
Estimates for n -widths of sets of smooth functions on complex spheres

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Abstract

In this work we investigate n -widths of multiplier operators Λ_* and Λ , defined for functions on the complex sphere Ω_d of \mathbb{C}^d , associated with sequences of multipliers of the type $\{\lambda_{m,n}^*\}_{m,n \in \mathbb{N}}$, $\lambda_{m,n}^* = \lambda(m+n)$ and $\{\lambda_{m,n}\}_{m,n \in \mathbb{N}}$, $\lambda_{m,n} = \lambda(\max\{m,n\})$, respectively, for a bounded function λ defined on $[0, \infty)$. If the operators Λ_* and Λ are bounded from $L^p(\Omega_d)$ into $L^q(\Omega_d)$, $1 \leq p, q \leq \infty$, and U_p is the closed unit ball of $L^p(\Omega_d)$, we study lower and upper estimates for the n -widths of Kolmogorov, linear, of Gelfand and of Bernstein, of the sets Λ_*U_p and ΛU_p in $L^q(\Omega_d)$. As application we obtain, in particular, estimates for the Kolmogorov n -width of classes of Sobolev, of finitely differentiable, infinitely differentiable and analytic functions on the complex sphere, in $L^q(\Omega_d)$, which are order sharp in various important situations.

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1 Introduction

In [2, 3, 10, 11, 12, 13, 14, 15, 16, 17, 18] techniques were developed to obtain estimates for n -widths of multiplier operators defined for functions on torus and on two-points homogeneous spaces \mathbb{M}^d : \mathbb{S}^d , $\mathbb{P}^d(\mathbb{R})$, $\mathbb{P}^d(\mathbb{C})$, $\mathbb{P}^d(\mathbb{H})$, $\mathbb{P}^{16}(\text{Cay})$. In this work we continue the development of methods of estimating n -widths of multiplier operators.

The studies on asymptotic estimates for Kolmogorov n -widths of Sobolev classes on the circle were performed by several important mathematicians such as Rudin, Stechkin, Gluskin, Ismagilov, Maiorov, Makovoz and Scholz in [6, 20, 21, 29, 31, 32], they were initiated by Kolmogorov [9] in 1936 and completed by Kashin [7, 8] in 1977. Several techniques were applied in these different cases, among them we highlight a technique of discretization due to Maiorov and the Borsuk theorem. Observing the historical evolution of the study of n -widths, it is possible to note that it has been an usual practice to use different techniques in proofs of lower and upper bounds and in estimates for classes of finitely and infinitely differentiable functions (see, e.g. [28]). Among the mathematicians who worked with n -widths of sets of analytic functions, we highlight Babenko, Fisher, Micchelli, Taikov and Tichomirov, who obtained estimates for sets of analytic functions in Hardy spaces in [1, 5, 33, 34]. Among the different tools used, we can emphasize the properties of the class of Blaschke products of degree m or less. In 1980, Melkman in [22] obtained estimates for a set of entire functions in $C[-T, T]$, using other technique. One of the goals of this work, is to give an unified treatment in the study of n -widths of sets of functions defined on complex spheres, determined by multiplier operators.

Let $\lambda : [0, \infty) \rightarrow \mathbb{C}$ be a bounded function and consider the sequences $\Lambda_* = (\lambda_{m,n}^*)_{m,n \in \mathbb{N}}$, $\lambda_{m,n}^* = \lambda(m+n)$ and $\Lambda = (\lambda_{m,n})_{m,n \in \mathbb{N}}$, $\lambda_{m,n} = \lambda(\max\{m,n\})$. Let Ω_d be the unit sphere of \mathbb{C}^d and for $m, n \in \mathbb{N}$, let $\mathcal{H}_{m,n}$ be the subspace of $L^2(\Omega_d)$ of all complex spherical harmonics of bi-degree (m, n) . Given $f \in L^2(\Omega_d)$, for each $m, n \in \mathbb{N}$, there is an unique $Y^{(m,n)} \in \mathcal{H}_{m,n}$ such that $f = \sum_{m,n} Y^{(m,n)}$ with convergence in the norm of $L^2(\Omega_d)$ (see Section 2 for definitions and properties). For $f \in L^2(\Omega_d)$ we define $\Lambda_*f = \sum_{m,n} \lambda_{m,n}^* Y^{(m,n)}$ and $\Lambda f = \sum_{m,n} \lambda_{m,n} Y^{(m,n)}$ where the convergence occurs in the norm of $L^2(\Omega_d)$. Let U_p be the closed unit ball of $L^p(\Omega_d)$, $1 \leq p \leq \infty$. In this paper we study asymptotic estimates for the n -widths of Kolmogorov, linear, of Gelfand and of Bernstein, of the sets Λ_*U_p and ΛU_p in $L^q(\Omega_d)$, for $1 \leq p, q \leq \infty$.

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Consider two Banach spaces X and Y . The norm of X will be denoted by $\|\cdot\|$ or $\|\cdot\|_X$ and the closed unit ball $\{x \in X : \|x\| \leq 1\}$ by B_X . Let A be a compact, convex and centrally symmetric subset of X . The Kolmogorov n -width of A in X is defined by

$$d_n(A, X) := \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|_X,$$

where X_n runs over all subspaces of X of dimension n . The linear n -width $\delta_n(A, X)$ is defined by

$$\delta_n(A, X) := \inf_{P_n} \sup_{x \in A} \|x - P_n(x)\|_X,$$

where P_n runs over all bounded linear operators $P_n : X \rightarrow X$ whose range has dimension n . The Gelfand n -width of A in X is defined by

$$d^n(A, X) = \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|_X,$$

where L^n runs over all subspaces of X of codimension n . The Bernstein n -width of A in X is defined by

$$b_n(A, X) = \sup_{X_{n+1}} \sup\{\lambda : \lambda B_X \cap X_{n+1} \subseteq A\},$$

where X_{n+1} is any $(n+1)$ -dimensional subspace of X . The following inequality is always valid:

$$b_n(A, X) \leq \min\{d_n(A, X), d^n(A, X)\}.$$

We define the n -widths of Kolmogorov, linear, Gelfand and Bernstein of an operator $T \in \mathcal{L}(X, Y)$, respectively, by

$$d_n(T) = d_n(T(B_X), Y), \quad \delta_n(T) = \delta_n(T(B_X), Y), \quad d^n(T) = d^n(T(B_X), Y), \quad b_n(T) = b_n(T(B_X), Y).$$

Consider $T \in \mathcal{L}(X, Y)$ and let T^* be its adjoint operator. If T is compact or Y is reflexive, then (see [28], p. 34)

$$d^n(T^*) = d_n(T). \tag{1}$$

In Section 2 we make a brief study about real and complex spherical harmonics. In Section 3 we study estimates for n -widths of sets of functions on the complex sphere Ω_d , associated with the multiplier operators of the type Λ_* . We prove that the n -widths of the sets $\Lambda_* U_p$ in $L^q(\Omega_d)$ can be obtained from n -widths of sets of functions associated with multiplier operators on the real sphere S^{2d-1} . As application we obtain estimates for the Kolmogorov n -width of Sobolev classes and of finitely differentiable, infinitely differentiable and analytic functions on the complex sphere.

In Section 4 and in the following sections we study n -widths of sets of functions on the complex sphere Ω_d associated with the multiplier operators of the type Λ . We note that in this case the results can not be transferred from results already obtained to the real sphere S^{2d-1} . In Section 4 we study estimates for Levy means of special norms on finite-dimensional Banach spaces which are associated with the multiplier sequences $(\lambda_{m,n})_{m,n \in \mathbb{N}}$.

In Section 5 we prove two theorems which provide us with lower and upper estimates for Kolmogorov n -widths of the sets ΛU_p in $L^q(\Omega_d)$, for general multiplier operators of the type Λ . The main tool used in these proofs are the results from Section 4.

In Section 6 we apply the results of Section 5 in the study of the Kolmogorov n -widths of the multiplier operators $\Lambda^{(1)}$ and $\Lambda^{(2)}$ associated with the functions $\lambda^{(1)}(t) = t^{-\gamma}(\ln t)^{-\xi}$ for $t > 1$, $\lambda^{(1)}(t) = 0$ for $0 \leq t \leq 1$, and $\lambda^{(2)}(t) = e^{-\gamma t^r}$ for $t \geq 0$, where $\gamma, \xi, r \in \mathbb{R}$, $\gamma, r > 0$ and $\xi \geq 0$, respectively. We have that the sets $\Lambda^{(1)} U_p$ and $\Lambda_*^{(1)} U_p$ are sets of finitely differentiable functions on Ω_d , and $\Lambda^{(2)} U_p$ and $\Lambda_*^{(2)} U_p$ are sets of infinitely differentiable functions if $0 < r < 1$ and of analytic functions if $r \geq 1$ (see [24, 25]). We prove that estimates for the Kolmogorov n -widths of the sets $\Lambda^{(1)} U_p$ and $\Lambda^{(2)} U_p$ in $L^q(\Omega_d)$ are sharp when $2 \leq p, q < \infty$.

We note that the order of decrease of the n -widths $d_n(\Lambda^{(1)} U_p, L^q)$ and $d_n(\Lambda_*^{(1)} U_p, L^q)$ is the same, but it is different for the n -widths $d_n(\Lambda^{(2)} U_p, L^q)$ and $d_n(\Lambda_*^{(2)} U_p, L^q)$.

In this work there are several universal constants which enter into the estimates. These positive constants are mostly denoted by the letters C, C_1, C_2, \dots . We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper. For ease of notation we will write $a_n \gg b_n$ for two sequences if $a_n \geq C b_n$ for $n \in \mathbb{N}$, $a_n \ll b_n$ if $a_n \leq C b_n$ for $n \in \mathbb{N}$, and $a_n \asymp b_n$ if $a_n \ll b_n$ and $a_n \gg b_n$. Also, we will write $(a)_+ = a$ if $a > 0$ and $(a)_+ = 0$ if $a \leq 0$.

2 Analysis on the spheres

We write $\mathbb{N} = \{0, 1, 2, \dots\}$. For $x, y \in \mathbb{R}^d$ we denote by $\langle x, y \rangle$ the usual inner product $\langle x, y \rangle = x_1 y_1 + \dots + x_d y_d$. Let

$$S^{d-1} = \{x \in \mathbb{R}^d : \langle x, x \rangle = 1\}$$

be the unit sphere in \mathbb{R}^d . Let τ_d be the Lebesgue measure on S^{d-1} , consider the usual Banach spaces $L^p(S^{d-1})$, $1 \leq p \leq \infty$ of τ_d -measurable complex functions on S^{d-1} and let $\widetilde{U}_p = \widetilde{U}_p(S^{d-1}) = \{\phi \in L^p(S^{d-1}) : \|\phi\|_p \leq 1\}$. We denote by \mathcal{H}_k the subspace of $L^2(S^{d-1})$ of all spherical harmonics of degree k , that is, the space of the restrictions to S^{d-1} of all polynomials $P(x_1, \dots, x_d)$ which are homogeneous of degree k and harmonic. Given $f \in L^2(S^{d-1})$, for each $k \in \mathbb{N}$, there exists a unique $Y^{(k)} \in \mathcal{H}_k$ such that

$$f = \sum_{k=0}^{\infty} Y^{(k)},$$

where the series converges to f in $L^2(S^{d-1})$. Let $\pi_k : L^2(S^{d-1}) \rightarrow \mathcal{H}_k$ be the harmonic projection operator, $\pi_k(f) = Y^{(k)}$. For each $x \in S^{d-1}$ and $k \in \mathbb{N}$, there exists a unique function $Z_x^{(k)} \in \mathcal{H}_k$, called the zonal harmonic of degree k with pole x , such that

$$\pi_k f(x) = \int_{S^{d-1}} f(y) Z_x^{(k)}(y) d\tau_d(y), \quad f \in L^2(S^{d-1}).$$

We have that $\mathcal{H}_k \perp \mathcal{H}_l$, for $k \neq l$, with respect to the usual inner product in $L^2(S^{d-1})$ and for $k \geq 2$,

$$d_k = \dim \mathcal{H}_k = \binom{d+k-1}{k} - \binom{d+k-3}{k-2}.$$

Let $\{Y_j\}_{j=1}^{d_k}$ be an orthonormal basis of \mathcal{H}_k . The following addition formula is known:

$$\sum_{j=1}^{d_k} \overline{Y_j^{(k)}(x)} Y_j^{(k)}(y) = Z_x^{(k)}(y) = \frac{d+2k-2}{\widetilde{\omega}_d(d-2)} P_k^{(d-2)/2}(\langle x, y \rangle), \quad x, y \in S^{d-1},$$

where $\widetilde{\omega}_d$ denotes the surface area of S^{d-1} and $P_k^{(d-2)/2}$ denotes the Gegenbauer polynomial of degree k . Let us denote by $\widetilde{Z}_d^{(k)}$ the function

$$\widetilde{Z}^{(k)}(t) = \widetilde{Z}_d^{(k)}(t) = \frac{d+2k-2}{\widetilde{\omega}_d(d-2)} P_k^{(d-2)/2}(t), \quad t \in [-1, 1].$$

Let $\overline{\Upsilon} = \overline{\Upsilon}_d : \mathbb{C}^d \rightarrow \mathbb{R}^{2d}$ be given by

$$\overline{\Upsilon}(z) = (x_1, y_1, \dots, x_d, y_d), \quad z_k = x_k + iy_k, \quad k = 1, \dots, d.$$

For $z, w \in \mathbb{C}^d$, we denote by $\langle z, w \rangle$ the usual inner product $\langle z, w \rangle = z_1 \overline{w}_1 + \dots + z_d \overline{w}_d$. Let

$$\Omega_d = \{z \in \mathbb{C}^d : \langle z, z \rangle = 1\}$$

be the unit sphere in \mathbb{C}^d . We denote by Υ or Υ_d , the restriction of $\overline{\Upsilon}$ to Ω_d . Since $\Upsilon(\Omega_d) = S^{2d-1}$ we can define the measure σ_d on the Borel σ -algebra $\mathcal{B}(\Omega_d)$ of Ω_d by $\sigma_d(E) = \tau_{2d}(\Upsilon(E))$, $E \in \mathcal{B}(\Omega_d)$. A measurable function $f : \Omega_d \rightarrow \mathbb{C}$ is σ_d -integrable if

$$\int_{\Omega_d} f d\sigma_d = \int_{S^{2d-1}} (f \circ \Upsilon^{-1}) d\tau_{2d}$$

is well defined. We denote by $L^p = L^p(\Omega_d)$, $1 \leq p \leq \infty$, the vector space of all measurable functions defined on Ω_d and with values in \mathbb{C} , satisfying

$$\|f\|_p := \left(\int_{\Omega_d} |f(z)|^p d\sigma_d(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \operatorname{ess\,sup}_{z \in \Omega_d} |f(z)| < \infty.$$

The space $L^2(\Omega_d)$ is a Hilbert space with the inner product given by

$$(f, g) = \int_{\Omega_d} f(z) \overline{g(z)} d\sigma_d(z), \quad f, g \in L^2(\Omega_d).$$

Let $U_p = U_p(\Omega_d) = \{\phi \in L^p(\Omega_d) : \|\phi\|_p \leq 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and let ν_α be the Borel measure on \mathbb{D} given by

$$\nu_\alpha(E) = \int_E (1 - |z|^2)^\alpha d\mu_1(z), \quad E \in \mathcal{B}(\mathbb{D}).$$

where $\alpha > -1$ and $\mu_1 \circ \bar{\Upsilon}_1^{-1}$ is the Lebesgue measure on \mathbb{R}^2 . For $f \in L^1(\Omega_d)$ and $K \in L^1(\mathbb{D}, \nu_\alpha)$, the convolution function $f * K$ is defined by

$$(f * K)(w) = \int_{\Omega_d} f(z) \overline{K(\langle z, w \rangle)} d\sigma_d(z), \quad w \in \Omega_d.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $z = (z_1, \dots, z_d) \in \mathbb{C}^d$. We write $|\alpha| = \alpha_1 + \dots + \alpha_d$, $z^\alpha = z_1^{\alpha_1} \dots z_d^{\alpha_d}$ and $\bar{z} = (\bar{z}_1, \dots, \bar{z}_d)$. We denote by $\mathbb{P}(\mathbb{C}^d)$ the vector space of all polynomials in the independent variables z and \bar{z} . If $p \in \mathbb{P}(\mathbb{C}^d)$ then there are $m, n \in \mathbb{N}$ such that

$$p(z) := p(z, \bar{z}) = \sum_{|\alpha| \leq m, |\beta| \leq n} p_{\alpha, \beta} z^\alpha \bar{z}^\beta, \quad z \in \mathbb{C}^d, \quad p_{\alpha, \beta} \in \mathbb{C}, \quad \alpha, \beta \in \mathbb{N}^d.$$

The vector subspace of $\mathbb{P}(\mathbb{C}^d)$ formed by the polynomials that are homogeneous of degree m in the variable z and of degree n in the variable \bar{z} , will be denoted by $\mathbb{P}_{m, n}(\mathbb{C}^d)$. The subspace of $\mathbb{P}_{m, n}(\mathbb{C}^d)$ of all polynomials that are in the kernel of the complex laplacian

$$\Delta_{(2d)} := 4 \sum_{j=1}^d \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

will be denoted by $\mathbb{H}_{m, n}(\mathbb{C}^d)$.

The space of the restrictions of the polynomials in $\mathbb{P}_{m, n}(\mathbb{C}^d)$ to the complex sphere Ω_d will be denoted by $\mathcal{P}_{m, n} = \mathcal{P}_{m, n}(\Omega_d)$ and of the restrictions of the polynomials in $\mathbb{H}_{m, n}(\mathbb{C}^d)$ to Ω_d by $\mathcal{H}_{m, n} = \mathcal{H}_{m, n}(\Omega_d)$. We have that $\mathcal{H}_{m, n} \perp \mathcal{H}_{r, s}$, for $(m, n) \neq (r, s)$, with respect to the usual inner product in $L^2(\Omega_d)$. The elements of $\mathcal{H}_{m, n}$ are called complex spherical harmonics of degree m in z and degree n in \bar{z} . Let $m, n \in \mathbb{N}$ and $m + n = k$. Since $\mathcal{H}_{m, n} \subset \{Y \circ \Upsilon : Y \in \mathcal{H}_k\}$ and there is an orthonormal basis of \mathcal{H}_k consisting of only real functions, then also there is an orthonormal basis of $\mathcal{H}_{m, n}$ consisting of only real functions. The dimension of $\mathcal{H}_{m, n}$ is given by

$$d_{m, n} = \dim \mathcal{H}_{m, n} = \binom{m + d - 1}{m} \binom{n + d - 1}{n} - \binom{m + d - 2}{m - 1} \binom{n + d - 2}{n - 1}.$$

We observe that there is an absolute constant $C > 0$ such that

$$\frac{(m + n)(mn)^{d-2}}{(d-1)!(d-2)!} \leq d_{m, n} \leq \frac{(m + n)(mn)^{d-2}}{(d-1)!(d-2)!} + C(m + n)m^{d-2}n^{d-3}, \quad (m, n) \in \mathbb{N}^2. \quad (2)$$

It is known (see [30], Chapter 12) that if $f \in L^2(\Omega_d)$ then there exists a unique decomposition

$$f = \sum_{m, n \in \mathbb{N}} Y^{(m, n)},$$

where $Y^{(m, n)} \in \mathcal{H}_{m, n}$ for $m, n \in \mathbb{N}$ and the above series converges to f in $L^2(\Omega_d)$. The projection operator $\pi_{m, n}$ from $L^2(\Omega_d)$ on $\mathcal{H}_{m, n}$ is defined by $\pi_{m, n} f = Y^{(m, n)}$. For every $w \in \Omega_d$ there exists a unique $Z_w^{(m, n)} \in \mathcal{H}_{m, n}$ that satisfies $(\pi_{m, n} f)(w) = (f, Z_w^{(m, n)})$, $f \in L^2(\Omega_d)$. The function $Z_w^{(m, n)}$ is called the zonal harmonic of degree (m, n) with pole w .

Next, we introduce some classes of disk polynomials. The disk polynomial of degree m in z and degree n in \bar{z} associated to the integer $d - 2$ is the polynomial $R_{m, n}^{d-2}$ given by

$$R_{m, n}^{d-2}(z) = \begin{cases} \frac{1}{P_n^{(d-2, m-n)}(1)} z^{m-n} P_n^{(d-2, m-n)}(2|z|^2 - 1), & \text{if } m \geq n, \\ \frac{1}{P_m^{(d-2, n-m)}(1)} \bar{z}^{n-m} P_m^{(d-2, n-m)}(2|z|^2 - 1), & \text{if } m < n, \end{cases}$$

where $P_k^{(d-2, m-n)}$ is the usual Jacobi polynomial of degree k associated to the pair of numbers $(d - 2, m - n)$. Let $\{Y_j\}_{j=1}^{d_{m, n}}$ be an orthonormal basis of $\mathcal{H}_{m, n}$. The following addition formula is known:

$$Z_w^{(m, n)}(z) = \sum_{k=1}^{d_{m, n}} \overline{Y_k(w)} Y_k(z) = \frac{d_{m, n}}{\omega_d} R_{m, n}^{d-2}(\langle z, w \rangle), \quad w, z \in \Omega_d,$$

where $\omega_d = \sigma_d(\Omega_d) = \tau_{2d}(S^{2d-1})$. In particular

$$Z_w^{(m, n)}(w) = \sum_{k=1}^{d_{m, n}} |Y_k(z)|^2 = \frac{d_{m, n}}{\omega_d}.$$

Let us denote by $\tilde{Z}^{(m,n)}$ the function

$$\tilde{Z}^{(m,n)}(t) = \frac{d_{m,n}}{\omega_d} R_{m,n}^{d-2}(t), \quad t \in \mathbb{D},$$

and let $e = e_d = (0, 0, \dots, 0, 1)$, $f \in L^2(\Omega_d)$. The convolution product $f * Z_e^{(m,n)}$ is given by

$$f * Z_e^{(m,n)}(w) = f * \tilde{Z}^{(m,n)}(w) = \int_{\Omega_d} f(z) \overline{\tilde{Z}_w^{(m,n)}}(z) d\sigma_d(z) = (\pi_{m,n} f)(w).$$

It follows a relationship between the real zonal harmonics on S^{2d-1} and the complex zonal harmonics on Ω_d , as a consequence, we obtain an expression relating polynomials on the disk and Gegenbauer polynomials.

Proposition 2.1. *For any $k \in \mathbb{N}$ and $x \in S^{2d-1}$, we have that*

$$Z_x^{(k)} \circ \Upsilon = \sum_{m+n=k} Z_w^{(m,n)}, \quad w = \Upsilon^{-1}(x).$$

Corollary 2.2. *Given $z, w \in \Omega_d$,*

$$\frac{2d+2k-2}{\omega_d(2d-2)} P_k^{(2d-2)/2}(\langle \Upsilon(z), \Upsilon(w) \rangle) = \sum_{m+n=k} \tilde{Z}^{(m,n)}(\langle z, w \rangle).$$

We denote $\hat{Z}_d^{(k)} = \sum_{m+n=k} \tilde{Z}^{(m,n)}$ and we have from the above corollary that $\tilde{Z}_{2d}^{(k)}(\langle \Upsilon(z), \Upsilon(w) \rangle) = \hat{Z}_d^{(k)}(\langle z, w \rangle)$, $z, w \in \Omega_d$. For a detailed study about real and complex spherical harmonics see [4, 23, 30].

3 Multiplier operators of type Λ_*

We denote by \mathcal{H} the vector space generated by the union $\bigcup_{m,n} \mathcal{H}_{m,n}$. The elements of \mathcal{H} are finite linear combinations of elements of the subspaces $\mathcal{H}_{m,n}$ and \mathcal{H} is dense in $L^p(\Omega_d)$ for $1 \leq p < \infty$. Also we denote by $\tilde{\mathcal{H}}$ the vector space generated by the union $\bigcup_k \mathcal{H}_k$ which is dense in $L^p(S^{2d-1})$ for $1 \leq p < \infty$.

Let $\Lambda = \{\lambda_{m,n}\}_{m,n \in \mathbb{N}}$ be a sequence of complex numbers and $1 \leq p, q \leq \infty$. If for any $\varphi \in L^p(\Omega_d)$ there is a function $f = \Lambda\varphi \in L^q(\Omega_d)$ with formal expansion in spherical harmonic

$$f \sim \sum_{m,n \in \mathbb{N}} \lambda_{m,n} \varphi * \tilde{Z}^{(m,n)},$$

such that $\|\Lambda\|_{p,q} = \sup\{\|\Lambda\varphi\|_q : \varphi \in U_p\} < \infty$, we say that the multiplier operator Λ is bounded from L^p into L^q with norm $\|\Lambda\|_{p,q}$. Now let $\tilde{\Lambda} = \{\lambda_k\}_{k \in \mathbb{N}}$ be a sequence of complex numbers and $1 \leq p, q \leq \infty$. If for any $\varphi \in L^p(S^{2d-1})$ there is a function $f = \tilde{\Lambda}\varphi \in L^q(S^{2d-1})$ with formal expansion in spherical harmonic

$$f \sim \sum_{k \in \mathbb{N}} \lambda_k \varphi * \tilde{Z}^{(k)},$$

such that $\|\tilde{\Lambda}\|_{p,q} = \sup\{\|\tilde{\Lambda}\varphi\|_q : \varphi \in \tilde{U}_p\} < \infty$, we say that the multiplier operator $\tilde{\Lambda}$ is bounded from L^p into L^q with norm $\|\tilde{\Lambda}\|_{p,q}$.

Lemma 3.1. *Let $\lambda : [0, \infty) \rightarrow \mathbb{C}$ be a bounded function, $\lambda_{m,n}^* = \lambda(m+n)$ and $\lambda_k = \lambda(k)$ for $m, n, k \in \mathbb{N}$. Consider the multiplier operators associated with the sequences $\Lambda_* = \{\lambda_{m,n}^*\}_{m,n \in \mathbb{N}}$ and $\tilde{\Lambda} = \{\lambda_k\}_{k \in \mathbb{N}}$ defined on \mathcal{H} and on $\tilde{\mathcal{H}}$ respectively. Let $1 \leq p, q \leq \infty$ and suppose that Λ_* is bounded from L^p to L^q . Then*

$$\Lambda_*\varphi = \left(\tilde{\Lambda}(\varphi \circ \Upsilon^{-1}) \right) \circ \Upsilon, \quad \varphi \in L^p(\Omega_d) \quad (3)$$

and

$$\Lambda_*U_p = (\tilde{\Lambda}\tilde{U}_p) \circ \Upsilon = \{\varphi \circ \Upsilon : \varphi \in \tilde{\Lambda}\tilde{U}_p\}. \quad (4)$$

Proof. For $\varphi \in L^2(\Omega_d)$ we get $\|\Lambda_*\varphi\|_2 \leq (\sup_{m,n \in \mathbb{N}} |\lambda_{m,n}^*|) \|\varphi\|_2$, then Λ_* is a bounded operator on $L^2(\Omega_d)$. Thus

$$\begin{aligned} \Lambda_*\varphi &= \sum_{k=0}^{\infty} \sum_{m+n=k} \lambda(m+n) \varphi * \tilde{Z}^{(m,n)} \\ &= \sum_{k=0}^{\infty} \lambda(k) \varphi * \left(\sum_{m+n=k} \tilde{Z}^{(m,n)} \right) \\ &= \sum_{k=0}^{\infty} \lambda(k) \varphi * \hat{Z}_d^{(k)}, \end{aligned}$$

where the above series converges in L^2 . On the other hand, from Corollary 2.2, for $w \in \Omega_d$ we have

$$\begin{aligned}\varphi * \hat{Z}_d^{(k)}(w) &= \int_{\Omega_d} \varphi(z) \overline{\hat{Z}_d^{(k)}(\langle z, w \rangle)} d\sigma_d(z) \\ &= \int_{\Omega_d} \varphi(z) \overline{Z_{\Upsilon(z)}^{(k)}(\Upsilon(w))} d\sigma_d(z) \\ &= \int_{S^{2d-1}} Z_{\Upsilon(w)}^{(k)}(x) (\varphi \circ \Upsilon^{-1})(x) d\tau_{2d}(x) \\ &= (\tilde{Z}_{2d}^{(k)} * (\varphi \circ \Upsilon^{-1}))(\Upsilon(w)).\end{aligned}$$

Thus

$$\Lambda_* \varphi = \left(\sum_{k=0}^{\infty} \lambda(k) \tilde{Z}_{2d}^{(k)} * (\varphi \circ \Upsilon^{-1}) \right) \circ \Upsilon = (\tilde{\Lambda}(\varphi \circ \Upsilon^{-1})) \circ \Upsilon, \quad \varphi \in L^2(\Omega_d).$$

For $2 \leq p \leq \infty$, we have that $L^\infty(\Omega_d) \subseteq L^p(\Omega_d) \subseteq L^2(\Omega_d)$, then $\Lambda_* \varphi = (\tilde{\Lambda}(\varphi \circ \Upsilon^{-1})) \circ \Upsilon$ for all $\varphi \in L^p(\Omega_d)$ with $2 \leq p \leq \infty$, in particular for $\varphi \in \mathcal{H}$. It follows for $1 \leq p < \infty$ that

$$\begin{aligned}\|\Lambda_*\|_{p,q} &= \sup\{\|\Lambda_*\varphi\|_q : \varphi \in U_p \cap \mathcal{H}\} \\ &= \sup\{\|(\tilde{\Lambda}(\varphi \circ \Upsilon^{-1})) \circ \Upsilon\|_q : \varphi \circ \Upsilon^{-1} \in \tilde{U}_p \cap \tilde{\mathcal{H}}\} \\ &= \sup\{\|\tilde{\Lambda}h\|_q : h \in \tilde{U}_p \cap \tilde{\mathcal{H}}\} \\ &= \|\tilde{\Lambda}\|_{p,q}.\end{aligned}$$

Thus $\tilde{\Lambda}$ is a bounded operator from $L^p(S^{2d-1})$ to $L^q(S^{2d-1})$.

Now consider $1 \leq p \leq 2$. Given $\varphi \in L^p(\Omega_d)$ and for each $n \in \mathbb{N}$, let $\varphi_n \in \mathcal{H}$ be such that $\varphi_n \rightarrow \varphi$ in $L^p(\Omega_d)$. Then $\Lambda_* \varphi_n \rightarrow \Lambda_* \varphi$ in $L^q(\Omega_d)$, $\Lambda_* \varphi_n = (\tilde{\Lambda}(\varphi_n \circ \Upsilon^{-1})) \circ \Upsilon$ and $\tilde{\Lambda}(\varphi_n \circ \Upsilon^{-1}) \rightarrow \tilde{\Lambda}(\varphi \circ \Upsilon^{-1})$ in $L^q(S^{2d-1})$. Therefore $\tilde{\Lambda}(\varphi_n \circ \Upsilon^{-1}) \circ \Upsilon \rightarrow \tilde{\Lambda}(\varphi \circ \Upsilon^{-1}) \circ \Upsilon$ in $L^q(\Omega_d)$, then $\Lambda_* \varphi = \tilde{\Lambda}(\varphi \circ \Upsilon^{-1}) \circ \Upsilon$ q.t.p. . Thus we proved (3) and the proof of (4) follows easily from (3). \square

Theorem 3.2. *Let Λ_* and $\tilde{\Lambda}$ as in Lemma 3.1. Then*

$$\begin{aligned}d_n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= d_n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})), \\ d^n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= d^n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})), \\ \delta_n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= \delta_n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})), \\ b_n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= b_n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})).\end{aligned}$$

Proof. By the Lemma 3.1

$$\begin{aligned}d_n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= \inf_{X_n} \sup_{f \in \Lambda_* U_p} \inf_{g \in X_n} \|f - g\|_{L^q(\Omega_d)} \\ &= \inf_{X_n} \sup_{f \in \Lambda_* U_p} \inf_{g \in X_n} \|f \circ \Upsilon^{-1} - g \circ \Upsilon^{-1}\|_{L^q(S^{2d-1})} \\ &= \inf_{\tilde{X}_n \circ \Upsilon} \sup_{f \in (\tilde{\Lambda} \tilde{U}_p) \circ \Upsilon} \inf_{g \in \tilde{X}_n \circ \Upsilon} \|f \circ \Upsilon^{-1} - g \circ \Upsilon^{-1}\|_{L^q(S^{2d-1})} \\ &= \inf_{\tilde{X}_n} \sup_{f \circ \Upsilon^{-1} \in \tilde{\Lambda} \tilde{U}_p} \inf_{g \circ \Upsilon^{-1} \in \tilde{X}_n} \|f \circ \Upsilon^{-1} - g \circ \Upsilon^{-1}\|_{L^q(S^{2d-1})} \\ &= \inf_{\tilde{X}_n} \sup_{\tilde{f} \in \tilde{\Lambda} \tilde{U}_p} \inf_{\tilde{g} \in \tilde{X}_n} \|\tilde{f} - \tilde{g}\|_{L^q(S^{2d-1})} \\ &= d_n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})),\end{aligned}$$

where X_n runs over all subspaces of $L^q(\Omega_d)$ of dimension n and \tilde{X}_n over all subspaces of $L^q(S^{2d-1})$ of dimension n ;

$$\begin{aligned}d^n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= \inf_{L^n} \sup_{f \in (\Lambda_* U_p) \cap L^n} \|f\|_{L^q(\Omega_d)} \\ &= \inf_{\tilde{L}^n \circ \Upsilon} \sup_{f \in ((\tilde{\Lambda} \tilde{U}_p) \circ \Upsilon) \cap (\tilde{L}^n \circ \Upsilon)} \|f \circ \Upsilon^{-1}\|_{L^q(S^{2d-1})} \\ &= \inf_{\tilde{L}^n} \sup_{f \circ \Upsilon^{-1} \in (\tilde{\Lambda} \tilde{U}_p) \cap \tilde{L}^n} \|f \circ \Upsilon^{-1}\|_{L^q(S^{2d-1})} \\ &= \inf_{\tilde{L}^n} \sup_{\tilde{f} \in (\tilde{\Lambda} \tilde{U}_p) \cap \tilde{L}^n} \|\tilde{f}\|_{L^q(S^{2d-1})} \\ &= d^n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})),\end{aligned}$$

where L^n runs over all subspaces of $L^q(\Omega_d)$ of codimension n and \tilde{L}^n over all subspaces of $L^q(S^{2d-1})$ of codimension n ;

$$\begin{aligned}
\delta_n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= \inf_{P_n} \sup_{f \in \Lambda_* U_p} \|f - P_n(f)\|_{L^q(\Omega_d)} \\
&= \inf_{\tilde{P}_n \circ \Upsilon} \sup_{f \in (\tilde{\Lambda} \tilde{U}_p) \circ \Upsilon} \|f \circ \Upsilon^{-1} - \tilde{P}_n(f \circ \Upsilon^{-1})\|_{L^q(S^{2d-1})} \\
&= \inf_{\tilde{P}_n} \sup_{f \circ \Upsilon^{-1} \in \tilde{\Lambda} \tilde{U}_p} \|f \circ \Upsilon^{-1} - \tilde{P}_n(f \circ \Upsilon^{-1})\|_{L^q(S^{2d-1})} \\
&= \inf_{\tilde{P}_n} \sup_{\tilde{f} \in \tilde{\Lambda} \tilde{U}_p} \|\tilde{f} - \tilde{P}_n(\tilde{f})\|_{L^q(S^{2d-1})} \\
&= \delta_n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})),
\end{aligned}$$

where P_n runs over all bounded linear operators $P_n : L^q(\Omega_d) \rightarrow L^q(\Omega_d)$ whose range has dimension n and \tilde{P}_n over all bounded linear operators $\tilde{P}_n : L^q(S^{2d-1}) \rightarrow L^q(S^{2d-1})$ whose range has dimension n ;

$$\begin{aligned}
b_n(\Lambda_* U_p(\Omega_d), L^q(\Omega_d)) &= \sup_{X_{n+1}} \sup\{\lambda : \lambda U_q \cap X_{n+1} \subseteq \Lambda_* U_p(\Omega_d)\} \\
&= \sup_{X_{n+1}} \sup\{\lambda : \lambda U_q \cap X_{n+1} \subseteq (\tilde{\Lambda} \tilde{U}_p) \circ \Upsilon\} \\
&= \sup_{\tilde{X}_{n+1} \circ \Upsilon} \sup\{\lambda : \lambda(\tilde{U}_q \circ \Upsilon) \cap (\tilde{X}_{n+1} \circ \Upsilon) \subseteq (\tilde{\Lambda} \tilde{U}_p) \circ \Upsilon\} \\
&= \sup_{\tilde{X}_{n+1}} \sup\{\lambda : \lambda \tilde{U}_q \cap \tilde{X}_{n+1} \subseteq \tilde{\Lambda} \tilde{U}_p\} = b_n(\tilde{\Lambda} \tilde{U}_p(S^{2d-1}), L^q(S^{2d-1})),
\end{aligned}$$

where X_{n+1} runs over all subspaces of $L^q(\Omega_d)$ of dimension $n+1$ and \tilde{X}_{n+1} over all subspaces of $L^q(S^{2d-1})$ of dimension $n+1$. \square

Remark 3.3. For $\gamma \in \mathbb{R}$, we denote $\Lambda^\gamma = \{\lambda_{m,n}^\gamma\}_{m,n \in \mathbb{N}}$, with $\lambda_{m,n}^\gamma = \lambda^\gamma(m+n)$, where the function $\lambda^\gamma : [0, \infty) \rightarrow \mathbb{R}$ is defined by $\lambda^\gamma(t) = (t(t+2d-2))^{-\gamma/2}$ if $t > 0$ and $\lambda(0) = 0$. The Sobolev space \overline{W}_p^γ on Ω_d with $\gamma > 0$, $1 \leq p \leq \infty$, is defined as the set

$$\overline{W}_p^\gamma = \{\varphi \in L^p(\Omega_d) : \Lambda^{-\gamma} \varphi \in L^p(\Omega_d)\} = \{c + \Lambda^\gamma \psi : c \in \mathbb{R}, \psi \in L^p(\Omega_d)\},$$

endowed with norm

$$\|\varphi\|_{\overline{W}_p^\gamma} = \|\Lambda^{-\gamma} \varphi\|_p + \|\varphi\|_p.$$

We define the Sobolev classe W_p^γ as the set

$$W_p^\gamma = \{\Lambda^\gamma \varphi : \varphi \in U_p\} = \Lambda^\gamma U_p, \quad \gamma > 0, \quad 1 \leq p \leq \infty.$$

Now let $\tilde{\Lambda}^\gamma = \{\lambda_k^\gamma\}_{k \in \mathbb{N}}$, $\lambda_k^\gamma = \lambda^\gamma(k)$. The Sobolev classe \widetilde{W}_p^γ on S^{2d-1} is given by

$$\widetilde{W}_p^\gamma = \{\tilde{\Lambda}^\gamma \varphi; \varphi \in \tilde{U}_p\} = \tilde{\Lambda}^\gamma \tilde{U}_p.$$

Applying the Lemma 3.1 for the operator Λ^γ we get

$$W_p^\gamma = \{\varphi \circ \Upsilon; \varphi \in \widetilde{W}_p^\gamma\},$$

and by Theorem 3.2 we have that

$$d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) = d_n(\widetilde{W}_p^\gamma(S^{2d-1}), L^q(S^{2d-1})), \quad (5)$$

$$d^n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) = d^n(\widetilde{W}_p^\gamma(S^{2d-1}), L^q(S^{2d-1})), \quad (6)$$

$$\delta_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) = \delta_n(\widetilde{W}_p^\gamma(S^{2d-1}), L^q(S^{2d-1})), \quad (7)$$

$$b_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) = b_n(\widetilde{W}_p^\gamma(S^{2d-1}), L^q(S^{2d-1})). \quad (8)$$

The estimates in the theorem below follow as a consequence of (5) and of estimates already obtained for the Kolmogorov n -width for the Sobolev classes \widetilde{W}_p^γ on the real sphere S^{2d-1} , which can be found in [2, 3, 11, 12, 14, 16, 18].

Theorem 3.4. For the Kolmogorov n -width of Sobolev classes on Ω_d , we have in particular the following estimates:

(a) if $1 \leq p = q \leq \infty$, $2 \leq q \leq p < \infty$, $\gamma > 0$,

$$d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \asymp n^{-\gamma/(2d-1)};$$

(b) if $1 \leq p = q \leq \infty$, $2 \leq q \leq p \leq \infty$, $\gamma > 0$,

$$n^{-\gamma/(2d-1)}(\ln n)^{-1/2} \ll d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \ll n^{-\gamma/(2d-1)};$$

(c) if $2 \leq q \leq p < \infty$, $\gamma/(2d-1) > 1/2$,

$$d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \asymp n^{-\gamma/(2d-1)};$$

(d) if $2 \leq q \leq p \leq \infty$, $\gamma/(2d-1) > 1/2$,

$$n^{-\gamma/(2d-1)} \ll d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \ll n^{-\gamma/(2d-1)}(\ln n)^{1/2};$$

(e) if $1 \leq p \leq q \leq 2$, $\gamma/(2d-1) > 1/p - 1/q$,

$$d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \asymp n^{-\gamma/(2d-1)+1/p-1/q};$$

(f) if $1 \leq p \leq q \leq 2$, $\gamma > 0$,

$$d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \asymp n^{-\gamma/(2d-1)};$$

(g) if $1 \leq p \leq q \leq 2$, $\gamma > 0$,

$$n^{-\gamma/(2d-1)}(\ln n)^{-1/2} \ll d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \ll n^{-\gamma/(2d-1)};$$

(h) if $1 \leq p \leq 2 \leq q < \infty$, $\gamma/(2d-1) > 1/p$,

$$d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \asymp n^{-\gamma/(2d-1)+1/p-1/2};$$

(i) if $1 \leq p \leq 2 \leq q \leq \infty$, $\gamma/(2d-1) > 1/p$,

$$n^{-\gamma/(2d-1)+1/p-1/2} \ll d_n(W_p^\gamma(\Omega_d), L^q(\Omega_d)) \ll n^{-\gamma/(2d-1)+1/p-1/2}(\ln n)^{1/2}.$$

Other estimates for the Kolmogorov n -widths of the Sobolev classes $W_p^\gamma(\Omega_d)$, as also to the n -widths of Gelfand, linear and Bernstein, can be obtained using (6)-(8).

Remark 3.5. For $\gamma, \xi \in \mathbb{R}$, $\gamma > 0$ and $\xi \geq 0$, we denote $\Lambda_*^{(1)} = \{\lambda_{m,n}^{(1),*}\}_{m,n \in \mathbb{N}}$, with $\lambda_{m,n}^{(1),*} = \lambda^{(1)}(m+n)$, where the function $\lambda^{(1)} : [0, \infty) \rightarrow \mathbb{R}$ is defined by $\lambda^{(1)}(t) = t^{-\gamma}(\ln t)^{-\xi}$ for $t > 1$ and $\lambda^{(1)}(t) = 0$ for $0 \leq t \leq 1$. We have that $\Lambda_*^{(1)}U_p$ is a set of finitely differentiable functions on Ω_d , in particular, $\Lambda_*^{(1)}U_p$ is a Sobolev-type class if $\xi = 0$. Now for $\gamma, r \in \mathbb{R}$, with $\gamma, r > 0$, we denote $\Lambda_*^{(2)} = \{\lambda_{m,n}^{(2),*}\}_{m,n \in \mathbb{N}}$ with $\lambda_{m,n}^{(2),*} = \lambda^{(2)}(m+n)$, where the function $\lambda^{(2)} : [0, \infty) \rightarrow \mathbb{R}$ is defined by $\lambda^{(2)}(t) = e^{-\gamma t^r}$ for $t \geq 0$. We have that $\Lambda_*^{(2)}U_p$ is a set of infinitely differentiable ($0 < r < 1$) or analytic ($r \geq 1$) functions on Ω_d .

The results of the next three theorems follow as a consequence of the Theorem 3.2 and of results proved in [18].

Theorem 3.6. If $1 \leq p \leq \infty$, $2 \leq q \leq \infty$ and $\gamma/(2d-1) > 1/p$, we have

$$d_n(\Lambda_*^{(1)}U_p, L^q) \ll n^{-\gamma/(2d-1)+(1/p-1/2)+}(\ln n)^{-\xi} \begin{cases} q^{1/2}, & \text{if } q < \infty, \\ (\ln n)^{1/2}, & \text{if } q = \infty, \end{cases}$$

and for $\gamma/(2d-1) > 1/p - 1/q$,

$$d_n(\Lambda_*^{(1)}U_p, L^q) \gg n^{-\gamma/(2d-1)}(\ln n)^{-\xi} \vartheta_n,$$

where

$$\vartheta_n = \begin{cases} 1, & \text{if } 1 \leq p \leq 2, 1 < q \leq 2, \\ 1, & \text{if } 2 \leq p < \infty, 2 \leq q \leq \infty, \\ 1, & \text{if } 1 \leq p \leq 2 \leq q \leq \infty, \\ (\ln n)^{-1/2}, & \text{if } 1 \leq p \leq 2, q = 1, \\ (\ln n)^{-1/2}, & \text{if } p = \infty, 2 \leq q \leq \infty. \end{cases}$$

In particular, if $\gamma/(2d-1) > 1/p$, $2 \leq p, q < \infty$,

$$d_n(\Lambda_*^{(1)}U_p, L^q) \asymp n^{-\gamma/(2d-1)}(\ln n)^{-\xi}.$$

Theorem 3.7. For $\gamma, r \in \mathbb{R}$ such that $\gamma > 0$, $0 < r \leq 1$ and for all $k \in \mathbb{N}$ we have that

$$d_k(\Lambda_*^{(2)}U_p, L^q) \gg e^{-\mathcal{R}_* k^{r/(2d-1)}} \vartheta_k,$$

where

$$\mathcal{R}_* = \gamma \left(\frac{(2d-1)!}{2} \right)^{r/(2d-1)},$$

and ϑ_k is given in the previous theorem.

Theorem 3.8. For $0 < r \leq 1$, $1 \leq p \leq \infty$, $2 \leq q \leq \infty$ and for all $k \in \mathbb{N}$, we have that

$$d_k(\Lambda_*^{(2)}U_p, L^q) \ll e^{-\mathcal{R}_* k^{r/(2d-1)}} k^{(1-r/(2d-1))(1/p-1/2)_+} \begin{cases} q^{1/2}, & \text{if } q < \infty, \\ (\ln k)^{1/2}, & \text{if } q = \infty, \end{cases}$$

where \mathcal{R}_* is the constant given in the previous theorem.

Other estimates for the Kolmogorov n -width for the operator $\Lambda_*^{(2)}$ when $r \in \mathbb{R}$, $r > 1$, can also be obtained from results in [18].

4 Estimates for Levy means

In this paper, from this point forward, we will consider only the multiplier operators Λ associated with the norm $|(m, n)| = \max\{m, n\}$.

Given $l, N, m, n, M_1, M_2 \in \mathbb{N}$, with $M_1 < M_2$, we consider the following notations:

$$A_l = \{(m, n) \in \mathbb{N}^2 : |(m, n)| \leq l\}, \quad A_{-1} = \emptyset, \quad a_l = \#(A_l \setminus A_{l-1})$$

$$\mathcal{H}_l = \bigoplus_{(m,n) \in A_l \setminus A_{l-1}} \mathcal{H}_{m,n}, \quad d_l = \dim \mathcal{H}_l = \sum_{(m,n) \in A_l \setminus A_{l-1}} d_{m,n}$$

$$\mathcal{T}_N = \bigoplus_{l=0}^N \mathcal{H}_l = \bigoplus_{(m,n) \in A_N} \mathcal{H}_{m,n}, \quad \mathcal{T}_{M_1, M_2} = \bigoplus_{(m,n) \in A_{M_2} \setminus A_{M_1}} \mathcal{H}_{m,n} \quad \text{and} \quad s = \dim \mathcal{T}_{M_1, M_2}.$$

In this section and in the following sections we will consider $\mathcal{H}_{m,n}$ and \mathcal{T}_{M_1, M_2} as real vector subspaces of the real vector spaces $L^p(\Omega_d, \mathbb{R})$, $1 \leq p \leq \infty$.

Proposition 4.1. There are positive constants C_1, C_2, C_3 such that

$$\frac{2(2d-1)}{d!(d-1)!} l^{2d-2} - C_1 l^{2d-3} \leq \dim \mathcal{H}_l \leq \frac{2(2d-1)}{d!(d-1)!} l^{2d-2} + C_2 l^{2d-3}, \quad (9)$$

$$\frac{2}{d!(d-1)!} N^{2d-1} \leq \dim \mathcal{T}_N \leq \frac{2}{d!(d-1)!} N^{2d-1} + C_3 N^{2d-2}. \quad (10)$$

In particular, $\dim \mathcal{H}_l \asymp l^{2d-2}$ and $\dim \mathcal{T}_N \asymp N^{2d-1}$.

Proof. The proofs of (9) and (10) are easily obtained using (2) and integration to estimate finite sums. \square

Let $E = (\mathbb{R}^n, \|\cdot\|)$ be a n -dimensional real Banach space with unit ball $B_E = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, and let $\|x\| = (\sum_{k=1}^n |x_k|^2)^{1/2}$ and $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$ be the euclidean norm and the inner product on \mathbb{R}^n . Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the euclidean unit sphere in \mathbb{R}^n . The Levy mean of the norm $\|\cdot\|$ on \mathbb{R}^n is defined by

$$M(\|\cdot\|) = M(\mathbb{R}^n, \|\cdot\|) = \left(\int_{S^{n-1}} \|x\|^2 d\mu(x) \right)^{1/2}.$$

where μ denotes the normalized Lebesgue measure on S^{n-1} .

Remark 4.2. For $M_1 + 1 \leq l \leq M_2$ let $A_l \setminus A_{l-1} = \{(m_j^l, n_j^l) : 1 \leq j \leq a_l\}$ such that $|(m_j^l, n_j^l)| \leq |(m_{j+1}^l, n_{j+1}^l)|$ for $1 \leq j \leq a_l - 1$ and let $\{Y_1^{(m_j^l, n_j^l)}, \dots, Y_{d_{m_j^l, n_j^l}}^{(m_j^l, n_j^l)}\}$ be an orthonormal basis of $\mathcal{H}_{m_j^l, n_j^l}$ consisting of only real functions. We denote $Y_i^{l,j} = Y_i^{(m_j^l, n_j^l)}$, $d_{l,j} = d_{m_j^l, n_j^l}$ and $\mathcal{H}_{l,j} = \mathcal{H}_{m_j^l, n_j^l}$. We consider the orthonormal basis

$$\{\xi_k\}_{k=1}^s = \bigcup_{l=M_1+1}^{M_2} \bigcup_{j=1}^{a_l} \{Y_1^{l,j}, \dots, Y_{d_{l,j}}^{l,j}\}$$

of \mathcal{T}_{M_1, M_2} endowed with the order $Y_1^{M_1+1,1}, \dots, Y_{d_{M_1+1,1}}^{M_1+1,1}, \dots, Y_1^{M_1+1, a_{M_1+1}}, \dots, Y_{d_{M_1+1, a_{M_1+1}}}^{M_1+1, a_{M_1+1}}, \dots, Y_1^{M_2,1}, \dots, Y_{d_{M_2,1}}^{M_2,1}, \dots, Y_1^{M_2, a_{M_2}}, \dots, Y_{d_{M_2, a_{M_2}}}^{M_2, a_{M_2}}$. Let $J : \mathbb{R}^s \rightarrow \mathcal{T}_{M_1, M_2}$ be the coordinate isomorphism which assigns to $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ the function

$$J(\alpha) = \sum_{k=1}^s \alpha_k \xi_k = \sum_{l=M_1+1}^{M_2} \sum_{j=1}^{a_l} \sum_{i=1}^{d_{l,j}} \alpha_i^{l,j} Y_i^{l,j}$$

where $(\alpha_1, \dots, \alpha_s) = (\alpha_1^{M_1+1,1}, \dots, \alpha_{d_{M_1+1,1}}^{M_1+1,1}, \dots, \alpha_1^{M_1+1, a_{M_1+1}}, \dots, \alpha_{d_{M_1+1, a_{M_1+1}}}^{M_1+1, a_{M_1+1}}, \dots, \alpha_1^{M_2,1}, \dots, \alpha_{d_{M_2,1}}^{M_2,1}, \dots, \alpha_1^{M_2, a_{M_2}}, \dots, \alpha_{d_{M_2, a_{M_2}}}^{M_2, a_{M_2}})$.

Consider a function $\lambda : [0, \infty) \rightarrow \mathbb{R}$, such that $\lambda(t) \neq 0$ for $t \geq 0$ and let $\Lambda = \{\lambda_{m,n}\}_{m,n \in \mathbb{N}}$ be the sequence of multipliers defined by $\lambda_{m,n} = \lambda(|(m,n)|)$. For $M_1 + 1 \leq l \leq M_2$ and $1 \leq j \leq a_l$, we write $\lambda_j^l = \lambda_{m_j^l, n_j^l} = \lambda(|(m_j^l, n_j^l)|)$.

Let $\Lambda_s = \{\tilde{\lambda}_k\}_{k=1}^s$ be the numerical sequence

$$\Lambda_s = \left\{ \underbrace{\lambda_1^{M_1+1}, \dots, \lambda_1^{M_1+1}}_{d_{M_1+1,1}}, \dots, \underbrace{\lambda_{a_{M_1+1}}^{M_1+1}, \dots, \lambda_{a_{M_1+1}}^{M_1+1}}_{d_{M_1+1, a_{M_1+1}}}, \dots, \underbrace{\lambda_1^{M_2}, \dots, \lambda_1^{M_2}}_{d_{M_2,1}}, \dots, \underbrace{\lambda_{a_{M_2}}^{M_2}, \dots, \lambda_{a_{M_2}}^{M_2}}_{d_{M_2, a_{M_2}}} \right\}.$$

Consider the multiplier operator Λ_s on \mathcal{T}_{M_1, M_2} defined by

$$\Lambda_s \left(\sum_{k=1}^s \alpha_k \xi_k \right) = \sum_{k=1}^s \tilde{\lambda}_k \alpha_k \xi_k.$$

We also denote by Λ_s the multiplier operator defined on \mathbb{R}^s by

$$\Lambda_s(\alpha_1, \dots, \alpha_s) = (\tilde{\lambda}_1 \alpha_1, \dots, \tilde{\lambda}_s \alpha_s).$$

For $\xi \in \mathcal{T}_{M_1, M_2}$ and $1 \leq p \leq \infty$, we define

$$\|\xi\|_{\Lambda_s, p} = \|\Lambda_s \xi\|_p,$$

and for $\alpha \in \mathbb{R}^s$ we define

$$\|\alpha\|_{(\Lambda_s, p)} = \|J(\alpha)\|_{\Lambda_s, p}.$$

The application $\mathcal{T}_{M_1, M_2} \ni \xi \mapsto \|\xi\|_{\Lambda_s, p}$ is a norm on \mathcal{T}_{M_1, M_2} and the application $\mathbb{R}^s \ni \alpha \mapsto \|\alpha\|_{(\Lambda_s, p)}$ is a norm on \mathbb{R}^s . We denote

$$B_{\Lambda_s, p}^s = B_{\Lambda, p}^s = \{\xi \in \mathcal{T}_{M_1, M_2} : \|\xi\|_{\Lambda_s, p} \leq 1\},$$

$$B_{(\Lambda_s, p)}^n = B_{(\Lambda, p)}^s = \{\alpha \in \mathbb{C}^s : \|\alpha\|_{(\Lambda_s, p)} \leq 1\}.$$

If Λ_s is the identity operator I , we will write $\|\cdot\|_{I, p} = \|\cdot\|_p$, $\|\cdot\|_{(I, p)} = \|\cdot\|_p$, $B_{I, p}^s = B_p^s$ and $B_{(I, p)}^s = B_p^s$.

Theorem 4.3. Let $\lambda : [0, \infty) \rightarrow \mathbb{R}$ be a positive and monotonic function, $s = \dim \mathcal{T}_{M_1, M_2}$ and consider the orthonormal system $\{\xi_k\}_{k=1}^s$ of \mathcal{T}_{M_1, M_2} and the multiplier operator Λ_s on \mathcal{T}_{M_1, M_2} as in Remark 4.2. If λ is non-increasing, then there is an absolute constant $C > 0$ such that:

(a) If $2 \leq p < \infty$, we have

$$s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_s, p)}) \leq C \omega_d^{1/p-1/2} p^{1/2} s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l \right)^{1/2}.$$

(b) If $p = \infty$, we have

$$s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_s, \infty)}) \leq C \omega_d^{-1/2} (\ln s)^{1/2} s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l \right)^{1/2}.$$

(c) If $1 \leq p \leq 2$, we have

$$\frac{\omega_d^{1/2}}{2} s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_s, p)}) \leq s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l \right)^{1/2}.$$

(d) If $p = 2$, we have

$$s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_s, p)}) \leq s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l \right)^{1/2}.$$

If λ is non-decreasing, then we obtain the estimates in (a), (b), (c) and (d) permuting $\lambda(l)$ for $\lambda(l-1)$.

Proof. Suppose λ a non-increasing function. For a continuous function f on S^{n-1} consider the function \tilde{f} defined on $\mathbb{R}^n \setminus \{0\}$ by $\tilde{f}(x) = \|x\|^2 f(x/\|x\|)$. It is known that

$$\int_{S^{n-1}} f(x) d\mu(x) = \frac{2\pi}{n} \int_{\mathbb{R}^n} \tilde{f}(x) d\gamma(x), \quad (11)$$

where $d\gamma(x) = e^{-\pi\|x\|^2} dx$ denotes the Gaussian measure on \mathbb{R}^n . Let $\{r_k\}_{k=1}^\infty$ be the sequence of Rademacher's functions given by $r_k(\theta) = \text{sign} \sin(2^k \pi \theta)$, $\theta \in [0, 1]$, $k \in \mathbb{N}$ and let

$$\delta_i^m(\theta) = m^{-1/2} (r_{(i-1)m+1}(\theta) + \dots + r_{im}(\theta)), \quad m \in \mathbb{N}, i = 1, 2, \dots, n.$$

It follows by Lemma 2.1 in Kwapien [19], p. 585, that if $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function satisfying $h(x_1, \dots, x_n) e^{-\sum_{k=1}^n |x_k|} \rightarrow 0$, uniformly when $\sum_{k=1}^n |x_k| \rightarrow \infty$, then

$$\int_{\mathbb{R}^n} h(x) d\gamma(x) = \lim_{m \rightarrow \infty} \int_0^1 h((2\pi)^{-1/2} (\delta_1^m(\theta), \dots, \delta_n^m(\theta))) d\theta. \quad (12)$$

Now, we consider $f(x) = \|x\|_{(\Lambda_s, p)}^2$, $x \in S^{s-1}$ and $h(x) = \tilde{f}(x) = f(x)$, $x \in \mathbb{R}^s$ applying (11) and (12), we obtain

$$\begin{aligned} \int_{S^{s-1}} \|x\|_{(\Lambda_s, p)}^2 d\mu(x) &= \frac{2\pi}{s} \lim_{m \rightarrow \infty} \int_0^1 \|(2\pi)^{-1/2} (\delta_1^m(\theta), \dots, \delta_s^m(\theta))\|_{(\Lambda_s, p)}^2 d\theta \\ &= s^{-1} \lim_{m \rightarrow \infty} \int_0^1 \left(\int_{\Omega_d} \left| \sum_{i=1}^s \tilde{\lambda}_i \delta_i^m(\theta) \xi_i(z) \right|^p d\sigma_d(z) \right)^{2/p} d\theta. \end{aligned} \quad (13)$$

Denoting $\tilde{\xi}_{(i-1)m+k}(z) = m^{-1/2} \xi_i(z)$, $z \in \Omega_d$ and $\tilde{\lambda}_{(i-1)m+k} = \tilde{\lambda}_i$, for $i = 1, \dots, s$; $k = 1, \dots, m$ and $m = 1, 2, \dots$ we obtain

$$\sum_{i=1}^s \tilde{\lambda}_i \delta_i^m(\theta) \xi_i(z) = \sum_{j=1}^{ms} r_j(\theta) \tilde{\lambda}_j \tilde{\xi}_j(z). \quad (14)$$

Hence, by (13) and (14),

$$\int_{S^{s-1}} \|x\|_{(\Lambda_s, p)}^2 d\mu(x) = s^{-1} \lim_{m \rightarrow \infty} \int_0^1 \left(\int_{\Omega_d} \left| \sum_{j=1}^{ms} r_j(\theta) \tilde{\lambda}_j \tilde{\xi}_j(z) \right|^p d\sigma_d(z) \right)^{2/p} d\theta. \quad (15)$$

Using the addition formula we get

$$\begin{aligned}
\sum_{j=1}^{ms} |\tilde{\lambda}_j \tilde{\xi}_j(z)|^2 &= \sum_{i=1}^s \sum_{k=1}^m \left| \tilde{\lambda}_{(i-1)m+k} \tilde{\xi}_{(i-1)m+k}(z) \right|^2 = \sum_{i=1}^s \sum_{k=1}^m \left| \tilde{\lambda}_i \right|^2 \left| m^{-1/2} \xi_i(z) \right|^2 \\
&= \sum_{l=M_1+1}^{M_2} \sum_{j=1}^{a_l} (\lambda_j^l)^2 \sum_{i=1}^{d_{l,j}} \left| Y_i^{l,j}(z) \right|^2 \\
&\leq \sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 \sum_{j=1}^{a_l} \frac{d_{l,j}}{\omega_d} \\
&= \sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 \frac{d_l}{\omega_d}
\end{aligned}$$

and analogously,

$$\sum_{j=1}^{2ms} \left| \tilde{\lambda}_j \tilde{\xi}_j(z) \right|^2 \geq \sum_{l=M_1+1}^{M_2} (\lambda(l))^2 \frac{d_l}{\omega_d}.$$

Therefore, from Jensen's inequality, Khintchine's inequality (see [27], p. 41) and (15), we obtain for $2 \leq p \leq \infty$,

$$\begin{aligned}
M(\|\cdot\|_{(\Lambda_s, p)}) &\leq s^{-1/2} \lim_{m \rightarrow \infty} \left(\left(\int_{\Omega_d} \int_0^1 \left| \sum_{j=1}^{ms} r_j(\theta) \tilde{\lambda}_j \tilde{\xi}_j(z) \right|^p d\theta d\sigma_d(z) \right)^{2/p} \right)^{1/2} \\
&\leq \gamma(p) s^{-1/2} \lim_{m \rightarrow \infty} \left(\int_{\Omega_d} \left(\sum_{j=1}^{ms} \left| \tilde{\lambda}_j \tilde{\xi}_j(z) \right|^2 \right)^{p/2} d\sigma_d(z) \right)^{1/p} \\
&\leq \gamma(p) s^{-1/2} \lim_{m \rightarrow \infty} \left(\int_{\Omega_d} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 \frac{d_l}{\omega_d} \right)^{p/2} d\sigma_d(z) \right)^{1/p} \\
&\leq C_1 p^{1/2} s^{-1/2} \omega_d^{1/p-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l \right)^{1/2},
\end{aligned}$$

where C_1 is obtained from the fact that $\gamma(p) \asymp p^{1/2}$, and hence we get the upper estimate in (a). On the other hand, for $p = 1$, it follows from Khintchine's inequality, Jensen's inequality and (15)

$$\begin{aligned}
M(\|\cdot\|_{(\Lambda_s, 1)}) &= s^{-1/2} \lim_{m \rightarrow \infty} \left(\int_0^1 \left(\int_{\Omega_d} \left| \sum_{j=1}^{ms} r_j(\theta) \tilde{\lambda}_j \tilde{\xi}_j(z) \right| d\sigma_d(z) \right)^2 d\theta \right)^{1/2} \\
&\geq s^{-1/2} \lim_{m \rightarrow \infty} \int_{\Omega_d} \int_0^1 \left| \sum_{j=1}^{ms} r_j(\theta) \tilde{\lambda}_j \tilde{\xi}_j(z) \right| d\theta d\sigma_d(z) \\
&\geq \beta(1) s^{-1/2} \lim_{m \rightarrow \infty} \int_{\Omega_d} \left(\sum_{j=1}^{ms} \left| \tilde{\lambda}_j \tilde{\xi}_j(z) \right|^2 \right)^{1/2} d\sigma_d(z) \\
&\geq \frac{1}{2} s^{-1/2} \omega_d^{1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2}.
\end{aligned}$$

Since the Levy mean is an increasing function of p , it follows that

$$M(\|\cdot\|_{(\Lambda_s, p)}) \geq M(\|\cdot\|_{(\Lambda_s, 1)}) \geq \frac{1}{2} s^{-1/2} \omega_d^{1/2} \left(\sum_{l=M_1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2}, \quad 1 \leq p \leq 2,$$

thus we obtain the lower estimate in (c).

Now, we will obtain the inequalities in (d). For $x \in \mathbb{R}^s$, we have

$$\begin{aligned} \|x\|_{(\Lambda_s, 2)}^2 &= \sum_{l=M_1+1}^{M_2} \sum_{j=1}^{a_l} |\lambda_j^l|^2 \sum_{i=1}^{d_{l,j}} (x_i^{l,j})^2 \|Y_i^{l,j}\|_2^2 \\ &\leq \sum_{l=M_1+1}^{M_2} \left(\sup_{1 \leq j \leq a_l} \lambda_j^l \right)^2 \sum_{j=1}^{a_l} \sum_{i=1}^{d_{l,j}} (x_i^{l,j})^2 \\ &\leq \sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 \sum_{j=1}^{a_l} \sum_{i=1}^{d_{l,j}} (x_i^{l,j})^2, \end{aligned}$$

and

$$\int_{S^{s-1}} x_i^2 d\mu(x) = \frac{1}{s} \sum_{i=1}^s \int_{S^{s-1}} x_i^2 d\mu(z) = \frac{1}{s} \int_{S^{s-1}} \|x\|^2 d\mu(x) = \frac{1}{s}, \quad i = 1, \dots, s.$$

Thus

$$\begin{aligned} \int_{S^{s-1}} \|x\|_{(\Lambda_s, 2)}^2 d\mu(z) &\leq \sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 \sum_{j=1}^{a_l} \sum_{i=1}^{d_{l,j}} \int_{S^{s-1}} (x_i^{l,j})^2 d\mu(x) \\ &= \frac{1}{s} \sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l, \end{aligned}$$

and consequently

$$M(\|\cdot\|_{(\Lambda_s, 2)}) \leq s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l-1))^2 d_l \right)^{1/2}.$$

Analogously we get

$$M(\|\cdot\|_{(\Lambda_s, 2)}) \geq s^{-1/2} \left(\sum_{l=M_1+1}^{M_2} (\lambda(l))^2 d_l \right)^{1/2}.$$

Therefore we obtain (d). Since the Levy mean $M(\|\cdot\|_{(\Lambda_s, p)})$ is an increasing function of p for $1 \leq p \leq \infty$, the lower estimates in (a),(b) and the upper estimate in (c) follow from (d).

Finally, we will find the upper estimate in (b). Any polynomial $t_s \in \mathcal{T}_{M_1, M_2}$, can be expressed as $t_s = D_{M_1, M_2} * t_s$, where $D_{M_1, M_2} = \sum_{(m,n) \in A_{M_2} \setminus A_{M_1}} Z_e^{(m,n)}$ and from Young's inequality, we get

$$\|t_s\|_\infty = \|D_{M_1, M_2} * t_s\|_\infty \leq \|D_{M_1, M_2}\|_\infty \|t_s\|_1,$$

and since $D_{M_1, M_2} = D_{M_1, M_2} * D_{M_1, M_2}$, then

$$\|D_{M_1, M_2}\|_\infty \leq \|D_{M_1, M_2}\|_2^2 = \frac{s}{\omega_d}$$

and thus $\|t_s\|_\infty \leq s \|t_s\|_1 / \omega_d$. Furthermore, if I denotes the identity operator, we have

$$\|I(t_s)\|_\infty \leq \frac{s}{\omega_d} \|t_s\|_1, \quad \|I(t_s)\|_\infty \leq \|t_s\|_\infty.$$

Applying the Riesz-Thorin Interpolation Theorem to the above inequalities, we obtain

$$\|t_s\|_\infty = \|I(t_s)\|_\infty \leq \left(\frac{s}{\omega_d} \right)^{1/p} \|t_s\|_p, \quad 1 \leq p \leq \infty. \quad (16)$$

Similarly, we can show that

$$\|t_s\|_p \leq \left(\frac{s}{\omega_d} \right)^{1/2-1/p} \|t_s\|_2, \quad 2 \leq p \leq \infty. \quad (17)$$

To obtain the upper estimate in (b), we apply (16) and the upper estimate in (a) with $p = \ln s$, and we get

$$\begin{aligned}
M(\|\cdot\|_{(\Lambda_s, \infty)}) &\leq \left(\frac{s}{\omega_d}\right)^{1/p} \left(\int_{S^{s-1}} \|x\|_{(\Lambda_s, p)}^2 d\mu(x)\right)^{1/2} \\
&\leq \left(\frac{s}{\omega_d}\right)^{1/p} C_1 p^{1/2} s^{-1/2} \omega_d^{1/p-1/2} \left(\sum_{l=M_1}^{M_2} (\lambda(l-1))^2 d_l\right)^{1/2} \\
&= eC_1 (\ln s)^{1/2} s^{-1/2} \omega_d^{-1/2} \left(\sum_{l=M_1}^{M_2} (\lambda(l-1))^2 d_l\right)^{1/2}.
\end{aligned}$$

If λ is a non-decreasing function, the proof is analogous. \square

5 Estimates for n -widths of general multiplier operators

Consider a n -dimensional Banach space $E = (\mathbb{R}^n, \|\cdot\|)$. The dual norm of $\|\cdot\|$ is defined by $\|x\|^0 = \sup\{|\langle x, y \rangle| : y \in B_E\}$, where $\langle x, y \rangle$ denotes the usual inner product of the elements $x, y \in \mathbb{R}^n$. The dual space $(\mathbb{R}^n, \|\cdot\|^0)$ of E will be denoted by E^0 .

Theorem 5.1. ([26]) *There exists an absolute constant $C > 0$ such that, for every $0 < \rho < 1$ and $n \in \mathbb{N}$, there exists a subspace $F_k \subseteq \mathbb{R}^n$, with $\dim F_k = k > \rho n$ and*

$$\|\|\alpha\|\| \leq CM (\|\cdot\|^0) (1 - \rho)^{-1/2} \|\alpha\|, \quad \alpha \in F_k.$$

Theorem 5.2. *Let $1 \leq q \leq p \leq 2$, $0 < \rho < 1$, $s = \dim \mathcal{T}_N$, $\mathcal{T}_N = \bigoplus_{l=0}^N \mathcal{H}_l$, $d_l = \dim \mathcal{H}_l$ and let $\lambda : [0, \infty) \rightarrow \mathbb{R}$ be a positive and non-increasing function with $\lambda(t) \neq 0$ for $t \geq 0$ and $\Lambda = \{\lambda_{m,n}\}_{m,n \in \mathbb{N}}$, $\lambda_{m,n} = \lambda(|(m,n)|)$. Then there is an absolute constant $C > 0$ such that*

$$\min\{d_{[\rho s - 1]}(\Lambda U_p, L^q), d^{[\rho s - 1]}(\Lambda U_p, L^q)\} \geq C(1 - \rho)^{1/2} s^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l\right)^{-1/2} \vartheta_{q,s}$$

where

$$\vartheta_{q,s} = \begin{cases} (1 - 1/q)^{1/2}, & q > 1, \\ (\ln s)^{-1/2}, & q = 1, \end{cases}$$

and $[\rho s - 1]$ denotes the integer part of the number $\rho s - 1$.

Proof. Let $x, y \in \mathbb{R}^s = J^{-1}\mathcal{T}_N$. The Hölder's inequality implies that

$$\begin{aligned}
\|x\|_{(\Lambda_s, q)}^0 &= \sup\{|\langle x, y \rangle| : y \in B_{(\Lambda_s, q)}^s\} \\
&= \sup\left\{\left|\int_{\Omega_d} (Jx)(Jy) d\sigma_d\right| : Jy \in \Lambda_s^{-1}(B_q^s)\right\} \\
&= \sup\left\{\left|\int_{\Omega_d} (\Lambda_s^{-1} Jx)(J\bar{y}) d\sigma_d\right| : \bar{y} \in B_{(q)}^s\right\} \\
&\leq \sup\{\|\Lambda_s^{-1} Jx\|_{q'} \|J\bar{y}\|_q : \bar{y} \in B_{(q)}^s\} \\
&\leq \|\Lambda_s^{-1} Jx\|_{q'} = \|x\|_{(\Lambda_s^{-1}, q')}
\end{aligned}$$

where $1/q + 1/q' = 1$ and $\Lambda_s^{-1} = \{\lambda_{m,n}^{-1}\}_{m,n \in \mathbb{N}}$. Taking $0 < \rho < 1$, the Theorem 5.1 guarantees the existence of a subspace $F_k \subseteq \mathbb{R}^s$, $\dim F_k = k > \rho s$, such that

$$\begin{aligned}
\|x\|_{(2)} = \|\|x\|\| &\leq C' M \left(\|\cdot\|_{(\Lambda_s, q)}^0\right) (1 - \rho)^{-1/2} \|x\|_{(\Lambda_s, q)}, \\
&\leq C' M \left(\|\cdot\|_{(\Lambda_s^{-1}, q')}\right) (1 - \rho)^{-1/2} \|x\|_{(\Lambda_s, q)},
\end{aligned}$$

for all $x \in F_k$. Thus, for $\epsilon = (C')^{-1}(1 - \rho)^{1/2} \left(M \left(\|\cdot\|_{(\Lambda_s^{-1}, q')} \right) \right)^{-1}$, we have that $\epsilon B_{(\Lambda_s, q)}^s \cap F_k \subseteq B_{(2)}^s$. From Theorem 4.3, since $2 \leq q' \leq \infty$, it follows that

$$\epsilon \geq C(1 - \rho)^{1/2} \begin{cases} (q')^{-1/2} s^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2}, & q' < \infty, \\ (s/\ln s)^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2}, & q' = \infty, \end{cases}$$

and since $1/q + 1/q' = 1$,

$$\epsilon \geq C(1 - \rho)^{1/2} \begin{cases} (1 - 1/q)^{1/2} s^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2}, & q > 1 \\ (s/\ln s)^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_k \right)^{-1/2}, & q = 1. \end{cases}$$

We have that $\Lambda_s = \Lambda|_{\mathcal{T}_N}$ and since $B_2^s \subset U_2$, then $\Lambda_s(B_2^s) \subset \Lambda(U_2)$, therefore

$$\begin{aligned} & \min\{d_{[\rho s-1]}(\Lambda U_p, L^q), d^{[\rho s-1]}(\Lambda U_p, L^q)\} \\ & \geq \min\{d_{[\rho s-1]}(\Lambda U_2, L^q), d^{[\rho s-1]}(\Lambda U_2, L^q)\} \\ & \geq b_{[\rho s-1]}(\Lambda U_2, L^q) \\ & \geq b_{[\rho s-1]}(\Lambda_s B_2^s, L^q). \end{aligned}$$

Taking $X_k = J(F_k)$ we obtain $\epsilon B_q^s \cap X_k \subseteq \Lambda_s B_2^s$. Thus by the definition of Bernstein n -width

$$b_{[\rho s-1]}(\Lambda_s B_2^s, L^q) \geq b_{k-1}(\Lambda_s B_2^s, L^q) \geq \epsilon$$

and consequently

$$\min\{d_{[\rho s-1]}(\Lambda U_p, L^q), d^{[\rho s-1]}(\Lambda U_p, L^q)\} \geq \epsilon,$$

thus concluding the proof of the theorem. \square

The next result follows applying Theorem 5.2, (1) and basic properties of n -widths.

Corollary 5.3. *In the conditions of Theorem 5.2, we have*

$$d_{[\rho s-1]}(\Lambda U_p, L^p) \geq C'(1 - \rho)^{1/2} s^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2} \kappa_s,$$

where

$$\kappa_s = \begin{cases} 1, & 1 \leq p \leq 2, 1 < q \leq 2, \\ 1, & 2 \leq p < \infty, 2 \leq q \leq \infty, \\ 1, & 1 \leq p \leq 2 \leq q \leq \infty, \\ (\ln s)^{-1/2}, & 1 \leq p \leq 2, q = 1, \\ (\ln s)^{-1/2}, & p = \infty, 2 \leq q \leq \infty. \end{cases}$$

Theorem 5.4. *Let $\lambda : (0, \infty) \rightarrow \mathbb{R}$ be a positive and non-increasing function and let $\Lambda = \{\lambda_{m,n}\}_{m,n \in \mathbb{N}}$, $\lambda_{m,n} = \lambda(|(m,n)|)$. Suppose $1 \leq p \leq 2 \leq q \leq \infty$ and that the multiplier operator Λ is bounded from L^1 to L^2 . Let $\{N_k\}_{k=0}^\infty$ and $\{m_k\}_{k=0}^M$ be sequences of natural numbers such that $N_k < N_{k+1}$, $N_0 = 0$ and $\sum_{k=0}^M m_k \leq \beta$. Then there exists an absolute constant $C > 0$ such that*

$$d_\beta(\Lambda U_p, L^q) \leq C \left(\sum_{k=1}^M \lambda(N_k) \varrho_{m_k} + \sum_{k=M+1}^\infty \lambda(N_k) (\theta_{N_k, N_{k+1}})^{1/p-1/q} \right),$$

where

$$\varrho_{m_k} = \frac{\theta_{N_k, N_{k+1}}^{1/p}}{(m_k)^{1/2}} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty, \end{cases} \quad \text{and} \quad \theta_{N_k, N_{k+1}} = \sum_{j=N_k+1}^{N_{k+1}} \dim \mathcal{H}_j, k \geq 1.$$

Proof. Let $\mathcal{T}_{M_1, M_2} = \bigoplus_{l=M_1+1}^{M_2} \mathcal{H}_l$, $s = \dim \mathcal{T}_{M_1, M_2} = \sum_{l=M_1+1}^{M_2} d_l$, $B_p^s = U_p \cap \mathcal{T}_{M_1, M_2}$, $B_{(p)}^s = J^{-1} B_p^s$, $\lambda(0) = 0$ and $0 < \rho < 1$. The Theorem 5.1 guarantees the existence of a subspace F_k of \mathbb{R}^s , with $\dim F_k = k > \rho s$, such that, for all $\alpha \in F_k$,

$$\|x\|_{(2)} \leq CM (\|\cdot\|_{(q)}) (1 - \rho)^{-1/2} \|x\|_{(q)}^0.$$

If $m = s - k$, then $(1 - \rho)^{-1/2} < (s/m)^{1/2}$. Applying the Theorem 4.3 for $\Lambda_s = Id$, we get

$$\|x\|_{(2)} \leq C \|x\|_{(q)}^0 \left(\frac{s}{m}\right)^{1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty. \end{cases}$$

Therefore, by (1)

$$\begin{aligned} d_m(B_2^s, L^q \cap \mathcal{T}_{M_1, M_2}) &= d^m \left((B_{(q)}^s)^0, (\mathbb{R}^s, \|\cdot\|_{(2)}) \right) \\ &\leq \sup_{x \in (B_{(q)}^s)^0 \cap F_k} \|x\|_{(2)} \\ &\leq C \left(\frac{s}{m}\right)^{1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty. \end{cases} \end{aligned} \quad (18)$$

Let $B_p^{N_k, N_{k+1}} = U_p \cap \mathcal{T}_{N_k, N_{k+1}}$ and for $f \in \Lambda U_p \subseteq L^2(\Omega_d)$, let $S_N(f) = \sum_{(m, n) \in A_N} f * \tilde{Z}^{(m, n)}$, $\phi_{N_k, N_{k+1}}(f) := S_{N_{k+1}}(f) - S_{N_k}(f)$, $k \geq 1$, $\phi_{N_0, N_1}(f) = S_{N_1}(f)$. Since $S_{N_k}(f) \rightarrow f$ in $L^2(\Omega_d)$, then

$$\lim_{j \rightarrow \infty} \left\| \sum_{k=j}^{\infty} \phi_{N_k, N_{k+1}}(f) \right\|_2 = 0.$$

Then it is easy to see that

$$\Lambda U_p \subseteq \bigoplus_{k=0}^{\infty} (\phi_{N_k, N_{k+1}} \circ \Lambda) U_p, \quad 1 \leq p \leq 2.$$

But, for each $\varphi \in U_p$, $\phi_{N_k, N_{k+1}}(\Lambda\varphi) = \Lambda(\phi_{N_k, N_{k+1}}(\varphi))$, therefore

$$\Lambda U_p \subseteq \bigoplus_{k=0}^{\infty} (\Lambda \circ \phi_{N_k, N_{k+1}}) U_p. \quad (19)$$

Now, given $\varphi \in U_p$,

$$\begin{aligned} \|(\Lambda \circ \phi_{N_k, N_{k+1}})\varphi\|_2 &\leq \left\| \sum_{l=N_k+1}^{N_{k+1}} \lambda(l-1) \sum_{(m, n) \in A_l \setminus A_{l-1}} \varphi * \tilde{Z}^{(m, n)} \right\|_2 \\ &\leq \lambda(N_k) \left\| \sum_{l=N_k+1}^{N_{k+1}} \sum_{(m, n) \in A_l \setminus A_{l-1}} \varphi * \tilde{Z}^{(m, n)} \right\|_2 \\ &= \lambda(N_k) \|\phi_{N_k, N_{k+1}}(\varphi)\|_2, \end{aligned} \quad (20)$$

and by Young's inequality

$$\|\phi_{N_k, N_{k+1}}(\varphi)\|_2 \leq \|\varphi\|_p \left\| \sum_{(m, n) \in A_{N_{k+1}} \setminus A_{N_k}} Z_e^{(m, n)} \right\|_{1/(3/2-1/p)}. \quad (21)$$

Using properties of the zonal harmonic functions, we have

$$\begin{aligned} \left\| \sum_{(m, n) \in A_{N_{k+1}} \setminus A_{N_k}} Z_e^{(m, n)} \right\|_2^2 &= \sum_{(m, n) \in A_{N_{k+1}} \setminus A_{N_k}} \int_{\Omega_d} Z_e^{(m, n)} \overline{Z_e^{(m, n)}} d\sigma_d \\ &= \sum_{(m, n) \in A_{N_{k+1}} \setminus A_{N_k}} Z_e^{(m, n)}(e) \\ &= \sum_{(m, n) \in A_{N_{k+1}} \setminus A_{N_k}} \frac{d_{m, n}}{\omega_d} = \frac{\theta_{N_k, N_{k+1}}}{\omega_d} \end{aligned}$$

and thus by (21), for $p = 1$ we obtain $\|\phi_{N_k, N_{k+1}}(\varphi)\|_2 \leq \omega_d^{-1/2} \theta_{N_k, N_{k+1}}^{1/2} \|\varphi\|_1$. Furthermore, for $p = 2$, and $\varphi \in U_2$ we get $\|\phi_{N_k, N_{k+1}}(\varphi)\|_2 \leq \|\varphi\|_2$. Applying the Riesz-Thorin Interpolation theorem to the last two inequalities, it follows for $1 \leq p \leq 2$ that

$$\|\phi_{N_k, N_{k+1}} \varphi\|_2 \leq \omega_d^{1/2-1/p} \theta_{N_k, N_{k+1}}^{1/p-1/2} \|\varphi\|_p \leq \omega_d^{1/2-1/p} \theta_{N_k, N_{k+1}}^{1/p-1/2}.$$

Hence, from (20) and (19),

$$\Lambda U_p \subseteq \bigoplus_{k=0}^{\infty} \omega_d^{1/2-1/p} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/2} B_2^{N_k, N_{k+1}}. \quad (22)$$

By (17) for $2 \leq q \leq \infty$, we have for $\varphi \in B_2^{N_k, N_{k+1}}$ that

$$\|\varphi\|_q = \|\phi_{N_k, N_{k+1}} \varphi\|_q \leq \omega_d^{1/q-1/2} \theta_{N_k, N_{k+1}}^{1/2-1/q} \|\phi_{N_k, N_{k+1}} \varphi\|_2 \leq \omega_d^{1/q-1/2} \theta_{N_k, N_{k+1}}^{1/2-1/q},$$

and therefore $B_2^{N_k, N_{k+1}} \subseteq \omega_d^{1/q-1/2} \theta_{N_k, N_{k+1}}^{1/2-1/q} B_q^{N_k, N_{k+1}}$. Consequently by (22)

$$\Lambda U_p \subseteq \bigoplus_{k=0}^M \omega_d^{1/2-1/p} \theta_{N_k, N_{k+1}}^{1/p-1/2} \lambda(N_k) B_2^{N_k, N_{k+1}} + \bigoplus_{k=M+1}^{\infty} \omega_d^{1/q-1/p} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/q} B_q^{N_k, N_{k+1}}. \quad (23)$$

Finally, using (18), (23) and properties of n -widths, we get

$$\begin{aligned} d_\beta(\Lambda U_p, L^q) &\leq \sum_{k=0}^M \omega_d^{1/2-1/p} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/2} d_{m_k}(B_2^{N_k, N_{k+1}}, L^q \cap \mathcal{T}_{N_k, N_{k+1}}) \\ &\quad + \sum_{k=M+1}^{\infty} \omega_d^{1/q-1/p} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/q} d_0(B_q^{N_k, N_{k+1}}, L^q \cap \mathcal{T}_{N_k, N_{k+1}}) \\ &\leq C \left(\sum_{k=0}^M \lambda(N_k) \varrho_{m_k} + \sum_{k=M+1}^{\infty} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/q} \right). \end{aligned}$$

□

Remark 5.5. We will improve the estimate obtained in the previous theorem, specifying the sequences N_k and m_k . We define $N_1 = N \in \mathbb{N}$ and

$$N_{k+1} = \min\{l \in \mathbb{N} : e\lambda(l) \leq \lambda(N_k)\}.$$

Let $\theta_{N_k, N_{k+1}}$ be as in Theorem 5.4. For $\epsilon > 0$ we define

$$M = \left\lceil \frac{\ln \theta_{N_1, N_2}}{\epsilon} \right\rceil, \quad m_k = \lceil e^{-\epsilon k} \theta_{N_1, N_2} \rceil + 1, \quad k = 1, \dots, M, \quad m_0 = \theta_{0, N}.$$

Hence,

$$\sum_{k=1}^M m_k = \sum_{k=1}^M ([e^{-\epsilon k} \theta_{N_1, N_2}] + 1) \leq M + \theta_{N_1, N_2} \sum_{k=0}^M (e^{-\epsilon})^k \leq C_\epsilon \theta_{N_1, N_2}, \quad (24)$$

where $C_\epsilon > 0$, depends only on ϵ . Applying the previous theorem for

$$\beta = m_0 + \sum_{k=1}^M m_k = \sum_{s=0}^N \dim \mathcal{H}_s + \sum_{k=1}^M m_k,$$

and writing $d_\beta = d_\beta(\Lambda U_p, L^q)$, we have

$$\begin{aligned} d_\beta &\leq C \sum_{k=1}^M \lambda(N_k) \frac{\theta_{N_k, N_{k+1}}^{1/p}}{m_k^{1/2}} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty, \end{cases} \\ &\quad + C \sum_{k=M+1}^{\infty} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/q} \\ &\leq eC\lambda(N) \sum_{k=1}^M e^{-k(1-\epsilon/2)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty, \end{cases} \\ &\quad + eC\lambda(N) \sum_{k=M+1}^{\infty} e^{-k} \theta_{N_k, N_{k+1}}^{1/p-1/q}. \end{aligned}$$

Definition 5.6. Let N_k, M and $\theta_{N_k, N_{k+1}}$ be as in the previous remark. We say that $\Lambda = \{\lambda_{m,n}\}_{m,n \in \mathbb{N}} \in K_{\epsilon,p}$, $\epsilon > 0$, $1 \leq p \leq 2$, if $\lambda(k+1) \leq \lambda(k)$, $N_{k+1} > N_k$, for all $k \in \mathbb{N}$ and if for all $N \in \mathbb{N}$, there exists a constant $C_{\epsilon,p}$, depending only on d, ϵ and p , such that

$$\sum_{k=1}^M e^{-k(1-\epsilon/2)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} \leq C_{\epsilon,p} \theta_{N_1, N_2}^{1/p-1/2}. \quad (25)$$

Corollary 5.7. Let $\Lambda = \{\lambda_{m,n}\}_{m,n \in \mathbb{N}}$ and $\theta_{N_k, N_{k+1}}$ be as in the previous theorem, and let $\epsilon > 0$, $N_k, M, \{m_k\}_{k=0}^M$ and β be as in the previous remark. Let $1 \leq p \leq 2 \leq q \leq \infty$ and suppose that $\Lambda \in K_{\epsilon,p}$, for a fixed $\epsilon > 0$. Then there exists a constant $C_{\epsilon,p} > 0$, such that

$$\begin{aligned} d_\beta(\Lambda U_p, L^q) &\leq C_{\epsilon,p} \lambda(N) \theta_{N_1, N_2}^{1/p-1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1 \leq k \leq M} (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty, \end{cases} \\ &+ C_{\epsilon,p} \lambda(N) \sum_{k=M+1}^{\infty} e^{-k} \theta_{N_k, N_{k+1}}^{1/p-1/q}. \end{aligned}$$

Remark 5.8. We note that the Theorems 5.2 and 5.4, and the Corollaries 5.3 and 5.7 hold if we consider a function λ such that $t \rightarrow |\lambda(t)|$ is a positive and non-increasing function. Simply we change $\lambda(t)$ by $|\lambda(t)|$.

6 Applications

Let $\lambda^{(1)}, \lambda^{(2)} : [0, \infty) \rightarrow \mathbb{R}$ be defined by $\lambda^{(1)}(t) = t^{-\gamma} (\ln t)^{-\xi}$ for $t > 1$, $\lambda^{(1)}(t) = 0$ for $0 \leq t \leq 1$, and $\lambda^{(2)}(t) = e^{-\gamma t^r}$, $t \geq 0$, where $\gamma, \xi, r \in \mathbb{R}$, $\gamma, r > 0$, $\xi \geq 0$. In this section we consider the multiplier operators associated with the sequences $\Lambda^{(1)} = (\lambda_{m,n}^{(1)})_{m,n \in \mathbb{N}}$ and $\Lambda^{(2)} = (\lambda_{m,n}^{(2)})_{m,n \in \mathbb{N}}$ where $\lambda_{m,n}^{(1)} = \lambda^{(1)}(|(m,n)|)$, $\lambda_{m,n}^{(2)} = \lambda^{(2)}(|(m,n)|)$, $|(m,n)| = \max\{m,n\}$.

Remark 6.1. We will prove that, if $\gamma > (2d-1)/2$, then the multiplier operator $\Lambda^{(1)}$ is bounded from L^1 to L^2 . Given $\varphi \in U_1$, for each $k \in \mathbb{N}^*$, let

$$\varphi_k = \sum_{l=1}^k \sum_{(m,n) \in A_l \setminus A_{l-1}} \varphi * \tilde{Z}^{(m,n)}.$$

We obtain that

$$\Lambda^{(1)} \varphi_k = \sum_{l=1}^k \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n} \varphi * \tilde{Z}^{(m,n)},$$

and thus since $d_l \asymp l^{2d-2}$, we obtain

$$\begin{aligned} \|\Lambda^{(1)} \varphi_k\|_2^2 &= \int_{\Omega_d} \Lambda^{(1)} \varphi_k \overline{\Lambda^{(1)} \varphi_k} d\sigma_d = \sum_{l=1}^k \sum_{(m,n) \in A_l \setminus A_{l-1}} |\lambda_{m,n}|^2 \|\varphi * \tilde{Z}^{(m,n)}\|_2^2 \\ &\leq \sum_{l=1}^k |\lambda(l-1)|^2 \sum_{(m,n) \in A_l \setminus A_{l-1}} \|\varphi * \tilde{Z}^{(m,n)}\|_2^2 \leq \sum_{l=1}^k |\lambda(l-1)|^2 \sum_{(m,n) \in A_l \setminus A_{l-1}} \frac{d_{m,n}}{\omega_d} \\ &= \frac{1}{\omega_d} \sum_{l=1}^k \lambda^2(l-1) d_l \leq C_1 \sum_{l=1}^k (l-1)^{-2\gamma} (\ln(l-1))^{-2\xi} l^{2d-2} \\ &\leq C_2 \sum_{l=1}^k l^{-2\gamma+2d-2} \leq C_3. \end{aligned}$$

Therefore

$$\|\Lambda^{(1)} \varphi\|_2 = \sup_k \|\Lambda^{(1)} \varphi_k\|_2 \leq (C_3)^{1/2},$$

and thus $\Lambda^{(1)}$ is bounded from L^1 to L^2 .

Theorem 6.2. For $1 \leq p \leq \infty$, $2 \leq q \leq \infty$, $\gamma > (2d-1)/p$ if $1 \leq p \leq 2$ and $\gamma > (2d-1)/2$ if $p \geq 2$, and for all $m \in \mathbb{N}$, we have

$$d_m(\Lambda^{(1)} U_p, L^q) \ll m^{-\gamma/(2d-1)+(1/p-1/2)+} (\ln m)^{-\xi} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln m)^{1/2}, & q = \infty. \end{cases} \quad (26)$$

Proof. Suppose $1 \leq p \leq 2 \leq q \leq \infty$, fix $\delta > 0$ and let $\lambda_1, \lambda_2 : (0, \infty) \rightarrow \mathbb{R}$ be defined by $\lambda_1(t) = t^{-\gamma}$ and $\lambda_2(t) = t^{-\gamma-\delta}$. Let $a > 1$ and let $b, b_1, b_2 \in \mathbb{R}$ such that $e\lambda(b) = \lambda(a)$, $e\lambda_1(b_1) = \lambda_1(a)$ and $e\lambda_2(b_2) = \lambda_2(a)$. We have that $b, b_1, b_2 > a$ and that $b_1 = e^{1/\gamma}a$, $b_2 = e^{1/(\gamma+\delta)}a$ and $b_1 = b(\ln b/\ln a)^{\xi/\gamma}$. Since $b > a$, $b_1 > b$ and $b > b_2$, we obtain $e^{1/(\gamma+\delta)}a < b < e^{1/\gamma}a$.

Taking $a = N_k$, we get $e^{1/(\gamma+\delta)}N_k < b < e^{1/\gamma}N_k$. But $N_k < b \leq N_{k+1} < b+1$ and therefore

$$e^{1/(\gamma+\delta)}N_k \leq N_{k+1} \leq e^{1/\gamma}N_k + 1, \quad k \geq 1. \quad (27)$$

Considering that $d_l = \dim \mathcal{H}_l \asymp l^{2d-2}$, integrating the function x^{2d-2} , we obtain

$$\theta_{N_k, N_{k+1}} \asymp \sum_{s=N_k+1}^{N_{k+1}} s^{2d-2} \leq \sum_{s=0}^{N_{k+1}-1} s^{2d-2} + N_{k+1}^{2d-2} \asymp N_{k+1}^{2d-1}. \quad (28)$$

Furthermore using (27), we get

$$\begin{aligned} \theta_{N_k, N_{k+1}} &\asymp \sum_{j=N_k+1}^{N_{k+1}} j^{2d-2} \\ &\gg N_{k+1}^{2d-1} - \frac{N_{k+1}^{2d-1}}{2^{2d-1}} \left(1 + \frac{N_k}{N_{k+1}}\right)^{2d-1} \\ &\geq N_{k+1}^{2d-1} \left[1 - 2^{-2d+1} \left(1 + (2d-1)e^{-1/\gamma+\delta}\right)\right] \\ &+ N_{k+1}^{2d-1} \left[2^{-2d+1} \left(\frac{(2d-1)(2d-2)}{2!} e^{-2/\gamma+\delta} + \dots + e^{-(2d+1)/(\gamma+\delta)}\right)\right] \\ &= N_{k+1}^{2d-1} (1 - C_{\gamma, \delta, d}). \end{aligned} \quad (29)$$

Thus, by (29) and (28), it follows that

$$\theta_{N_k, N_{k+1}} \asymp N_{k+1}^{2d-1}, \quad k \geq 1, \quad (30)$$

and using (27) we obtain

$$e^{k/(\gamma+\delta)}N \leq N_{k+1} \leq C_\gamma e^{k/\gamma}N, \quad k \geq 1. \quad (31)$$

Let $M = \lceil \epsilon^{-1} \ln \theta_{N_1, N_2} \rceil$ be as in Remark 5.5. We have that $M \asymp \epsilon^{-1} \ln N \asymp \epsilon^{-1} \ln s$. By (30) and (31),

$$\begin{aligned} \sigma &= \sum_{k=M+1}^{\infty} e^{-k} (\theta_{N_k, N_{k+1}})^{1/p-1/q} \ll \sum_{k=M+1}^{\infty} e^{-k} \left(e^{k/\gamma}N\right)^{(2d-1)(1/p-1/q)} \\ &= N^{(2d-1)(1/p-1/q)} \sum_{k=M+1}^{\infty} e^{-k} e^{k(2d-1)(1/p-1/q)/\gamma} = N^{(2d-1)(1/p-1/q)} \sum_{k=M+1}^{\infty} e^{-k(1-(2d-1)(1/p-1/q)/\gamma)}. \end{aligned}$$

Since $M \asymp \epsilon^{-1} \ln N$, then there exists a positive constant C such that

$$\sigma \ll N^{(2d-1)(1/p-1/q)} \sum_{k=\lceil C\epsilon^{-1} \ln N \rceil}^{\infty} e^{-k(1-(2d-1)(1/p-1/q)/\gamma)}, \quad (32)$$

and since $1 \leq p \leq 2 \leq q \leq \infty$ and $\gamma > (2d-1)(1/p-1/q)$, we have that $1 - (1/p-1/q)(2d-1)/\gamma > 0$ and thus

$$\begin{aligned} \sum_{k=\lceil C\epsilon^{-1} \ln N \rceil}^{\infty} e^{-k(1-(2d-1)(1/p-1/q)/\gamma)} &\asymp \frac{e^{-(1-(2d-1)(1/p-1/q)/\gamma)C\epsilon^{-1} \ln N}}{1 - e^{-1-(2d-1)(1/p-1/q)/\gamma}} \\ &\ll N^{-C(1-(2d-1)(1/p-1/q)/\gamma)/\epsilon}. \end{aligned}$$

Therefore we obtain by (32) that

$$\sigma \ll N^{(2d-1)(1/p-1/q) - C(1-(2d-1)(1/p-1/q)/\gamma)/\epsilon}.$$

So, taking $0 < \epsilon < C(1 - (2d-1)(1/p-1/q)/\gamma)/(2d-1)(1/p-1/q)$, we get

$$\sigma \ll 1. \quad (33)$$

Next we will prove that $\Lambda^{(1)} \in K_{\epsilon,p}$ for some $\epsilon > 0$. Remembering that $M \asymp \epsilon^{-1} \ln N$, by (30) and (31), it follows that

$$\begin{aligned} \sum_{k=1}^M e^{-k(1-\epsilon/2)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} &\leq \sum_{k=1}^{[C\epsilon^{-1} \ln N]} e^{-k(1-\epsilon/2)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} \\ &\ll \sum_{k=1}^{[C\epsilon^{-1} \ln N]} e^{-k(1-\epsilon/2)} \frac{(e^{k/\gamma} N)^{(2d-1)/p}}{(e^{1/(\gamma+\delta)} N)^{(2d-1)/2}} \\ &\ll N^{(2d-1)(1/p-1/2)} \sum_{k=1}^{[C\epsilon^{-1} \ln N]} e^{-k(1-\epsilon/2-(2d-1)/\gamma p)}. \end{aligned}$$

Since $\gamma/(2d-1) > 1/p$, we have $t = -(1-\epsilon/2-(2d-1)/\gamma p) < 0$ if $0 < \epsilon/2 < 1 - (2d-1)/\gamma p$ and consequently

$$\sum_{k=1}^M e^{-k(1-\epsilon/2)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} \leq C_{\epsilon,p} N^{(2d-1)(1/p-1/2)}.$$

By (31) there exists a constant C_1 , such that $C_1 e^{(2d-1)/(\gamma+\delta)} N^{2d-1} \leq \theta_{N_1, N_2}$ and then

$$\sum_{k=1}^M e^{-k(1-\epsilon/2)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} \leq C'_{\epsilon,p} \theta_{N_1, N_2}^{1/p-1/2},$$

that is, $\Lambda^{(1)} \in K_{\epsilon,p}$. Now, by Remark 6.1 we have that $\Lambda^{(1)}$ is bounded from $L^1(\Omega_d)$ to $L^2(\Omega_d)$, and hence from Corollary 5.7 and (33), we obtain

$$d_\beta(\Lambda^{(1)} U_p, L^q) \ll \lambda(N) \theta_{N_1, N_2}^{1/p-1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1 \leq k \leq M} (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty, \end{cases} + \lambda(N).$$

For $1 \leq k \leq M$, using (30), (31), the definition of M , and the fact that $s \asymp N^{2d-1}$, we obtain

$$\begin{aligned} \theta_{N_k, N_{k+1}} &\ll (e^{k/\gamma} N)^{2d-1} \leq (e^{M/\gamma} N)^{2d-1} \leq e^{(2d-1)(C \ln N)/\gamma\epsilon} N^{2d-1} \\ &= N^{(2d-1)+(2d-1)C/\gamma\epsilon} = (N^{2d-1})^{1+C/\gamma\epsilon} \ll s^{1+C/\gamma\epsilon}, \end{aligned}$$

and thus $\ln(\theta_{N_k, N_{k+1}}) \ll \ln(s^{1+C/\gamma\epsilon}) = (1+C/\gamma\epsilon) \ln s \ll \ln s$. Furthermore, by (30) and (31), it follows that $\theta_{N_1, N_2}^{1/p-1/2} \ll N^{(2d-1)(1/p-1/2)}$ and therefore

$$\begin{aligned} d_\beta(\Lambda^{(1)} U_p, L^q) &\ll \lambda(N) N^{(2d-1)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty, \end{cases} \\ &= N^{-\gamma} N^{(2d-1)(1/p-1/2)} (\ln N)^{-\xi} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty. \end{cases} \end{aligned}$$

Now, since $s \asymp N^{2d-1}$, then $N^{-\gamma} \asymp s^{-\gamma/(2d-1)}$, $N^{(2d-1)(1/p-1/2)} \asymp s^{(1/p-1/2)}$ and $(\ln N)^{-\xi} \asymp (\ln N^d)^{-\xi} \asymp (\ln s)^{-\xi}$. Consequently

$$d_\beta(\Lambda^{(1)} U_p, L^q) \ll s^{-\gamma/(2d-1)+(1/p-1/2)} (\ln s)^{-\xi} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty. \end{cases} \quad (34)$$

From Remark 5.5, $\beta \asymp s + \sum_{j=1}^M e^{-\epsilon j} \theta_{N_1, N_2}$ and keeping in mind that $s \asymp N^{2d-1}$ and $M \asymp \epsilon^{-1} \ln N$, by (30) and (31) we get

$$\beta \ll N^{2d-1} + e^{(2d-1)/\gamma} N^{2d-1} \sum_{j=1}^{[C\epsilon^{-1} \ln N]} (e^{-\epsilon})^j \ll N^{2d-1},$$

that is, there exists a constant $C_2 \in \mathbb{N}$ such that $\beta \leq C_2 N^{2d-1}$. Given $m \in \mathbb{N}$, let $N \in \mathbb{N}$ such that $C_2 N^{2d-1} \leq m \leq C_2(N+1)^{2d-1}$. It follows by (34) that

$$d_m(\Lambda^{(1)} U_p, L^q) \ll m^{-\gamma/(2d-1)+(1/p-1/2)} (\ln m)^{-\xi} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln m)^{1/2}, & q = \infty, \end{cases}$$

for $1 \leq p \leq 2$. The case $2 \leq p \leq \infty$, follows because $\Lambda(U_p) \subset \Lambda(U_2)$. \square

Theorem 6.3. For $\gamma > (2d - 1)/2$ and for all $m \in \mathbb{N}$, we have

$$d_m(\Lambda^{(1)}U_p, L^q) \gg m^{-\gamma/(2d-1)}(\ln m)^{-\xi} \kappa_m, \quad (35)$$

where

$$\kappa_m = \begin{cases} 1, & 1 \leq p \leq 2, 1 < q \leq 2, \\ 1, & 2 \leq p < \infty, 2 \leq q \leq \infty, \\ 1, & 1 \leq p \leq 2 \leq q \leq \infty, \\ (\ln m)^{-1/2}, & 1 \leq p \leq 2, q = 1, \\ (\ln m)^{-1/2}, & p = \infty, 2 \leq q \leq \infty. \end{cases}$$

Proof. We have that $d_l \asymp l^{2d-2}$ and $s = \dim \mathcal{T}_N \asymp N^{2d-1}$, and thus

$$\begin{aligned} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2} &\asymp \left(\sum_{l=1}^N (l^{-\gamma} (\ln l)^{-\xi})^{-2} l^{2d-2} \right)^{-1/2} \geq \left(\sum_{l=1}^N N^{2\gamma+2d-2} (\ln N)^{2\xi} \right)^{-1/2} \\ &\asymp s^{-\gamma/(2d-1)-1/2} (\ln s)^{-\xi}. \end{aligned}$$

Applying the Corollary 5.7 with $\rho = 1/2$, we obtain

$$\begin{aligned} d_{\lfloor (s-2)/2 \rfloor}(\Lambda^{(1)}U_p, L^q) &\geq C(1-1/2)^{1/2} s^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2} \kappa_s \\ &\gg s^{1/2} s^{-\gamma/(2d-1)-1/2} (\ln s)^{-\xi} \kappa_s = s^{-\gamma/(2d-1)} (\ln s)^{-\xi} \kappa_s. \end{aligned}$$

Finally, taking $m = \lfloor s/3 \rfloor \leq \lfloor (s-2)/2 \rfloor$ we have that $m \asymp s$ and thus

$$d_m(\Lambda^{(1)}U_p, L^q) \geq d_{\lfloor (s-2)/2 \rfloor}(\Lambda^{(1)}U_p, L^q) \geq s^{-\gamma/(2d-1)} (\ln s)^{-\xi} \kappa_s \gg m^{-\gamma/(2d-1)} (\ln m)^{-\xi} \kappa_m. \quad \square$$

Remark 6.4. We will prove that the multiplier operator $\Lambda^{(2)}$ is bounded from L^p to L^q , for $1 \leq p, q \leq \infty$. We will show that this result is true for $p = 1$ and $q = \infty$ and the other cases will follow immediately from inequalities between norms. Given $\varphi \in U_1$, we have that

$$\Lambda^{(2)}\varphi \sim \sum_{l=1}^{\infty} \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n} \varphi * \tilde{Z}^{(m,n)}$$

and thus

$$\|\Lambda^{(2)}\varphi\|_{\infty} \leq \sup_{\omega \in \Omega_d} \sum_{l=1}^{\infty} \left| \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n} \varphi * \tilde{Z}^{(m,n)}(\omega) \right| \leq \sum_{l=1}^{\infty} \|a_l\|_{\infty},$$

where $a_l(\omega) = \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n} \varphi * \tilde{Z}^{(m,n)}(\omega)$. Let

$$D_l(\omega) = \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n} Z_e^{(m,n)}(\omega), \quad \overline{D}_l(\omega) = \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n}^{1/2} Z_e^{(m,n)}(\omega).$$

We have that $D_l = \overline{D}_l * \overline{D}_l$ and $a_l = \varphi * D_l$. It follows by Young's inequality that

$$\begin{aligned} \|D_l\|_{\infty} &= \|\overline{D}_l * \overline{D}_l\|_{\infty} \leq \|\overline{D}_l\|_2 \|\overline{D}_l\|_2 \\ &= \sum_{(m,n) \in A_l \setminus A_{l-1}} \lambda_{m,n} \frac{d_{m,n}}{\omega_d} \\ &\leq \frac{\lambda(l-1)}{\omega_d} d_l \leq C_1 \lambda(l-1) l^{2d-2} \end{aligned}$$

and hence

$$\|a_l\|_{\infty} = \|\varphi * D_l\|_{\infty} \leq \|\varphi\|_1 \|D_l\|_{\infty} \leq C_1 \lambda(l-1) l^{2d-2}.$$

Moreover there is a constant C_2 such that $e^{-\gamma l^r} l^{2d} \leq C_2$, for all $l \in \mathbb{N}$ and therefore

$$\|a_l\|_\infty \leq C_1 \lambda (l-1) l^{2d-2} = C_1 e^{-\gamma(l-1)^r} l^{2d} l^{-2} \leq C_3 e^{-\gamma(l-1)^r} (l-1)^{2d} l^{-2} \leq C_4 l^{-2}.$$

Thus

$$\|\Lambda^{(2)}\varphi\|_\infty \leq \sum_{l=1}^{\infty} \|a_l\|_\infty \leq \sum_{l=1}^{\infty} C_4 l^{-2} = C_5,$$

so $\Lambda^{(2)}$ is bounded from $L^1(\Omega_d)$ to $L^\infty(\Omega_d)$.

Theorem 6.5. Consider the sequences $\phi_k = \dim \mathcal{T}_k$, $\psi_k = \phi_k - \phi_k^{1-r/(2d-1)} - 1$ and κ_k as in Theorem 6.3. Then

$$d_{[\psi_k]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\phi_k^{r/(2d-1)}} \kappa_k, \quad r > 0, \quad k \in \mathbb{N}, \quad (36)$$

$$d_{[\psi_k]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\psi_k^{r/(2d-1)}} \kappa_k, \quad 0 < r \leq 2d-1, \quad k \in \mathbb{N}, \quad (37)$$

$$d_{\phi_k-1}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\phi_k^{r/(2d-1)}} \kappa_k, \quad r > 2d-1, \quad k \in \mathbb{N}, \quad (38)$$

$$d_k(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}k^{r/(2d-1)}} \kappa_k, \quad 0 < r \leq 1, \quad k \in \mathbb{N}, \quad (39)$$

where

$$\mathcal{R} = \gamma \left(\frac{d!(d-1)!}{2} \right)^{r/(2d-1)}.$$

Proof. For $\theta, \eta, r > 0$ and $\eta \geq r-1$, we have

$$\sum_{k=1}^N e^{\theta k^r} k^\eta = \sum_{k=1}^N \left(\frac{1}{\theta r} k^{1-r+\eta} \right) e^{\theta k^r} \theta r k^{r-1} \ll N^{1-r+\eta} \int_1^{N+1} e^{\theta x^r} \theta r x^{r-1} dx \leq N^{1-r+\eta} e^{\theta(N+1)^r}. \quad (40)$$

Since there is a positive constant C_1 such that $(N+1)^r \leq N^r + C_1 N^{-1}$, then

$$e^{\theta(N+1)^r} \leq e^{\theta C_1} e^{\theta N^r} = C_2 e^{\theta N^r}.$$

Let us fix $N \in \mathbb{N}$ and let $s = \dim \mathcal{T}_N$, $D = 2/d!(d-1)!$. By the Proposition 4.1 we have

$$DN^{2d-1} \leq s \leq DN^{2d-1} + C_3 N^{2d-2}.$$

From the above estimates, we obtain $N^r \leq D^{-r/(2d-1)} s^{r/(2d-1)}$ and $N^r \geq D^{-r/(2d-1)} s^{r/(2d-1)} - C_4 N^{r-1}$ for a positive constant C_4 and for $r > 0$. Therefore, for $\mathcal{R} = \gamma D^{-r/(2d-1)}$ and the fact that $s \asymp N^{2d-1}$, we get

$$-\mathcal{R} s^{r/(2d-1)} \leq -\gamma N^r \leq -\mathcal{R} s^{r/(2d-1)} + C_5 s^{(r-1)/(2d-1)}, \quad (41)$$

for $r > 0$. The above estimates also hold if we put $N+1$ in the place of N . If $\rho = 1 - s^{-r/(2d-1)}$, using (40), we get

$$(1-\rho)^{1/2} s^{1/2} \left(\sum_{l=1}^N (\lambda(l))^{-2} d_l \right)^{-1/2} \gg s^{-r/2(2d-1)+1/2} \left(N^{1-r+2d-2} e^{2\gamma N^r} \right)^{-1/2} \gg e^{-\gamma N^r},$$

and hence it follows by Corollary 5.7 and (41) that

$$d_{[s-s^{1-r/(2d-1)}-1]}(\Lambda^{(2)}U_p, L^q) = d_{[\rho s-1]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\gamma N^r} \kappa_s \gg e^{-\mathcal{R} s^{r/(2d-1)}} \kappa_s.$$

The above estimate says that, for $r > 0$

$$d_{[\psi_N]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\phi_N^{r/(2d-1)}} \kappa_{\phi_N},$$

but, since $\phi_N = \dim \mathcal{T}_N \asymp N^{2d-1}$, we have that $\kappa_{\phi_N} \asymp \kappa_N$ and therefore we get (36). Furthermore, if $r > 2d-1$, then $[\psi_N] = [\phi_N - \phi_N^{1-r/(2d-1)} - 1] = \phi_N - 1$ and thus (38) follows from (36).

Consider $0 < r \leq 2d-1$. Then

$$\left(1 - s^{-r/(2d-1)} \right)^{r/(2d-1)} = 1 - s^{-r/(2d-1)} \frac{r}{2d-1} \left(1 + \frac{2d-1-r}{2!(2d-1)} s^{-r/(2d-1)} + \dots \right) = 1 - s^{-r/(2d-1)} S_s,$$

and there exists a positive constant C_r such that $0 < S_s \leq C_r$ for all $s \in \mathbb{N}$. Therefore

$$\begin{aligned} (\psi_N + 1)^{r/(2d-1)} &= \left(s - s^{1-r/(2d-1)} \right)^{r/(2d-1)} = s^{r/(2d-1)} \left(1 - s^{-r/(2d-1)} \right)^{r/(2d-1)} \\ &\geq s^{r/(2d-1)} \left(1 - C_r s^{-r/(2d-1)} \right) = s^{r/(2d-1)} - C_r = \phi_N^{r/(2d-1)} - C_r, \end{aligned}$$

thus

$$e^{-\mathcal{R}(\psi_N+1)^{r/(2d-1)}} \leq e^{-\mathcal{R}(\phi_N^{r/(2d-1)} - C_r)} \ll e^{-\mathcal{R}\phi_N^{r/(2d-1)}}. \quad (42)$$

and from (36) we get

$$d_{[\psi_N]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}(\psi_N+1)^{r/(2d-1)}} \kappa_N. \quad (43)$$

Now let $0 < r \leq 1$. Using the Mean Value Theorem we get $(N+1)^r - N^r = r(N+c)^{r-1} < rN^{r-1} \leq r$, and then $1 < e^{-\gamma N^r} / e^{-\gamma(N+1)^r} \leq e^{\gamma r}$. From (36) and (41) it follows that

$$\begin{aligned} d_{[\psi_{N+1}]}(\Lambda^{(2)}U_p, L^q) &\gg e^{-\gamma(N+1)^r} \kappa_{N+1} \gg e^{-\gamma N^r} \kappa_N \\ &\gg e^{-\mathcal{R}s^{r/(2d-1)}} \kappa_N \\ &= e^{-\mathcal{R}\phi_N^{r/(2d-1)}} \kappa_N, \end{aligned}$$

and using (42) we obtain

$$d_{[\psi_{N+1}]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\phi_N^{r/(2d-1)}} \kappa_N \gg e^{-\mathcal{R}(\psi_N+1)^{r/(2d-1)}} \kappa_N. \quad (44)$$

If $k \in \mathbb{N}$ and $[\psi_N] < k \leq [\psi_{N+1}]$, then from (44)

$$\begin{aligned} d_k(\Lambda^{(2)}U_p, L^q) &\gg d_{[\psi_{N+1}]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}(\psi_N+1)^{r/(2d-1)}} \kappa_N \\ &\gg e^{-\mathcal{R}\psi_N^{r/(2d-1)}} \kappa_N \geq e^{-\mathcal{R}k^{r/(2d-1)}} \kappa_N \\ &\gg e^{-\mathcal{R}k^{r/(2d-1)}} \kappa_k \end{aligned}$$

and thus we proved (39).

Finally, if $0 < r \leq 2d-1$, we get $e^{\mathcal{R}l^{r/(2d-1)}} \asymp e^{\mathcal{R}(l+1)^{r/(2d-1)}}$, $l \in \mathbb{N}$ and thus it follows from (43) that

$$d_{[\psi_N]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}(\psi_N+1)^{r/(2d-1)}} \kappa_N \gg e^{-\mathcal{R}\psi_N^{r/(2d-1)}} \kappa_N,$$

that is, we proved (37). \square

Theorem 6.6. *Let ϕ_k be as in Theorem 6.5. Then for $0 < r \leq 1$, $1 \leq p \leq \infty$, $2 \leq q \leq \infty$ and for all $k \in \mathbb{N}$, we have*

$$d_k(\Lambda^{(2)}U_p, L^q) \ll e^{-\mathcal{R}k^{r/(2d-1)}} k^{(1-r/(2d-1))(1/p-1/2)+} \begin{cases} 1, & 2 \leq q < \infty, \\ (\ln k)^{1/2}, & q = \infty, \end{cases} \quad (45)$$

and for all $r > 1$ and all $k \in \mathbb{N}$,

$$d_{\phi_k}(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma k^r} \begin{cases} k^{(2d-2)(1/p-1/q)}, & 1 \leq p \leq 2 \leq q \leq \infty, \\ k^{(2d-2)(1/2-1/q)}, & 2 \leq p, q \leq \infty, \end{cases} \quad (46)$$

where \mathcal{R} is the constant given in Theorem 6.5.

Proof. Suppose $0 < r \leq 1$ and let $\lambda(x) = e^{-\gamma x^r}$. For $k \in \mathbb{N}$, let $x_k \in \mathbb{R}$, $e\lambda(x_k) = \lambda(N_k)$. Then $x_k = (N_k^r + 1/\gamma)^{1/r}$ and $N_k < x_k \leq N_{k+1} < x_k + 1$. Therefore

$$(N_k^r + 1/\gamma)^{1/r} \leq N_{k+1} \leq (N_k^r + 1/\gamma)^{1/r} + 1. \quad (47)$$

We have that

$$\begin{aligned} N_{k+1}^r &\leq (N_k^r + 1/\gamma) \left[1 + r(N_k^r + 1/\gamma)^{-1/r} + \frac{r(r-1)}{2!} (N_k^r + 1/\gamma)^{-2/r} + \dots \right] \\ &\leq (N_k^r + 1/\gamma) \left[1 + C_1(N_k^r + 1/\gamma)^{-1/r} \right] = (N_k^r + 1/\gamma) + C_1(N_k^r + 1/\gamma)^{1-1/r} \end{aligned}$$

and since $0 < r \leq 1$, then $(N_k^r + 1/\gamma)^{1-1/r} \leq 1$ and therefore $N_{k+1}^r \leq N_k^r + C_1$. Thus

$$N_k^r + \frac{1}{\gamma} \leq N_{k+1}^r \leq N_k^r + C_1$$

and using that $N_1 = N$, we show that

$$N^r + \frac{k}{\gamma} \leq N_{k+1}^r \leq N^r + C_1 k. \quad (48)$$

By (47), $N_{k+1} \asymp (N_k^r + 1/\gamma)^{1/r} = N_k (1 + N_k^{-r}/\gamma)^{1/r}$, and then

$$N_{k+1} \asymp N_k \left(1 + \frac{1}{r\gamma} N_k^{-r} + \frac{1-r}{2r^2\gamma^2} N_k^{-2r} + \frac{(1-r)(1-2r)}{6r^3\gamma^3} N_k^{-3r} + \dots \right)^{1/r}.$$

Thus $N_{k+1} - N_k \asymp N_k^{1-r}$. On the other hand

$$\theta_{N_k, N_{k+1}} \asymp N_k^{2d-r-1}, \quad (49)$$

consequently, from (48), (49) and for $\epsilon > 0$ and k sufficiently large, we obtain

$$\begin{aligned} \left(\frac{\theta_{N_k, N_{k+1}}}{\theta_{N_1, N_2}} \right)^{r/(2d-r-1)} &\asymp \left(\frac{N_k^{2d-r-1}}{N^{2d-r-1}} \right)^{r/(2d-r-1)} = N^{-r} N_k^r \\ &\leq 1 + C_1 k N^{-r} \leq 1 + C_1 k. \end{aligned}$$

Hence, for any $\delta > 0$, there exists $C_2 > 0$ such that

$$\left(\frac{\theta_{N_k, N_{k+1}}}{\theta_{N_1, N_2}} \right)^{r/(2d-r-1)} \leq C_2 e^{\delta k}, \quad k \geq 0,$$

and taking $\delta' = \delta(2d-r-1)/r$, we get

$$\frac{\theta_{N_k, N_{k+1}}}{\theta_{N_1, N_2}} \leq C_2' e^{\delta' k}, \quad k \geq 0. \quad (50)$$

For $1 \leq p \leq 2$,

$$\sum_{k=1}^M e^{-k(1-\epsilon)} \left(\frac{\theta_{N_k, N_{k+1}}}{\theta_{N_1, N_2}} \right)^{1/p} \leq (C_2')^{1/p} \sum_{k=1}^M e^{-k(1-\epsilon)} e^{\delta' k/p} \leq (C_2')^{1/p} \sum_{k=1}^{\infty} e^{-k(1-\epsilon-\delta'/p)} \leq C_3,$$

where ϵ, p, δ are chosen satisfying $\epsilon + \delta'/p < 1$. Thus

$$\sum_{k=1}^M e^{-k(1-\epsilon)} \frac{\theta_{N_k, N_{k+1}}^{1/p}}{\theta_{N_1, N_2}^{1/2}} = \sum_{k=1}^M e^{-k(1-\epsilon)} \left(\frac{\theta_{N_k, N_{k+1}}}{\theta_{N_1, N_2}} \right)^{1/p} \theta_{N_1, N_2}^{1/p-1/2} \leq C_3 \theta_{N_1, N_2}^{1/p-1/2},$$

and then $\Lambda^{(2)} \in K_{2\epsilon, p}$. Therefore it follows from Corollary 5.7 that

$$\begin{aligned} d_\beta(\Lambda^{(2)} U_p, L^q) &\leq C_{\epsilon, p} \lambda(N) \theta_{N_1, N_2}^{1/p-1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1 \leq k \leq M} (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty, \end{cases} \\ &+ C_{\epsilon, p} \lambda(N) \sum_{k=M+1}^{\infty} e^{-k} \theta_{N_k, N_{k+1}}^{1/p-1/q}. \end{aligned} \quad (51)$$

Furthermore, by (49) and (50) we have that

$$\begin{aligned} \lambda(N) \sum_{k=M+1}^{\infty} e^{-k} \theta_{N_k, N_{k+1}}^{1/p-1/q} &= \lambda(N) \theta_{N_1, N_2}^{1/p-1/q} \sum_{k=M+1}^{\infty} e^{-k} \left(\frac{\theta_{N_k, N_{k+1}}}{\theta_{N_1, N_2}} \right)^{1/p-1/q} \\ &\leq C_4 \lambda(N) N^{(2d-r-1)(1/p-1/q)} \sum_{k=M+1}^{\infty} e^{-k} e^{\delta'' k} \\ &= C_4 \lambda(N) N^{(2d-r-1)(1/p-1/q)} \left(e^{-(1-\delta'')} \right)^{M+1} \frac{1}{1 - e^{-(1-\delta'')}} \\ &\leq C_6 \lambda(N) N^{(2d-r-1)(1/p-1/q)} \left(e^{-(1-\delta'')} \right)^M, \end{aligned}$$

where $\delta'' = \delta'(1/p-1/q)$, $\delta'' < 1$. We have that $M \asymp \epsilon^{-1} \ln \theta_{N_1, N_2}$ and $\theta_{N_1, N_2} \asymp N^{2d-r-1}$, and hence

$$\lambda(N) \sum_{k=M+1}^{\infty} e^{-k} \theta_{N_k, N_{k+1}}^{1/p-1/q} \leq C_6 \lambda(N) N^{(2d-r-1)(1/p-1/q)} \left(e^{-(1-\delta'')} \right)^{C_5 (\ln \theta_{N_1, N_2})/\epsilon} \leq C_7 \lambda(N),$$

if $0 < \epsilon < C_5(1 - \delta'')/(1/p - 1/q)$. Then it follows by (49) and (51) that

$$d_\beta(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma N^r} N^{(2d-r-1)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1 \leq k \leq M} (\ln \theta_{N_k, N_{k+1}})^{1/2}, & q = \infty. \end{cases} \quad (52)$$

By (48) and (49), we get $\theta_{N_k, N_{k+1}} \asymp N_k^{2d-r-1} = (N_k^r)^{(2d-r-1)/r} \leq (N^r + C_1(k-1))^{(2d-r-1)/r}$ and considering that $M \leq \epsilon^{-1} \ln \theta_{N_1, N_2}$, $\theta_{N_1, N_2} \asymp N^{2d-r-1}$ and $N^{2d-1} \asymp s$ we get $\ln \theta_{N_k, N_{k+1}} \asymp \ln s$. Therefore by (52)

$$d_\beta(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma N^r} N^{(2d-r-1)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty. \end{cases}$$

From Remark 5.5, (24) and (49), we get

$$\beta = m_0 + \sum_{k=1}^M m_k = \sum_{s=0}^M \dim \mathcal{H}_s + \sum_{k=1}^M m_k \leq s + C_\epsilon \theta_{N_1, N_2} \leq s + C_8 N^{2d-r-1},$$

and since $N^{2d-1} \asymp s$, it follows that $\beta \leq s + C_9 s^{1-r/(2d-1)}$ and then

$$d_{[s+C_9 s^{1-r/(2d-1)}]}(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma N^r} N^{(2d-r-1)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\ln s)^{1/2}, & q = \infty. \end{cases} \quad (53)$$

Let $\mathcal{T}_N = s + C_9 s^{1-r/(2d-1)}$. By (41) we have $-\gamma N^r \leq -\mathcal{R} s^{r/(2d-1)} + C_{10} s^{(r-1)/(2d-1)}$ and hence $C_{10} s^{(r-1)/(2d-1)} < C_{11}$. Thus

$$\begin{aligned} -\gamma N^r + \mathcal{R} \tau_N^{r/(2d-1)} &\leq -\mathcal{R} s^{r/(2d-1)} + \mathcal{R} \tau_N^{r/(2d-1)} + C_{11} \\ &= -\mathcal{R} \phi_N^{r/(2d-1)} + \mathcal{R} \phi_N^{r/(2d-1)} \left(1 + C_9 \phi_N^{-r/(2d-1)}\right)^{r/(2d-1)} + C_{11} \\ &= \mathcal{R} \phi_N^{r/(2d-1)} \left(-1 + \left(1 + C_9 \phi_N^{-r/(2d-1)}\right)^{r/(2d-1)}\right) + C_{11}. \end{aligned}$$

Taking N sufficiently large, we have $|C_9 \phi_N^{-r/(2d-1)}| < 1$ and hence we obtain

$$\begin{aligned} &\left(1 + C_9 \phi_N^{-r/(2d-1)}\right)^{r/(2d-1)} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(r/(2d-1)) (r/(2d-1) - 1) \cdots (r/(2d-1) - k + 1)}{k!} \left(C_9 \phi_N^{-r/(2d-1)}\right)^k \\ &= 1 + \frac{C_9 r}{2d-1} \phi_N^{-r/(2d-1)} S_N, \end{aligned}$$

where $0 \leq S_N \leq C_r$ and therefore $-\gamma N^r + \mathcal{R} \tau_N^{r/(2d-1)} \leq C_{12}$. Let $l \in \mathbb{N}$, $[\tau_N] \leq l \leq [\tau_{N+1}]$. We have $1 < e^{-\gamma N^r} / e^{-\gamma(N+1)^r} \leq e^{\gamma r}$, so, using (53), we obtain

$$\begin{aligned} d_l(\Lambda^{(2)}U_p, L^q) &\leq d_{[\tau_N]}(\Lambda^{(2)}U_p, L^q) \\ &\leq e^{-\gamma N^r} N^{(2d-r-1)(1/p-1/2)} \begin{cases} 1, & 2 \leq q < \infty, \\ (\ln \phi_N)^{1/2}, & q = \infty, \end{cases} \\ &\ll e^{-\gamma(N+1)^r} (N^{2d-1})^{(2d-r-1)(1/p-1/2)/(2d-1)} \begin{cases} 1, & 2 \leq q < \infty, \\ (\ln \phi_N)^{1/2}, & q = \infty, \end{cases} \\ &\ll e^{-\mathcal{R} l^r/(2d-1)} l^{(1-r/(2d-1))(1/p-1/2)} \begin{cases} 1, & 2 \leq q < \infty, \\ (\ln l)^{1/2}, & q = \infty, \end{cases}, \end{aligned}$$

proving (45) for $1 \leq p \leq 2 \leq q \leq \infty$. The case $2 \leq p \leq \infty$ follows because $\Lambda U_p \subset \Lambda U_2$.

Now for $r > 1$ we will apply the Theorem 5.4. We have $-\gamma(k+1)^r + \gamma k^r \leq -\gamma r k^{r-1}$ for $k \geq 1$ and therefore $e\lambda(k+1) \leq e^{-\gamma r k^{r-1} + 1} \lambda(k)$. Let $a \in \mathbb{N}$ such that $e^{-\gamma r k^{r-1} + 1} \leq 1$, for all $k \geq a$. Consider $N \geq a$, $N_0 = 0$, $N_1 = N$, $N_{k+1} = N + k$, $M = 0$, $\beta = m_0 = n = \phi_N$. Applying Theorem 5.4 for $1 \leq p \leq 2 \leq q \leq \infty$ we obtain

$$d_{\phi_N}(\Lambda^{(2)}U_p, L^q) \ll \sum_{k=1}^{\infty} \lambda(N_k) \theta_{N_k, N_{k+1}}^{1/p-1/q} = \sum_{k=1}^{\infty} \lambda(N+k-1) \theta_{N_k, N_{k+1}}^{1/p-1/q}.$$

Since $\lambda(N+k) \leq e^{-k}\lambda(N)$, for $1 \leq p \leq 2 \leq q \leq \infty$ we obtain

$$\begin{aligned}
d_{\phi_N}(\Lambda^{(2)}U_p, L^q) &\ll \lambda(N) \sum_{k=1}^{\infty} e^{-(k-1)} \theta_{N_k, N_{k+1}}^{1/p-1/q} \ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} \theta_{N_k, N_{k+1}}^{1/p-1/q} \\
&\ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} (\dim \mathcal{H}_{N+k})^{1/p-1/q} \\
&\ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} ((N+k)^{2d-2})^{1/p-1/q} \\
&\ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} (Nk)^{(2d-2)(1/p-1/q)} \\
&\ll \lambda(N) N^{(2d-2)(1/p-1/q)} = e^{-\gamma N^r} N^{(2d-2)(1/p-1/q)}.
\end{aligned}$$

Now for $2 \leq p, q \leq \infty$, we obtain

$$d_{\phi_N}(\Lambda^{(2)}U_p, L^q) \leq d_{\phi_N}(\Lambda^{(2)}U_2, L^q) \ll e^{-\gamma N^r} N^{(2d-2)(1/2-1/q)},$$

and this proves (46). □

References

- [1] K.I. Babenko, *Best approximations of a class of analytic functions*, Izv. Akad. Nauk SSSR 22 (1958) 631–640.
- [2] B. Bordin, A. Kushpel, J. Levesley, S. Tozoni, *n-Widths of multiplier operators on two-point homogeneous spaces*, in: C. Chui, L. L., Schumaker (Eds.), *Approximation Theory IX, Vol. 1, Theoretical Aspects*, Vanderbilt University Press, Nashville, 1998, pp. 22-30.
- [3] B. Bordin, A. Kushpel, J. Levesley, S. Tozoni, *Estimates of n-widths of Sobolev's classes on compact globally symmetric spaces of rank 1*, J. Funct. Anal. 202 (2003) 307-326.
- [4] F. Dai, Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer Monographs in Mathematics, Springer, New York, 2013.
- [5] S.D. Fisher, C.A. Micchelli, *The n-width of sets of analytic functions*, Duke Math. J. 47 (1980) 789-801.
- [6] R.S. Ismagilov, *Diameters of sets in normed linear spaces, and the approximation of functions by trigonometric polynomials*, Uspehi Mat. Nauk 29 (1974) 161-178.
- [7] B.S. Kashin, *On Kolmogorov diameters of octahedra*, Sov. Math. Dokl. 15 (1974) 304-307.
- [8] B.S. Kashin, *Diameters of some finite-dimensional sets and classes of smooth functions*, Math. USSR Izv. 11 (1977) 317-333.
- [9] A. Kolmogorov, *Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse*, Ann. of Math. 37 (1936) 107–110.
- [10] A. Kushpel, *Estimates of Bernstein's widths and their analogs*, Ukrainian Math. J. 45 (1) (1993) 59-65.
- [11] A. Kushpel, *Levy means associated with two-point homogeneous spaces and applications*, in: 49th Seminário Brasileiro de Análise, Campinas, SP, (1999), pp. 807-823.
- [12] A. Kushpel, *Estimates of n-widths and ϵ -entropy of Sobolev's sets on compact globally symmetric spaces of rank 1*, in: 50th Seminário Brasileiro de Análise, São Paulo, SP, (1999), pp. 53-66.
- [13] A. Kushpel, *Optimal Approximation on S^d* , J. Complexity 16 (2000) 424-458.
- [14] A. Kushpel, *n-Widths of Sobolev's classes on compact globally symmetric spaces of rank 1*, in: K. Kopotun, T. Lyche, M. Neamtu (Eds.), *Trends in Approximation Theory*, Vanderbilt University Press, Nashville, 2001, pp. 203-212.
- [15] A. Kushpel, R. Stábile, S. Tozoni, *Estimates for n-widths of sets of smooth functions on the torus \mathbb{T}^d* , J. Approx. Theory 183 (2014), 45-71.
- [16] A. Kushpel, S. Tozoni, *Sharp orders of n-widths of Sobolev's classes on compact globally symmetric spaces of rank 1*, in: 54th Seminário Brasileiro de Análise, São José do Rio Preto, SP, (2001), pp. 293-303.

- [17] A. Kushpel, S. Tozoni, *On the problem of optimal reconstruction*, J. Fourier Anal. Appl. 13 (4) (2007) 459-475.
- [18] A. Kushpel, S. Tozoni, *Entropy and widths of multiplier operators on two-point homogeneous spaces*, Constr. Approx. 35 (2012) 137-180.
- [19] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. 44 (1972), 583-595.
- [20] V.E. Maiorov, *Discretization of the problem of diameters*, Uspehi Mat. Nauk 30 (1975) 179-180.
- [21] Ju. I. Makovoz, *Diameters of Sobolev classes and splines deviating least from zero*, Mat. Zametki 26 (1979) 805-812.
- [22] A.A. Melkman, *n-Widths and optimal interpolation of time and band limited functions. II*, SIAM J. Math. Anal. 16 (1985) 803-813.
- [23] V.A. Menegatto, C.P. Oliveira, *Annihilating properties of convolution operators on complex spheres*, Anal. Math. 31 (2005) 13-30.
- [24] M. Morimoto, *Analytic functionals on the sphere and their Fourier-Borel transformations*, Complex Analysis, Banach Center Publ. 11 (1983), PWN-Polish Scientific Publishers, Warsaw, 223-250.
- [25] M. Morimoto, *Analytic functionals on the sphere*, Translations of Mathematical Monographs 178, Amer. Math. Soc., Providence, 1998.
- [26] A. Pajor, N. Tomczak-Jaegermann, *Subspaces of small codimension of finite-dimensional Banach spaces*, Proc. Amer. Math. Soc. 97 (1986), 637-642.
- [27] A. Pietsch, *Operator Ideals*, North-Holland Publ. Co., Amsterdam, 1980.
- [28] A. Pinkus, *n-Widths in Approximation Theory*, Springer-Verlag, Berlin, 1985.
- [29] W. Rudin, *L^2 -approximation by partial sums of orthogonal developments*, Duke Math. J. 19 (1952) 1-4.
- [30] W. Rudin, *Function Theory in the Unit Ball of \mathbb{C}^n* , Springer-Verlag, New York, 1980.
- [31] R. Scholz, *Durchmesserabschätzungen für die Einheitskugel des Sobolev-Raumes $W_q^\gamma(U)$ in $L_p(U)$* , Appl. Anal. 5 (1976) 257-264.
- [32] S.R. Stechkin, *The best approximation of given classes of function*, Uspehi Mat. Nauk 9 (1954) 133-134.
- [33] L.V. Taikov, *Best approximation in the mean of certain classes of analytic functions*, Mat. Zametki 1 (1967) 155-162.
- [34] V.M. Tichomirov, *Diameters of sets in function spaces and the theory of best approximations*, Uspehi Mat. Nauk 15 (1960) 81-120.