Dynamical obstruction to the existence of continuous sub-actions for interval maps with regularly varying property

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Abstract

In ergodic optimization theory, the existence of sub-actions is an important tool in the study of the so-called optimizing measures. For transformations with regularly varying property, we highlight a class of moduli of continuity which is not compatible with the existence of continuous sub-actions. Our result relies fundamentally on the local behavior of the dynamics near a fixed point and applies to interval maps that are expanding outside an neutral fixed point, including Manneville-Pomeau and Farey maps.

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1 Introduction

Let $T : X \to X$ be a continuous surjective map on a compact metric space X. Suppose that $f : X \to \mathbb{R}$ is a continuous function (called *potential*). Let $M(X,T)$ denote the set of T-invariant Borel probability measures on X. As usual the maximum ergodic average is defined as

$$m(f,T) := \max_{\mu \in M(X,T)} \int f \, d\mu.$$

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Given a potential $f : X \to \mathbb{R}$, a function $u : X \to \mathbb{R}$ is said to be a *sub-action* for f if it satisfies the cohomological inequality

$$f + u - u \circ T \leq m(f, T).$$

The existence of sub-actions for a potential f plays an important role in the study of measures μ in $M(X,T)$ that maximize (or minimize) the average $\int_X f d\mu$. The study of these measures gave rise to the ergodic optimization (see [Jen06, Jen18, Gar17] and references therein).

The existence of continuous sub-actions is guaranteed when the map is uniformly expanding and the potentials have Hölder modulus of continuity (see [CLT01] for the context of expanding transformations of the circle). For related studies on the existence of sub-actions, see [LT03, LT05, LRR07, GLT09], and see also [Sou03, BraF07, Bra08, Mor09] for results in one-dimensional dynamics.

For transitive expanding dynamics, generic continuous potentials do not admit bounded measurable sub-actions (see [BJ02, Theorem C] and for details [Gar17, Appendix]). Surprisingly there are few cases in the literature about specific examples of non-existence of continuous sub-actions. An example is provided by Morris [Mor07, Proposition 2] in the context of shift spaces.

Our theorem highlights a dynamical obstruction on the existence of continuous sub-actions. It seems that Morris [Mor09] was the first to notice this kind of phenomenon. Although our result holds for interval dynamics, we are convinced that such an obstruction must occur in a similar way for multidimensional settings. Precisely, we deal with interval maps with a regularly varying property and we identify an associated class of moduli of continuity whose members do not always admit continuous sub-actions. We present our theorem in the following subsection.

1.1 Statement of the result

Let $[0,1]$ be endowed with the standard metric on $\mathbb{R}$. Our dynamical setting will be interval maps $T : [0,1] \to [0,1]$, defined for x close enough to 0 as an invertible function of the form $T(x) := x(1 \pm V(x))$, where for some $\sigma > 0$, the continuous and increasing function $V : (0,+\infty) \to (0,1)$ satisfies

$$\lim_{x \to 0} \frac{V(tx)}{V(x)} = t^{\sigma}, \text{ for all } t > 0. \tag{1}$$

The function V is said to be *regularly varying at* 0 *with index* σ.

By a *modulus of continuity*, we mean a continuous and non-decreasing function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying $\lim_{\epsilon \to 0} \omega(\epsilon) = \omega(0) = 0$. Let $\mathcal{M}$ denote the family of concave modulus of continuity. For a given $\omega \in \mathcal{M}$, we denote by $\mathcal{C}^{\omega}([0,1])$ the space of functions $\varphi : [0,1] \to \mathbb{R}$ with a multiple of ω as modulus of continuity: $|\varphi(x) - \varphi(y)| \leq C\omega(d(x,y))$ for some constant $C > 0$, for all $x, y \in [0,1]$.

Theorem 1. *Let $T : [0,1] \to [0,1]$ be an interval map such that, for x close to 0, T is invertible and has the form $T(x) := x(1 \pm V(x))$, where the continuous and increasing function $V : (0,+\infty) \to (0,1)$ is regularly varying at 0 with index $\sigma > 0$.*

Suppose that $\omega \in \mathcal{M}$ satisfies

$$\liminf_{x\to 0} \frac{\omega(x)}{V(x)} > 0. \tag{2}$$

Then there exists a function $f \in \mathcal{C}^{\omega}([0,1])$, with $m(f,T) = \int f\, d\delta_0 = f(0)$, that does not admit continuous sub-action.

In the following Subsection, we give examples of applications of this theorem. We gather in Section 2 preliminary results. In Section 3, we present the proof of Theorem 1.

1.2 Examples

A trivial example of elements of $\mathcal{M}$ are the functions $\omega(h) = Ch^{\alpha}$ with $\alpha \in (0,1]$, which describe α-Hölder continuous functions. The family $\mathcal{M}$ also includes the minimal concave majorants ω_0 of non-decreasing subadditive functions $\omega : [0,+\infty) \to [0,+\infty)$, with $\lim_{h\to 0}\omega(h) = \omega(0) = 0$. Following [Med01] these concave majorants are infinitely differentiable on $(0,+\infty)$. Moreover, if $\omega'(0) < \infty$ then $\omega_0(h) = \omega'(0)h$ on some neighborhood of 0.

Another example of members of $\mathcal{M}$ are the functions $\omega(h) = h\left(\log\left(\frac{1}{h^k}\right)+1\right)$ (for $k > 0$ and h small enough), which describe locally Hölder continuous functions. A more general class of modulus of continuity in $\mathcal{M}$ is defined as follows: for $0 \le \alpha < 1$ and $\beta \ge 0$ with $\alpha + \beta > 0$, consider $\omega_{\alpha,\beta} : [0,+\infty) \to [0,+\infty)$ given as

$$\omega_{\alpha,\beta}(h) := \begin{cases} h^{\alpha}(-\log h)^{-\beta}, & 0 < h < h_0, \\ h_0^{\alpha}(-\log h_0)^{-\beta}, & h \ge h_0, \end{cases} \tag{3}$$

where h_0 is taken small enough so that $\omega_{\alpha,\beta}$ is concave. Note that $\omega_{\alpha,0}$ is reduced to the Hölder continuity, and $\omega_{0,\beta}$ for $\beta > 0$ determines a class that is larger than local Hölder continuity – see property (4).

Remark 1. *Let $\omega_{\alpha,\beta} : [0,+\infty) \to [0,+\infty)$ be the modulus of continuity defined in (3). It is easy to see that for every $\epsilon > \alpha$,*

$$\lim_{h\to 0} \frac{\omega_{\alpha,\beta}(h)}{h^{\epsilon}} = +\infty. \tag{4}$$

Note that $\mathcal{M}$ includes many functions besides the previous examples for the simple fact that for each pair $\omega_1, \omega_2 \in \mathcal{M}$, we have $\omega_1 \circ \omega_2 \in \mathcal{M}$. However, we are interested in a class of modulus of continuity whose behavior near 0 satisfies condition (2), which is dictated by the dynamics.

Let $V : (0,+\infty) \to (0,1)$ be a continuous and increasing function which is regularly varying at 0 with index $\sigma > 0$. Consider the modulus of continuity $\omega_{\alpha,\beta}$ defined in (3) with $0 \le \alpha < \min\{\sigma, 1\}$ and $\beta \ge 0$ such that $\alpha + \beta > 0$. Thanks to property (4), the condition $\liminf_{x\to 0} \frac{\omega_{\alpha,\beta}(x)}{V(x)} > 0$ holds whenever $\liminf_{x\to 0} \frac{x^{\sigma}}{V(x)} > 0$. Therefore, we obtain the following corollary.

Corollary 1. *Let $T : [0,1] \to [0,1]$ be an interval map such that in a neighborhood of the origin T is invertible and has the form $T(x) = x(1 \pm V(x))$, where $V : (0,+\infty) \to (0,1)$ is a continuous, increasing and regularly varying function at 0 with index $\sigma > 0$ that satisfies $\liminf_{x\to 0} \frac{x^\sigma}{V(x)} > 0$. Let $\omega_{\alpha,\beta}(x)$ be defined as in (3). Then, for $\alpha = \sigma$ and $\beta = 0$ or for $0 \le \alpha < \min\{\sigma, 1\}$ and $\beta \ge 0$ with $\alpha + \beta > 0$, there is a function $f \in \mathcal{C}^{\omega_{\alpha,\beta}}([0,1])$ which does not admit continuous sub-action.*

Examples of this kind of dynamics include Manneville-Pomeau interval map: for a given $s > 0$, $T_s : [0,1] \to [0,1]$ is defined as

$$T_s(x) := x(1 + x^s) \mod 1.$$

Note that $T_s'(x) \ge 1$ for all x with equality only at $x = 0$. Let c be the unique point in $(0,1)$ such that $T_s(c) = 1$ and $T_s|_{[0,c]} : [0,c] \to [0,1]$ is a diffeomorphism. Let us denote $U_s : [0,1] \to [0,c]$ the corresponding inverse branch. Note that $U_s'(x) \le 1$ for all x and U_s is concave, so that $cx \le U_s(x) \le x$. If we write $U_s(x) = x(1 - V(x))$, then $0 \le V(x) \le 1 - c$. Moreover, by using the identity $T_s \circ U_s = \mathrm{Id}$, we have $V(x) = x^s(1 - V(x))^{s+1}$ for all $x \ne 0$. Hence $\lim_{x\to 0} V(x) = 0$,

$$\lim_{x\to 0} \frac{V(tx)}{V(x)} = \lim_{x\to 0} t^s \left(\frac{1 - V(tx)}{1 - V(x)}\right)^{s+1} = t^s \text{ and } \lim_{x\to 0} \frac{x^s}{V(x)} = \lim_{x\to 0} \frac{1}{(1 - V(x))^{s+1}} = 1.$$

It is not difficult to argue that V is increasing. Then Corollary 1 applies to U_s as well.

Corollary 2. *Let $s \in (0,1)$ and $T_s(x) = x + x^{1+s}$ for x close enough to 0. Denote U_s the corresponding inverse branch. Let $\omega_{\alpha,\beta}(x)$ be defined as in (3), where either $\alpha \in [0, \min\{s,1\})$ and $\beta \ge 0$ with $\alpha + \beta > 0$ or $\alpha = s$ and $\beta = 0$. Then there are functions $f, g \in \mathcal{C}^{\omega_{\alpha,\beta}}([0,1])$ which do not admit continuous sub-actions with respect to T_s and U_s, respectively.*

The above corollary is an extension of Morris' result [Mor09], which established that for $T_s(x) = x + x^{1+s} \mod 1$, there is $f \in \mathcal{C}^{\omega_{s,0}}([0,1])$ that does not admit continuous sub-action.

Another one-parameter family of maps on the interval $[0,1]$ with indifferent fixed point at $x = 0$ is defined as follows: for $\rho \in (0,1]$, let $F_\rho : [0,1] \to [0,1]$ be given as

$$F_\rho(x) = \begin{cases} \frac{x}{(1-x^\rho)^{1/\rho}} & \text{if } 0 \le x \le 2^{-1/\rho} \\ \frac{(1-x^\rho)^{1/\rho}}{x} & \text{if } 2^{-1/\rho} < x \le 1. \end{cases}$$

Note that Farey map corresponds to the special case $\rho = 1$. For any $\rho \in (0,1]$, the first inverse branch has an explicit expression: $G_\rho(x) = \frac{x}{(1+x^\rho)^{1/\rho}}$. Note then that the functions $V(x) = \frac{1}{(1-x^\rho)^{1/\rho}} - 1$ and $W(x) = 1 - \frac{1}{(1+x^\rho)^{1/\rho}}$ are continuous, increasing, regularly varying with index ρ, and satisfy $\lim_{x\to 0} \frac{x^\rho}{V(x)} = \lim_{x\to 0} \frac{x^\rho}{W(x)} = \rho > 0$. Clearly, $F_\rho(x) = x(1 + V(x))$ and $G_\rho(x) = x(1 - W(x))$.

Corollary 3. *For $\rho \in (0,1]$, let $F_\rho(x) = \frac{x}{(1-x^\rho)^{1/\rho}}$ and $G_\rho(x) = \frac{x}{(1+x^\rho)^{1/\rho}}$ for x close to 0. Let $\omega_{\alpha,\beta}(x)$ be defined as in (3), where either $\alpha \in [0,\rho)$ and $\beta \geq 0$ with $\alpha+\beta > 0$ or $\alpha = \rho$ and $\beta = 0$. Then there are functions $f, g \in \mathcal{C}^{\omega_{\alpha,\beta}}([0,1])$ which do not admit continuous sub-actions with respect to F_ρ and G_ρ, respectively.*

As a final example of application of our theorem, let

$$T(x) = \begin{cases} x + \frac{2}{\log 2}\, x^2 |\log x| & \text{if } 0 \leq x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

Note that $V(x) = \frac{2}{\log 2}\, x|\log x|$ is a regularly varying function with index 1. For $k > 0$, the concave modulus of continuity defined for h sufficiently small as $\omega(h) = h\left(\log\left(\frac{1}{h^k}\right) + 1\right)$ clearly satisfies $\lim_{x\to 0} \frac{\omega(x)}{V(x)} = \frac{2k}{\log 2} > 0$. Recalling that such a modulus describes locally Hölder continuous functions, we have the following result.

Corollary 4. *With respect to a dynamics that behaves as $T(x) = x + \frac{2}{\log 2}\, x^2|\log x|$ for x sufficiently small, there exist locally Hölder continuous functions that do not admit continuous sub-actions.*

2 Preliminaries

2.1 Some facts about modulus of continuity

Recall that $\mathcal{M}$ denotes the family of concave modulus of continuity. Note that, given a non-identically null $\omega \in \mathcal{M}$, then $([0,1], \omega \circ d)$ is a metric space. Indeed, the subadditivity of ω follows from its concavity and thus, since ω is non-decreasing, we obtain the triangle inequality:

$$\omega(d(x,y)) \leq \omega(d(x,z)) + \omega(d(z,y)) \quad \forall\, x,y,z \in [0,1].$$

In particular, a function $\varphi : [0,1] \to \mathbb{R}$ with modulus of continuity $\omega \in \mathcal{M}$ is nothing else than a Lipschitz function with respect to the metric $\omega \circ d$.

We will use the following property.

Lemma 1. *Let $\omega \in \mathcal{M}$. For any positive constant χ, we have*

$$\frac{\chi}{1+\chi}\omega(h) \leq \omega(\chi h) \leq (\chi+1)\omega(h).$$

Proof. Since ω is subadditive, we have for all positive integer $n \geq 1$, $\omega(nh) \leq n\omega(h)$. For a positive constant χ, by monotonicity of ω, we see that

$$\omega(\chi h) \leq \omega(\lceil \chi \rceil h) \leq \lceil \chi \rceil \omega(h) \leq (\chi+1)\omega(h),$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Then, we also obtain

$$\omega(\chi h) \geq \frac{1}{\frac{1}{\chi}+1}\omega(h) = \frac{\chi}{1+\chi}\omega(h).$$

□

2.2 Local behavior near a fixed point

Given $\sigma > 0$, a measurable function $V : (0, +\infty) \to (0, +\infty)$ is said to be *regularly varying at* 0 *with index* σ if condition (1) holds. A regularly varying function can be represented in the form $V(x) = x^\sigma \mathcal{V}(x)$, where the function $\mathcal{V}$ satisfies $\lim_{x\to 0} \frac{\mathcal{V}(tx)}{\mathcal{V}(x)} = 1$, for all $t > 0$. Similarly a measurable function $V : (0, +\infty) \to (0, +\infty)$ is *regularly varying at* ∞ *with index* $\sigma \in \mathbb{R}$ if the function $x \mapsto V(\frac{1}{x})$ is regularly varying at 0. For properties of regularly varying functions, we refer to [Sen76] and [Aar97]. See also [Kar33] for details concerning the original literature.

Recall that near to origin the dynamics is supposed invertible and defined as $T(x) = x(1 \pm V(x))$. Let $(w_n)_{n=0}^{+\infty} \subset [0, 1]$ be a sequence of points obtained by choosing w_0 close enough to 0 and by defining $w_{n+1} = T^{\mp 1}(w_n)$, $n \geq 0$. In clear terms, for $x \mapsto x(1+V(x))$ we take pre-images, and for $x \mapsto x(1-V(x))$ we consider future iterates. Note that in both cases $w_n \to 0$ as $n \to \infty$. A sequence of iteration times will also play a central role in our construction. More precisely, let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers such that for some $\gamma \in (0, 1)$,

$$\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = \gamma. \tag{5}$$

The study of the behavior close to 0 can be done in a similar way for both $x \mapsto x(1 + V(x))$ and $x \mapsto x(1 - V(x))$. From now on in this subsection, we look at the case $T(x) = x(1-V(x))$. We will point out in the end similarities and particularities to the other case

We write $\alpha_j \sim \beta_j$ whenever $\frac{\alpha_j}{\beta_j} \to 1$ as $j \to \infty$. The next lemma summarizes the main properties concerning the asymptotic behavior of the sequences $(w_n = T(w_{n-1}))$ and (n_k).

Lemma 2. *The following properties hold*

(i)

$$w_n \sim \frac{1}{\sigma^{1/\sigma} b(n)}, \qquad \textit{where } b^{-1}(x) := \frac{1}{V(\frac{1}{x})}; \tag{6}$$

(ii)

$$d(w_n, w_{n+1}) \sim \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{nb(n)}; \tag{7}$$

(iii)

$$\frac{n_k}{n_{k+1}} \sim \gamma^{1+1/\sigma} \frac{b(n_{k+1})}{b(n_k)}. \tag{8}$$

Proof. To verify Part (iii), we first note that $\frac{b^{-1}(tx)}{b^{-1}(x)} = \frac{V(1/x)}{V(1/tx)} \to \frac{1}{(1/t)^\sigma} = t^\sigma$ as $x \to \infty$, which means that b^{-1} is regularly varying at ∞ with index σ. Hence, its inverse, the increasing function b, is regularly varying at ∞ with index $1/\sigma$ (for details, see [Sen76]).

We set $b(y) = y^{1/\sigma}\mathcal{B}(y)$, where $\lim_{y\to\infty} \frac{\mathcal{B}(ty)}{\mathcal{B}(y)} = 1$, for every $t > 0$. The function $\mathcal{B}$ has the following representation (for a proof, see [Sen76, Theorem 1.2]): there exist $Y > 0$ and measurable functions $\Theta : [Y, \infty) \to \mathbb{R}$, $\varepsilon : [Y, \infty) \to (-\frac{\sigma}{2}, \frac{\sigma}{2})$, with $\Theta(y) \to \theta \in \mathbb{R}^+$ as $y \to \infty$ and $\varepsilon(t) \to 0$ as $t \to \infty$, such that

$$\mathcal{B}(y) = \Theta(y) e^{\int_Y^y \frac{\varepsilon(t)}{t} dt} \qquad \forall\, y \geq Y.$$

Then

$$\log \frac{\mathcal{B}(n_k)}{\mathcal{B}(n_{k+1})} = \log \frac{\Theta(n_k)}{\Theta(n_{k+1})} + \int_{n_{k+1}}^{n_k} \frac{\varepsilon(t)}{t} dt \qquad \text{and}$$

$$\big(\sup_{[n_k, +\infty)} \varepsilon\big) \log \frac{n_k}{n_{k+1}} \leq \int_{n_{k+1}}^{n_k} \frac{\varepsilon(t)}{t} dt \leq \big(\inf_{[n_k, +\infty)} \varepsilon\big) \log \frac{n_k}{n_{k+1}}$$

ensure that $\frac{\mathcal{B}(n_k)}{\mathcal{B}(n_{k+1})} \to 1$ as $k \to +\infty$. Therefore

$$\frac{n_k b(n_k)}{n_{k+1} b(n_{k+1})} = \Big(\frac{n_k}{n_{k+1}}\Big)^{1+1/\sigma} \frac{\mathcal{B}(n_k)}{\mathcal{B}(n_{k+1})} \to \gamma^{1+1/\sigma} \text{ as } k \to \infty.$$

Part (i) follows from [Aar97, Lemma 4.8.6] which is deduced using that

$$b^{-1}\Big(\frac{1}{w_n}\Big) \sim n\sigma. \tag{9}$$

The asymptotic equivalence (9) implies that $V(w_n) = 1/b^{-1}\big(\frac{1}{w_n}\big) \sim \frac{1}{n\sigma}$, so it follows that $d(w_n, w_{n+1}) = w_n V(w_n) \sim \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{nb(n)}$ and therefore Part (ii) holds. $\square$

Remark 2. *Since b is a continuous and increasing function and since we consider the standard metric on $\mathbb{R}$, by the asymptotic equivalence (7), there exists a constant $C_0 > 1$ such that for every $i \leq j$,*

$$(j-i) C_0^{-1} \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{j\, b(j)} \leq d(w_i, w_j) \leq (j-i) C_0 \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{i\, b(i)}. \tag{10}$$

The next lemma provides us estimates on the cardinality of future iterates that stay within suitable intervals.

Lemma 3. *Let us consider $(w_{n_k})_{k=1}^{+\infty}$ a subsequence of $(w_n)_{n=0}^{+\infty}$, where $(n_k)_{k \geq 1}$ is an increasing sequence satisfying (5) and $T^{n_k - n_{k-1}}(w_{n_{k-1}}) = w_{n_k}$. For $k \geq 1$, denote*

$$R_k := \frac{1}{3C_0^3} \frac{n_{k-1} b(n_{k-1})}{n_k b(n_k)} d(w_{n_k}, w_{n_{k-1}}).$$

Then, for $z \in [w_{n_k} + R_k, w_{n_{k-1}}]$ and k large enough,

$$\begin{aligned} \#\big\{0 \leq j < n_k - n_{k-1} : R_k \leq d(T^j(z), w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\big\} &\geq \\ \geq C_1 n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}),& \end{aligned}$$

where $C_1 := \frac{1}{4}(C_0^{-1} - C_0^{-2})\sigma^{1+1/\sigma} > 0$. In particular, there is $C_2 > 0$ such that, for k sufficiently large,

$$\#\big\{0 \leq j < n_k - n_{k-1} : R_k \leq d(T^j(w_{n_{k-1}}), w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\big\} \geq \frac{C_2}{V(w_{n_k})}.$$

Proof. Let $\ell \geq 1$ be such that $w_{n_{k-1}+\ell} < z \leq w_{n_{k-1}+(\ell-1)}$. Note that a nonnegative integer j such that

$$R_k \leq d(w_{n_{k-1}+\ell+j}, w_{n_k}) \quad \text{and} \quad d(w_{n_{k-1}+(\ell-1)+j}, w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}}) \tag{11}$$

belongs to $\{j : R_k \leq d(T^j(z), w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\}$. Moreover, thanks to (10), any $j \geq 0$ such that

$$\begin{aligned} R_k \leq (n_k - n_{k-1} - \ell - j) C_0^{-1} \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{n_k b(n_k)} \quad \text{and} \\ (n_k - n_{k-1} - (\ell - 1) - j) C_0 \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{n_{k-1} b(n_{k-1})} \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}}) \end{aligned} \tag{12}$$

satisfies (11). Denoting $\kappa := n_k - n_{k-1} - \ell$, there are exactly

$$\lfloor \kappa - C_0 \sigma^{1+1/\sigma} n_k b(n_k) R_k \rfloor - \lceil \kappa + 1 - \frac{1}{3} C_0^{-1} \sigma^{1+1/\sigma} n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}) \rceil + 1$$

nonnegative integers j that fulfill (12). Therefore, we have

$$\begin{aligned} \#\{j : R_k \leq d(T^j(z), w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\} \geq \\ \geq \frac{1}{3} C_0^{-1} \sigma^{1+1/\sigma} n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}) - C_0 \sigma^{1+1/\sigma} n_k b(n_k) R_k - 2 \\ = \frac{1}{3} (C_0^{-1} - C_0^{-2}) \sigma^{1+1/\sigma} n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}) - 2. \end{aligned}$$

Note that, from Remark 2 and Lemma 2, as $k \to \infty$

$$\sigma^{1+1/\sigma} n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}) \geq C_0^{-1} n_k \Big(1 - \frac{n_{k-1}}{n_k}\Big) \frac{n_{k-1} b(n_{k-1})}{n_k b(n_k)} \to \infty.$$

Hence, ignoring at most finitely many initial terms of (n_k) if necessary, we obtain

$$\#\{j : R_k \leq d(T^j(z), w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\} \geq C_1 n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}).$$

In particular, for $z = w_{n_{k-1}}$, from (10) we have

$$\begin{aligned} d(w_{n_k}, w_{n_{k-1}})^\sigma \#\{j : R_k \leq d(T^j(w_{n_{k-1}}), w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\} \geq \\ \geq C_1 d(w_{n_k}, w_{n_{k-1}})^{\sigma+1} n_{k-1} b(n_{k-1}) \\ \geq C_1 \Big[(n_k - n_{k-1}) C_0^{-1} \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{n_k b(n_k)}\Big]^{\sigma+1} n_{k-1} b(n_{k-1}) \\ = \frac{C_1}{C_0^{\sigma+1} \sigma^{(\sigma+1)^2/\sigma}} \Big(1 - \frac{n_{k-1}}{n_k}\Big)^{\sigma+1} \frac{n_{k-1} b(n_{k-1})}{n_k b(n_k)} \frac{n_k}{b(n_k)^\sigma}. \end{aligned}$$

Note now that, from (6) and (9),

$$\frac{n}{b(n)^\sigma} \sim \sigma n w_n^\sigma \sim \frac{w_n^\sigma}{V(w_n)}.$$

Denote thus $C_1' := \frac{1}{2}\frac{C_1}{C_0^{\sigma+1}\sigma^{(\sigma+1)^2/\sigma}}(1-\gamma)^{\sigma+1}\gamma^{1+1/\sigma} > 0$. Following the previous estimate and the above asymptotic equivalence, from (5) and (8), for k large enough,

$$\#\{j : R_k \le d(T^j(w_{n_{k-1}}), w_{n_k}) \le \frac{1}{3}d(w_{n_{k-1}}, w_{n_k})\} \ge \frac{C_1'}{V(w_{n_k})}\frac{w_{n_k}^\sigma}{d(w_{n_k}, w_{n_{k-1}})^\sigma}.$$

Note now that, from Remark 2 and Lemma 2, for k sufficiently large,

$$d(w_{n_k}, w_{n_{k-1}}) \le \left(1 - \frac{n_{k-1}}{n_k}\right) C_0 \frac{1}{\sigma}\frac{n_k b(n_k)}{n_{k-1}b(n_{k-1})}\frac{1}{\sigma^{1/\sigma}b(n_k)} \le 2(1-\gamma)C_0\frac{1}{\sigma}\frac{1}{\gamma^{1+1/\sigma}}w_{n_k}.$$

We obtain thus a constant $C_1'' > 0$ such that $\frac{w_{n_k}^\sigma}{d(w_{n_k}, w_{n_{k-1}})^\sigma} \ge C_1''$ whenever k is large enough, which completes the proof with $C_2 := C_1' C_1''$. $\square$

Comments on local behavior near to origin for $x \mapsto x(1 + V(x))$. In this case, we deal with a sequence of past iterates $(w_n = T(w_{n+1}))$, where $T(x) = x(1+V(x))$ in a neighborhood of 0. It is not a surprise that asymptotic equivalences are exactly the same as in the statement of Lemma 2. One may show easily such a fact with minor adjustments in the proof and an appropriate version of [Aar97, Lemma 4.8.6], which can be obtained repeating almost verbatim original arguments. The statement of Lemma 3 for this case obviously requires contextual changes since the sequences are now related by $T^{n_k - n_{k-1}}(w_{n_k}) = w_{n_{k-1}}$. If one follows the same lines of proof, one will conclude that for $z \in [w_{n_k}, w_{n_{k-1}} - R_k]$ and k large enough,

$$\begin{aligned}\#\{0 \le j < n_k - n_{k-1} : R_k \le d(T^j(z), w_{n_{k-1}}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\} \ge \\ \ge C_1 n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}),\end{aligned}$$

and in particular for k sufficiently large,

$$\#\{0 \le j < n_k - n_{k-1} : R_k \le d(T^j(w_{n_k}), w_{n_{k-1}}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\} \ge \frac{C_2}{V(w_{n_k})}. \tag{13}$$

3 Proof of Theorem 1

We will present in details the proof of Theorem 1 when $T(x) = x(1 - V(x))$ for x close to 0. In the end, we will comment on the small changes of arguments required to prove the theorem in the case $x \mapsto x(1 + V(x))$. Hence, let $(w_{n_k})_{k=1}^{+\infty}$ be a subsequence of future iterates $(w_n = T^n(w_0))_{n=0}^{+\infty}$, where $w_0 \in (0, 1)$ is a point close enough to 0 and $(n_k)_{k\ge1}$ is an increasing sequence such that $\lim_{k\to+\infty} \frac{n_k}{n_{k+1}} = \gamma$ for some $\gamma \in (0, 1)$.

Define then

$$S := \{w_{n_k}\}_{k=1}^{+\infty} \cup \{0\}.$$

For every $k>1$, set

$$I_k = \Big(\frac{1}{5}(3w_{n_k}+2w_{n_{k+1}}), \frac{1}{5}(3w_{n_k}+2w_{n_{k-1}})\Big) \qquad \text{and}$$
$$J_k = \Big(\frac{1}{3}(w_{n_k}+2w_{n_{k+1}}), \frac{1}{3}(2w_{n_k}+w_{n_{k+1}})\Big),$$

and denote $Y := (w_{n_1},1] \cup \bigcup_k J_k$. Since $\{Y, I_k\ (k>1)\}$ is an open cover of $((0,1],\omega\circ d)$, we may consider a partition of unity subordinate to it (see Figure 1).

Precisely, let $\{\varphi_Y, \varphi_k : ((0,1],\omega\circ d) \to [0,1]\ (k>1)\}$ be a family of Lipschitz continuous functions such that $\varphi_Y + \sum_k \varphi_k = 1$, with $\mathrm{Supp}(\varphi_Y) \subset Y$ and $\mathrm{Supp}(\varphi_k) \subset I_k$. In particular, ω is a modulus of continuity of φ_Y and of φ_k $(k>1)$.

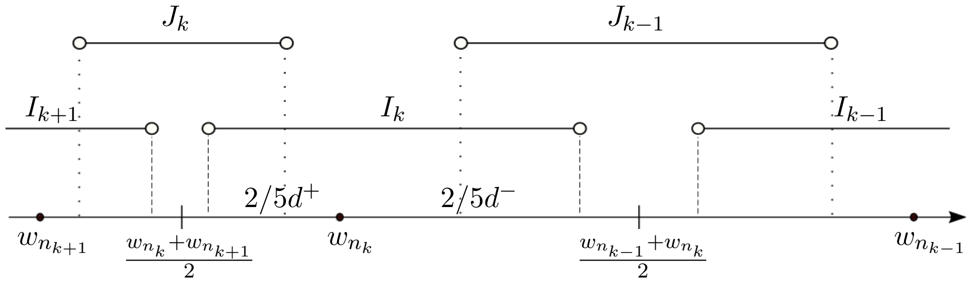

Figure 1: $d^- := d(w_{n_k}, w_{n_{k-1}})$, $d^+ := d(w_{n_k}, w_{n_{k+1}})$

For $\xi>0$, define

$$\Phi(x) := \begin{cases} \varphi_k(x), & x\in I_k,\ k=1 \mod 3 \\ -\xi\varphi_k(x), & x\in I_k,\ k=2 \mod 3 \\ 0, & \text{otherwise,} \end{cases}$$

and consider $f:[0,1]\to\mathbb{R}$ given as

$$f(x) := \Phi(x)\,\omega(d(x,S)). \tag{14}$$

This function clearly vanishes on S. Moreover, f has ω as modulus of continuity. We will show that, for ξ large enough, f does not admit a continuous sub-action.

We have $T^{m_k}(w_{n_{k-1}}) = w_{n_k}$, where $m_k := n_k - n_{k-1}$, and

$$S_{m_k} f(w_{n_{k-1}}) = \sum_{j=0}^{m_k-1} f\left(T^j\left(w_{n_{k-1}}\right)\right) = \sum_{j=0}^{m_k-1} \Phi(w_{n_{k-1}+j})\,\omega(d(w_{n_{k-1}+j},S)).$$

Recall the definition of R_k in the statement of Lemma 3. Note that, for k large enough, $[w_{n_k}, w_{n_k}+R_k) \subset \left[w_{n_k}, \frac{1}{3}(2w_{n_k}+w_{n_{k-1}})\right) \subset I_k$. Besides, by construction $\varphi_k \equiv 1$ on $\left[\frac{1}{3}(2w_{n_k}+w_{n_{k+1}}), \frac{1}{3}(2w_{n_k}+w_{n_{k-1}})\right]$. Therefore, if $k=1 \mod 3$ is sufficiently large, from Lemma 3 we get

$$\begin{aligned} S_{m_k} f(w_{n_{k-1}}) &\geq \#\big\{j : R_k \leq d(w_{n_{k-1}+j}, w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\big\}\omega(R_k) \\ &\geq \frac{C_2}{V(w_{n_k})}\omega(R_k). \end{aligned}$$

We will show that for k sufficiently large, $\frac{\omega(R_k)}{V(w_{n_k})}$ is bounded from below by a positive constant. As a matter of fact, by the definition of R_k and (8),

$$\lim_{k\to\infty}\frac{R_k}{d(w_{n_k},w_{n_{k-1}})}=\frac{1}{3}\frac{\gamma^{1+1/\sigma}}{C_0^3}.$$

For $C_3:=\frac{1}{4}\frac{\gamma^{1+1/\sigma}}{C_0^3}>0$, using the monotonicity of ω and Lemma 1, we have that for a sufficiently large k,

$$\omega(R_k)\geq\frac{C_3}{1+C_3}\omega(d(w_{n_k},w_{n_{k-1}})).$$

Moreover, from Remark 2 and Lemma 2, we see that for k sufficiently large,

$$d(w_{n_k},w_{n_{k-1}})\geq C_0^{-1}\frac{1}{\sigma}\left(1-\frac{n_{k-1}}{n_k}\right)\frac{1}{\sigma^{1/\sigma}b(n_k)}\geq\frac{1}{2}C_0^{-1}\frac{1}{\sigma}(1-\gamma)w_{n_k}.$$

Then, for $C_4:=\frac{1}{2}C_0^{-1}\frac{1}{\sigma}(1-\gamma)>0$, we obtain

$$\frac{\omega(R_k)}{V(w_{n_k})}\geq\frac{C_3}{1+C_3}\frac{C_4}{1+C_4}\frac{\omega(w_{n_k})}{V(w_{n_k})}.$$

Therefore, thanks to hypothesis (2), we conclude that there exists a constant $C_5>0$ such that, for $k=1 \mod 3$ large enough,

$$S_{m_k}f(w_{n_{k-1}})>C_5.$$

We will show in Subsection 3.1 that $m(f,T)=0$ for ξ large enough. Let us assume this fact for a moment and argue that the inequality

$$f\leq u\circ T-u$$

is impossible for every continuous function $u:[0,1]\to\mathbb{R}$. Suppose the opposite happens. Then, if $k=1 \mod 3$ is sufficiently large, we have shown that

$$\begin{aligned}u(w_{n_k})=u\left(T^{m_k}\left(w_{n_{k-1}}\right)\right)&\geq S_{m_k}f(w_{n_{k-1}})+u(w_{n_{k-1}})\\&> C_5+u(w_{n_{k-1}}).\end{aligned}$$

Since u is continuous at 0, by letting $k\to+\infty$, we get a contradiction.

3.1 A condition for $m(f,T)=0$

It remains to argue that, for ξ large enough, $m(f,T)=0$. Since $f(0)=0$ and δ_0 is T-invariant, clearly $m(f,T)\geq\int f d\delta_0=f(0)=0$. If ξ is sufficiently large, by choosing a suitable constant $\gamma\in(0,1)$ and an appropriate initial point w_0 close enough to 0, we will show that for each x there is $n(x)$ such that $S_{n(x)}f(x)\leq 0$. From Birkhoff's ergodic theorem, we thus conclude that $m(f,T)\leq 0$, which completes the proof.

We first choose $\gamma\in(0,1)$ satisfying

$$\gamma^{1+1/\sigma}>\frac{6}{7}. \tag{15}$$

Note now that, replacing w_0 by w_{n_0} with n_0 large enough, we may assume that the constant C_0 in Remark 2 is as close as we want to 1. Thus, we suppose henceforth that

$$1 < C_0^2 \leq \frac{7}{6}\gamma^{1+1/\sigma}. \tag{16}$$

Furthermore, thanks to (8), if n_0 is sufficiently large, we may also assume that

$$\frac{1}{2}\gamma^{1+1/\sigma} \leq \frac{n_k b(n_k)}{n_{k+1} b(n_{k+1})} \quad \forall\, k \geq 0. \tag{17}$$

If $x \in [0,1] \setminus \bigcup_{k=1 \mod 3} I_k$, just take $n(x) = 1$, since $f(x) \leq 0$. Suppose then $x \in I_k$ for some $k = 1 \mod 3$. Define

$$p(x) := \min\{p \geq 1 : T^p(x) \notin I_k\}.$$

Note that

$$S_{p(x)} f(x) \leq \#\{j \geq 0 : T^j(x) \in I_k\}\, \omega\Big(\frac{2}{5}\max\{d(w_{n_{k+1}}, w_{n_k}), d(w_{n_k}, w_{n_{k-1}})\}\Big).$$

Let us estimate the cardinality in the right term. Denote

$$L_k := \Big\lceil \frac{3}{7} C_0 \sigma^{1+1/\sigma} n_k b(n_k) d(w_{n_k}, w_{n_{k-1}}) \Big\rceil.$$

From Remark 2, we have $d(w_{n_k}, w_{n_k - L_k}) \geq L_k C_0^{-1} \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{n_k b(n_k)} > \frac{2}{5} d(w_{n_k}, w_{n_{k-1}})$, which means that $w_{n_k - L_k}$ is greater than the right endpoint of I_k. Thanks to (15), (16) and (17),

$$\frac{3}{7} C_0 \sigma^{1+1/\sigma} n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_k}) \leq \frac{3}{7} C_0^2 \frac{n_{k+1} b(n_{k+1})}{n_k b(n_k)} (n_{k+1} - n_k) \leq n_{k+1} - n_k,$$

so that $L_{k+1} \leq n_{k+1} - n_k$. Hence, a similar reasoning shows that $w_{n_k + L_{k+1}}$ is smaller than the left endpoint of I_k. Therefore, by the monotonicity of T, we obtain

$$\begin{aligned} \#\{j : T^j(x) \in I_k\} &\leq (L_k - 1) + (L_{k+1} - 1) \\ &\leq \frac{3}{7} C_0 \sigma^{1+1/\sigma} n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_{k-1}}). \end{aligned}$$

We have shown that

$$S_{p(x)} f(x) \leq \frac{3}{7} C_0 \sigma^{1+1/\sigma} n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_{k-1}})\, \omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big). \tag{18}$$

Now, for $y \in \big[w_{n_{k+1}} + R_{k+1}, \frac{1}{5}(3w_{n_k} + 2w_{n_{k+1}})\big]$, denote

$$q(y) := \min\{q \geq 1 : d(T^q(y), w_{n_{k+1}}) < R_{k+1}\}.$$

Clearly,

$$S_{q(y)} f(y) \leq -\xi\, \#\big\{j \geq 0 : R_{k+1} \leq d(T^j(y), w_{n_{k+1}}) \leq \frac{1}{3} d(w_{n_{k+1}}, w_{n_k})\big\}\, \omega(R_{k+1}).$$

Thanks to Lemma 3, we obtain that

$$S_{q(y)}f(y) \leq -\xi\, C_1 n_k b(n_k) d(w_{n_k}, w_{n_{k+1}}) \omega(R_{k+1}). \tag{19}$$

We claim that, whenever ξ is sufficiently large, for $n(x) := p(x) + q(T^{p(x)}(x))$ one has $S_{n(x)}f(x) \leq 0$. Thanks to (18) and (19), it is enough to prove that

$$\sup_k \frac{n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_{k-1}}) \omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big)}{n_k b(n_k) d(w_{n_k}, w_{n_{k+1}}) \omega(R_{k+1})} < \infty.$$

Recalling the asymptotic equivalence (8), we just have to show that both suprema

$$\sup_k \frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{d(w_{n_k}, w_{n_{k+1}})} \quad \text{and} \quad \sup_k \frac{\omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big)}{\omega(R_{k+1})}$$

are finite. With respect to the first one, from (10) it is immediate that

$$\begin{aligned}
\frac{d(w_{n_k}, w_{n_{k-1}})}{d(w_{n_{k+1}}, w_{n_k})} &\leq \frac{C_0 (n_k - n_{k-1}) 1/\big[\sigma^{1+1/\sigma} n_{k-1} b(n_{k-1})\big]}{C_0^{-1} (n_{k+1} - n_k) 1/\big[\sigma^{1+1/\sigma} n_{k+1} b(n_{k+1})\big]} \\
&= C_0^2 \, \frac{1 - \frac{n_{k-1}}{n_k}}{\frac{n_{k+1}}{n_k} - 1} \, \frac{n_{k+1} b(n_{k+1})}{n_k b(n_k)} \, \frac{n_k b(n_k)}{n_{k-1} b(n_{k-1})},
\end{aligned} \tag{20}$$

which ensures $\frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{d(w_{n_k}, w_{n_{k+1}})} = 1 + \frac{d(w_{n_k}, w_{n_{k-1}})}{d(w_{n_{k+1}}, w_{n_k})}$ is bounded from above. With respect to the second one, note first that, thanks to (20),

$$\frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{R_{k+1}} = 3C_0^3 \, \frac{n_{k+1} b(n_{k+1})}{n_k b(n_k)} \, \frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{d(w_{n_k}, w_{n_{k+1}})}$$

is bounded from above. Hence, there exists a positive constant C_6 such that $d(w_{n_{k+1}}, w_{n_{k-1}}) \leq C_6 R_{k+1}$. By the monotonicity of ω and Lemma 1, we obtain

$$\frac{\omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big)}{\omega(R_{k+1})} \leq C_6 + 1 < \infty.$$

The proof is complete.

Comments on the proof of Theorem 1 for $x \mapsto x(1+V(x))$. We consider now a subsequence (w_{n_k}) that fulfills $w_{n_{k-1}} = T^{n_k - n_{k-1}}(w_{n_k})$, where $T(x) = x(1+V(x))$ in a neighborhood of 0. Note that orbits are moving monotonically away from the origin, that is, they are moving to the right instead of to the left as in the previous case. This merely produces a, let us say, *reflexive effect* on our arguments, exchanging the roles of indices $k = 1 \mod 3$ and $k = 2 \mod 3$. In practical terms, we define Φ for this case as

$$\Phi(x) := \begin{cases} -\xi \varphi_k(x), & x \in I_k,\ k = 1 \mod 3 \\ \varphi_k(x), & x \in I_k,\ k = 2 \mod 3 \\ 0, & \text{otherwise.} \end{cases}$$

Introducing f as in (14) and supposing by a moment that $m(f,T)=0$, we apply the same strategy to show that f does not admit continuous sub-action. In fact, for $k=2 \mod 3$ sufficiently large, using (13) one estimates the number of iterates that remain in the interval $[\frac{1}{3}(2w_{n_k}+w_{n_{k+1}}), w_{n_k}-R_{k+1}]$ to conclude that $S_{m_{k+1}}f(w_{n_{k+1}})$ is bounded from below by a positive constant and thus to reach a contradiction. In order to show that, for the same choice of parameters (15), (16), and (17), $m(f,T)=0$ whenever ξ is sufficiently large, suitable adjustments are required to obtain that for $x \in I_k$ with $k=2 \mod 3$, there is $n(x)$ such that $S_{n(x)}f(x) \leq 0$. Similarly to the previous case, the key observation is that such a Birkhoff sum may be bounded from above by the difference of two terms, the first one takes into account the iterates that remain in I_k, the second one considers iterates that remain in $[\frac{1}{3}(2w_{n_{k-1}}+w_{n_k}), w_{n_{k-1}}-R_k]$, and their ratio is uniformly bounded.

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