Optimal Approximation by $sk$-Splines on the Torus

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Abstract

Fixed a continuous kernel $K$ on the $d$-dimensional torus, we consider a generalization of the univariate $sk$-spline to the torus, associated with the kernel $K$. It is proved an estimate which provides the rate of convergence of a given function by its interpolating $sk$-splines, in the norm of $L^q$ for functions of the type $f = K \ast \varphi$ where $\varphi \in L^p$ and $1 \leq p \leq 2 \leq q \leq \infty$, $1/p - 1/q \geq 1/2$. The rate of convergence is obtained for functions $f$ in Sobolev classes and this rate gives optimal error estimate of the same order as best trigonometric approximation, in a special case.

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1 Introduction

The sk-splines are a natural generalization of the polynomial splines and of the Lsplines of Micchelli [11]. The sk-splines were introduced and their basic theory developed by A. K. Kushpel. The latest results about convergence of sk-splines on the circle in the spaces $L^q$, were obtained by Kushpel in [6, 7]. For an overview of approximation by sk-splines see [11].

In Section 3 we introduce the concept of sk-spline on the torus $T^d$ and we show some basic results. In Section 4 we define the fundamental sk-spline, interpolating sk-splines and we find conditions for the existence and uniqueness of interpolating sk-splines of a given function. We show that the interpolating sk-spline can be obtained from the fundamental sk-spline. In Section 5 we prove Theorem 5.7 which provides the rate of convergence of a given function by its interpolating sk-spline. The rate of convergence is given in the norm of $L^q(T^d)$ for functions of the type $f = K \ast \varphi$ where $\varphi \in L^p(T^d)$ and $K$ is a fixed kernel, for $1 \leq p \leq 2 \leq q \leq \infty$ where $p^{-1} - q^{-1} \geq 2^{-1}$. The most important result for our applications is the Corollary 5.9.

Consider fixed a kernel $K$. Given $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ let $x_k = (x_{k_1}, \ldots, x_{k_d})$, $x_{k_l} = \pi k_l / n_l$. We denote by $sk_n(f, \cdot)$ the unique interpolating sk-spline of a function $f$ with set of knots and interpolating points $\Lambda_n = \{x_k : 0 \leq k_l \leq 2n_l - 1, 1 \leq l \leq d\}$. In the last section we prove that if $\gamma \in \mathbb{R}$, $\gamma > d$,

$$K(x) = \sum_{1 \in \mathbb{Z}^d \setminus \{0\}} |l|^{-\gamma} e^{i l \cdot x}, \quad x \in T^d,$$  \hspace{1cm} (1)

where $|\cdot|$ is the norm $|\cdot|_2$ or $|\cdot|_\infty$ on $\mathbb{R}^d$, $n = (n, \ldots, n) \in \mathbb{N}^d$, $1 \leq p \leq 2 \leq q \leq \infty$, with $1/p - 1/q \geq 1/2$, then there is a positive constant $C_{p,q}$, independent of $n \in \mathbb{N}$, such that

$$\sup_{f \in K \ast U_p} ||f - sk_n(f, \cdot)||_q \leq C_{p,q} n^{-\gamma + d(1/p - 1/q)}.$$

(2)
The set $K \ast U_p = \{ K \ast \varphi : \varphi \in U_p \}$, where $U_p$ is the unit ball of $L^p(\mathbb{T}^d)$, is a Sobolev class on the torus $\mathbb{T}^d$.

It follows from [10] and [13] that for $1 \leq p \leq 2 \leq q < \infty$, the $(2n)^d$-width of Kolmogorov of $K \ast U_p$ verifies

$$d_{(2n)^d}(K \ast U_p, L^q) \asymp n^{-\gamma + d(1/p-1/2)}.$$ (3)

For $n = (n, \ldots, n) \in \mathbb{N}^d$, the dimension of the space $SK(\Lambda_n)$ of interpolating $sk$-splines on $\Lambda_n$ is $(2n)^d$. Comparing (2) and (3) we can see that the rate of convergence by interpolating $sk$-splines is as good as the rate of convergence by subspaces of trigonometric polynomials of the dimension of $SK(\Lambda_n)$ on Sobolev classes, when $p = 1$ and $q = 2$, that is, the rate of convergence is optimal in the sense of $n$-widths. The construction of the optimal interpolating $sk$-splines is given in Theorem 4.5. $SK(\Lambda_n)$ is an optimal subspace for the Kolmogorov $(2n)^d$-width of the Sobolev class $K \ast U_1$ in $L^2$.

In [4, 5] the convergence of $sk$-splines for functions in anisotropic Sobolev classes on the torus was studied. These studies were improved in [3, 2]. The best result was obtained in [2]. It was proved an almost optimal estimate, in the sense of best approximation by trigonometric polynomials, for functions in Sobolev classes, by $sk$-splines, optimal up to a logarithmic factor. In [9] it was obtained a similar result for the case $p = q = 1$.

2 Preliminaries

If $(a_n)$ and $(b_n)$ are sequences, we write $a_n \gg b_n$ to indicate that there is a constant $C_1 > 0$ such that $a_n \geq C_1 b_n$ for all $n \in \mathbb{N}$ and we write $a_n \ll b_n$ to indicate that there is a constant $C_2 > 0$ such that $a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$. We write $a_n \asymp b_n$ to indicate that $a_n \ll b_n$ and $a_n \gg b_n$.

The $d$-dimensional torus $\mathbb{T}^d$ is defined as the product of $d$ copies of the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, or $\mathbb{T}^d = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \times \cdots \times \mathbb{R}/2\pi\mathbb{Z}$. We can
identify \( \mathbb{T}^d \) with the \( d \)-dimensional cube \([-\pi, \pi]^d\) and also with the cartesian product \( S^1 \times \cdots \times S^1 \), of \( d \) times the unitary circle \( S^1 = \{ e^{it} : t \in [-\pi, \pi] \} \).

We will consider \( \mathbb{T}^d \) endowed with the normalized Lebesgue measure \( d\nu(x) = (1/(2\pi)^d) \, dx_1 dx_2 \cdots dx_d \), where \( (1/2\pi) \, dt \) is the normalized Lebesgue measure on \( S^1 \).

For \( \mathbf{l} = (l_1, \ldots, l_d), \mathbf{k} = (k_1, \ldots, k_d), \mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d \) and \( \mathbf{x} = (x_1, \ldots, x_d), \mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d \), we denote \( \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_d y_d \); \( \mathbf{l} \equiv \mathbf{k} \pmod{\mathbf{j}} \) if there is \( \mathbf{p} \in \mathbb{Z}^d \) such that \( \mathbf{l} - \mathbf{k} = \mathbf{p} \mathbf{j} \); \( \mathbf{0} = (0, 0, \ldots, 0) \); \( \mathbf{1} = (1, 1, \ldots, 1) \); \( |\mathbf{x}|_p = (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{1/p} \) for \( 1 \leq p < \infty \); \( |\mathbf{x}|_\infty = \max_{1 \leq j \leq d} |x_j| \).

In this paper we consider an arbitrary norm \( \mathbf{x} \to |\mathbf{x}| \) on \( \mathbb{R}^d \) and we denote by \( |\mathbf{l}| \) the norm of the element \( \mathbf{l} \in \mathbb{Z}^d \).

We denote by \( L^p = L^p(\mathbb{T}^d), \ 1 \leq p \leq \infty \), the vector space of all measurable functions \( f \) defined on \( \mathbb{T}^d \) and with values in \( \mathbb{C} \), satisfying

\[
||f||_p = \left( \int_{\mathbb{T}^d} |f(x)|^p d\nu(x) \right)^{1/p} < \infty, \ 1 \leq p < \infty,
\]

\[
||f||_\infty = \text{ess sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.
\]

We write \( U_p = \{ f \in L^p(\mathbb{T}^d) : ||f||_p \leq 1 \} \).

Given \( f \in L^1(\mathbb{T}^d) \) we define the Fourier series of the function \( f \) by

\[
\sum_{\mathbf{m} \in \mathbb{Z}^d} \hat{f}(\mathbf{m}) e^{i\mathbf{m} \cdot \mathbf{x}}.
\]

where

\[
\hat{f}(\mathbf{m}) = \int_{\mathbb{T}^d} f(x) e^{-i\mathbf{m} \cdot \mathbf{x}} d\nu(x).
\]

The convolution product of two functions \( f \) and \( g \) in \( L^1(\mathbb{T}^d) \), denoted by \( f \ast g \), is defined by

\[
f \ast g(x) = \int_{\mathbb{T}^d} f(x-y) g(y) d\nu(y).
\]
If \( 1 \leq p, q \leq \infty \), \( f \in L^q(\mathbb{T}^d) \) and \( g \in L^p(\mathbb{T}^d) \), then the Young Inequality says that \( f * g \in L^s(\mathbb{T}^d) \), where \( 1/s = 1/p + 1/q - 1 \), and

\[
||f * g||_s \leq ||f||_q ||g||_p.
\]

Let \( (a_l)_{l \in \mathbb{Z}^d} \) be a sequence of real numbers such that \( a_1 = a_{-1} \) for every \( l \in \mathbb{Z}^d \) and

\[
\sum_{l \in \mathbb{Z}^d} |a_l| < \infty.
\]

Consider the kernel \( K(x) \) given by

\[
K(x) = \sum_{l \in \mathbb{Z}^d} a_l e^{i l \cdot x}.
\]

We have that \( K \) is a real function, continuous and even. We consider the convolution operator defined for \( f \in L^1(\mathbb{T}^d) \) by

\[
Tf(x) = K * f(x), \quad x \in \mathbb{T}^d.
\]

\( T \) is a bounded linear operator from \( L^p(\mathbb{T}^d) \) to \( L^q(\mathbb{T}^d) \), for \( 1 \leq p, q \leq \infty \). For \( f \in L^1(\mathbb{T}^d) \) we have

\[
Tf(x) = \sum_{l \in \mathbb{Z}^d} a_l \hat{f}(l) e^{i l \cdot x},
\]

and the norm of \( T \) as an operator from \( L^p \) to \( L^q \) is the norm of \( T \) as an operator from \( L^{p'} \) to \( L^{q'} \), that is \( ||T||_{p,q} = ||T||_{q',p'} \), for every \( p, q \in \mathbb{R}, 1 \leq p, q \leq \infty \), where \( p' \) and \( q' \) satisfy \( 1/p + 1/p' = 1/q + 1/q' = 1 \). We denote

\[
K * U_p = \{K * f : f \in U_p\}.
\]

For \( l, N \in \mathbb{N} \) we define

\[
A_l = \{k \in \mathbb{Z}^d : |k|_2 \leq l\}, \quad A^*_l = \{k \in \mathbb{Z}^d : |k|_{\infty} \leq l\},
\]

\[
\mathcal{H}_l = [e^{i k \cdot x} : k \in A_l \setminus A_{l-1}], \quad \mathcal{H}^*_l = [e^{i k \cdot x} : k \in A^*_l \setminus A^*_{l-1}],
\]

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where \( A_{-1} = \mathbb{A}_{-1} = \emptyset \) and \([f_j : j \in \Gamma]\) denotes the vector space generated by the functions \( f_j : \mathbb{T}^d \rightarrow \mathbb{C} \), with \( j \) in the set of indexes \( \Gamma \). We denote by \( \mathcal{H} \) the vector space generated by the family \( \{e^{ik \cdot x} : k \in \mathbb{Z}^d\} \) which is dense in \( L^p(\mathbb{T}^d) \) for \( 1 \leq p < \infty \). Then by [1] and [12], there are positive constants \( C_1, C_2 \) and \( C_3 \) satisfying
\[
\frac{2\pi^d/2}{\Gamma(d/2)} l^{d-1} - C_2 l^{d-2} \leq \dim \mathcal{H}_l \leq \frac{2\pi^d/2}{\Gamma(d/2)} l^{d-1} + C_1 l^{d-2}.
\]
It is easily to verify that there is a positive constant \( C \) such that for every \( l, N \in \mathbb{N} \),
\[
\dim \mathcal{H}_l^* = (2l + 1)^d - (2(l - 1) + 1)^d \approx l^{d-1}.
\]
In particular \( \dim \mathcal{H}_l \approx \dim \mathcal{H}_l^* \approx l^{d-1} \).

Consider a function \( \lambda : [0, \infty) \rightarrow \mathbb{R} \) and let \( x \rightarrow |x| \) be a norm on \( \mathbb{R}^d \).

For each \( k \in \mathbb{Z}^d \) we define \( \lambda_k = \lambda(|k|) \). We denote by \( \Lambda \) the linear operator defined for \( \varphi \in \mathcal{H} \) by
\[
\Lambda \varphi = \sum_{k \in \mathbb{Z}^d} \lambda_k \hat{\varphi}(k) e^{ik \cdot x}.
\]

Let \( \Lambda = \{\lambda_k\}_{k \in \mathbb{Z}^d} \), \( \lambda_k \in \mathbb{C} \), and \( 1 \leq p, q \leq \infty \). If for any \( \varphi \in L^p(\mathbb{T}^d) \) there is a function \( f = \Lambda \varphi \in L^q(\mathbb{T}^d) \) with formal Fourier expansion given by
\[
f \sim \sum_{k \in \mathbb{Z}^d} \lambda_k \hat{\varphi}(k) e^{ik \cdot x}
\]
such that \( ||\Lambda||_{p,q} = \sup\{||\Lambda \varphi||_q : \varphi \in U_p\} < \infty \), we say that \( \Lambda \) is a bounded multiplier operator from \( L^p(\mathbb{T}^d) \) into \( L^q(\mathbb{T}^d) \), with norm \( ||\Lambda||_{p,q} \).

## 3 Basic results

Let \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \). For \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) we denote \( x_{kl} = \pi k_l/n_l \), \( 1 \leq l \leq d \) and \( x_k = (x_{k_1}, \ldots, x_{k_d}) \). We also denote
\[
\Omega_n = \{j = (j_1, \ldots, j_d) \in \mathbb{Z}^d : 0 \leq j_l \leq 2n_l - 1, 1 \leq l \leq d\},
\]
\[ \Lambda_n = \{ x_k : k \in \Omega_n \}, \quad N = \# \Omega_n = \# \Lambda_n = 2^d n_1 n_2 \cdots n_d. \]

The real vector space of all continuous functions \( f : \mathbb{T}^d \to \mathbb{R} \) endowed with the norm of the uniform convergence will be denoted by \( C(\mathbb{T}^d) \).

For a fixed kernel \( K \in C(\mathbb{T}^d) \), a \( sk \)-spline on \( \Lambda_n \) is a function represented in the form

\[ sk_n(x) = c + \sum_{k \in \Omega_n} c_k K(x - x_k), \]

where the coefficients \( c, c_k \in \mathbb{R}, \ k \in \Omega_n \), satisfy the condition

\[ \sum_{k \in \Omega_n} c_k = 0. \]

The points \( x_k \) are the knots of the \( sk \)-spline \( sk_n(x) \).

The real vector space of all \( sk \)-splines on \( \Lambda_n \), associated with the kernel \( K \) will be denoted by \( SK(\Lambda_n) \). As the vector space \( V \) generated by the set of functions \( \{ 1, K(x - x_k), \ k \in \Omega_n \} \) has dimension at most \( N + 1 \) and \( SK(\Lambda_n) \) is the subspace of \( V \) formed by the functions whose coefficients satisfy the condition \( \sum_{k \in \Omega_n} c_k = 0 \), then \( \dim SK(\Lambda_n) \leq N \).

The next four lemmas will not be proved, because they are of simple verification.

**Lemma 3.1.** Let \( l \in \mathbb{Z}^d \). Then

\[
\sum_{k \in \Omega_n} e^{i l \cdot x_k} = \begin{cases} 
N, & l \equiv 0 \mod(2n), \\
0, & \text{otherwise},
\end{cases} \quad \text{and} \quad \int_{\mathbb{T}^d} e^{i l \cdot x} d\nu(x) = \begin{cases} 
0, & l \neq 0, \\
1, & l = 0.
\end{cases}
\]

**Lemma 3.2.** For every \( l \in \mathbb{Z}^d \),

\[
\sum_{k \in \Omega_n} \cos(1 \cdot x_k) = \begin{cases} 
N, & l \equiv 0 \mod(2n), \\
0, & \text{otherwise},
\end{cases} \quad \text{and} \quad \sum_{k \in \Omega_n} \sin(1 \cdot x_k) = 0.
\]
Lemma 3.3. For every \( l, j \in \mathbb{Z}^d \) we have that

\[
\sum_{k \in \Omega_n} (\cos(j \cdot x_k)) (\cos(l \cdot x_k)) = \begin{cases} 
N, & 1 + j \equiv 0 \mod(2n) \text{ and } 1 - j \equiv 0 \mod(2n), \\
N/2, & 1 + j \equiv 0 \mod(2n) \text{ or } 1 - j \equiv 0 \mod(2n), \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\sum_{k \in \Omega_n} (\sin(j \cdot x_k)) (\sin(l \cdot x_k)) = \begin{cases} 
N/2, & 1 + j \equiv 0 \mod(2n) \text{ and } 1 - j \equiv 0 \mod(2n), \\
-N/2, & 1 + j \equiv 0 \mod(2n) \text{ and } 1 - j \not\equiv 0 \mod(2n), \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\sum_{k \in \Omega_n} (\cos(j \cdot x_k)) (\sin(l \cdot x_k)) = 0.
\]

Definition 3.4. For \( K \in C(\mathbb{T}^d) \), \( j \in \mathbb{Z}^d \) and \( x \in \mathbb{T}^d \), we define

\[
\lambda_j(x) = \sum_{k \in \Omega_n} e^{ij \cdot x_k} K(x - x_k),
\]

\[
\rho_j(x) = \frac{2}{N} \Re(\lambda_j(x)) = \frac{2}{N} \sum_{k \in \Omega_n} (\cos(j \cdot x_k)) K(x - x_k)
\]

and

\[
\sigma_j(x) = \frac{2}{N} \Im(\lambda_j(x)) = \frac{2}{N} \sum_{k \in \Omega_n} (\sin(j \cdot x_k)) K(x - x_k).
\]

Lemma 3.5. Let \( p, j \in \mathbb{Z}^d \). Then for every \( x \in \mathbb{T}^d \),

\[
\rho_{2np+j}(x) = \rho_j(x), \quad \rho_{2np-j}(x) = \rho_j(x), \quad \rho_{-j}(x) = \rho_j(x),
\]

\[
\sigma_{2np+j}(x) = \sigma_j(x), \quad \sigma_{2np-j}(x) = -\sigma_j(x), \quad \sigma_{-j}(x) = -\sigma_j(x).
\]
Theorem 3.6. Consider a kernel $K$ given by $K(x) = \sum_{l \in \mathbb{Z}^d} a_l e^{i l \cdot x}$, where $(a_l)_{l \in \mathbb{Z}^d}$ is a sequence of real numbers such that $\sum_{l \in \mathbb{Z}^d} |a_l| < \infty$ and $a_1 = a_{-1}$ for every $l \in \mathbb{Z}^d$. Then $K$ is a real function, continuous, even and for every $j \in \mathbb{Z}^d$,

$$
\rho_j(x) = \sum_{p \in \mathbb{Z}^d} (a_{2np+j} \cos((2np + j) \cdot x) + a_{2np-j} \cos((2np - j) \cdot x)),
$$

$$
\sigma_j(x) = \sum_{p \in \mathbb{Z}^d} (a_{2np+j} \sin((2np + j) \cdot x) - a_{2np-j} \sin((2np - j) \cdot x)).
$$

Proof: For each $m \in \mathbb{N}$, let $K_m(x) = \sum_{|l| \leq m} a_l e^{i l \cdot x}$. We have that $(K_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{T}^d)$ and as $C(\mathbb{T}^d)$ is complete, there is a function $K \in C(\mathbb{T}^d)$ such that $K_m \to K$ uniformly. We have that

$$
K(x) = \sum_{l \in \mathbb{Z}^d} a_l \cos(l \cdot x)
$$

and $K$ is a real and even function. Fix $j \in \mathbb{Z}^d$ and let

$$
A_j = \{l \in \mathbb{Z}^d : l + j \equiv 0 \mod(2n)\} = \{2np - j : p \in \mathbb{Z}^d\},
$$

$$
B_j = \{l \in \mathbb{Z}^d : l - j \equiv 0 \mod(2n)\} = \{2np + j : p \in \mathbb{Z}^d\}.
$$

From Lemma 3.3, we have that

$$
\sum_{k \in \Omega_n} (\cos(j \cdot x_k))(\cos(l \cdot x_k)) = \begin{cases} 
N, & l \in A_j \cap B_j; \\
N/2, & l \in A_j \Delta B_j; \\
0, & l \in (A_j \cup B_j)^c 
\end{cases}
$$

(5)

$$
\sum_{k \in \Omega_n} (\sin(j \cdot x_k))(\sin(l \cdot x_k)) = \begin{cases} 
N/2, & l \in B_j \setminus A_j; \\
-N/2, & l \in A_j \setminus B_j; \\
0, & l \in (A_j \Delta B_j)^c.
\end{cases}
$$

(6)
Now using (4) we obtain
\[
\lambda_j(x) = \sum_{k \in \Omega_n} e^{ij \cdot x_k} K(x - x_k)
\]
\[
= \sum_{l \in \mathbb{Z}^d} a_l \sum_{k \in \Omega_n} (\cos(j \cdot x_k)) (\cos(l \cdot (x - x_k)))
\]
\[
+ i \sum_{l \in \mathbb{Z}^d} a_l \sum_{k \in \Omega_n} (\sin(j \cdot x_k)) (\cos(l \cdot (x - x_k))).
\]  
(7)

Thus by (7), from Lemma 3.3 and from (5)
\[
\rho_j(x) = \frac{2}{N} \sum_{l \in \mathbb{Z}^d} a_l \sum_{k \in \Omega_n} (\cos(j \cdot x_k)) (\cos(l \cdot x - l \cdot x_k))
\]
\[
= \frac{2}{N} \sum_{l \in \mathbb{Z}^d} a_l (\cos(1 \cdot x)) \sum_{k \in \Omega_n} (\cos(j \cdot x_k)) (\cos(l \cdot x_k))
\]
\[
= \frac{2}{N} \sum_{l \in A_j \cap B_j} N a_l \cos(1 \cdot x) + \frac{2}{N} \sum_{l \in A_j \Delta B_j} N a_l \cos(1 \cdot x)
\]
\[
= \sum_{2np+j \in A_j \cap B_j, p \in \mathbb{Z}^d} a_{2np+j} \cos((2np + j) \cdot x)
\]
\[
+ \sum_{2np-j \in A_j \cap B_j, p \in \mathbb{Z}^d} a_{2np-j} \cos((2np - j) \cdot x)
\]
\[
+ \sum_{2np+j \in A_j \Delta B_j, p \in \mathbb{Z}^d} a_{2np+j} \cos((2np + j) \cdot x)
\]
\[
+ \sum_{2np-j \in A_j \Delta B_j, p \in \mathbb{Z}^d} a_{2np-j} \cos((2np - j) \cdot x)
\]
\[
= \sum_{p \in \mathbb{Z}^d} a_{2np+j} \cos((2np + j) \cdot x) + \sum_{p \in \mathbb{Z}^d} a_{2np-j} \cos((2np - j) \cdot x).
\]

In an analogous way, using (7), Lemma 3.3 and (6) we obtain
\[
\sigma_j(x) = \sum_{p \in \mathbb{Z}^d} a_{2np+j} \sin((2np + j) \cdot x) - \sum_{p \in \mathbb{Z}^d} a_{2np-j} \sin((2np - j) \cdot x),
\]

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and this concludes the proof. \qed

4 Fundamental sk-spline

Definition 4.1. Suppose that $\rho_j(0) \neq 0$ for all $j \in \Omega_n, j \neq 0$. We define $\tilde{s}_k_n$ by

$$\tilde{s}_k_n(x) = \frac{1}{N} + \frac{1}{N} \sum_{j \in \Omega_n^*} \rho_j(x)$$

where $\Omega_n^* = \Omega_n \setminus \{(0, \ldots, 0)\}$.

Lemma 4.2. The function $\tilde{s}_k_n$ is a sk-spline.

Proof: We have, by the definition of $\rho_j(x)$ that

$$\tilde{s}_k_n(x) = \frac{1}{N} + \frac{1}{N} \sum_{j \in \Omega_n^*} \frac{\rho_j(x)}{\rho_j(0)}$$

$$= \frac{1}{N} + \sum_{k \in \Omega_n} \left( \frac{2}{N^2} \sum_{j \in \Omega_n^*} \frac{1}{\rho_j(0)} (\cos(j \cdot x_k)) \right) K(x - x_k)$$

$$= \frac{1}{N} + \sum_{k \in \Omega_n} c_k K(x - x_k)$$

and by Lemma 3.2

$$\sum_{k \in \Omega_n} c_k = \frac{2}{N^2} \sum_{j \in \Omega_n^*} \frac{1}{\rho_j(0)} \sum_{k \in \Omega_n} \cos(j \cdot x_k) = 0.$$

Thus $\tilde{s}_k_n$ is a sk-spline by definition. \qed

The sk-spline $\tilde{s}_k_n$ will be called fundamental sk-spline.

Lemma 4.3. If $\rho_j(0) \neq 0$ for all $j \in \Omega_n^*$, then the sk-spline $\tilde{s}_k_n$ satisfy

$$\tilde{s}_k_n(x_k) = \begin{cases} 1, & k = 0, \\ 0, & k \in \Omega_n^*. \end{cases}$$
Proof: From Theorem 3.6 we have that
\[
\rho_j(x_l) = \sum_{p \in \mathbb{Z}^d} (a_{2np+j} \cos((2np+j) \cdot x_l) + a_{2np-j} \cos((2np-j) \cdot x_l)) \\
= \sum_{p \in \mathbb{Z}^d} (a_{2np+j} \cos(j \cdot x_l) + a_{2np-j} \cos((2np-j) \cdot x_l)) \\
= (\cos(j \cdot x_l)) \sum_{p \in \mathbb{Z}^d} (a_{2np+j} + a_{2np-j}) \\
= (\cos(j \cdot x_l))\rho_j(0),
\]
then by Lemma 3.2
\[
\tilde{s}_k^n(x_k) = \frac{1}{N} + \frac{1}{N} \sum_{j \in \Omega^*_n} (\cos(j \cdot x_k))\rho_j(0) = \frac{1}{N} \sum_{j \in \Omega_n} \cos(k \cdot x_j) = \begin{cases} 1, & k = 0, \\ 0, & k \in \Omega^*_n, \end{cases}
\]
and then we proved the lemma.

\[\square\]

Definition 4.4. Let \(f\) be a function defined on \(\mathbb{T}^d\) and let \(\{y_j : j \in \Omega_n\} \subset \mathbb{T}^d\).
If there are constants \(c^*, c^*_k \in \mathbb{R}\), such that
\[
sk_n(f, y_j) = c^* + \sum_{k \in \Omega_n} c^*_k K(y_j - x_k) = f(y_j), \quad j \in \Omega_n,
\]
we say that the sk-spline
\[
sk_n(f, x) = c^* + \sum_{k \in \Omega_n} c^*_k K(x - x_k)
\]
is an interpolating sk-spline of \(f\) with knots \(x_k\) and interpolation points \(y_k\).

Theorem 4.5. Suppose \(\rho_j(0) \neq 0\) for any \(j \in \Omega^*_n\). Then for any function \(f\) defined on \(\mathbb{T}^d\), there is an unique interpolating sk-spline of \(f\) with knots and interpolation points \(x_k, k \in \Omega_n\), that can be written in the form
\[
sk_n(f, x) = \sum_{k \in \Omega_n} f(x_k) \tilde{s}_k^n(x - x_k).
\]
Proof: Let $c_k, k \in \Omega_n$ be the coefficients of the $sk$-spline $\tilde{s}_k$ that were obtained in the proof of Lemma 4.2 and let

$$s_k(f, x) = \sum_{k \in \Omega_n} f(x_k) + \sum_{l \in \Omega_n} \left( \sum_{k \in \Omega_n} c_l f(x_k) \right) K(x - x_l) = d + \sum_{l \in \Omega_n} d_l K(x - x_l).$$

Since $\sum_{l \in \Omega_n} c_l = 0$, it follows that $\sum_{l \in \Omega_n} d_l = 0$, and thus $s_k(f, \cdot)$ is a $sk$-spline. Applying Lemma 4.3 we obtain that

$$s_k(f, x_l) = \sum_{k \in \Omega_n} f(x_k) \tilde{s}_k(x_l - x_k) = f(x_l)$$

for any $l \in \Omega_n$. Then we can conclude that $s_k(f, \cdot)$ is an interpolating $sk$-spline of $f$ with knots and interpolation points $x_k$.

Let $\{w_j : 1 \leq j \leq N\}$ be an enumeration of $\Lambda_n$. Then for every function $f$ on $\mathbb{T}^d$, there are constants $c_1, c_2, \ldots, c_{N+1} \in \mathbb{R}$ satisfying $\sum_{l=1}^N c_l = 0$, such that

$$s_k(f, x) = c_{N+1} + \sum_{l=1}^N c_l K(x - w_l)$$

is an interpolating $sk$-spline of $f$. Given $y_1, y_2, \ldots, y_N \in \mathbb{R}$, let

$$\alpha_j = \left( \prod_{1 \leq l \leq N, l \neq j} |w_j - w_l|^2 \right)^{-1}, \ 1 \leq j \leq N,$$

and $g : \mathbb{T}^d \to \mathbb{R}$ defined by

$$g(x) = \sum_{j=1}^N y_j \alpha_j \prod_{1 \leq l \leq N, l \neq j} |x - w_l|^2, \ x \in \mathbb{T}^d.$$ 

Thus $g(w_k) = y_k$, for $1 \leq k \leq N$. Then there are $c_1, \ldots, c_N, c_{N+1} \in \mathbb{R}$ such that the $sk$-spline

$$s_k(x) = c_{N+1} + \sum_{l=1}^N c_l K(x - w_l)$$

for any $x \in \mathbb{T}^d$.
is an interpolating $sk$-spline of $g$, that is,
\[ c_{N+1} + \sum_{l=1}^{N} c_l K(w_k - w_l) = g(w_k) = y_k, \quad 1 \leq k \leq N. \]

Let
\[
\tilde{K} = \begin{pmatrix}
K(w_1 - w_1) & \cdots & K(w_1 - w_N) & 1 \\
\vdots & \ddots & \vdots \\
K(w_N - w_1) & \cdots & K(w_N - w_N) & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix},
\]
\[ \mathbf{C} = (c_1, c_2, \ldots, c_{N+1}), \quad \mathbf{Y} = (y_1, y_2, \ldots, y_{N+1}) \in \mathbb{R}^{N+1} \text{ and} \]
\[
\tilde{K} = \begin{pmatrix}
K & \mathbf{u} \\
\mathbf{u}' & 0
\end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}, \quad \mathbf{C}^t = \begin{pmatrix}
c_1 \\
\vdots \\
c_{N+1}
\end{pmatrix}, \quad \mathbf{Y}^t = \begin{pmatrix}
y_1 \\
\vdots \\
y_{N+1}
\end{pmatrix},
\]
where $\mathbf{u}$ is a $N \times 1$ matrix. Let $W = \{ \mathbf{C} = (c_1, c_2, \ldots, c_{N+1}) \in \mathbb{R}^{N+1} : \sum_{l=1}^{N} c_l = 0 \}$. Then for every $\mathbf{Y} \in \mathbb{R}^{N+1}$ with $y_{N+1} = 0$, there is $\mathbf{C} \in W$ such that $\tilde{K} \mathbf{C}^t = \mathbf{Y}^t$.

Now we consider the linear map $T : W \rightarrow \mathbb{R}^{N+1}$ defined by $T(\mathbf{C}) = (\tilde{K} \mathbf{C})^t$. We can conclude that $T$ is injective, since $T$ is linear and $\dim W = N = \dim \text{Im}(T)$.

Let $f$ be a function on $\mathbb{T}^d$ and suppose that there are $\mathbf{C} = (c_1, c_2, \ldots, c_{N+1})$, $\bar{\mathbf{C}} = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{N+1}) \in W$ such that
\[
sk_n(f, \mathbf{x}) = c_{N+1} + \sum_{l=1}^{N} c_l K(\mathbf{x} - \mathbf{w}_l) \quad \text{and} \quad \sk_n(f, \mathbf{x}) = \bar{c}_{N+1} + \sum_{l=1}^{N} \bar{c}_l K(\mathbf{x} - \mathbf{w}_l)
\]
are two interpolating $sk$-splines of $f$. If $F = (f(w_1), \ldots, f(w_N), 0)$, then we have $T(\mathbf{C'}) = (\tilde{K} \mathbf{C'})^t = F$ and $T(\bar{\mathbf{C}}') = (\tilde{K} \bar{\mathbf{C}}')^t = F$. Since $T$ is injective, it follows that $\mathbf{C} = \bar{\mathbf{C}}$, that is, $\sk_n(f, \mathbf{x}) = \sk_n(f, \mathbf{x})$, for all $\mathbf{x} \in \mathbb{T}^d$. \qed
Remark 4.6. Let $K$ be a kernel satisfying the conditions of the Theorem 3.6 and $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$. Suppose $\rho_j(0) \neq 0$ for all $j \in \Omega_n$. Then the vector space $SK(\Lambda_n)$ of all sk-splines on $\Lambda_n$ and associated with the kernel $K$ has dimension $N = 2^d n_1 n_2 \cdots n_d$. In particular if $n_1 = n_2 = \cdots = n_d = n$ we have $\dim(SK(\Lambda_n)) = (2n)^d$.

5 Approximation by sk-splines

In this section we will prove the main result of this paper, the Theorem 5.7. This theorem says how a function of the type $f = K \ast \phi$, for $\phi \in L^p(\mathbb{T}^d)$, can be approximated by the sk-splines $sk_n(f, \cdot)$ in the space $L^q(\mathbb{T}^d)$, where $1 \leq p \leq 2 \leq q \leq \infty$ with $1/p - 1/q \geq 1/2$. But for our applications, the most interesting result is the Corollary 5.9, since its hypothesis can be easily verified.

In all results of this section, we consider a kernel $K$ as in the Theorem 3.6 and such that $\rho_j(0) \neq 0$ for all $n \in \mathbb{N}^d$ and $j \in \Omega_n$.

The following result can be easily verified.

Lemma 5.1. For $j \in \Omega_n$ and $l \in \mathbb{Z}^d$ we have that $\rho_l(x - x_j) = \rho_l(x) \cos(l \cdot x_j) + \sigma_l(x) \sin(l \cdot x_j)$.

Lemma 5.2. For every $l \in \mathbb{Z}^d$, $l \not\equiv 0 \mod(2n)$ and $x \in \mathbb{T}^d$,

$$\sum_{j \in \Omega_n} e^{i l \cdot x_j} \tilde{sk}_n(x - x_j) = \frac{\lambda_l(x)}{\rho_l(0)}.$$

Proof: Firstly we will prove the result for the real part. Consider the sets $A_1$ and $B_1$ introduced in the proof of Theorem 3.6.

Using Lemma 5.1 together with Lemmas 3.2 and 3.3, the equation (5)
and the fact that $1 \not\equiv 0 \mod(2n)$, we have that

$$
\sum_{j \in \Omega_n} (\cos(1 \cdot x_j)) s^k_n(x - x_j) = \sum_{j \in \Omega_n} (\cos(1 \cdot x_j)) \left\{ \frac{1}{N} + \frac{1}{N} \sum_{k \in \Omega_n} \frac{\rho_k(x - x_j)}{\rho_k(0)} \right\}
$$

$$
= \frac{1}{N} \sum_{k \in \Omega_n} \frac{\rho_k(x)}{\rho_k(0)} \sum_{j \in \Omega_n} (\cos(1 \cdot x_j))(\cos(k \cdot x_j))
$$

$$
= \frac{1}{N} \sum_{k \in \Omega_n \cap (A_1 \cap B_1)} \frac{\rho_k(x)}{\rho_k(0)} \sum_{j \in \Omega_n} (\cos(1 \cdot x_j))(\cos(k \cdot x_j))
$$

$$
+ \frac{1}{N} \sum_{k \in \Omega_n \cap (A_1 \Delta B_1)} \frac{\rho_k(x)}{\rho_k(0)} \sum_{j \in \Omega_n} (\cos(1 \cdot x_j))(\cos(k \cdot x_j))
$$

$$
= \frac{1}{N} \sum_{k \in \Omega_n \cap (A_1 \cap B_1)} \frac{\rho_k(x)}{\rho_k(0)} + \frac{1}{2} \sum_{k \in \Omega_n \cap (A_1 \Delta B_1)} \frac{\rho_k(x)}{\rho_k(0)}. \tag{8}
$$

If $k \in B_1$ then there is $p \in \mathbb{Z}^d$ such that $k = 2np + 1$ and thus $\rho_k(x) = \rho_{2np+1}(x) = \rho_1(x)$ by Lemma 3.5. In an analogous way, if $k \in A_1$ we can conclude that $\rho_k(x) = \rho_1(x)$. Then, by (8) we have that

$$
\sum_{j \in \Omega_n} (\cos(1 \cdot x_j)) s^k_n(x - x_j) = \frac{\rho_1(x)}{\rho_1(0)} \left( \#(\Omega^*_n \cap (A_1 \cap B_1)) + \frac{1}{2} \#(\Omega^*_n \cap (A_1 \Delta B_1)) \right).
$$

(9)

Let $l = (l_1, \ldots, l_d) \in \mathbb{Z}^d$. For each $1 \leq j \leq d$, there is an unique $q_j$ and an unique $r_j$ satisfying $q_j, r_j \in \mathbb{Z}, 0 \leq r_j \leq 2n_j - 1$ and $l_j = 2n_jq_j + r_j$. Then $l = 2nq + r$ where $q = (q_1, \ldots, q_d) \in \mathbb{Z}^d$ and $r = (r_1, \ldots, r_d) \in \Omega_n$, so

$$
B_1 = \{2np + 1 : p \in \mathbb{Z}^d\} = \{2n(p + q) + r : p \in \mathbb{Z}^d\} = \{2np + r : p \in \mathbb{Z}^d\} = B_r.
$$

In an analogous way we have $A_1 = \{2np - r : p \in \mathbb{Z}^d\} = A_r$. As $\rho_1(x) = \rho_r(x)$ by Lemma 3.5, we obtain by (9) that

$$
\sum_{j \in \Omega_n} (\cos(1 \cdot x_j)) s^k_n(x - x_j) = \frac{\rho_r(x)}{\rho_r(0)} \left( \#(\Omega^*_n \cap (A_r \cap B_r)) + \frac{1}{2} \#(\Omega^*_n \cap (A_r \Delta B_r)) \right).
$$

Then it is enough to prove the result for $l \in \Omega^*_n$. 

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Let \( l = (l_1, \ldots, l_d), k = (k_1, \ldots, k_d) \in \Omega_n^* \). Then \( l - k \equiv 0 \mod (2n) \) if and only if \( k = l \), and \( l + k \equiv 0 \mod (2n) \) if and only if \( k_j = l_j = 0 \) and \( k_j = 2n_j - l_j \) if \( l_j \neq 0 \). Then \( k \in \Omega_n^* \cap (A_1 \cap B_1) \) if and only if \( k = l \) and \( l_j \in \{0, n_j\} \) for all \( 1 \leq j \leq d \); \( k \in \Omega_n^* \cap (B_1 \setminus A_1) \) if and only if \( k = l \) and \( l_j \not\in \{0, n_j\} \) for some \( 1 \leq j \leq d \); \( k \in \Omega_n^* \cap (A_1 \setminus B_1) \) if and only if \( k_j = l_j = 0 \), \( k_j = 2n_j - l_j \) if \( l_j \neq 0 \) and \( l_j \not\in \{0, n_j\} \) for some \( 1 \leq j \leq d \). Let

\[
\mathcal{A} = \{ l \in \Omega_n^* : l_j \in \{0, n_j\} \text{ for all } 1 \leq j \leq d \},
\]

\[
\mathcal{B} = \{ l \in \Omega_n^* : l_j \not\in \{0, n_j\} \text{ for all } 1 \leq j \leq d \}.
\]

Then \( \Omega_n^* = \mathcal{A} \cup \mathcal{B} \), \( \mathcal{A} \cap \mathcal{B} = \emptyset \) and

\[
\#(\Omega_n^* \cap (A_1 \cap B_1)) = \begin{cases} 1, & l \in \mathcal{A}, \\ 0, & l \in \mathcal{B}, \end{cases}
\]

\[
\#(\Omega_n^* \cap (A_1 \Delta B_1)) = \begin{cases} 0, & l \in \mathcal{A}, \\ 2, & l \in \mathcal{B}. \end{cases}
\]

Then it follow from \([9]\) that for all \( l \in \Omega_n^* \) we have

\[
\sum_{j \in \Omega_n} (\cos(l \cdot x_j)) \tilde{s}k_n(x - x_j) = \frac{\rho_l(x)}{\rho_l(0)}.
\]

(10)

In an analogous way, for the imaginary part, we obtain that for all \( l \in \mathbb{Z}^d \), \( l \neq 0 \),

\[
\sum_{j \in \Omega_n} (\sin(l \cdot x_j)) \tilde{s}k_n(x - x_j) = \frac{\sigma_l(x)}{\rho_l(0)},
\]

and this concludes the proof.

\[\Box\]

**Remark 5.3.** Let \( | \cdot | \) be a norm on \( \mathbb{R}^d \) and let \( K \in C(\mathbb{T}^d) \) be a kernel as in Theorem 3.6, such that \( a_l = a_k \) if \( l, k \in \mathbb{Z}^d \) and \( |l| = |k| \). Given \( k = (k_1, \ldots, k_d), p = (p_1, \ldots, p_d), i = (i_1, \ldots, i_d) \in \mathbb{Z}^d \), let \( \overline{k} = ((-1)^{i_1}k_1, \ldots, (-1)^{i_d}k_d) \)

and \( \overline{p} = ((-1)^{i_1}p_1, \ldots, (-1)^{i_d}p_d) \). Then \( |2np + \overline{k}| = |2np + k| \) and so \( a_{2np + \overline{k}} = a_{2np + k} \).

Given \( j = (j_1, \ldots, j_d) \in (\mathbb{N} \cup \{0\})^d = \{0, 1, 2, \ldots \}^d \), let

\[
D_j = \{ p = (p_1, \ldots, p_d) \in \mathbb{Z}^d : |p_i| = j_i, \ i = 1, 2, \ldots, d \}.
\]
Then
\[
\sum_{p \in \mathbb{Z}^d} a_{2np+k} = \sum_{j \in (\mathbb{N} \cup \{0\})^d} \sum_{p \in D_j} a_{2np+k} = \sum_{j \in (\mathbb{N} \cup \{0\})^d} \sum_{p \in D_j} a_{2np+k}.
\]

Remark 5.4. Consider a kernel \( K \) given by \( K(x) = \sum_{l \in \mathbb{Z}^d} a_l e^{il \cdot x} \), such that \( a_1 \geq 0 \), for all \( l \in \mathbb{Z}^d \) and \( \sum_{p \in \mathbb{Z}^d} a_{2np-k} \leq Ca_{2n-k} \), for all \( n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d \) and every \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \), with \( 0 \leq k_j \leq n_j \), for \( j = 1, 2, \ldots, d \), where \( C \) is a positive constant independent of \( n \) and \( k \). Then \( \sum_{l \in \mathbb{Z}^d} a_l < \infty \). If \( a_1 = a_{-1} \) for all \( l \in \mathbb{Z}^d \), then by Theorem 3.6, the kernel \( K \) is a real, continuous and even function.

Lemma 5.5. Let \( | \cdot | \) be a norm on \( \mathbb{R}^d \) and let \( K \) be the kernel given by \( K(x) = \sum_{l \in \mathbb{Z}^d} a_l e^{il \cdot x} \), where \( (a_l)_{l \in \mathbb{Z}^d} \) is a sequence with \( a_l = a_k \) if \( |l| = |k| \) and \( a_l \geq a_k > 0 \) if \( |l| \geq |k| \), for \( l, k \in \mathbb{Z}^d \). Suppose that there is a positive constant \( C \) such that for every \( n \in \mathbb{N}^d \) and all \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \), with \( 0 \leq k_j \leq n_j \) for \( j = 1, 2, \ldots, d \), we have \( \sum_{p \in \mathbb{Z}^d} a_{2np-k} \leq Ca_{2n-k} \). Let
\[
\theta_{\mathbf{n},1}(x) = e^{i1 \cdot x} - \sum_{j \in \Omega_n} e^{i l \cdot x_j} \tilde{s}_{k_n}(x - x_j).
\]

Then for \( l \in \mathbb{Z}^d \), \( \bar{l} = (|l_1|, \ldots, |l_d|) \),
\[
|\theta_{\mathbf{n},1}(x)| \leq \left\{ \begin{array}{ll}
4C^{a_{2n-\bar{l}}} a_1, & 0 < |l| \leq |n|, \\
|e^{i1 \cdot x} - 1|, & \text{for } l \equiv 0 \mod (2n), \\
4, & \text{for all } l.
\end{array} \right.
\]

Proof: Let \( \mu_{\mathbf{n},1}(x) \) be the real part of \( \theta_{\mathbf{n},1}(x) \). For \( l \equiv 0 \mod (2n) \), by the Definition 4.1, Lemmas 3.2 and 5.1 we have
\[
\mu_{\mathbf{n},1}(x) = \cos(1 \cdot x) - \sum_{j \in \Omega_n} (\cos(1 \cdot x_j)) \tilde{s}_{k_n}(x - x_j) = \cos(1 \cdot x) - 1.
\]
For \( l \not\equiv 0 \mod (2n) \), using Lemma 5.2 we have
\[
\mu_{n,l}(x) = \cos(1 \cdot x) - \frac{\rho_l(x)}{\rho_l(0)} = \frac{\rho_l(0) \cos(1 \cdot x) - \rho_l(x)}{\rho_l(0)}.
\]

Thus, as in the Theorem 3.6 we have \( \rho_l(x) \leq \rho_l(0) \) for all \( x \) and for all \( l \in \mathbb{Z}^d \), then
\[
|\mu_{n,l}(x)| = \left| \frac{\rho_l(0) \cos(1 \cdot x) - \rho_l(x)}{\rho_l(0)} \right| \leq \frac{2\rho_l(0)}{\rho_l(0)} = 2.
\]

Suppose now that \( 0 < |l| \leq |n| \) and let \( \tilde{l} = (|l_1|, \ldots, |l_d|) \). Thus using the hypothesis and the Remark 5.3 we obtain
\[
|\mu_{n,l}(x)| = \left| \frac{\rho_l(0) \cos(1 \cdot x) - \rho_l(x)}{\rho_l(0)} \right| \leq \frac{2\rho_l(0)}{2a_1} \leq 2C \frac{a_{2n-1}}{a_1}
\]

The imaginary part of \( \theta_{n,l}(x) \) is given by
\[
\phi_{n,l}(x) = \sin(1 \cdot x) - \sum_{j \in \Omega_n} (\sin (1 \cdot x_j)) \tilde{s}_n (x - x_j).
\]

The estimate for the imaginary part is analogous to the real part. Considering the estimates obtained for \( \mu_{n,l}(x) \) and \( \phi_{n,l}(x) \), we obtain the desired estimate for \( \theta_{n,l}(x) \).

\[\square\]

**Lemma 5.6.** Let \( K \) be a kernel as in Lemma 5.5. Then for each \( 1 \leq p < \infty \), there is a positive constant \( C \), depending only on \( p \), such that
\[
\sum_{l \in \mathbb{Z}^d} a_l^p |\theta_{n,l}(x)|^p \leq C \sum_{|l| \geq |n|} a_l^p.
\]

**Proof:** Since \( a_1 \geq a_k > 0 \) if \( |k| \geq |l|, \ k, l \in \mathbb{Z}^d \), using Lemma 5.5 and taking
\[ \tilde{l} = (|l_1|, \ldots, |l_d|) \] we have

\[
\sum_{l \in \mathbb{Z}^d} a_l^p |\theta_{n,1}(x)|^p \leq \sum_{0 < |l| \leq |n|} a_l^p |\theta_{n,1}(x)|^p + \sum_{|l| \geq |n|} a_l^p |\theta_{n,1}(x)|^p
\]

\[
\leq \sum_{0 < |l| \leq |n|} a_l^p 4pC^p \left( \frac{a_{2n} - 1}{a_1} \right)^p + \sum_{|l| \geq |n|} a_l^p 4p
\]

\[
= 4pC^p \sum_{0 < |l| \leq |n|} a_l^p 2n - 1 + 4p \sum_{|l| \geq |n|} a_l^p.
\]

For each \( l \in \mathbb{Z}^d, l \neq 0 \), let \( D_l = \{ k = (k_1, \ldots, k_d) \in \mathbb{Z}^d : |k_j| = |l_j|, 1 \leq j \leq d \} \). Then

\[
\sum_{l \in \mathbb{Z}^d} a_l^p |\theta_{n,1}(x)|^p \leq 4pC^p \sum_{0 < |l| \leq |n|} \sum_{k \in D_l} a_{2n-k}^p + 4p \sum_{|l| \geq |n|} a_l^p
\]

\[
\leq 4pC^p 2d \sum_{0 < |l| \leq |n|} a_{2n-1}^p + 4p \sum_{|l| \geq |n|} a_l^p
\]

\[
\leq C_1 \left( \sum_{|n| \leq |j| \leq 3|n|} a_j^p + \sum_{|l| \geq |n|} a_l^p \right) \leq 2C_1 \sum_{|l| \geq |n|} a_l^p,
\]

completing the proof of the lemma. \( \square \)

**Theorem 5.7.** Let \(| \cdot |\) be a norm on \( \mathbb{R}^d \) and let \( K \) be a kernel given by

\[
K(x) = \sum_{l \in \mathbb{Z}^d} a_l e^{il \cdot x},
\]

where \((a_l)_{l \in \mathbb{Z}^d}\) is a sequence that satisfies \( a_1 = a_k \) if \(|l| = |k|\) and \( a_1 \geq a_k > 0 \) if \(|k| \geq |l|\), for \( k, l \in \mathbb{Z}^d \). Suppose that

\[
\sum_{p \in \mathbb{Z}^d} a_{2np-k} \leq Ca_{2n-k},
\]

for all \( n \in \mathbb{N}^d \) and all \( k = (k_1, \ldots, k_d) \) with \( 0 \leq k_j \leq n_j \) for \( j = 1, 2, \ldots, d \), where \( C \) is a positive constant that is independent of \( n \) and \( k \). Then for
1 ≤ p ≤ 2 ≤ q ≤ ∞, with \( p^{-1} - q^{-1} ≥ 2^{-1} \), we have

\[
\sup_{f \in K \ast U_p} \| f - sk_n(f, \cdot) \|_q \leq C \left( \sum_{|l| \geq |n|} a_1^{q(p-q)} \right)^{p^{-1} - q^{-1}}
\]

**Proof:** Let \( p \in \mathbb{R} \), \( 1 ≤ p ≤ 2 \) and let \( p' \) such that \( 1/p + 1/p' = 1 \). Given \( f \in K \ast U_p \), \( \phi \in U_p \) such that \( f = K \ast \phi \), by Theorem 4.5,

\[
\sigma_n (f, x) = f(x) - sk_n(f, x)
\]

\[
= \int_{\mathbb{T}^d} \left( K(x - y) - \sum_{k \in \Omega_n} K(x_k - y) \tilde{s_k n}(x - x_k) \right) \phi(y) d\nu(y)
\]

\[
= \int_{\mathbb{T}^d} \Phi_n(x, y) \phi(y) d\nu(y),
\]

where \( \Phi_n(x, y) = K(x, y) - \sum_{k \in \Omega_n} K(x_k - y) \tilde{s_k n}(x - x_k) \). Thus by Hölder inequality we have that

\[
|f(x) - sk_n(f, x)| ≤ \|\phi\|_1 \|\Phi_n(x, \cdot)\|_{p'}. \tag{11}
\]

Since \( 1 ≤ p ≤ 2 \), it follows from Hausdorff-Young inequality that

\[
\|\Phi_n(x, \cdot)\|_{p'} ≤ \left( \sum_{l \in \mathbb{Z}^d} \left| b_l \right|^p \right)^{1/p},
\]

where for \( l \in \mathbb{Z}^d \), \( b_l = \int_{\mathbb{T}^d} \Phi_n(x, y) e^{-il \cdot y} d\nu(y) \). By Lemma 3.1 we have that

\[
\int_{\mathbb{T}^d} K(x - y) e^{-il \cdot y} d\nu(y) = \sum_{j \in \mathbb{Z}^d} a_j e^{il \cdot x} \int_{\mathbb{T}^d} e^{-i(l+j) \cdot y} d\nu(y) = a_l e^{-il \cdot x}
\]

and in an analogous way

\[
\int_{\mathbb{T}^d} \left( \sum_{k \in \Omega_n} K(x_k - y) \tilde{s_k n}(x - x_k) e^{-il \cdot y} \right) d\nu(y) = \sum_{k \in \Omega_n} a_l e^{-il \cdot x_k} \tilde{s_k n}(x - x_k).
\]

Thus

\[
b_l = a_l \left( e^{-il \cdot x} - \sum_{k \in \Omega_n} e^{-il \cdot x_k} \tilde{s_k n}(x - x_k) \right) = a_l \theta_{n, -1}(x) = a_l \theta_{n, -1}(x).
\]
Using Lemma 5.6 we obtain
\[ \|\Phi_n(x, \cdot)\|_{p'} \leq \left( \sum_{l \in \mathbb{Z}^d} a_{-1}^p |\theta_{n,-1}(x)|^p \right)^{1/p} \leq C \left( \sum_{|l| \geq |n|} a_1^p \right)^{1/p}. \] (12)

For \( \phi \in L^p(\mathbb{T}^d) \) we define
\[ T\phi(x) = \int_{\mathbb{T}^d} \Phi_n(x, y) \phi(y) d\nu(y). \]

By inequalities (11) and (12) we conclude that \( T \) is a bounded operator from \( L^p(\mathbb{T}^d) \) to \( L^\infty(\mathbb{T}^d) \) and that
\[ \|T\|_{p,\infty} \leq C \left( \sum_{|l| \geq |n|} a_1^p \right)^{1/p}. \] (13)

By duality, \( T \) is bounded from \( L^1(\mathbb{T}^d) \) to \( L^{p'}(\mathbb{T}^d) \) and
\[ \|T\|_{1,p'} \leq C \left( \sum_{|l| \geq |n|} a_1^p \right)^{1/p}. \] (14)

Applying the Riesz-Thorin Interpolation Theorem we have \( 1 \leq (p_t^{-1} - q_t^{-1})^{-1} \leq p \) and
\[ \|T\|_{p_t,q_t} \leq C \left( \sum_{|l| \geq |n|} a_1^{q_t p_t(p_t - q_t)^{-1}} \right)^{p_t^{-1} - q_t^{-1}}. \]

If \( 1 \leq r \leq 2, 2 \leq s \leq \infty \) and \( 1/r - 1/s \geq 1/2 \), then there are \( 0 \leq t \leq 1 \) and \( 1 \leq p \leq 2 \) such that \( 1/r = 1 - t + t/p \) and \( 1/s = (1-t)/p' \), that is, \( r = p_t \) and \( s = q_t \).

**Lemma 5.8.** Let \( a : [0, +\infty) \to \mathbb{R} \) be a decreasing and positive function and \( |\cdot| = |\cdot|_p \) for some \( 1 \leq p \leq \infty \). For each \( p \in \mathbb{Z}^d \), let \( a_p = a(|p|) \). Suppose that there is a constant \( c_1 > 0 \) such that for each \( n \in \mathbb{N}^d \),
\[ \sum_{p \in \mathbb{Z}^d} a_{2np} \leq c_1 a_{2n}. \] (15)

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Then there is a constant $c_2 > 0$ such that for each $n \in \mathbb{N}^d$ and $k \in \mathbb{Z}^d$ with $|k| \leq |n|$, we have

$$\sum_{p \in \mathbb{Z}^d} a_{2np-k} \leq c_2 a_{2n-k}. \quad (16)$$

**Proof:** Fix $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. By Remark 5.3 it is enough to consider $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $|k| \leq |n|$ and $0 \leq k_j \leq n_j$, for each $j = 1, 2, \ldots, d$. Let $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d$. For each $1 \leq j \leq d$, if

$$\psi_j(p_j) = \tilde{p}_j = \begin{cases} p_j - 1 & , p_j > 0, \\ p_j & , p_j \leq 0, \end{cases}$$

we define $\psi(p) = \tilde{p} = (\psi_1(p_1), \ldots, \psi_d(p_d))$, and $\psi$ is well defined as a function from $\mathbb{Z}^d$ to $\mathbb{Z}^d$. As $|2n_j p_j - k_j| \geq |2n_j \tilde{p}_j|$ we have

$$|2np - k|_p \geq (|2n_1 \tilde{p}_1|^p + \cdots + |2n_d \tilde{p}_d|^p)^{1/p} = |2n \tilde{p}|_p, \ 1 \leq p < \infty,$$

$$|2np - k|_\infty \geq \max\{|2n_1 \tilde{p}_1|, \ldots, |2n_d \tilde{p}_d|\} = |2n \tilde{p}|_\infty.$$ 

Thus $|2np - k| \geq |2n \tilde{p}|$ and consequently $a_{2np-k} \leq a_{2n \tilde{p}}$. Since the cardinality of $\psi^{-1}(\{k\})$ is at most $2^d$ for all $k \in \mathbb{Z}^d$, by (15)

$$\sum_{p \in \mathbb{Z}^d} a_{2np-k} \leq \sum_{p \in \mathbb{Z}^d} a_{2n \tilde{p}} \leq 2^d \sum_{p \in \mathbb{Z}^d} a_{2n \tilde{p}} \leq 2^d c_1 a_{2n} \leq 2^d c_1 a_{2n-k},$$

and this concludes the proof. \[\square\]

The next result is consequence of Theorem 5.7 and Lemma 5.8.

**Corollary 5.9.** Let $a : [0, +\infty) \to \mathbb{R}$ be a decreasing and positive function and $|\cdot| = |\cdot|_p$ for some $1 \leq p \leq \infty$. For each $p \in \mathbb{Z}^d$ let $a_p = a(|p|)$. Consider the kernel $K$ given by

$$K(x) = \sum_{l \in \mathbb{Z}^d} a_{l} e^{i l \cdot x},$$

such that

$$\sum_{p \in \mathbb{Z}^d} a_{2np} \leq Ca_{2n}.$$
where $C$ is a positive constant independent of $n \in \mathbb{N}^d$. Then there is a positive constant $\overline{C}$, such that for each $1 \leq p \leq 2 \leq q \leq \infty$, with $p^{-1} - q^{-1} \geq 2^{-1}$ and all $n \in \mathbb{N}^d$, we have

$$\sup_{f \in K^*U_p} \|f - sk_n(f, \cdot)\|_q \leq \overline{C} \left( \sum_{|l| \geq |n|} a_{q}^{p(q-p)^{-1}} \right)^{p^{-1}-q^{-1}}.$$  

6 Approximation of finitely differentiable functions

Theorem 6.1. For $\gamma \in \mathbb{R}$, $\gamma > d$, let

$$K(x) = \sum_{l \in \mathbb{Z}^d \setminus \{0\}} |l|^{-\gamma} e^{i l \cdot x}, \ x \in \mathbb{T}^d,$$

where $|\cdot| = |\cdot|_2$ or $|\cdot| = |\cdot|_\infty$. For $n \in \mathbb{N}$, let $n = (n, \ldots, n) \in \mathbb{N}^d$. Then, for $1 \leq p \leq 2 \leq q \leq \infty$, with $1/p - 1/q \geq 1/2$, there is a positive constant $C_{p,q}$, independent of $n \in \mathbb{N}$, such that

$$\sup_{f \in K^*U_p} \|f - sk_n(f, \cdot)\|_q \leq C_{p,q} n^{-\gamma + d(1/p - 1/q)}. \quad (17)$$

Proof: Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Using the function $f(x) = (x - 1)^\alpha / x^\alpha$, $x \geq 2$ we obtain

$$(j - 1)^{-\alpha} \leq 2^\alpha j^{-\alpha}, \ j \geq 2. \quad (18)$$

Fix $n = (n, n, \ldots, n)$ and for each $j \in \mathbb{N}$ let $B_j = \{l \in \mathbb{Z}^d : j - 1 \leq |l| < j\}$. Then $\mathbb{Z}^d = \bigcup_{j=1}^\infty B_j$. If $p \in B_j$, then $j - 1 \leq |p| < j$ and thus $j^{-\gamma} < |p|^{-\gamma} \leq (j - 1)^{-\gamma}$. Let $a_1 = |l|^{-\gamma}$ for $l \in \mathbb{Z}^d \setminus \{0\}$ and $a_0 = 0$. We have dim $\mathcal{H}_l \approx$ dim $\mathcal{H}_l^* \approx l^{d-1}$ and then the cardinality of $B_j$ satisfies

$$\#B_j \leq C j^{d-1}, \ j \in \mathbb{N}, \quad (19)$$
where $C$ is a positive constant independent of $j$. Since $2np = 2p$, by (18) and (19)

$$
\sum_{p \in \mathbb{Z}^d} a_{2np} = \sum_{j=2}^{\infty} \sum_{p \in B_j} |2np|^{-\gamma} \leq \sum_{j=2}^{\infty} \sum_{p \in B_j} (2n)^{-\gamma} (j-1)^{-\gamma}
$$

$$
\leq C \sum_{j=2}^{\infty} (2n)^{-\gamma} j^{d-1} (j-1)^{-\gamma} \leq 2^\gamma C (2n)^{-\gamma} \sum_{j=2}^{\infty} j^{d-1-\gamma}.
$$

Since $\gamma > d$, then $d - 1 - \gamma < 0$ and $\sum_{j=2}^{\infty} j^{d-1-\gamma} \leq \int_1^{\infty} t^{d-1-\gamma} dt$. Thus, since $d - \gamma < 0$,

$$
\sum_{p \in \mathbb{Z}^d} a_{2np} \leq 2^\gamma C (2n)^{-\gamma} \lim_{m \to \infty} \int_1^{m} t^{d-1-\gamma} dt = \frac{2^\gamma C |1|^{\gamma}}{\gamma - d} a_{2n} = C_1 a_{2n}. \quad (20)
$$

Therefore the hypothesis of Corollary 5.9 is satisfied.

Let $r = p^{-1} - q^{-1}$ and $s = r^{-1}$. Then using (18) and (19)

$$
\sum_{|l| \geq |n|} (a_1)^s \leq \sum_{j=|n|+1}^{\infty} \sum_{l \in B_j} (j-1)^{-s\gamma} \leq C \sum_{j=|n|}^{\infty} (j+1)^{d-1} j^{-s\gamma} \leq 2^{d-1} C \sum_{j=|n|}^{\infty} j^{d-1-s\gamma}.
$$

We have $1 \leq p \leq 2$ and thus $r = 1/p - 1/q \leq 1$ and $s \geq 1$. Then $d - 1 - s\gamma < 0$ and therefore $j^{d-1-s\gamma} \leq \int_{j-1}^{j} t^{d-1-s\gamma} dt$. We obtain that

$$
\sum_{|l| \geq |n|} (a_1)^s \leq 2^{d-1} C \sum_{j=|n|}^{\infty} \int_{j-1}^{j} t^{d-1-s\gamma} dt = -2^{d-1} C \frac{|n| - 1)^{d-s\gamma}}{d - s\gamma}
$$

$$
\leq \frac{2^{d-1} C}{s\gamma - d} |n|^{d-s\gamma} = C_2 |n|^{d-s\gamma}.
$$

Applying the Corollary 5.9 we have

$$
\sup_{f \in K^* U_p} \|f - sk_n(f, \cdot)\|_q \leq C_3 \left( \sum_{|l| \geq |n|} (a_1)^s \right)^{\gamma} \leq C_4 |n|^{-\gamma + d(p^{-1} - q^{-1})},
$$

for $1 \leq p \leq 2 \leq q \leq \infty$, with $p^{-1} - q^{-1} \geq 2^{-1}$. 

\[\square\]
References


