ON THE PERIODIC SOLUTIONS OF A GENERALIZED SMOOTH AND NON-SMOOTH PERTURBED PLANAR DOUBLE PENDULUM WITH SMALL OSCILLATIONS

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Abstract. We provide sufficient conditions for the existence of periodic solutions of the smooth and non-smooth perturbed planar double pendulum with small oscillations having equations of motion

$$\ddot{\theta}_1 = -a\theta_1 + \theta_2 + \varepsilon\left(F_1(t,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) + F_2(t,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2)\mathrm{sgn}(\dot{\theta_1})\right),$$
$$\ddot{\theta}_2 = b\theta_1 - b\theta_2 + \varepsilon\left(F_3(t,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) + F_4(t,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2)\mathrm{sgn}(\dot{\theta_2})\right),$$

where $a > 1, b > 0$ and ε are real parameters. Here the parameter ε is small and the smooth functions F_i for $i = 1, 2, 3, 4$ define the perturbation which are periodic functions in t and in resonance p_i:q_i with some of the periodic solutions of the unperturbed double pendulum, being p_i and q_i relatively prime positive integers.

1. Introduction and statement of the main results

We consider a system of two point masses m_1 and m_2 moving in a fixed plane, in which the distance between a point (called pivot) and m_1 and the distance between m_1 and m_2 are fixed, and equal to l_1 and l_2 respectively. We assume the masses do not interact. We allow gravity to act on the masses m_1 and m_2. This system is called the planar *double pendulum*.

The position of the double pendulum is determined by the two angles θ_1 and θ_2 shown in Figure 1. We consider only the motion in the vicinity of the equilibrium $\theta_1 = \theta_2 = 0$, i.e. we are only interested in small oscillations around this equilibrium. Expanding the Lagrangian of this system to second order in θ_1 and θ_2 and their time derivatives, the corresponding Lagrange equations of motion are

$$\begin{aligned}(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1 &= 0,\\ m_2 l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 + m_2 g\theta_2 &= 0,\end{aligned} \tag{1}$$

where g is the acceleration of the gravity. For more details on these equations of motion see [2]. Here the dot denotes derivative with respect to the time t.

The authors in [5] have studied the persistence of periodic solutions of system (1) perturbed smoothly in the particular case thus $m_1 = m_2$ and $l_1 = l_2$. Now m_1, m_2, l_1 and l_2 can take arbitrary positive values and we shall study the periodic orbits of system (1) which persist with smooth and non-smooth perturbations.

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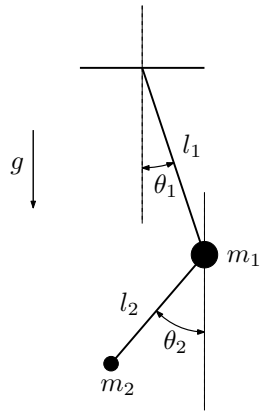

FIGURE 1. The planar double pendulum.

Dividing the equations of system (1) by $m_2 l_1$ and denoting $l = l_2/l_1 > 0$, $m = (m_1 + m_2)/m_2 > 1$, system (1) becomes

$$\begin{aligned} m\,\ddot{\theta}_1 + l\,\ddot{\theta}_2 + \frac{mg}{l_1}\theta_1 &= 0, \\ \ddot{\theta}_1 + l\,\ddot{\theta}_2 + \frac{g}{l_1}\theta_2 &= 0. \end{aligned} \tag{2}$$

Isolating $\ddot{\theta}_1$ and $\ddot{\theta}_2$ of equations (2), we obtain that the following equations of motion for the double pendulum with small oscillations

$$\begin{aligned} \ddot{\theta}_1 &= -\frac{mg}{l_1(m-1)}\theta_1 + \frac{g}{l_1(m-1)}\theta_2, \\ \ddot{\theta}_2 &= \frac{mg}{ll_1(m-1)}\theta_1 - \frac{mg}{ll_1(m-1)}\theta_2. \end{aligned} \tag{3}$$

Taking a new time τ given by the rescaling $\tau = \sqrt{g/(l_1(m-1))}\,t$ and denoting $a = m > 1$ and $b = m/l > 0$, the equations of motion (3) become

$$\begin{aligned} \theta_1'' &= -a\theta_1 + \theta_2, \\ \theta_2'' &= b\theta_1 - b\theta_2, \end{aligned} \tag{4}$$

where now the prime denotes derivative with respect to the new time τ.

The objective of this paper is to provide a system of non-linear and non-smooth equations whose simple zeros provide periodic solutions of the smooth and non-smooth perturbed planar double pendulum with equations of motion

$$\begin{aligned} \theta_1'' &= -a\theta_1 + \theta_2 + \varepsilon\left(F_1(t,\theta_1,\theta_1',\theta_2,\theta_2') + F_2(t,\theta_1,\theta_1',\theta_2,\theta_2')\mathrm{sgn}(\theta_1')\right), \\ \theta_2'' &= b\theta_1 - b\theta_2 + \varepsilon\left(F_3(t,\theta_1,\theta_1',\theta_2,\theta_2') + F_4(t,\theta_1,\theta_1',\theta_2,\theta_2')\mathrm{sgn}(\theta_2')\right), \end{aligned} \tag{5}$$

where ε is a small parameter. Here the smooth functions F_i for $i = 1, 2, 3, 4$ define the perturbation. These functions are periodic in τ and in resonance p_i:q_i with some of the periodic solutions of the unperturbed double pendulum, being p_i and q_i relatively prime positive integers and the function $\mathrm{sgn}(z)$ denotes the sign function,

i.e.

$$\operatorname{sgn}(z) = \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

Note that the functions F_i for $i = 1, 2, 3, 4$ can be taken in a certain way arbitrary, i.e., only observing some hypothesis. It makes us able to provide, in a physical context, real meaning of these functions, for example, friction for F_1 and F_3 and excitation for F_2 and F_4.

In order to present our results we need some preliminary definitions and notations.

The unperturbed system (4) has a unique singular point, the origin with eigenvalues $\pm\omega_1\, i, \pm\omega_2\, i$, where

$$\omega_1 = \frac{\sqrt{a+b-\sqrt{\Delta}}}{\sqrt{2}}, \quad \omega_2 = \frac{\sqrt{a+b+\sqrt{\Delta}}}{\sqrt{2}},$$

with $\Delta = (a-b)^2 + 4b > 0$. Consequently this system in the phase space $(\theta_1, \theta_1', \theta_2, \theta_2')$ has two planes filled with periodic solutions except the origin. The periods of such periodic orbits are

$$T_1 = \frac{2\pi}{\omega_1} \quad \text{or} \quad T_2 = \frac{2\pi}{\omega_2},$$

according they belong to the plane associated to the eigenvectors with eigenvalues $\pm\omega_1\, i$ or $\pm\omega_2\, i$, respectively. We shall study which of these periodic solutions persist for the perturbed system (5) when the parameter ε is sufficiently small and the perturbed functions F_i for $i = 1, 2, 3, 4$ have period either $p_i T_1/q_i$, or $p_i T_2/q_i$, where p_i and q_i are relatively prime positive integers.

Let $Y_{X_0,Y_0}(\tau)$ be the periodic function

$$Y_{X_0,Y_0}(\tau) = Y_0 \cos(\omega_1\, \tau) - X_0 \sin(\omega_1\, \tau),$$

then we define the non-linear and non-smooth functions

(6)

$$\begin{aligned} \mathcal{F}_1(X_0, Y_0) = &\int_0^{pT_1} \sin(\omega_1\, \tau) \left(2b\bar{F}_1 + \bar{F}_3 \left(a - b + \sqrt{\Delta}\right)\right) d\tau \\ &+ \int_0^{pT_1} \sin(\omega_1\, \tau) \left(2b\bar{F}_2 + \bar{F}_4 \left(a - b + \sqrt{\Delta}\right)\right) \operatorname{sgn}(Y_{X_0,Y_0}(\tau))\, d\tau, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_2(X_0, Y_0) = &\int_0^{pT_1} \cos(\omega_1\, \tau) \left(2b\bar{F}_1 + \bar{F}_3 \left(a - b + \sqrt{D}\right)\right) d\tau \\ &+ \int_0^{pT_1} \cos(\omega_1\, \tau) \left(2b\bar{F}_2 + \bar{F}_4 \left(a - b + \sqrt{D}\right)\right) \operatorname{sgn}(Y_{X_0,Y_0}(\tau))\, d\tau. \end{aligned}$$

with p the least common multiple among the p_i's for $i = 1, 2, 3, 4$, and

$$\bar{F}_i = F_i(\tau, A_1, B_1, C_1, D_1) \tag{7}$$

for $i = 1, 2, 3, 4$ with

$$A_1 = \frac{\left(-a+b+\sqrt{\Delta}\right)}{2\,b\,\omega_1}\left(X_0\cos\left(\omega_1\,\tau\right)+Y_0\sin\left(\omega_1\,\tau\right)\right),$$
$$B_1 = \frac{\left(-a+b+\sqrt{\Delta}\right)}{2b}\left(Y_0\cos\left(\omega_1\,\tau\right)-X_0\sin\left(\omega_1\,\tau\right)\right),$$
$$C_1 = \frac{1}{\omega_1}\left(X_0\cos\left(\omega_1\,\tau\right)+Y_0\sin\left(\omega_1\,\tau\right)\right),$$
$$D_1 = Y_0\cos\left(\omega_1\,\tau\right)-X_0\sin\left(\omega_1\,\tau\right).$$

Note that if $2b\bar{F}_2+\bar{F}_4\left(a-b+\sqrt{D}\right)=0$, then the functions $\mathcal{F}_1(X_0,Y_0)$ and $\mathcal{F}_2(X_0,Y_0)$ are smooth.

A zero (X_0^*,Y_0^*) of the system of the non-linear and non-smooth functions

$$\mathcal{F}_1(X_0,Y_0)=0,\quad \mathcal{F}_2(X_0,Y_0)=0, \tag{8}$$

such that

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(X_0,Y_0)}\right|_{(X_0,Y_0)=(X_0^*,Y_0^*)}\right)\neq 0,$$

is called a *simple zero* of system (8).

Our main result on the periodic solutions of the non-smooth perturbed double pendulum (5) which bifurcate from the periodic solutions of the unperturbed double pendulum (4) with period T_1 traveled p times is the following.

Theorem 1. *Assume that the functions F_i of the non-smooth perturbed double pendulum with equations of motion (5) are periodic in τ of period p_iT_1/q_i with p_i and q_i relatively prime positive integers. If p is the least common multiple among the p_i's for $i=1,2,3,4$, then for $\varepsilon\neq 0$ sufficiently small and for every simple zero $(X_0^*,Y_0^*)\neq(0,0)$ of the non-linear and non-smooth system (8), the non-smooth perturbed double pendulum (5) has a periodic solution $(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon))$ tending, when $\varepsilon\to 0$, to the periodic solution $(\theta_1(\tau),\theta_2(\tau))$, traveled p times, of the unperturbed double pendulum (4) given by*

$$\left(\frac{-a+b+\sqrt{\Delta}}{2\,b\,\omega_1}\left(X_0^*\cos\left(\omega_1\,\tau\right)+Y_0^*\sin\left(\omega_1\,\tau\right)\right),\frac{1}{\omega_1}\left(X_0^*\cos\left(\omega_1\,\tau\right)+Y_0^*\sin\left(\omega_1\,\tau\right)\right)\right). \tag{9}$$

Theorem 1 is proved in section 2. Its proof is based in the averaging theory for computing periodic solutions, see the appendix.

We provide an application of Theorem 1 in the following corollary, which will be proved in section 3.

Corollary 2. *If $F_1=\alpha\cos(\omega_1\,t)$, $F_2=\beta\theta_2\dot{\theta}_2$, $F_3=\gamma\sin(\omega_1\,t)$ and $F_4=0$, with $\alpha=\omega_1/(2b\pi)$, $\beta=3\omega_1^2/(8b)$ and $\gamma=\omega_1/(\pi(a-b+\sqrt{\Delta}))$. Then the differential system (5) for $\varepsilon\neq 0$ sufficiently small has one periodic solution $(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon))$ tending, when $\varepsilon\to 0$, to the periodic solution $(\theta_1(\tau),\theta_2(\tau))$ of the unperturbed double pendulum (4) given by (9) with*

$$(X_0^*,Y_0^*)=\left(2^{-\frac{1}{4}},2^{-\frac{1}{4}}\right).$$

Another application of Theorem 1 has been proved in [5] with the following statement

Corollary 3. *Assume $a = b = 2$. If $F_1 = 0$, $F_2 = 0$, $F_3 = \left(1-\theta_1^2\right)\sin\left(\sqrt{2-\sqrt{2}}\,t\right)$ and $F_4 = 0$, then the differential system* (5) *for $\varepsilon \neq 0$ sufficiently small has two periodic solutions $(\theta_1(\tau,\varepsilon),\theta_2(\tau,\varepsilon))$ tending, when $\varepsilon \to 0$, to the two periodic solutions $(\theta_1(\tau),\theta_2(\tau))$ of the unperturbed double pendulum* (4) *given by* (9) *with*

$$(X_0^*, Y_0^*) = \left(2\sqrt{2\left(2-\sqrt{2}\right)}, 0\right) \quad \textit{and} \quad (X_0^*, Y_0^*) = \left(0, 2\sqrt{\frac{2}{3}\left(2-\sqrt{2}\right)}\right),$$

respectively.

Now let $W^{Z_0,W_0}(\tau)$ be the periodic function

$$W^{Z_0,W_0}(\tau) = W_0\cos(\omega_2\,\tau) - Z_0\sin(\omega_2\,\tau),$$

then we define the non-linear and non-smooth functions

(10)

$$\begin{aligned}\mathcal{F}^1(Z_0,W_0) = &\int_0^{pT_2} \sin(\omega_2\,\tau)\left(-2b\bar{F}_1 + \bar{F}_3\left(-a+b+\sqrt{D}\right)\right)\,d\tau \\ &+ \int_0^{pT_2} \sin(\omega_2\,\tau)\left(2b\bar{F}_2 + \bar{F}_4\left(-a+b+\sqrt{D}\right)\right)\,\mathrm{sgn}(W^{Z_0,W_0}(\tau))\,d\tau,\end{aligned}$$

$$\begin{aligned}\mathcal{F}^2(Z_0,W_0) = &\int_0^{pT_2} \cos(\omega_2\,\tau)\left(-2b\bar{F}_1 + \bar{F}_3\left(-a+b+\sqrt{D}\right)\right)\,d\tau \\ &+ \int_0^{pT_2} \cos(\omega_2\,\tau)\left(2b\bar{F}_2 + \bar{F}_4\left(-a+b+\sqrt{D}\right)\right)\,\mathrm{sgn}(W^{Z_0,W_0}(\tau))\,d\tau.\end{aligned}$$

with p the least common multiple among the p_i's for $i = 1,2,3,4$, and

$$\bar{F}_i = F_i(\tau, A_2, B_2, C_2, D_2)$$

for $i = 1,2,3,4$ with

$$\begin{aligned}A_2 &= -\frac{\left(a-b+\sqrt{\Delta}\right)}{2\,b\,\omega_2}\left(Z_0\cos(\omega_2\,\tau) + W_0\sin(\omega_2\,\tau)\right), \\ B_2 &= -\frac{\left(a-b+\sqrt{\Delta}\right)}{2b}\left(W_0\cos(\omega_2\,\tau) - Z_0\sin(\omega_2\,\tau)\right), \\ C_2 &= \frac{1}{\omega_2}\left(Z_0\cos(\omega_2\,\tau) + W_0\sin(\omega_2\,\tau)\right), \\ D_2 &= W_0\cos(\omega_2\,\tau) - Z_0\sin(\omega_2\,\tau).\end{aligned}$$

Note that if $2b\bar{F}_2 + \bar{F}_4\left(-a+b+\sqrt{D}\right) = 0$, then the functions $\mathcal{F}^1(Z_0,W_0)$ and $\mathcal{F}^2(Z_0,Z_0)$ are smooth.

Consider the non-linear and non-smooth system

$$\mathcal{F}^1(Z_0,W_0) = 0, \quad \mathcal{F}^2(Z_0,W_0) = 0. \tag{11}$$

Our main result on the periodic solutions of the non-smooth perturbed double pendulum (5) which bifurcate from the periodic solutions of the unperturbed double pendulum (4) with period T_2 traveled p times is the following.

Theorem 4. *Assume that the functions F_i of the non-smooth perturbed double pendulum with equations of motion* (5) *are periodic in τ of period $p_i T_2/q_i$ with p_i and q_i relatively prime positive integers. If p is the least common multiple among the p_i's for $i = 1, 2, 3, 4$, then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(Z_0^*, W_0^*) \neq (0, 0)$ of the non-linear and non-smooth system* (11)*, the non-smooth perturbed double pendulum* (5) *has a periodic solution $(\theta_1(\tau, \varepsilon), \theta_2(\tau, \varepsilon))$ tending, when $\varepsilon \to 0$, to the periodic solution $(\theta_1(\tau), \theta_2(\tau))$, traveled p times, of the unperturbed double pendulum* (4) *given by*

(12)
$$\left(-\frac{a-b+\sqrt{\Delta}}{2\,b\,\omega_2}\left(Z_0^*\cos\left(\omega_2\,\tau\right)+W_0^*\sin\left(\omega_2\,\tau\right)\right),\frac{1}{\omega_2}\left(Z_0^*\cos\left(\omega_2\,\tau\right)+W_0^*\sin\left(\omega_2\,\tau\right)\right)\right).$$

Theorem 4 is also proved in section 2.

We provide an application of Theorem 4 in the following corollary, which will be proved in section 3.

Corollary 5. *If $F_1 = \alpha \sin(\omega_2\, t)$, $F_2 = \beta\theta_2\dot{\theta}_2$, $F_3 = \gamma\cos(\omega_2\, t)$ and $F_4 = 0$, with $\alpha = -\omega_2/(2b\pi)$, $\beta = 3\omega_2^2/(8b)$ and $\gamma = \omega_2/(\pi(-a+b+\sqrt{\Delta}))$. Then the differential equation* (5) *for $\varepsilon \neq 0$ sufficiently small has one periodic solutions $(\theta_1(\tau, \varepsilon), \theta_2(\tau, \varepsilon))$ tending, when $\varepsilon \to 0$, to the periodic solutions $(\theta_1(\tau), \theta_2(\tau))$ of the unperturbed double pendulum* (4) *given by* (12) *with*

$$(Z_0^*, W_0^*) = \left(2^{-\frac{1}{4}}, 2^{-\frac{1}{4}}\right).$$

Another application of Theorem 4 has been proved in [5] with the following statement

Corollary 6. *Assume $a = b = 2$. If $F_1 = \theta_2' + \theta_1^2\cos\left(\sqrt{2+\sqrt{2}}\,t\right)$, $F_2 = 0$, $F_3 = 0$ and $F_4 = 0$, then the differential equation* (5) *for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(\theta_1(\tau, \varepsilon), \theta_2(\tau, \varepsilon))$ tending, when $\varepsilon \to 0$, to the periodic solution $(\theta_1(\tau), \theta_2(\tau))$ of the unperturbed double pendulum* (4) *given by* (12) *with* $(Z_0^*, W_0^*) = \left(0, -8\left(2+\sqrt{2}\right)\right)$.

2. Proofs of Theorems 1 and 4

Introducing the variables $(x, y, z, w) = (\theta_1, \theta_1', \theta_2, \theta_2')$ we write the differential system of the non-smooth perturbed double pendulum (5) as a first–order differential system defined in $\mathbb{R}^4$. Thus we have the differential system

$$\begin{aligned} x' &= y, \\ y' &= -ax + z + \varepsilon\left(F_1(\tau, x, y, z, w) + F_2(\tau, x, y, z, w)\,\mathrm{sgn}(y)\right), \\ z' &= w, \\ w' &= bx - bz + \varepsilon\left(F_3(\tau, x, y, z, w) + F_4(\tau, x, y, z, w)\,\mathrm{sgn}(w)\right). \end{aligned} \tag{13}$$

System (13) with $\varepsilon = 0$ is equivalent to the unperturbed double pendulum system (4), called in what follows simply by the *unperturbed system*. Otherwise we have the *perturbed system*.

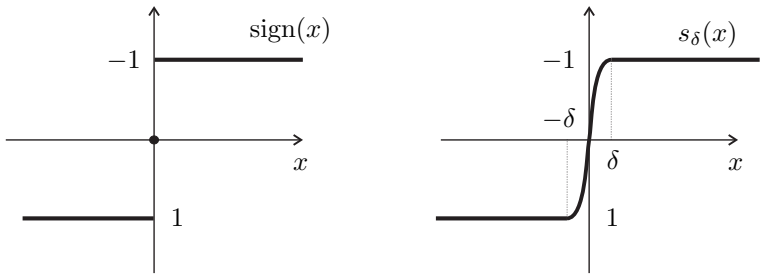

FIGURE 2. The functions $\operatorname{sign}(x)$ and $s_\delta(x)$.

Instead of working with the discontinuous differential system (13) we shall work with the smooth differential system

$$
\begin{aligned}
&x' = y,\\
&y' = -ax + z + \varepsilon\left(F_1(\tau,x,y,z,w) + F_1(\tau,x,y,z,w)\, s_\delta(y)\right),\\
&z' = w,\\
&w' = bx - bz + \varepsilon\left(F_3(\tau,x,y,z,w) + F_4(\tau,x,y,z,w)\, s_\delta(w)\right).
\end{aligned}
\tag{14}
$$

where $s_\delta(x)$ is the smooth function defined in Figure 2, such that

$$\lim_{\delta\to 0} s_\delta(x) = \operatorname{sgn}(x).$$

We shall write system (14) in such a way that the linear part at the origin will be in its real normal Jordan form. Then, doing the change of variables $(\tau,x,y,z,w) \to (\tau,X,Y,Z,W)$ given by

(15)

$$
\begin{pmatrix} X\\ Y\\ Z\\ W \end{pmatrix} = \begin{pmatrix}
\dfrac{b\omega_1}{\sqrt{\Delta}} & 0 & \dfrac{\omega_1\left(a-b+\sqrt{\Delta}\right)}{2\sqrt{\Delta}} & 0\\
0 & \dfrac{b}{\sqrt{\Delta}} & 0 & \dfrac{a-b+\sqrt{\Delta}}{2\sqrt{\Delta}}\\
-\dfrac{b\omega_2}{\sqrt{\Delta}} & 0 & \dfrac{\omega_2\left(-a+b+\sqrt{\Delta}\right)}{2\sqrt{\Delta}} & 0\\
0 & -\dfrac{b}{\sqrt{\Delta}} & 0 & \dfrac{-a+b+\sqrt{\Delta}}{2\sqrt{\Delta}}
\end{pmatrix}
\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix},
$$

the differential system (14) becomes

(16)

$$
\begin{aligned}
&X' = \omega_1 Y,\\
&Y' = -\omega_1 X + \varepsilon\frac{1}{2\sqrt{\Delta}}\left(2b\left(\tilde{F}_1 + \tilde{F}_2\, s_\delta(\mathcal{B})\right) + \left(a-b+\sqrt{\Delta}\right)\left(\tilde{F}_3 + \tilde{F}_4\, s_\delta(\mathcal{D})\right)\right),\\
&Z' = \omega_2 W,\\
&W' = -\omega_2 Z + \varepsilon\frac{1}{2\sqrt{\Delta}}\left(-2b\left(\tilde{F}_1 + \tilde{F}_2\, s_\delta(\mathcal{B})\right) + \left(-a+b+\sqrt{\Delta}\right)\left(\tilde{F}_3 + \tilde{F}_4\, s_\delta(\mathcal{D})\right)\right),
\end{aligned}
$$

where $\tilde{F}_i(\tau, X, Y, Z, W) = F_i(\tau, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ for $i = 1, 2, 3, 4$ with

$$\begin{aligned}
\mathcal{A} &= \frac{\left(-a+b+\sqrt{\Delta}\right)}{2\,b\,\omega_1} X - \frac{\left(a-b+\sqrt{\Delta}\right)}{2\,b\,\omega_2} Z,\\
\mathcal{B} &= \frac{\left(-a+b+\sqrt{\Delta}\right)}{2b} Y - \frac{\left(a-b+\sqrt{\Delta}\right)}{2b} W,\\
\mathcal{C} &= \frac{1}{\omega_1} X + \frac{1}{\omega_2} Z,\\
\mathcal{D} &= Y + W.
\end{aligned}$$

Note that the linear part of the differential system (16) at the origin is in its real normal Jordan form.

Lemma 7. *The periodic solutions of the differential system* (16) *with* $\varepsilon = 0$ *are*

$$\begin{aligned}
&X_{X_0,Y_0}(\tau) = X_0 \cos(\omega_1 \tau) + Y_0 \sin(\omega_1 \tau),\\
&Y_{X_0,Y_0}(\tau) = Y_0 \cos(\omega_1 \tau) - X_0 \sin(\omega_1 \tau),\\
&Z_{X_0,Y_0}(\tau) = 0,\\
&W_{X_0,Y_0}(\tau) = 0,
\end{aligned} \tag{17}$$

of period T_1*, and*

$$\begin{aligned}
&X^{Z_0,W_0}(\tau) = 0,\\
&Y^{Z_0,W_0}(\tau) = 0,\\
&Z^{Z_0,W_0}(\tau) = Z_0 \cos(\omega_2 \tau) + W_0 \sin(\omega_2 \tau),\\
&W^{Z_0,W_0}(\tau) = W_0 \cos(\omega_2 \tau) - Z_0 \sin(\omega_2 \tau),
\end{aligned} \tag{18}$$

of period T_2*.*

Proof. Since system (16) with $\varepsilon = 0$ is a linear differential system, the proof follows easily. □

Proof of Theorem 1. Assume that the functions F_i of the non-smooth perturbed double pendulum with equations of motion (5) are periodic in τ of period $p_i T_1/q_i$ with p_i and q_i relatively prime positive integers. Then we can think that system (5) is periodic in τ of period pT_1, with p the least common multiple among the p_i's for $i = 1, 2, 3, 4$,. Thinking in this way the differential system and the periodic solutions (17) have the same period pT_1.

It is well known that a Poincaré map defined in a smooth differential system is smooth. So the Poincaré maps associated to the periodic orbits of the differential system (14) are smooth. The Poincaré maps associated to the periodic solutions of the non-smooth differential system (13), which are perturbations of the periodic solutions (17) are also smooth. Indeed, such Poincaré maps are compositions of two smooth Poincaré maps, one from $y = 0$ to itself following the orbits of the system (13) in the region $y > 0$, and the other from $y = 0$ to itself following the orbits of the system (13) in the region $y < 0$. In a similar way it follows that the Poincaré maps associated to the periodic solutions of the non-smooth differential system (13), which are perturbations of the periodic solutions (18) are also smooth.

We can use Theorem 8 (see the appendix) for computing some of the periodic solutions of the smooth systems. The periodic solutions are zeros of the displacement function, which is the Poincaré map associated to periodic solutions minus the identity. In fact, the non-linear function (24) whose zeros can provide periodic

solutions, is the first term of order ε of the displacement function. See for more details the proof of Theorem 8 in [1].

Since the Poincaré maps associated to periodic solutions of system (13), coming from the perturbed periodic solutions (17) or (18), are smooth and these Poincaré maps are the limit of the Poincaré maps associated to the smooth system (14), for which we can use Theorem 8, it follows that we also can use Theorem 8 for computing some of the periodic solutions of the non-smooth system (13). In other words, we can apply Theorem 8 to the smooth systems (14) and then pass to the limit, when $\delta \to 0$, the function (24) for obtaining a function whose zeros can give periodic solutions of the non-smooth system (13).

We shall apply Theorem 8 of the appendix to the differential system (16). We note that system (16) can be written as system (21) taking

$$\mathbf{x} = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}, \quad t = \tau, \quad G_0(t, \mathbf{x}) = \begin{pmatrix} \omega_1 Y, \\ -\omega_1 X, \\ \omega_2 W, \\ -\omega_2 Z \end{pmatrix},$$

$$G_1(t, \mathbf{x}) = \begin{pmatrix} 0 \\ \dfrac{b}{\sqrt{\Delta}} \left(\tilde{F}_1 + \tilde{F}_2 \, s_\delta(\mathcal{B}) \right) + \dfrac{a - b + \sqrt{\Delta}}{2\sqrt{\Delta}} \left(\tilde{F}_3 + \tilde{F}_4 \, s_\delta(\mathcal{D}) \right) \\ 0 \\ -\dfrac{b}{\sqrt{\Delta}} \left(\tilde{F}_1 + \tilde{F}_2 \, s_\delta(\mathcal{B}) \right) + \dfrac{-a + b + \sqrt{\Delta}}{2\sqrt{\Delta}} \left(\tilde{F}_3 + \tilde{F}_4 \, s_\delta(\mathcal{D}) \right) \end{pmatrix}$$

$$\text{and} \quad G_2(t, \mathbf{x}, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We shall study which periodic solutions (17) of the unperturbed system (16) with $\varepsilon = 0$ can be continued to periodic solutions of the perturbed system (16) for $\varepsilon \neq 0$ sufficiently small.

We shall describe the different elements which appear in the statement of Theorem 8 in the particular case of the differential system (16). Thus we have that $\Omega = \mathbb{R}^4$, $k = 2$ and $n = 4$. Let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. We take the open and bounded subset V of the plane $Z = W = 0$ as

$$V = \{(X_0, Y_0, 0, 0) \in \mathbb{R}^4 : r_1 < \sqrt{X_0^2 + Y_0^2} < r_2\}.$$

As usual $\mathrm{Cl}(V)$ denotes the closure of V. If $\alpha = (X_0, Y_0)$, then we can identify V with the set

$$\{\alpha \in \mathbb{R}^2 : r_1 < ||\alpha|| < r_2\},$$

here $|| \cdot ||$ denotes the Euclidean norm of $\mathbb{R}^2$. The function $\beta : \mathrm{Cl}(V) \to \mathbb{R}^2$ is $\beta(\alpha) = (0, 0)$. Therefore, in our case the set

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)) \,, \ \alpha \in \mathrm{Cl}(V)\} = \{(X_0, Y_0, 0, 0) \in \mathbb{R}^4 : r_1 \leq \sqrt{X_0^2 + Y_0^2} \leq r_2\}.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider the periodic solution $\mathbf{x}(\tau, \mathbf{z}_\alpha) = (X(\tau), Y(\tau), 0, 0)$ given by (17) with period pT_1.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\tau)$ of the linear differential system (16) with $\varepsilon = 0$ associated to the T–periodic solution $\mathbf{z}_\alpha = (X_0, Y_0, 0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^4$, we get that $M(\tau) = M_{\mathbf{z}_\alpha}(\tau)$ is equal to

$$\begin{pmatrix} \cos(\omega_1 \tau) & \sin(\omega_1 \tau) & 0 & 0 \\ -\sin(\omega_1 \tau) & \cos(\omega_1 \tau) & 0 & 0 \\ 0 & 0 & \cos(\omega_2 \tau) & \sin(\omega_2 \tau) \\ 0 & 0 & -\sin(\omega_2 \tau) & \cos(\omega_2 \tau) \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\tau)$ does not depend of the particular periodic solution $\mathbf{x}(\tau, \mathbf{z}_\alpha)$. Since the matrix

$$M^{-1}(0) - M^{-1}(pT_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) & \sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) \\ 0 & 0 & -\sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) & 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) \end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 8 because the determinant

$$\begin{vmatrix} 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) & \sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) \\ -\sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) & 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) \end{vmatrix} = 4\sin^2\left(\frac{p\pi\,\omega_2}{\omega_1}\right) \neq 0,$$

we can apply Theorem 8 to system (16).

Now $\xi : \mathbb{R}^4 \to \mathbb{R}^2$ is $\xi(X, Y, Z, W) = (X, Y)$. We calculate, when $\delta \to 0$, the function

$$\mathcal{G}(X_0, Y_0) = \mathcal{G}(\alpha) = \xi\left(\frac{1}{pT_1}\int_0^{pT_1} M_{\mathbf{z}_\alpha}^{-1}(\tau) G_1(\tau, \mathbf{x}(\tau, \mathbf{z}_\alpha)) d\tau\right),$$

and we obtain

(19)

$$\begin{aligned} \mathcal{G}_1(X_0, Y_0) = &-\frac{1}{2\sqrt{\Delta}pT_1}\int_0^{pT_1}\Bigg[\sin(\omega_1\tau)\left(2b\left(\bar{F}_1 + \bar{F}_2\,\mathrm{sgn}\left(\frac{a+b+\sqrt{\Delta}}{2b}Y_{X_0,Y_0}(\tau)\right)\right)\right. \\ &\left. + \left(a - b + \sqrt{\Delta}\right)\left(\bar{F}_3 + \bar{F}_4\,\mathrm{sgn}(Y_{X_0,Y_0}(\tau))\right)\right)\Bigg] d\tau, \\ \mathcal{G}_2(X_0, Y_0) = &\frac{1}{2\sqrt{\Delta}pT_1}\int_0^{pT_1}\Bigg[\cos(\omega_1\tau)\left(+2b\left(\bar{F}_1 + \bar{F}_2\,\mathrm{sgn}\left(\frac{a+b+\sqrt{\Delta}}{2b}Y_{X_0,Y_0}(\tau)\right)\right)\right. \\ &\left. + \left(a - b + \sqrt{\Delta}\right)\left(\bar{F}_3 + \bar{F}_4\,\mathrm{sgn}(Y_{X_0,Y_0}(\tau))\right)\right)\Bigg] d\tau, \end{aligned}$$

where the functions of $\bar{F}_i$ for $i = 1, 2, 3, 4$ are the ones given in (7). Note that $-a + b + \sqrt{\Delta} > 0$, then

$$\mathrm{sgn}\left(\frac{-a+b+\sqrt{\Delta}}{2b}Y_{X_0,Y_0}(\tau)\right) = \mathrm{sgn}\left(Y_{X_0,Y_0}(\tau)\right),$$

denoting by $K = 1/(2\sqrt{\Delta}pT_1)$, the system (19) becomes

$$\begin{aligned}\mathcal{G}_1(X_0,Y_0) =& -K\int_0^{pT_1}\sin(\omega_1\tau)\,\bar{F}_3\left(a-b+\sqrt{\Delta}\right)+2b\bar{F}_1\,d\tau\\ &-K\int_0^{pT_1}\sin(\omega_1\tau)\left(\bar{F}_4\left(a-b+\sqrt{\Delta}\right)+2b\bar{F}_2\right)\operatorname{sgn}(Y_{X_0,Y_0}(\tau))\,d\tau,\end{aligned}$$

$$\begin{aligned}\mathcal{G}_2(X_0,Y_0) =& K\int_0^{pT_1}\cos(\omega_1\tau)\,\bar{F}_3\left(a-b+\sqrt{D}\right)+2b\bar{F}_1\,d\tau\\ &K\int_0^{pT_1}\cos(\omega_1\tau)\left(\bar{F}_4\left(a-b+\sqrt{D}\right)+2b\bar{F}_2\right)\operatorname{sgn}(Y_{X_0,Y_0}(\tau))\,d\tau.\end{aligned}$$

Then, by Theorem 8 we have that for every simple zero $(X_0^*, Y_0^*) \in V$ of the system of non-linear and non-smooth functions

$$\mathcal{G}_1(X_0,Y_0)=0 \quad , \quad \mathcal{G}_2(X_0,Y_0)=0, \tag{20}$$

we have a periodic solution $(X,Y,Z,W)(\tau,\varepsilon)$ of system (16) such that

$$(X,Y,Z,W)(0,\varepsilon)\to(X_0^*,Y_0^*,0,0) \quad \text{as } \varepsilon\to 0.$$

Note that system (20) is equivalent to system (8), because both equations only differs in a non–zero multiplicative constant.

Going back through the change of coordinates (15) we get a periodic solution $(x,y,z,w)(\tau,\varepsilon)$ of system (17) such that

$$\begin{pmatrix} x(\tau,\varepsilon)\\ y(\tau,\varepsilon)\\ z(\tau,\varepsilon)\\ w(\tau,\varepsilon)\end{pmatrix}\to\begin{pmatrix}\dfrac{-a+b+\sqrt{\Delta}}{2\,b\,\omega_1}\left(X_0^*\cos(\omega_1\tau)+Y_0^*\sin(\omega_1\tau)\right)\\ \dfrac{-a+b+\sqrt{\Delta}}{2b}\left(Y_0^*\cos(\omega_1\tau)-X_0^*\sin(\omega_1\tau)\right)\\ \dfrac{1}{\omega_1}\left(X_0^*\cos(\omega_1\tau)+Y_0^*\sin(\omega_1\tau)\right)\\ Y_0^*\cos(\omega_1\tau)-X_0^*\sin(\omega_1\tau)\end{pmatrix}$$

as $\varepsilon\to 0$.

Consequently we obtain a periodic solution $(\theta_1,\theta_2)(\tau,\varepsilon)$ of system (5) such that

$$(\theta_1,\theta_2)(\tau,\varepsilon)\to\begin{pmatrix}\dfrac{-a+b+\sqrt{\Delta}}{2\,b\,\omega_1}\left(X_0^*\cos(\omega_1\tau)+Y_0^*\sin(\omega_1\tau)\right)\\ \dfrac{1}{\omega_1}\left(X_0^*\cos(\omega_1\tau)+Y_0^*\sin(\omega_1\tau)\right)\end{pmatrix}$$

as $\varepsilon\to 0$. Hence Theorem 1 is proved. □

Proof of Theorem 4. This proof is completely analogous to the proof of Theorem 1. □

3. Proofs of corollaries

To obtain the expression of the functions given in (6) and (10) we have to study the sign change of the functions $Y_{X_0,Y_0}(\tau)$ and $W^{Z_0,W_0}(\tau)$ respectively for $\tau \in [0, pT_1]$ and $\tau \in [0, pT_2]$.

Note that $Y_{X_0,Y_0}(\tau_n) = 0$ for

$$\tau_n = \frac{1}{\omega_1}\left(\pi n + \arctan\left(\frac{Y_0}{X_0}\right)\right).$$

If $X_0Y_0 > 0$, then $\tau_n \in [0, pT_1]$ only for $n = 0, 1, \cdots, p+1$, and if $X_0Y_0 < 0$, then $\tau_n \in [0, pT_1]$ only for $n = 1, 2, \cdots, p+2$. We know that for all $\tau \in [t_n, t_{n+1}]$ the function $Y_{X_0,Y_0}(\tau)$ has the same sign and different sign for any $\tau \in [t_{n-1}, t_n]$, thus the integral can be computed using the partitions $\{0,\, \tau_n,\, pT_1\,; n = 0, 1, \cdots, p+1\}$ and $\{0,\, \tau_n,\, pT_1\,; n = 1, 2, \cdots, p+2\}$ as the limits of integration respectively for $X_0Y_0 > 0$ and $X_0Y_0 < 0$.

The study of the sign change of the function $W^{Z_0,W_0}(\tau)$ for $\tau \in [0, pT_2]$ and $Z_0W_0 \neq 0$ is completely analogous.

Proof of Corollary 2. Studying the sign change of the function $Y_{X_0,Y_0}(\tau)$ for $\tau \in [0, T_1]$ we conclude that under the assumptions of Corollary 2 the non-linear and non-smooth functions (6), for $X_0Y_0 \neq 0$, becomes

$$\mathcal{F}_1(X_0, Y_0) = \begin{cases} 1 + X_0Y_0\sqrt{1 + \dfrac{Y_0^2}{X_0^2}} & \text{if} \quad X_0Y_0 > 0, \\[2ex] 1 - X_0Y_0\sqrt{1 + \dfrac{Y_0^2}{X_0^2}} & \text{if} \quad X_0Y_0 < 0, \end{cases}$$

$$\mathcal{F}_2(X_0, Y_0) = \begin{cases} 1 + X_0^2\sqrt{1 + \dfrac{Y_0^2}{X_0^2}} & \text{if} \quad X_0Y_0 > 0, \\[2ex] 1 - X_0^2\sqrt{1 + \dfrac{Y_0^2}{X_0^2}} & \text{if} \quad X_0Y_0 < 0. \end{cases}$$

We do not care about $X_0Y_0 = 0$, since we are only interested in the simple zeros of the system (8).

This system has the following two solutions

$$(X_0^*, Y_0^*) = \left(\pm 2^{-\frac{1}{4}}, \pm 2^{-\frac{1}{4}}\right).$$

But the solutions which differs in a sign are different initial conditions of the same periodic solution of the unperturbed double pendulum. Moreover, it easy to check that these solutions are simple. So, by Theorem 1 we have one periodic solution of the non-smooth perturbed double pendulum. This completes the proof of the corollary. □

Proof of Corollary 5. This proof is completely analogous to the proof of Corollary 2. □

Appendix: Basic results on averaging theory

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\dot{\mathbf{x}}(t) = G_0(t,\mathbf{x}) + \varepsilon G_1(t,\mathbf{x}) + \varepsilon^2 G_2(t,\mathbf{x},\varepsilon), \tag{21}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $G_0, G_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $G_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}}(t) = G_0(t,\mathbf{x}), \tag{22}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of the system (22) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} G_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{23}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (23), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_k)$.

We assume that there exists a k–dimensional submanifold $\mathcal{Z}$ of Ω filled with T–periodic solutions of (22). Then an answer to the problem of bifurcation of T–periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (21) is given in the following result.

Theorem 8. *Let V be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha,\beta(\alpha)),\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha)$ of (22) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (23) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k\times(n-k)$ zero matrix, and in the lower right corner a $(n-k)\times(n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{G} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{G}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t,\mathbf{x}(t,\mathbf{z}_\alpha))dt\right). \tag{24}$$

If there exists $a \in V$ with $\mathcal{G}(a) = 0$ and $\det\left((d\mathcal{G}/d\alpha)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t,\varepsilon)$ of system (21) such that $\varphi(0,\varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 8 goes back to Malkin [3] and Roseau [4], for a shorter proof see [1].

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