Abstract

Since techniques used to address the Nivat’s conjecture usually rely on Morse-Hedlund Theorem, an improved version of this classical result may mean a new step towards a proof for the conjecture. In this paper, we consider an alphabetical version of the Morse-Hedlund Theorem. Following methods highlighted by Cyr and Kra [1], we show that, for a configuration \( \eta \in \mathcal{A}^{\mathbb{Z}^2} \) that contains all letters of a given finite alphabet \( \mathcal{A} \), if its complexity with respect to a quasi-regular set \( U \subset \mathbb{Z}^2 \) (a finite set whose convex hull on \( \mathbb{R}^2 \) is described by pairs of edges with identical size) is bounded from above by \( \frac{1}{2}|U| + |\mathcal{A}| - 1 \), then \( \eta \) is periodic.

Keywords:
Combinatorics on words, Formal languages, Symbolic dynamics.

1. Introduction

Fixed a finite alphabet \( \mathcal{A} \) (with at least two elements), for each \( n \in \mathbb{N} \), the \( n \)-complexity of an infinite sequence \( \mathcal{E} = (\xi_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \), denoted by \( P_\mathcal{E}(n) \), is defined to be the number of distinct words of the form \( \xi_i \xi_{i+1} \cdots \xi_{i+n-1} \) appearing in \( \xi \). In 1938, Morse and Hedlund [2] proved one of the most famous results in symbolic dynamics which establishes a connection between periodic sequences (sequences for which there is an integer \( m \geq 1 \) such that \( \xi_{i+m} = \xi_i \) for all \( i \in \mathbb{Z} \)) and complexity. More specifically, they proved that \( \mathcal{E} \in \mathcal{A}^{\mathbb{Z}} \) is periodic if, and only if, there exists \( n \in \mathbb{N} \) such that \( P_\mathcal{E}(n) \leq n \).

A natural extension of the complexity function to higher dimensions is obtained when we consider, instead of words, blocks of symbols. More precisely, the \( n_1 \times \cdots \times n_d \)-complexity of a configuration \( \mathcal{N} = (\eta_g)_{g \in \mathbb{Z}^d} \in \mathcal{A}^{\mathbb{Z}^d} \), denoted by \( P_\mathcal{N}(n_1, \ldots, n_d) \), is defined to be the number of distinct \( n_1 \times \cdots \times n_d \) blocks of symbols appearing in \( \eta \). Of course periodicity also has a natural higher dimensional generalization: \( \eta \in \mathcal{A}^{\mathbb{Z}^d} \) is periodic if there exists a vector \( h \in (\mathbb{Z}^d)^* \), called period of \( \eta \), such that \( \eta_{g+h} = \eta_g \) for all \( g \in \mathbb{Z}^d \). A configuration that is not periodic is said to be aperiodic.

The Nivat’s Conjecture [3] is the natural generalization of the Morse-Hedlund Theorem for the two-dimensional case.