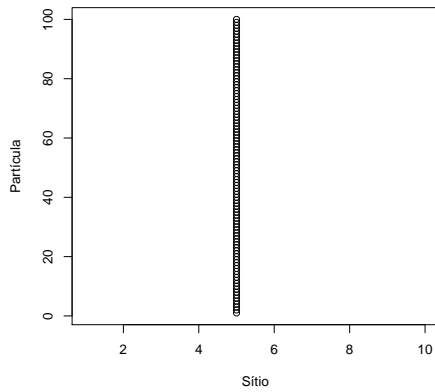
Figure 8 shows the site where the particles are adsorbed considering an initial configuration with a random amount of particles in the first 8 sites. For this particular case the initial setting was  $x = (2, 19, 21, 17, 29, 14, 25, 10, 0, \dots)$



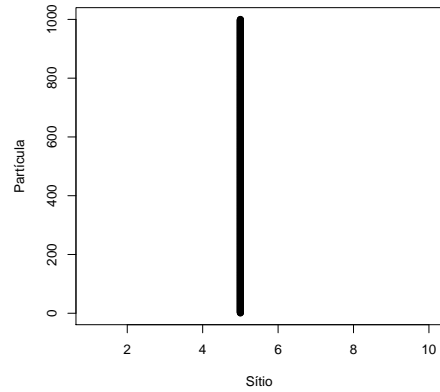
(a)



(b)



(c)



(d)

Figure 8: Particle Adsorption when  $\lambda_i = |\sin(i)|$ , initial configuration  $x = (2, 19, 21, 17, 29, 14, 25, 10, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  (d)  $n=1000$ .

Considering the initial configuration of Figure 8 we can see that the first 1000 particles are adsorbed by site 5 which is also a local maximum.

### 3.1.4 Constant Lambdas

Something interesting occurs here. As in [2], it can happen that all the particles are observed by two neighbor sites. The Figure 9 shows the sites where each particle is adsorbed.

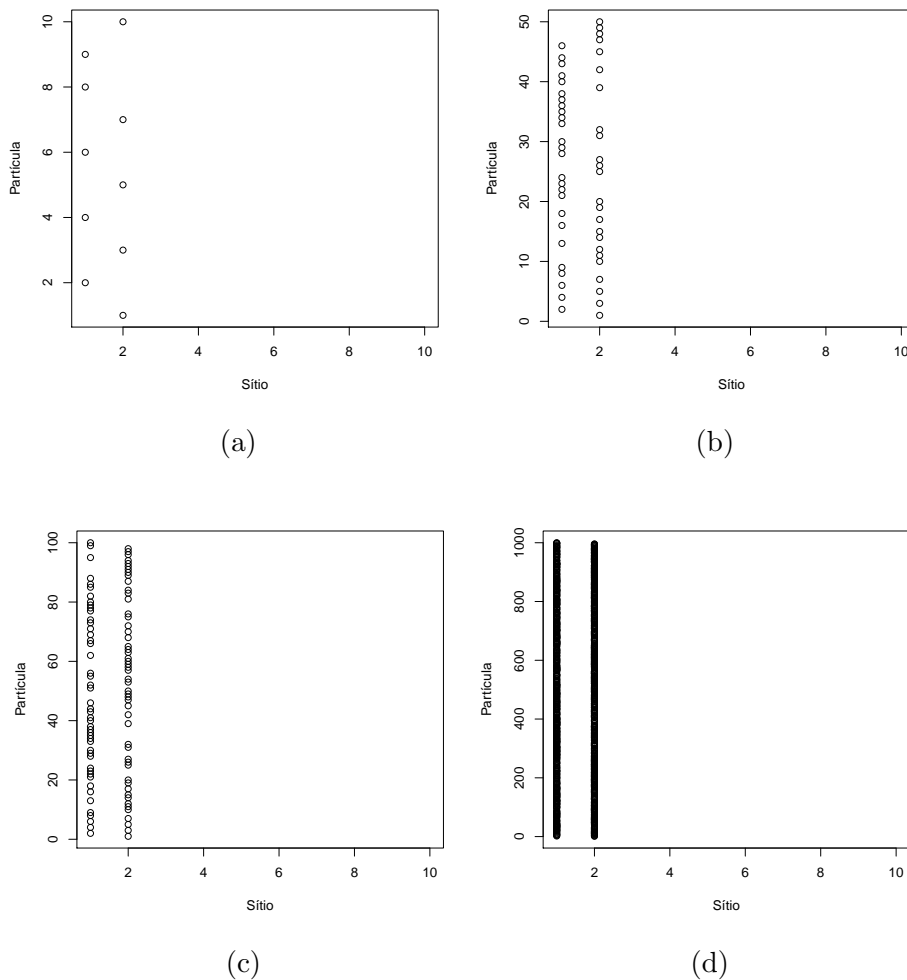
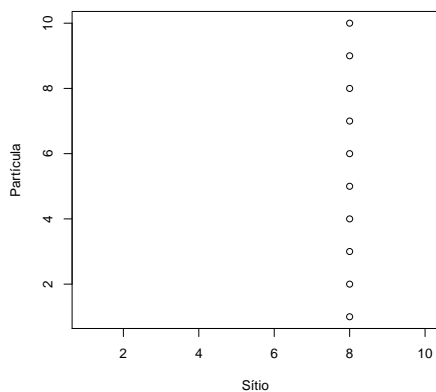


Figure 9: Particle Adsorption when  $\lambda_i = 1$ , initial configuration  $x = (1, 0, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  (d)  $n=1000$ .

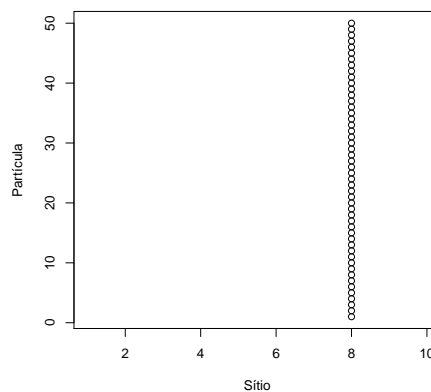
On the above figure all the particles are absorbed by sites 1 and 2, nothing

at 3. The probability that the next particle will be absorbed by 1 or by 2 are equal.

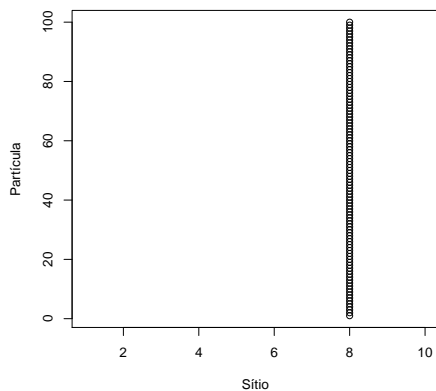
Figure 10 shows the sites where each particle is adsorbed considering the initial configuration where the first 12 sites have particles.



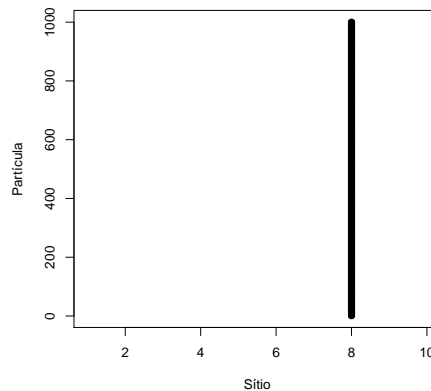
(a)



(b)



(c)



(d)

Figure 10: Particle adsorption when  $\lambda_i = 1$  and initial configuration  $x = (28, 4, 22, 25, 11, 21, 13, 23, 30, 3, 14, 21, 0, \dots)$ : (a)  $n=100$  (b)  $n=1000$ .

In Figure 10 we can see that all particles are adsorbed by site 8. We concluded from the simulations that two sites with similar adsorption prob-

abilities adsorb all particles from instant  $t$  and if a site is much more likely than other sites to adsorb the first particle, then it will adsorb all particles from instant  $t$ .

## 3.2 Polynomial Adsorption Rate

The next simulations illustrate the adsorption in the polynomial model. That is

$$\Gamma_i(x) := \begin{cases} (\lambda_i U_i(x))^\alpha, & U_i(x) \neq 0 \\ 0, & U_i(x) = 0, \end{cases}$$

considering different formulas for the parameters  $\lambda_i$  and  $\alpha$ .

The figure 11 shows how particles are adsorbed when adsorption occurs at a polynomial rate where  $\lambda = 10$  and  $\alpha > 1$ .

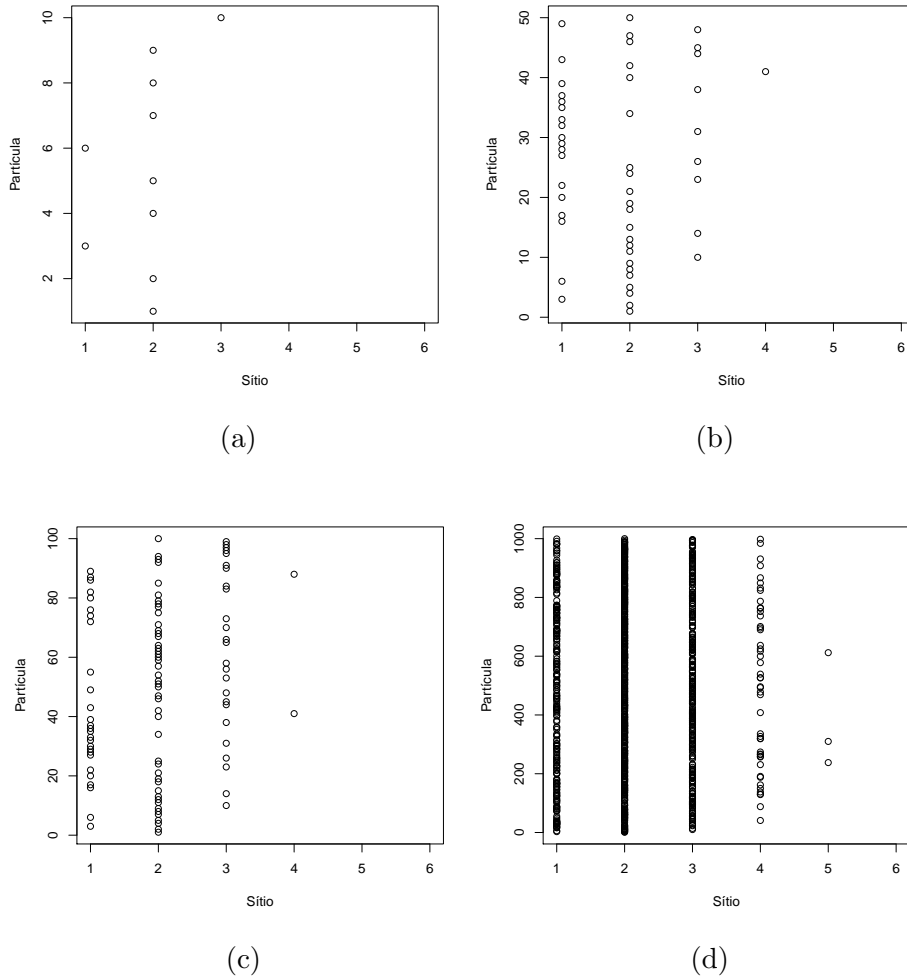
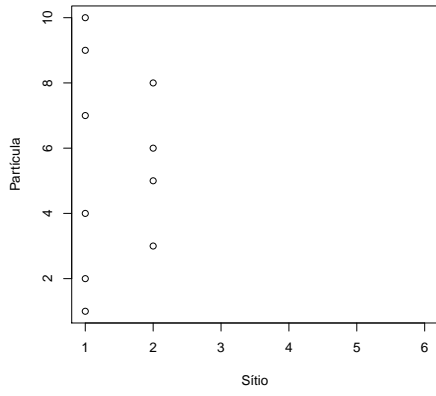


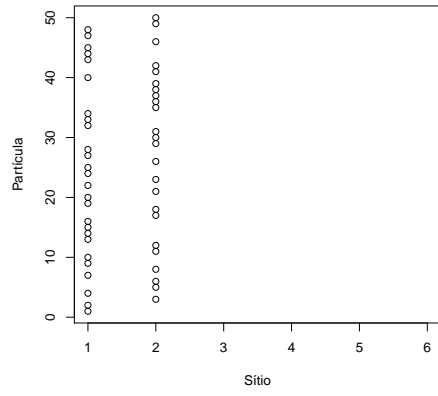
Figure 11: Particle Adsorption when  $\lambda_i = 10$ ,  $\alpha = 2$ , initial configuration  $x = (1, 0, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

Considering the initial configuration of Figure 11 we can see that the first 1000 particles are adsorbed by sites 1,2,3 and 4, due to the initial configuration. According to the theorem 1.2 explosion occurs so in this case the explosion occurs at the first sites.

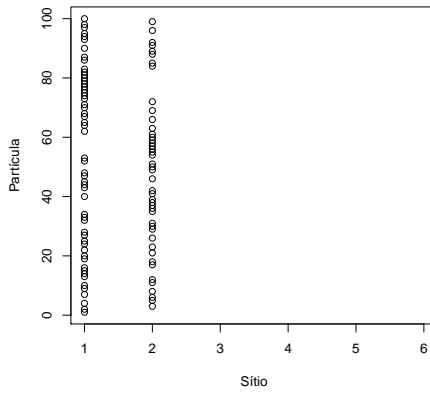
The figure 12 shows how particles are adsorbed when adsorption occurs at a polynomial rate where  $\lambda = 20$  and  $\alpha > 1$ .



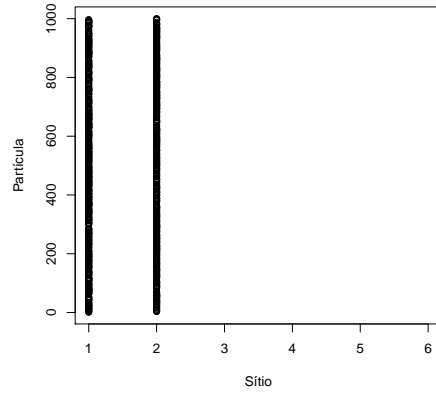
(a)



(b)



(c)



(d)

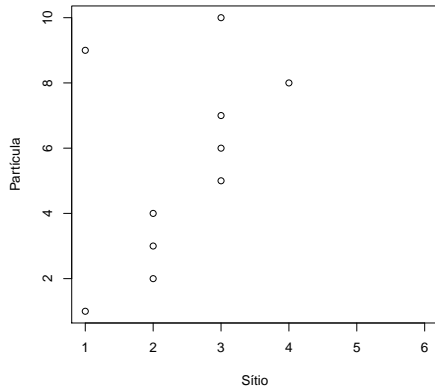
Figure 12: Particle Adsorption when  $\lambda_i = 20$ ,  $\alpha = 2$ , initial configuration  $x = (1, 0, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

We can see in the figure 12 that the first particles are adsorbed by sites 1 and 2.

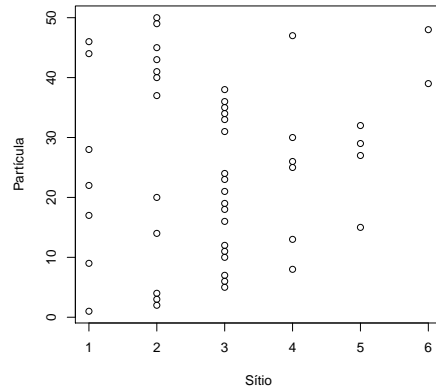
The next figure show the adsorption in the polynomial model with increasing lambdas and  $\alpha > 1$ .



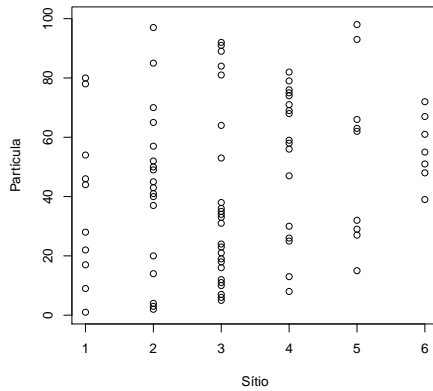




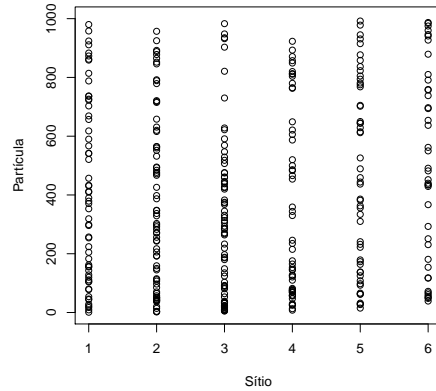
(a)



(b)



(c)



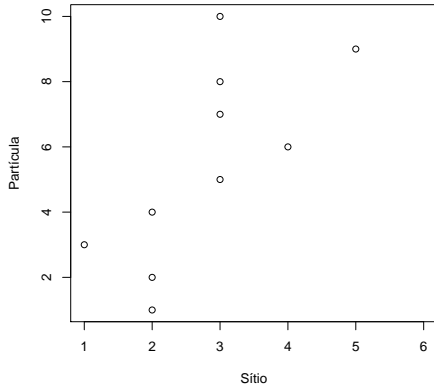
(d)

Figure 14: Particle Adsorption when  $\lambda_i = 10$ ,  $\alpha = 0, 2$ , initial configuration  $x = (1, 0, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

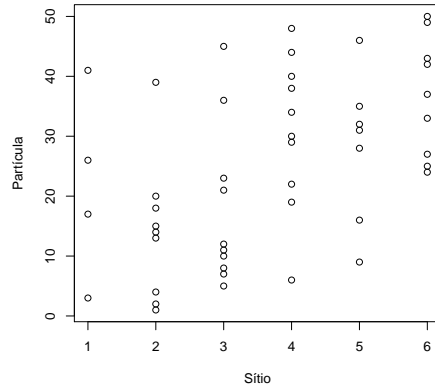
Considering the initial configuration of Figure 11 we can see that the first 1000 particles are adsorbed by sites 1,2,3,4,5 and 6, due to the initial configuration. According to the theorem 1.2 no explosion occurs if lambdas is limited and  $\alpha \leq 1$ .

The Figure 15 show the adsorption in the polynomial model with  $\alpha \leq 1$ , but  $\lambda_i$  is increasing at a polynomial rate with a degree greater than or equal

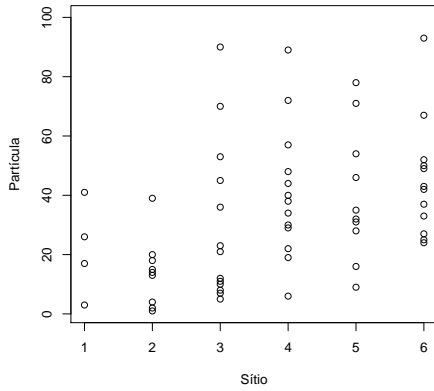
to  $\frac{1}{\alpha}$



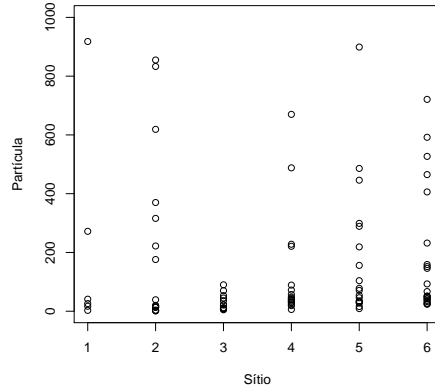
(a)



(b)



(c)



(d)

Figure 15: Particle Adsorption when  $\lambda_i = i^10$ ,  $\alpha = 0, 2$ , initial configuration  $x = (1, 0, 0, \dots)$ , and the transition moments (a)  $n=10$  (b)  $n=50$  (c)  $n=100$  e (d)  $n=1000$ .

We can see in the Figure 15 that horizontal explosion occurs.

## Acknowledgments

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