ON SPACE MAXIMAL CURVES

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Abstract. Any maximal curve $\mathcal{X}$ is equipped with an intrinsic embedding $\pi : \mathcal{X} \to \mathbf{P}^r$ which reveal outstanding properties of the curve. By dealing with the contact divisors of the curve $\pi(\mathcal{X})$ and tangent lines, in this paper we investigate the first positive element that the Weierstrass semigroup at rational points can have whenever $r = 3$ and $\pi(\mathcal{X})$ is contained in a cubic surface.

1. Introduction

Throughout this paper, $\mathbf{F}$ stands for the finite field $\mathbb{F}_{q^2}$ of order q^2. A projective, geometrically irreducible, non-singular algebraic curve $\mathcal{X}$ defined over $\mathbf{F}$ of genus $g = g(\mathcal{X})$ is said to be $\mathbf{F}$-*maximal* if the number of its $\mathbf{F}$-rational points attains the Hasse-Weil upper bound; that is,

$$\#\mathcal{X}(\mathbf{F}) = q^2 + 1 + 2q \cdot g \,.$$

Apart from being interesting mathematical objects by their own, these curves have been extensively studied as they are of great interest in Coding Theory, Cryptography and related areas; see for example the books [24], [14], [16].

Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve of genus g. Then the numerator of the Zeta function of $\mathcal{X}$ is the polynomial $L(t) = (1+qt)^{2g}$ and hence $h(t) = t^{2g}L(t^{-1}) = (t+q)^{2g}$ is the characteristic polynomial of certain endomorphism $\tilde{\Phi}$ on the Jacobian $\mathcal{J}$ of $\mathcal{X}$. This map is uniquely determined by the $\mathbf{F}$-Frobenius morphism $\Phi : \mathcal{X} \to \mathcal{X}$ in such a way that $\iota \circ \Phi = \tilde{\Phi} \circ \iota$, where $\iota : \mathcal{X} \to \mathcal{J}$ is the natural embedding given by $P \mapsto [P - P_0]$ with $P_0 \in \mathcal{X}(\mathbf{F})$. It turns out that $\tilde{\Phi}$ is semisimple and so the following linear equivalence (sometimes called the *fundamental equivalence*) on $\mathcal{X}$ arises (see [14, Thm. 10.1, Thm. 9.79]):

$$(q+1)P_0 \sim qP + \Phi(P)\,, \quad P \in \mathcal{X}\,. \tag{1.1}$$

This suggests the study of the (complete) linear series $\mathcal{D}_{\mathcal{X}} := |(q+1)P_0|$ (sometimes called the *Frobenius linear series* of $\mathcal{X}$) whose definition clearly does not depend on the choice of the $\mathbf{F}$-rational point P_0. As a matter of fact, several arithmetical and geometrical properties of maximal curves are revealed through this linear series (loc. cit.). In particular, $\mathcal{D}_{\mathcal{X}}$

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is very ample [8, Prop. 1.9], [18, Thm. 2.5] which means that the morphism associated to $\mathcal{D}_\mathcal{X}$

$$\pi_{\mathcal{D}_\mathcal{X}} : \mathcal{X} \to \mathbf{P}^r \tag{1.2}$$

is an embedding, where $r = r(\mathcal{X}) \geq 2$ is the projective dimension of $\mathcal{D}_\mathcal{X}$ (sometimes called the *Frobenius dimension* of $\mathcal{X}$), and $\mathbf{P}^r$ is the projective r-space over the algebraic closure of $\mathbf{F}$.

It is well-known that the Hermitian curve $\mathcal{H}$ defined by $v^{q+1} = u^{q+1} + 1$ is $\mathbf{F}$-maximal of genus $g(\mathcal{H}) = g_0 := q(q-1)/2$; see [24, Ex. 6.3.6]. Indeed, among $\mathbf{F}$-maximal curves, the curve $\mathcal{H}$ admits the following characterization.

Proposition 1.1. ([23], [27], [9]) *If $\mathcal{X}$ is $\mathbf{F}$-maximal, the following sentences are equivalent:*

1. *$\mathcal{X}$ is $\mathbf{F}$-isomorphic to the Hermitian curve $\mathcal{H}$;*
2. *$g(\mathcal{X}) = g_0$;*
3. *$g(\mathcal{X}) > (q-1)^2/4$;*
4. *$r(\mathcal{X}) = 2$;*
5. *There exists $P \in \mathcal{X}(\mathbf{F})$ such that the first positive element of $H(P)$, the Weierstrass semigroup at P, equals q.*

From (1.2) any $\mathbf{F}$-maximal curve $\mathcal{X}$ is $\mathbf{F}$-isomorphic to a non-degenerate curve of degree $q+1$ in $\mathbf{P}^r$, $r = r(\mathcal{X})$. Thus the classical Castelnuovo's genus bound can be used to explain partially Proposition 1.1 as it gives the first general constrain between $g(\mathcal{X})$ and the pair (q, r) (see [14, Cor. 10.25]):

$$g(\mathcal{X}) \leq F(q,r) := \begin{cases} [(q-(r-1)/2)^2 - 1/4]/2(r-1)\,, & \text{if } r \text{ is even}\,, \\ [(q-(r-1)/2)^2]/2(r-1)\,, & \text{if } r \text{ is odd}\,. \end{cases} \tag{1.3}$$

Notice that the function $F(q,r)$ satisfies $F(q,r) \leq F(q,s)$ for $r \geq s$; in particular, $g(\mathcal{X}) \leq F(q,3) = (q-1)^2/4$ provided that $r(\mathcal{X}) \geq 3$. Then, with $g_1 := \lfloor (q-1)^2/4 \rfloor$, by Proposition 1.1 the *spectrum for the genera* of $\mathbf{F}$-maximal curves, namely the set

$$\mathbf{M}(q^2) := \{g \in \mathbb{N}_0 : \text{there is an } \mathbf{F}\text{-maximal curve of genus } g\}\,,$$

satisfies

$$\mathbf{M}(q^2) \subseteq [0, g_1] \cup \{g_0\}\,. \tag{1.4}$$

We recall that g_0 is the well-known Ihara's bound on the genus of $\mathbf{F}$-maximal curves [17]. One of the main problems in Curve Theory Over Finite Fields is the computation of $\mathbf{M}(q^2)$; in general one cannot expect to give a full answer to this matter but improvements on (1.4) can be expected as far as improvements on Castelnuovo's genus bound of curves in $\mathbf{P}^r$ are known.

In view of Proposition 1.1 it is natural to investigate *space* $\mathbf{F}$*-maximal curves* with respect to $\mathcal{D}_{\mathcal{X}}$; that is, those with $r(\mathcal{X}) = 3$. Here a natural way of bounding $g(\mathcal{X})$, which generalizes Castelnuovo's method, is by looking at the degree $d \geq 2$ of surfaces $S \subseteq \mathbf{P}^3$ such that $\pi(\mathcal{X}) \subseteq S$ where $\pi = \pi_{\mathcal{X}}$ is as in (1.2); cf. [13], [21]. We have the following Halphen-Ballico result (see [3]) which deals with the case of quadrics. Let g_1 be as in (1.4) and set $g_2 := \lfloor (q^2 - q + 4)/6 \rfloor$; then

(1.5) $\pi(\mathcal{X})$ is contained in a quadric in $\mathbf{P}^3$ provided that $g_2 < g(\mathcal{X}) \leq g_1$.

Now the $\mathbf{F}$-maximal property of $\mathcal{X}$ implies certain constrains on the first positive element $m_1(P)$ of the Weierstrass semigroup $H(P)$ at some $P \in \mathcal{X}(\mathbf{F})$, and (1.4) admits the folloing improvement [18]:

$$\mathbf{M}(q^2) \subseteq [0, g_2] \cup \{g_1\} \cup \{g_0\} . \tag{1.6}$$

An analogue of Proposition 1.1 emerges, namely

Proposition 1.2. ([8], [1], [19], [18]) *Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve. The following sentences are equivalent:*

1. *$\mathcal{X}$ is isomorphic to a quotient of $\mathcal{H}$ by certain involution;*
2. *$g(\mathcal{X}) = g_1$;*
3. *$\pi(\mathcal{X})$ is contained in a quadric;*
4. *There exists $P \in \mathcal{X}(\mathbf{F})$ such that the first positive element of $H(P)$, the Weierstrass semigroup at P, equals $\lfloor (q+1)/2 \rfloor$.*

The starting points of our result are in fact Propositions 1.1, 1.2 above. Under condition (2.1) below, the main result in this paper is Corollary 2.6, where a hypothesis on a cubic surface is considered; in this way a weak version of the aforementioned propositions is obtained. We always assume $q > 7$; cf. [2].

We do point out that our approach follows closely the works by Cossidente-Korchmáros-Torres [5, Sect. 3], [4, Sect. 5], Korchmáros-Torres [18], Fanali-Giulietti [7] and Arakelian-Tafazolian-Torres [2].

Conventions. $\mathbf{P}^s$ stands for the projective s-space over the algebraic closure of the base field. For a point P in a curve, $H(P)$ denotes the Weierstrass semigroup at P; $m_1(P)$ is the first positive element of $H(P)$.

2. Maximal curves and cubic surfaces

Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve, $P_0 \in \mathcal{X}(\mathbf{F})$ and $\mathcal{D} = \mathcal{D}_{\mathcal{X}} = |(q+1)P_0|$ the liner series introduced in Section 1; i.e., it is the set of effective divisors on $\mathcal{X}$ which are linearly equivalent to the divisor $(q+1)P_0$. We always assume $g(\mathcal{X}) > 0$; taking into consideration (1.6) and Propositions 1.1, 1.2 above, we also assume:

$$r(\mathcal{X}) = 3 \qquad \text{and} \qquad g(\mathcal{X}) \leq g_2 = \lfloor (q^2 - q + 4)^2/6 \rfloor . \tag{2.1}$$

Remark 2.1. Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve. From (1.3) and Proposition 1.1, a sufficient condition to have $r(\mathcal{X})=3$ is that $(q-1)(q-2)/6<g(\mathcal{X})\le g_1=\lfloor (q-1)^2/4\rfloor$.

Let $\pi=\pi_{\mathcal{D}}:\mathcal{X}\to\mathbf{P}^3$ be the morphism associated to $\mathcal{D}$.

Definition 2.2. For $P\in\mathcal{X}$, a non-negative integer j is called an $(\mathcal{D},P)$-order if there is $D\in\mathcal{D}$ such that the coefficient $v_P(D)$ of P in D equals j.

Now let $P\in\mathcal{X}(\mathbf{F})$. Relation (1.1) implies the following behaviour for elements of $H(P)$:

$$m_0(P)=0<m_1(P)<m_2(P)<m_3(P)=q+1\,.$$

Thus for each $i=0,1,2,3$ there are rational functions on $\mathcal{X}$, $h_i:\mathcal{X}\to\mathbf{P}^1$ such that $\mathrm{div}(h_i)=D_i-m_i(P)P$, $P\notin\mathrm{Supp}(D_i)$ with $\mathrm{div}(h_3)=(q+1)P-(q+1)P_0$, $P\neq P_0$. For $P=P_0$ we put $h_3=1$. Then

$$\mathrm{div}(h_ih_3)+(q+1)P_0=D_i+(q+1-m_i(P))P\in\mathcal{D}$$

and the $(\mathcal{D},P)$-orders do satisfy (cf. [8, Prop. 1.5(iii)])

$$j_i(P)=q+1-m_{3-i}(P)\,,\quad i=0,1,2,3\,; \tag{2.2}$$

therefore at $P\in\mathcal{X}(\mathbf{F})$, $j_3(P)=q+1$ and the first positive element $m_1(P)$ of $H(P)$ and $j_2(P)$ are related to each other by the equation

$$m_1(P)=q+1-j_2(P)\,. \tag{2.3}$$

Remark 2.3. For the linear system $\mathcal{D}$ above and any $P\in\mathcal{X}$, the $(\mathcal{D},P)$ orders can be ordered as a sequence $j_0(P)<j_1(P)<j_2(P)<j_3(P)\le q+1$ with $j_0(P)=0$ as $\mathcal{D}$ is base-point-free. Relation (1.1) shows that 1 and q are $(\mathcal{D},P)$-orders for $P\notin\mathcal{X}(\mathbf{F})$. Thus for such points $j_1(P)=1$ and $j_3(P)=q$ (as $g(\mathcal{X})>0$).

Now $j_3(P)$ is the intersection multiplicity of the curve $\pi(\mathcal{X})\subseteq\mathbf{P}^3$ and the osculating hyperplane at $\pi(P)$ (cf. [25]); in addition, (1.1) also shows that $\pi(\mathbf{\Phi}(P))$ belongs to this hyperplane and we have the following key observation due to Stöhr and Voloch [25, Cor. 2.6]: Let $\nu_2:=q$ and $P\in\mathcal{X}(\mathbf{F})$. Then $j_3(P)-j_1(P)\ge\nu_2$; in particular, for $P\in\mathcal{X}(\mathbf{F})$, $j_1(P)=1$, and so $m_2(P)=q$ by (2.2).

Lemma 2.4. *Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve satisfying (2.1) and let $P\in\mathcal{X}(\mathbf{F})$.*

1. *If $q>3$, then $j_2(P)\notin\{(q+3)/2,(2q+3)/3,(2q+2)/3,q-1,q\}$;*
2. *$j_2(P)\notin\{(q+1)/2,(q+2)/2\}$.*
3. *If q is even and $j_2(P)=q/2$, then $g(\mathcal{X})\le q^2/8$.*

Proof. We have $m_1(P)=q+1-j_2(P)$; see (2.3).

(1) Since $2m_1(P)\ge m_2(P)$ and $m_2=q$ by Remark 2.3, then $j_2(P)\le(q+2)/2$. If any of the values in (1) were allowed, then $q\le 3$.

(2) Suppose $j_2 = (q+1)/2$ (resp. $j_2 = (q+2)/2$). Then $m_1(P) = (q+1)/2$ (resp. $m_1(P) = q/2$) and hence $g(\mathcal{X}) = \lfloor (q-1)^2/4 \rfloor$ by [18, Thm. 1].

(3) If $j_2(P) = q/2$, $m_1(P) = (q+2)/2$ by Remark 2.3; then $g(\mathcal{X}) \le g(H)$ where H is the semigroup generated by $(q+2)/2, q, q+1$ and $g(H) = \#(\mathbb{N}_0 \setminus S)$ is the genus of H. This number can be computed by the method of Rosales and García-Sánchez in [22]; i.e., $g(H) = q^2/8$ and the result follows. □

Theorem 2.5. *Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve satisfying (2.1). Suppose that $\pi(\mathcal{X})$ is contained in a cubic surface S.*

1. *For $P \in \mathcal{X}(\mathbf{F})$, $j_2(P) \in \{2, 3, q/2, (q+1)/3, (q+2)/3, (q+3)/3\}$;*
2. *If q is even and $g(\mathcal{X}) > q^2/8$, then $j_2(P) \ne q/2$.*

Proof. Let $j_0 = 0 < j_1 = 1 < j_2 < j_3 = q+1$ be the $(\mathcal{D}, P)$-orders with $j_2 = j_2(P)$ and $v = v_P$ the valuation at P. Then π can be defined by $(f_0 : f_1 : f_2 : f_3)$ such that $v(f_i) = j_i$; in particular, $\pi(P) = (1:0:0:0)$ and throughout we assume $f_0 = 1$. Let the cubic surface S be defined by

$$\begin{array}{rcl} F(X_0,X_1,X_2,X_3) & = & a_{000}X_0^3 + a_{001}X_0^2X_1 + a_{002}X_0^2X_2 + a_{003}X_0^2X_3 + a_{111}X_1^3 \\ & + & a_{110}X_1^2X_0 + a_{112}X_1^2X_2 + a_{113}X_1^2X_3 + a_{222}X_2^3 + a_{220}X_2^2X_0 \\ & + & a_{221}X_2^2X_1 + a_{223}X_2^2X_3 + a_{333}X_3^3 + a_{330}X_3^2X_0 + a_{331}X_3^2X_1 \\ & + & a_{332}X_3^2X_2 + a_{012}X_0X_1X_2 + a_{013}X_0X_1X_3 + a_{023}X_0X_2X_3 \\ & + & a_{123}X_1X_2X_3\,. \end{array}$$

Then $F(1, f_1, f_2, f_3) = 0$ and $a_{000} = 0$. Now the valuation at P of the functions

$$f_1, f_2, f_3, f_1^3, f_1^2, f_1^2f_2, f_1^2f_3, f_2^3, f_2^2, f_2^2f_1, f_2^2f_3, f_3^3,$$
$$f_3^2, f_3^2f_1, f_3^2f_2, f_1f_2, f_1f_3, f_2f_3, f_1f_2f_3$$

are respectively

$$1, j_2, j_3, 3, 2, 2+j_2, 2+j_3, 3j_2, 2j_2, 1+2j_2, 2j_2+j_3, 3j_3,$$
$$2j_3, 1+2j_3, j_2+2j_3, 1+j_2, 1+j_3, j_2+j_3, 1+j_2+j_3\,.$$

Then the valuation property of v implies $a_{001} = 0$. Let $j_2 > 3$ so that $a_{111} = a_{110} = 0$ (recall that $q > 7$). We have $j_2 + 2 < j_3$, otherwise $j_2 \in \{q, q-1\}$ which is not possible by Lemma 2.4. Thus

$$j_2 < j_2{+}1 < j_2{+}2 < j_3 < j_3{+}1 < j_3{+}2 < j_3{+}j_2 < j_3{+}j_2{+}1 < 2j_3 < 2j_3{+}1 < 2j_3{+}j_2 < 3j_3\,.$$

Since $2j_2 < 2j_2 + 1 < 3j_2 < 2j_2 + j_3$, the valuation property of v implies the following cases:

1. Either $2j_2 \in \{j_3, j_3+1, j_3+2\}$, or
2. $2j_2 + 1 = j_3$, or
3. $3j_2 \in \{j_3, j_3+1, j_3+2, 2j_3, 2j_3+1\}$.

By Lemma 2.4 $2j_2 \neq j_3, j_3+1, j_3+2$, $3j_2 \neq 2j_3$, $3j_2 \neq 2j_3+1$, and $2j_2+1 \neq j_3$ whenever $g > q^2/8$.

Therefore $j_2 \in \{2, 3, (q+1)/3, (q+2)/3, (q+1)/3\}$ and the proof follows. □

Now we can state the main result in this paper.

Corollary 2.6. *Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve as in Theorem 2.5. Then the multiplicity $m_1(P)$ of the Weierstrass semigroup $H(P)$ at $P \in \mathcal{X}(\mathbf{F})$ do satisfy*

$$m_1(P) \in \{(q+2)/2, (2q+2)/3, (2q+1)/3, 2q/3, q-2, q-1\}\,.$$

In addition, if q is even and $g(\mathcal{X}) > q^2/8$, then $m_1(P) \neq (q+2)/2$.

Proof. It follows from (2.3) and the theorem above. □

Remark 2.7. Notation as in Remark 2.3. For the linear series $\mathcal{D}$, a basic result is that for almost $P \in \mathcal{X}$, the sequence $j_0(P < j_1(P) < j_2(P) < j_3(P)$ is constant (so called *order sequence of* $\mathcal{D}$) cf. [25, p. 5]). In Remark 2.3 we noticed that $j_0(P) = 0$, $j_1(P) = 1$, $j_3(P) = q$ for $P \not\in \mathcal{X}(\mathbf{F})$ and thus the order sequence of $\mathcal{D}$ is of type $0 < 1 < \epsilon_2 < q$.

By the proof of [2, Prop. 3.1], $\epsilon_2 = 2$ provided that

$$g(\mathcal{X}) > \begin{cases} (q^2+1)(q-4)/2(4q-1)\,, & \text{whenever } q \not\equiv 0 \pmod 3\,, \\ g > (q^2+1)(q-3)/2(3q-1)\,, & \text{otherwise}\,. \end{cases}$$

This forces $g(\mathcal{X}) \geq (q^2-2q+3)/6\,(*)$ (see [5, Remark 3.3], [2, Prop. 3.1]).

Now for $P \in \mathcal{X}(\mathbf{F})$ the Weierstrass semigroup $H(P)$ contains the semigroup generated by $m, q, q+1$, where $m = m_1(P) = q+1-j_2(P)$ (cf. 2.3); hence $g(\mathcal{X}) \leq g(H)$ (the genus of H). Then by using heavy arithmetical computations from [10, Sect. 2] and by taking into consideration restriction $(*)$ above, Corollary 2.6 was already proved in [5, Cor. 3.5] whithout the hypothesis regarding the cubic surface.

Remark 2.8. Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve such that (2.1) holds; in particular, we identify $\mathcal{X}$ with a non-degenerate projective curve in $\mathbf{P}^3$ and we can apply the aforementioned Castelnuovo and Halphen-Ballico results as they are true in positive characteristic [3]. We look forward a result of type: There exists a polynomial (of one indeterminate) $A(x) \in \mathbb{Q}[x]$ such that

$$g(\mathcal{X}) > A(q+1) \quad \text{implies} \quad \pi(\mathcal{X}) \subseteq S\,, \tag{2.4}$$

where S is a surface of degree $d \leq 3$ (Then we shall assume $d = 3$ by Propositions 1.1, 1.2.)

Remark 2.9. In the literature, for a non-degenerate projective space curve $\mathcal{C}$ of degree $q+1$ over the complex numbers, there are available results of type (2.4) which in fact appear as particular cases of a vast theory that generalize the aforementioned Castelnuovo and Halphen results; see Eisenbud-Harris book [13, Thm. 3.22, p. 117].

Let q be large, says $q \geq 107$. If

$$g(\mathcal{C}) > B(q+1) := \begin{cases} \dfrac{q(q+2)}{8}, & \text{if} \qquad q \equiv 0, 2 \pmod 4, \\ \dfrac{q^2+2q-3}{8}, & \text{if} \qquad q \equiv 1 \pmod 4, \\ \dfrac{q^2+2q+9}{8}, & \text{otherwise}, \end{cases}$$

then there exists a surface S of degree 2 or 3 such that $\mathcal{C} \subseteq S$.

Question 2.10. Is Remark 2.9 true in positive characteristic?

3. Examples

In this section we illustrate Corollary 2.6. Notation as above; in particular, $\mathcal{H}$ is the Hermitian curve over $\mathbf{F} = \mathbf{F}_{q^2}$ defined by $v^{q+1} = u^{q+1} + 1$. Let $\pi : \mathcal{H} \to \mathbf{P}^2$ be a non-trivial morphism over $\mathbf{F}$ and $\mathcal{X}$ the non-singular model of the plane curve $\pi(\mathcal{H})$; then π can be lifted to a morphism $\mathcal{H} \to \mathcal{X}$, which we still denote by π. In this case, the curve $\mathcal{X}$ is also $\mathbf{F}$-maximal (see e.g. [20]).

Example 3.1. (cf. [7, Sect. 5]) Let $q \equiv 2 \pmod 3$ and $\pi : \mathcal{H} \to \mathbf{P}^2$ be the morphism given by $\pi = (x : y : 1) := (u^3 : uv : 1)$. Then the plane curve $\pi(\mathcal{H})$ is defined by

$$y^{q+1} = x^{(q+1)/3}(x^{(q+1)/3} + 1)\,,$$

and by applying the Riemann-Hurwitz formula to the function $x : \mathcal{X} \to \mathbf{P}^1$, where $\mathcal{X}$ is the non-singular model of $\pi(\mathcal{H})$, we find that $\mathcal{X}$ is $\mathbf{F}$-maximal of genus $g(\mathcal{X}) = g_2 = (q^2 - q + 4)/6$ (cf. [11], [4, Prop. 2.1]). We notice that $r(\mathcal{X}) = 3$ by Remark 2.1 above.

Next we shall compute the Weierstrass semigroup $H(P)$ at certain points of $\mathcal{X}$; we start by computing some principal divisors on $\mathcal{X}$ via tools from [24].

(a) There are $(q+1)/3$ points in $x^{-1}(\infty)$, say P_i, $i = 1, \ldots, (q+1)/3$.
Set $D_\infty := P_1 + \ldots + P_{(q+1)/3}$.

(b) There are $(q+1)/3$ points in $x^{-1}(0)$, say Q_i, $i = 1, \ldots, (q+1)/3$.
Set $D_0 := Q_1 + \ldots + Q_{(q+1)/3}$. Then $\mathrm{div}(x) = 3D_0 - 3D_\infty$.

(c) Let $a \in \mathbf{F}$ such that $a^{(q+1)/3} = -1\,(*)$. There is just point R_a over $x^{-1}(a)$ and $\mathrm{div}(x-a) = (q+1)R_a - 3D_\infty$. Set $D := \sum_{i/i^{(q+1)/3}=-1} R_i$.
Then $\mathrm{div}(x^{(q+1)/3} + 1) = (q+1)D - (q+1)D_\infty$.

From (a), (b), (c), $\mathrm{div}(y) = D_0 + D - 2D_\infty$, and for $a \in \mathbf{F}$ as in $(*)$ above

$$\mathrm{div}((x-a)^{-1}) = 3D_\infty - (q+1)R_a\,, \quad \mathrm{div}(y(x-a)^{-1}) = D_0 + D' + D_\infty - qR_a \quad \text{and}$$

$$\mathrm{div}(y^3x^{-1}(x-a)^{-1}) = 3D' - (q-2)R_a\,,$$

where $D' = D - R_a$. It follows that $H(R_a) \supseteq H := \langle q-2, q, q+1 \rangle$ so that $g(\mathcal{X}) \leq g(H)$. We have that the sequence $q-2, q+1, q$ is telescopic and so $g(H) = (q^2 - q + 4)/6$ (see e.g. [15, Prop. 5.35]). Therefore

Claim. $H(R_a) = H$ and $m_1(R_a) = q-2$ (this also shows that $r(\mathcal{X}) = 3$).

Moreover by Remark 2.7 the order sequence of $\mathcal{X}$ is $0 < 1 < 2 < q$ and thus there is also a point $P \in \mathcal{X}(\mathbf{F})$ with $m_1(P) = q-1$ (see [5, Lemma 3.7]).

Remark 3.2. We can construct explicit and outstanding AG one-point codes based on the curve in Example 3.1 by taking into consideration the telescopic property of $H(R_a)$; cf. [15, Sect. 5], [26, Sect. 5].

Example 3.3. Let $q \equiv 2 \pmod 3$. Here we point out properties of an arbitrary $\mathbf{F}$-maximal curve $\mathcal{X}$ of genus $g(\mathcal{X}) = g_2 = (q^2-q+4)/6$. We have $r(\mathcal{X}) = 3$ by Remark 2.1, and that $0 < 1 < 2 < q$ is the order sequence of $\mathcal{D} = \mathcal{D}_{\mathcal{X}}$ by Remark 2.7. Then by [5, Lemma 3.7] there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $m_1(\bar{P}) = q-1$, or $j_2(\bar{P}) = 2$ by (2.3).

Claim. There is $P \in \mathcal{X}(\mathbf{F})$ such that $j_2(P) > 2$.

Indeed, otherwise [18, Lemma 7] would imply $g = (q^2-2q+3)/6$, a contradiction.

Let $\pi : \mathcal{X} \to \mathbf{P}^3$ be the morphism associated to $\mathcal{D}$. We are led to the following questions.

(A) Is $\pi(\mathcal{X})$ contained in a cubic surface? (This would be true if the answer to Question 2.10 is affirmative)

(B) Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve. Then $g(\mathcal{X}) = g_2$ if and only if $\pi(\mathcal{X})$ is contained in a cubic surface and there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $j_2(\bar{P}) > 2$?

Question (B) above is related to the following result which is a consequence of the proof of [18, Thm. 1] and [Lemma 7]KT.

Remark 3.4. With g_1 as in (1.4), for an $\mathbf{F}$-maximal curve $\mathcal{X}$ we have that $g(\mathcal{X}) = g_1$ if and only if $\pi(\mathcal{X})$ is contained in a quadric and there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $j_2(\bar{P}) > 2$.

Example 3.5. Let $q \equiv 2 \pmod 3$. We investigate $\mathbf{F}$-maximal curves of genus $g(\mathcal{X}) = g_3 = g_2 - 1 = (q^2-q-2)/6$ which were constructed in [5]. To start with, $r(\mathcal{X}) = 3$ by Remark 2.1 and the order seqeunce of $\mathcal{D}$ is $0 < 1 < 2 < q$ by Remark 2.7. In particular, there is $P \in \mathcal{X}(\mathbf{F})$ such that $m_1(P) = q-1$ by [5, Lemma 3.7].

We further assume the following properties:

(a) $\pi(\mathcal{X})$ is contained in a cubic surface;
(b) $\pi : \mathcal{H} \to \mathcal{X}$ is Galois of degree three.

(The aforementioned curves in [5] satisfy these properties)

Claim. There is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $m_1(\bar{P}) = (2q+2)/3$.

Proof of the Claim. By the Riemann-Hurwitz relation there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ which is totally ramified for π. Let $Q = \pi^{-1}(\bar{P}) \in \mathcal{H}$. The first six positive elements of the Weierstrass semigroup at Q are $q, q+1, 2q, 2q+1, 2q+2, 3q$. Now let $m = m_1(\bar{P}) < q < q+1$ be the first three positive elements of $H(\bar{P})$. Then $3m \in \{q, q+1, 2q, 2q+1, 2q+2\}$ and so

$m \in \{(q+1)/3, (2q+2)/3$. We eliminate the case $m = (q+1)/3$ by Corollary 2.6 and the Claim follows.

Example 3.6. Let $q \not\equiv 2 \pmod 3$ and $\mathcal{X}$ be an $\mathbf{F}$-maximal curve of genus $g(\mathcal{X}) = g_2 = (q^2-q)/6$; hence $r(\mathcal{X}) = 3$ by Remark 2.1 and the order sequence of $\mathcal{D}$ is $0 < 1 < 2 < q$ by Remark 2.7. by Remark 2.7. In particular, there is $P \in \mathcal{X}(\mathbf{F})$ such that $m_1(P) = q-1$ by [5, Lemma 3.7]. We notice that examples of such curves do exist: see e.g. [11], [4, Prop. 2.1].

Let us assume properties (a) and (b) in Example 3.5 (indeed, the aforementioned examples satisfy these hypotheses).

Claim. If $q \equiv 1 \pmod 3$ (resp. $q \equiv 0 \pmod 3$), then there exists $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $m_1(\bar{P}) = (2q+1)/3$ (resp. $m_1(\bar{P}) = 2q/3$).

Arguing as in Example 3.5 there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ such that $3m \in \{q, q+1, 2q, 2q+1, 2q+2\}$ with $m = m_1(\bar{P})$.

If $q \equiv 1 \pmod 3$, $m = (2q+1)/3$.

If $q \equiv 0 \pmod 3$, either $m = q/3$ or $m = 2q/3$. The former case is eliminated by Corollary 2.6.

Example 3.7. Here we present an $\mathbf{F}$-maximal curve $\mathcal{X}$ with $r(\mathcal{X}) = 3$ such that $\pi(\mathcal{X})$ cannot be contained in a cubic surface, where π is the morphism associated to $\mathcal{D}$. Indeed, we consider the so-called GK-curve [12] whose Weierstrass semigroups at rational points were computed in [7]. This curve is defined over $\mathbf{F} = \mathbb{F}_{q^2}$ with $q = \ell^3$. For $\ell > 2$ this is the first example of an $\mathbf{F}$-maximal curve that cannot be dominated by $\mathcal{H}$ (loc. cit.)

On this curve there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ such that $m_1(\bar{P}) = \ell^3 - \ell^2 + \ell$ [12, Sect. 4], and therefore, according to Corollary 2.6, $\pi(\mathcal{X})$ cannot be contained in a cubic. We notice that the genus of $\mathcal{X}$ is $g(\mathcal{X}) = \frac{1}{2}(\ell^5 - 2\ell^3 + \ell^2)/2$ and so it does not satisfies Remark 2.9. Further examples can be found in [26].

We end this paper with the following:

Question 3.8. Let $\mathcal{X}$ be an $\mathbf{F}$-maximal curve with $r(\mathcal{X}) = 3$. Suppose that $\pi(\mathcal{X}) \subseteq S$, where S is a surface of degree $d \geq 2$. Let $P \in \mathcal{X}(\mathbf{F})$ and suppose $g(\mathcal{X})$ large enough. Then $m_1(P) = (q+1) - \frac{q+i}{d}$ or $m_1(P) = q - j$ for some $i = 1, \ldots, d$, $j = 2, \ldots, d$. Are all these cases possible?

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