ON THE HAMILTONIAN STRUCTURE OF NORMAL FORMS AT ELLIPTIC EQUILIBRIA OF REVERSIBLE VECTOR FIELDS IN $\mathbb{R}^4$

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Abstract. This paper addresses the question whether normal forms of smooth reversible vector fields in $\mathbb{R}^4$ at an elliptic equilibrium possess a formal Hamiltonian structure. In the non-resonant case we establish a formal conjugacy between reversible and Hamiltonian normal forms. In the case of non-semi-simple $1:1$ resonancewe establish a weaker form of equivalence, namely that of a formal orbital equivalence to a Hamiltonian normal form that involves an additional time-reparametrization of orbits.

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1. Introduction and main result

The similarity between certain aspects of the dynamical behaviour of time-reversible and Hamiltonian dynamical systems, explored already by Poincaré and Birkhoff, has attracted much attention. Many results that hold for Hamiltonian sytems, such as KAM theory and Lyapunov center theorems, have been shown to hold also for time-reversible systems, see for instance [1, 2, 3, 6, 7, 8, 13, 20] and references therein. At the same time, as should be expected, there are also many differences between Hamiltonian and reversible systems, see for instance [13, 19].

In this paper, we address the question whether normal forms of reversible vector fields in $\mathbb{R}^4$ at an elliptic equilibrium point *formally* have a Hamiltonian structure. That is, given a reversible vector field with equilibrium 0 and linear part with two pairs of purely imaginary eigenvalues, we investigate whether there exists a change of coordinates that renders truncations of the Taylor expansion of the vector field at any given order to be Hamiltonian.

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The question we address here is antipodal to the one addressed before in [14, 18], concerning the formal reversibility of normal forms at fixed points of symplectic maps of the plane (which turns out to be almost always true). Some semi-global obstructions for reversibility of two-dimensional Hamiltonian vector fields and symplectic diffeomorphisms were presented in [12].

Recall that a vector field $X : \mathbb{R}^4 \to \mathbb{R}^4$ has a time-reversal symmetry $\varphi \in \mathrm{Diffeo}(\mathbb{R}^4)$ if $\varphi_*(X) = -X$, i.e. if $x(t)$ is a solution of $\dot{x} = X(x)$ then so is $\varphi x(-t)$. We also say that X is *φ-reversible*, or simply *reversible*. In this paper we focus on reversible vector fields without additional symmetries and thus assume that $\langle \phi \rangle \simeq \mathbb{Z}_2$ (and thus that $\varphi^2 = \mathrm{id}$). In the neighbourhood of a φ-invariant equilibrium point, say 0 satisfying $X(0) = 0$, by Bochner's Theorem [17] it then follows that there exist local coordinates in which φ is linear and orthogonal. In this paper we will always assume that we start with such coordinates in the neighbourhood of the equilibrium. From the assumptions on the eigenvalues of the linear part of the vector field we then obtain that near an elliptic equilibrium point $\dim \mathrm{Fix}(\varphi) = 2$ where $\mathrm{Fix}(\varphi) := \{x \in \mathbb{R}^4 \mid \varphi(x) = x\}$.

A vector field X is called *Hamiltonian* if there exists a non-degenerate skew-symmetric bilinear form ω and a function $H : \mathbb{R}^4 \to \mathbb{R}$ such that $\omega(X(x), x) = dH(x)$. By the Darboux Theorem, locally (for instance near an equilibrium point) one can always find coordinates such that $X(x) = J\nabla_x H(x)$, where

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},$$

and I_2 denotes the 2×2 unit matrix.

The categories of equivalence that we consider in this paper are *formal conjugacy* and *formal orbital equivalence*. We say that two vector fields are formally conjugate if there exists a formal change of coordinates transforming one vector field to the other (to any given order, without concerning the convergence of the transformation in the limit where the order goes to infinity). Two vector fields X and Y are said to be formally orbitally equivalent if there is a smooth function $f : \mathbb{R}^4 \to \mathbb{R}$ with no zeros near 0, so that $f \cdot X$ (X multiplied by f) is formally conjugate to Y. The multiplication by f has the interpretation of a time-reparametrization of the orbits of X. Formal conjugacy between two vector fields implies formal orbital equivalence but not vice versa.

Given a reversible vector field X with equilibrium 0 and derivative $DX(0)$ with eigenvalues $(\pm\alpha i, \pm\beta i)$, we say that the equilibrium of X has a $p : q$ $(p, q \in \mathbb{Z})$ resonance if $q\alpha - p\beta = 0$. Without loss of generality we may take $\alpha, \beta > 0$ and $\alpha \leq \beta$ so that $p, q \in \mathbb{N}$ with $p \leq q$ and $\gcd(p, q) = 1$. Throughout this paper we will assume that vector fields are C^∞.

The main results of this paper are summarized in the following theorem:

Theorem 1.1. *Let X be a reversible vector field in $\mathbb{R}^4$ with equilibrium 0 such that $DX(0)$ has two pairs of purely imaginary eigenvalues $(\pm\alpha i, \pm\beta i)$, with $\alpha, \beta > 0$. Then,*

(i) *If $\alpha : \beta \notin \mathbb{Q}$, X is generically formally conjugated to an integrable reversible Hamiltonian vector field. Moreover, generically this normal form is formally orbitally five-jet determined: the vector field can be reduced to a polynomial of degree five by the combination of a change of coordinates and rescaling of time.*

(ii) *If* $\alpha : \beta = 1 : 1$ *and* $DX(0)$ *is not semi-simple (non-semi-simple* $1 : 1$ *resonance*[1]*) then* X *is generically not formally conjugate to a Hamiltonian vector field. But* X *is always formally orbitally equivalent to an integrable Hamiltonian vector field.*

In Section 3 we state more detailed results in from which one can immediately deduce the senses of genericity we refer to. The case $p : q \neq 1 : 1$ will be treated in a forthcoming paper, as its normal form is much more complicated, due to the appearance of terms of even power, for some values of p, q.

The non-semi-simple $1 : 1$ resonant case was considered previously by Van der Meer *et al.* [16]. There, a similarity between the reversible and Hamiltonian cases was found after reduction by the S^1-equivariance of the normal form that is generated by the semi-simple part of the derivative at $1 : 1$ resonance. Our results provide an alternative point of view, and illustrate that the reversible normal form is formally orbitally equivalent to a Hamiltonian vector field.

Similarities between the elliptic points in four-dimensional reversible and Hamiltonian vector fields have also been observed in the context of local bifurcation theory. Indeed, when using Lyapunov-Schmidt reduction, it can be shown that the reduced bifurcation equations for subharmonic branching at $p : q$ resonances in the reversible, Hamiltonian and reversible Hamiltonian contexts are identical, and thus give rise to identical branching patterns of periodic solutions [8]. Our results thus illustrate that the formal equivalence of branching patterns does not imply that the corresponding vector fields are formally conjugate.

Finally, we note that it is of interest to address the question of this paper also in the opposite direction, namely if the normal for of an elliptic equilbrium of a Hamiltonian vector field in $\mathbb{R}^4$ is formally reversible. This clearly holds in the non-resonant case (as in the simpler case in $\mathbb{R}^2$), but in the presence of resonances this problem is still open.

2. Preliminaries

In this section we recall some general results and techniques for the normalization of vector fields near an equilibrium solution.

Consider a vector field X on $\mathbb{R}^n$ with equilibrium 0. We are interested in establising coordinates in terms of which (finite order truncations of) Taylor expansions have special properties. We recall that the existence of a coordinate transformations between two vector fields is a conjugacy relation between these two vector fields. We say that two smooth vector fields X and Y are *formally conjugate* if for each order $k \geq 1$ there exists a coordinate transformation ϕ such that the vector field X and vector field $\phi_*(Y)$ (conjugate to Y) have the same Taylor series expansion up to degree k. It is well known that a formal conjugacy does not always imply a true conjugacy, due to the possibility of divergence of the coordinate transformations as $k \to \infty$.

Our starting point will often be the well-established result that coordinates can always be chosen in such a way that the nonlinear terms of a finite order Taylor expansion of the vector field commute with the transpose of the linear part of the vector field. This implies in particular that the resulting Taylor expansion

[1] We note that generically , in one-parameter families of reversible vector fields, non-semi-simple $1 : 1$ resonances arise persistenty. The semi-simple $1 : 1$ resonance has higher codimension and is not considered here.

commutes with the (closure of) the group generated by the semi-simple part (in the sense of Jordan-Chevalley decomposition) of the linear part of the vector field.

Let $L = DX(0)$ denote the linear part of the vector field X. Then the Jordan-Chevalley decomposition theorem asserts that J can be written uniquely as the sum $L = S + N$ where S is semi-simple, N is nilpotent and $[S, N] = 0$.

The following normal form theorem is a reversible version of a classical result by Belitskii [4].

Theorem 2.1 ([11])**.** *Let X be a φ-reversible vector field, where φ acts linearly and $\varphi^2 = \mathrm{Id}$. Let moreover X have equilibrium 0 and linear part $L = DX(0)$ with Jordan-Chevalley decomposition $L = S + N$. Then X is formally conjugate, by a φ-equivarant coordinate transformation, to a φ-reversible vector field $\tilde{X}$ with linear part L, satisfying $[(\tilde{X} - L), L^T] = 0$, where L^T denotes the transpose of L.*

Corollary 2.2 (Formal normal form symmetry)**.** The normal form $\tilde{X}$ from Theorem 2.1 is reversible-equivariant with respect to the group $G \rtimes \mathbb{Z}_2(\varphi)$ with $G = \overline{\{\exp(St) \mid t \in \mathbb{R}\}}$.

The proof of Theorem 2.1 relies on the analysis of the effect of coordinate transformations that are derived from the flow of a vector field.

Assume that X and Y are vector fields such that

$$j^k[X, Y] = 0, \quad \text{and} \quad j^1 Y = 0, \tag{2.1}$$

where $[X, Y]$ denotes the Lie bracket of X and Y, and $j^k X$ denotes the k-jet of X (equivalence class of vector fields with the same kth order Taylor expansion as X). Let ϕ_Y^t denote the time-t flow of Y and $\tilde{X} = (\phi_Y^t)_* X$, then we have

$$j^{k+1}(\tilde{X}) = j^{k+1}X + t j^{k+1}[X, Y]. \tag{2.2}$$

This implies that if X and $\tilde{X}$ are vector fields such that $j^k X = j^k \tilde{X}$ and

$$j^{k+1}[X, Y] = j^{k+1}(\tilde{X} - X) \tag{2.3}$$

is solvable with respect to a vector field Y such that $j^1 Y = 0$, then there is a diffeomorphism ϕ such that

$$j^{k+1}\phi_* X = j^{k+1}\tilde{X}. \tag{2.4}$$

For normalization purposes we usually assume that the linear part of the vector field has been normalized (often to Jordan normal form), and focus on coordinate transformations with linear part the identity (which are formally identical to the time-one maps of flows of vector fields).

In this paper we carefully examine how within the above framework we can find Y so that $j^{k+1}\tilde{X}$ is Hamiltonian, which requires an effort that goes well beyond the proof of Theorem 2.1, where at each order k it suffices to consider only vector fields Y that are homogeneous of degree k.

Since there also exists a (reversible-equivariant) Hamiltonian version of Theorem 2.1 we can moreover assume without loss of generality that Y is $G \rtimes \mathbb{Z}_2(\varpi)$ equivariant. Namely, if we would find Y without these properties we could obtain an additional normalization hat preserves the Hamiltonian structure to yield a $G \rtimes \mathbb{Z}_2(\varphi)$ reversible-equivariant (and Hamiltonian) normal form. It then follows [11] that such a normal form can also be obtained by a symmetry preserving ($G \rtimes \mathbb{Z}_2(\varpi)$ equivariant) coordinate transformation.

3. Detailed statement of the results

In this section, we present in more detail the results summarized in Theorem 1.1. In each individual case we identify explicitly the Hamiltonian structure of the normal forms. We will often use the identification $\mathbb{C}^2 \cong \mathbb{R}^4$, in coordinates $z_j = x_j + iy_j$, $j = 1, 2$. The starting point of our study is usually the normal form of Theorem 2.1, that is characterized by the fact that the nonlinear terms commute with the transpose of the linear part of the vector field. By Corollary 2.2 this implies that this normal form is equivariant with respect to the (Lie) group generated by the semi-simple part of the linear part of the vector field, yielding in the context of this paper a *formal symmetry group* of the form $S^1 \times S^1$ (in the non-resonance case) or S^1 (in the case of $p : q$ resonance). Because of this symmetry, it is useful to introduce the variables $A := z_1\overline{z_1}$, $B := z_2\overline{z_2}$, $C := z_1^q\overline{z_2}^p$ and $D := \overline{C}$ (which are invariant under the relevant group actions).

3.1. **Non-resonant case.** Let X be a φ-reversible vector field, with $\varphi(z_1, z_2) = (i\bar{z}_1, -i\bar{z}_2)$, $(z_1, z_2) \in \mathbb{C}^2$, and $X(0) = 0$, so that $DX(0)$ has eigenvalues $(\pm\alpha i, \pm\beta i)$, with $\alpha : \beta \notin \mathbb{Q}$. Then our starting point is the reversible-equivariant normal form

$$X = \left(\alpha i z_1 + i z_1 \sum_{j+l=1}^{\infty} a_{j,l} A^j B^l\right) \frac{\partial}{\partial z_1} + \left(\beta i z_2 + i z_2 \sum_{j+l=1}^{\infty} b_{j,l} A^j B^l\right) \frac{\partial}{\partial z_2}, \tag{3.1}$$

where due to the φ-reversibility all parameters $a_{j,l}$ and $b_{j,l}$ are real. It turns out that the sign of $a_{0,1}b_{1,0}$ is invariant under changes of coordinates that do not change the linear part of the vector field.

Theorem 3.1 (Non-resonant formal conjugacy)**.** *Let X be φ-reversible and non-resonant, as detailed above. If, with refererence to (3.1), $a_{0,1}b_{1,0} \neq 0$, then X is formally conjugate to a φ-reversible Hamiltonian vector field with symplectic form $\omega(z_1, z_2) = dz_1 \wedge \varepsilon dz_2$, with $\varepsilon := \operatorname{sgn}(a_{0,1}b_{1,0})$, and Hamiltonian $H = \alpha A + \varepsilon\beta B + h(A, B)$, where $h(0, 0) = \partial_A h(0, 0) = \partial_B h(0, 0) = 0$.*

In the orbital equivalence setting, the sign of $a_{0,1}b_{1,0}$ is no longer invariant. Nevertheless, it turns out that, in terms of (3.1), the inequalities

$$a_{1,0}\beta - b_{1,0}\alpha \neq 0, \quad a_{0,1}\beta - b_{0,1}\alpha \neq 0 \tag{3.2}$$

are invariant.

Theorem 3.2 (Non-resonant orbital equivalence)**.** *Let X be a φ-reversible vector field given by (3.1) where α and β satisfy (3.2). Then X is formally orbitally equivalent to a ϕ-reversible Hamiltonian vector field with symplectic form $\omega = dz_1 \wedge dz_2$ and Hamiltonian $H = \frac{\alpha}{2}A + \frac{\beta}{2}B + a\frac{A^2}{2} + b\frac{B^2}{2} + c\frac{A^3}{6}$. where $a = \pm 1$, $b = \pm 1$, and $c \in \mathbb{R}$.*

3.2. **Non-semi-simple** $1 : 1$ **resonance.** Our starting point is the S^1-equivariant reversible Belitskii normal form

$$\begin{aligned} X = \; & (\alpha i z_1 + z_2 + z_1 f_1(A, B, C, D) + z_2 f_2(A, B, C, D)) \tfrac{\partial}{\partial z_1} + \\ & (\alpha i z_2 + z_1 g_1(A, B, C, D) + z_2 g_2(A, B, C, D)) \tfrac{\partial}{\partial z_2} \end{aligned} \tag{3.3}$$

where f_j and g_j, $j = 1, 2$, have no constant or linear parts. The functions f and g moreover satisfy some conditions imposed by the φ-reversibility (where without loss of generality we take φ as in the non-resonant case).

Theorem 3.3. *Let X be a $1:1$ non-semi-simple resonant reversible vector field. Then its normal form is formally orbitally equivalent to a Hamiltonian vector field with symplectic form ω and Hamiltonian $H = (y_1^2 + y_2^2)/2 + v + f(A, v)$, where $v = x_1y_2 - x_2y_1$, $A = x_1^2 + x_2^2$, and f satisfies $f(0, 0) = \partial_A f(0, 0) = \partial_v f(0, 0) = 0$.*

However, the normal form is generically not formally conjugate to a Hamiltonian vector field.

Remark 3.4. To be more precise about the final claim of Theorem 3.3, let us give a simple computational argument. Consider the 3-jet of equation (3.3), reduced by the Belitiskii normal form and reversibility. We can write this system in real coordinates as

$$\begin{cases} \dot{x}_1 = -x_2 + y_1 - x_2(a_1A + a_2v) \\ \dot{x}_2 = x_1 + y_2 + x_1(a_1A + a_2v) \\ \dot{y}_1 = -y_2 - y_2(a_1A + a_2v) + x_1(a_3v + a_4A) \\ \dot{y}_2 = y_1 + y_1(a_1A + a_2v) + x_2(a_3v + a_4A), \end{cases}$$

where $A = x_1^2 + x_2^2$ and $v = x_1y_2 - x_2y_1$. Define $H = H_2 + H_3 + H_4$ and $Y = J\nabla H$, where H_k is a homogeneous polynomial of degree k and J is the canonical sympletic matrix. If X is conjugate to some hamiltonian vector field, then there exist such H and a change of coordinates $\Psi = Id + \psi$, with $\psi = o(3)$, such that

$$j^3(D\Psi(x)X(x)) = j^3(Y(\Psi(x))). \tag{3.4}$$

In turns out that a solution can be obtained if and only if $a_3 = -2a_1$. If this condition holds, at the next (5th) order a similar condition arises.

4. Proofs

In this section we present the proofs for the results stated in Section 3, and summarized in Theorem 1.1.

4.1. **Non-resonant case.**

4.1.1. *Formal conjugacy: proof of Theorem 3.1.* We only consider the case $a_{01}b_{10} > 0$. The proof in case $a_{01}b_{10} < 0$ is similar.

To prove the theorem we need to show that for any fixed $k \geq 0$,

$$X^{(2k+3)} = \begin{array}{c} -x_2 \sum_{i+j=k+1} (a_{i,j} A^i B^j \frac{\partial}{\partial x_1} + x_1 \sum_{i+j=k+1} a_{i,j} A^i B^j \frac{\partial}{\partial x_2} + \\ -y_2 \sum_{i+j=k+1} b_{i,j} A^i B^j \frac{\partial}{\partial y_1} + y_1 \sum_{i+j=k+1} b_{i,j} A^i B^j \frac{\partial}{\partial y_2} \end{array}$$

can be so normalized that the following compatibility relations hold.

$$(j+1)a_{k-j,j+1} = (k-j+1)b_{k-j+1,j}, \quad j = 0, 1, \ldots, k. \tag{4.1}$$

Namely, if these compatibility conditions are satisfied then there exists a generating function of the form $H_k = \sum_{i+j=k+2} h_{i,j} A^i B^j$, where $h_{i,j}$ will be choosen in an adequate way.

We prove things order-by-order. As the lowest order of resonant terms, $X^{(3)}$ can be normalized to satisfy (4.1). Namely, in terms of (3.1), we can put $a_{0,1} = b_{1,0}$ by applying linear scalings of z_1 and z_2, recalling the assumption that $a_{0,1}b_{1,0} > 0$. Moreover, one can scale variables so that $a_{0,1} = b_{1,0} = \pm 1$.

Next we show that the normalization of $X^{(2k+3)}$ can be done for any $k>0$. Following the methodology of Section 2 we have to show the solvability of Y from the following homological equation

$$[X^{(3)},Y]=\tilde{X}^{(2k+3)}-X^{(2k+3)}, \tag{4.2}$$

where $\tilde{X}=J\nabla H_k$ for some polynomial Hamiltonian $H_k:\mathbb{R}^4\to\mathbb{R}$ of degree $2k+4$.

Note that

$$\tilde{X}^{(2k+3)} = \begin{array}{l} -x_2\sum_{i+j=k+1}(i+1)h_{i+1,j}A^iB^j\frac{\partial}{\partial x_1}+x_1\sum_{i+j=k+1}(i+1)h_{i+1,j}A^iB^j\frac{\partial}{\partial x_2}+ \\ -y_2\sum_{i+j=k+1}(j+1)h_{i,j+1}A^iB^j\frac{\partial}{\partial y_1}+y_1\sum_{i+j=k+1}(j+1)h_{i,j+1}A^iB^j\frac{\partial}{\partial y_2} \end{array}$$

Writing Y in the form

$$Y=\begin{pmatrix} x_1\left(\alpha_{k,0}A^k+\alpha_{k-1,1}A^{k-1}B+\ldots+\alpha_{1,k-1}AB^{k-1}+\alpha_{0,k}B^k\right)\\ x_2\left(\alpha_{k,0}A^k+\alpha_{k-1,1}A^{k-1}B+\ldots+\alpha_{1,k-1}AB^{k-1}+\alpha_{0,k}B^k\right)\\ y_1\left(\beta_{k,0}A^k+\beta_{k-1,1}A^{k-1}B+\ldots+\beta_{1,k-1}AB^{k-1}+\beta_{0,k}B^k\right)\\ y_2\left(\beta_{k,0}A^k+\beta_{k-1,1}A^{k-1}B+\ldots+\beta_{1,k-1}AB^{k-1}+\beta_{0,k}B^k\right)\end{pmatrix}$$

the left side of (4.2) can be calculated explicitly:

$$[X^{(3)},Y]=\begin{pmatrix} -2x_2\left(\gamma_0A^{k+1}+\gamma_1A^kB+\ldots+\gamma_kAB^k+\gamma_{k+1}B^{k+1}\right)\\ 2x_1\left(\gamma_0A^{k+1}+\gamma_1A^kB+\ldots+\gamma_kAB^k+\gamma_{k+1}B^{k+1}\right)\\ -2y_2\left(\delta_0A^{k+1}+\delta_1A^kB+\ldots+\delta_kAB^k+\delta_{k+1}B^{k+1}\right)\\ 2y_1\left(\delta_0A^{k+1}+\delta_1A^kB+\ldots+\delta_kAB^k+\delta_{k+1}B^{k+1}\right)\end{pmatrix}, \tag{4.3}$$

where $\gamma_0=\alpha_{k,0}a_{1,0}$, $\gamma_{k+1}=\beta_{k,0}a_{0,1}$, $\delta_0=\alpha_{k,0}a_{0,1}$, $\delta_{k+1}=\beta_{0,k}b_{0,1}$ and $\gamma_j=\alpha_{k-j,j}a_{1,0}+\beta_{k-j+1,j-1}a_{0,1}$, $\delta_j=\alpha_{k-j,j}a_{0,1}+\beta_{k-j+1,j-1}b_{0,1}$ for $j=1,\ldots,k$.

The right side of (4.2) is:

$$\tilde{X}^{(2k+3)}-X^{(2k+3)}=\begin{pmatrix} -x_2\left(\sum_{i+j=k+1}[(i+1)h_{i+1,j}-a_{i,j}]A^iB^j\right)\\ x_1\left(\sum_{i+j=k+1}[(i+1)h_{i+1,j}-a_{i,j}]A^iB^j\right)\\ -y_2\left(\sum_{i+j=k+1}[(j+1)h_{i,j+1}-b_{i,j}]A^iB^j\right)\\ y_1\left(\sum_{i+j=k+1}[(j+1)h_{i,j+1}-b_{i,j}]A^iB^j\right)\end{pmatrix} \tag{4.4}$$

Now comparing (4.3) and (4.4), we see that to solve $[X^{(3)},Y]=\tilde{X}^{(2k+3)}-X^{(2k+3)}$ is equivalent to solve, for α's, β's and h's, the following system of equations:

$$\left\{\begin{array}{lcl} 2\alpha_{k,0}a_{1,0} & = & (k+2)h_{k+2,0}-a_{k+1,0}\\ 2\beta_{k,0}a_{0,1} & = & h_{1,k+1}-a_{0,k+1}\\ 2\alpha_{k,0}a_{0,1} & = & h_{k+1,1}-b_{k+1,0}\\ 2\beta_{0,k}b_{0,1} & = & (k+2)h_{0,k+2}-b_{0,k+1}\\ 2[\alpha_{k-1,1}a_{1,0}+\beta_{k,0}a_{0,1}] & = & (k+1)h_{k+1,1}-a_{k,1}\\ & \vdots & \\ 2[\alpha_{0,k}a_{1,0}+\beta_{1,k-1}a_{0,1}] & = & 2h_{2,k}-a_{1,k}\\ 2[\alpha_{k-1,1}a_{0,1}+\beta_{k,0}b_{0,1}] & = & 2h_{k,2}-b_{1,k}\\ & \vdots & \\ 2[\alpha_{0,k}a_{0,1}+\beta_{1,k-1}b_{0,1}] & = & (k+1)h_{k-1,2}-b_{k,1} \end{array}\right.$$

We rewrite this system as

$$\begin{cases} 2\alpha_{k,0}a_{1,0} & = & (k+2)h_{k+2,0} - a_{k+1,0} \\ 2\alpha_{k,0}a_{0,1} & = & h_{k+1,1} - b_{k+1,0} \\ 2\beta_{k,0}a_{0,1} & = & h_{1,k+1} - a_{0,k+1} \\ 2\beta_{0,k}b_{0,1} & = & (k+2)h_{0,k+2} - b_{0,k+1} \\ 2[\alpha_{k-1,1}a_{1,0} + \beta_{k,0}a_{0,1}] & = & (k+1)h_{k+1,1} - a_{k,1} \\ 2[\alpha_{k-1,1}a_{0,1} + \beta_{k,0}b_{0,1}] & = & 2h_{k,2} - b_{1,k} \\ & \vdots & \\ 2[\alpha_{0,k}a_{1,0} + \beta_{1,k-1}a_{0,1}] & = & 2h_{2,k} - a_{1,k} \\ 2[\alpha_{0,k}a_{0,1} + \beta_{1,k-1}b_{0,1}] & = & (k+1)h_{k-1,2} - b_{k,1} \end{cases}$$

As $a_{0,1} \neq 0$, the above system has solution. For $k = 1$, one solution is:

$$\begin{aligned} \alpha_{1,0} &= 1/4\,(-2\,b_{2,0}a_{0,1} + a_{1,1}a_{0,1} - a_{1,0}b_{1,1} + 2\,a_{1,0}a_{0,2})/a_{0,1}^2, \\ \alpha_{0,1} &= 1/2\,(2\,a_{0,2} - b_{1,1})/a_{0,1}, \\ h_{3,0} &= 1/12\,(-2\,a_{1,0}b_{2,0}a_{0,1} + a_{1,0}a_{1,1}a_{0,1} - a_{1,0}^2b_{1,1} + 2\,a_{1,0}^2a_{0,2} + 2\,a_{2,0}a_{0,1}^2)/a_{0,1}^2, \\ h_{2,1} &= 1/4\,(a_{1,1}a_{0,1} - a_{1,0}b_{1,1} + 2\,a_{1,0}a_{0,2})/a_{0,1}, \\ h_{1,2} &= 1/2\,a_{0,2}, \\ h_{0,3} &= 1/6\,b_{0,2}. \end{aligned}$$

4.1.2. *Formal orbital equivalence: Theorem 3.2.* Formal conjugacy, as established in the previous section, implies formal orbital equivalence. It remains to be shown that in the formal orbital equivalence setting the resulting Hamiltonian can generically be chosen to be polynomial . The aim is to show that by a combination of a coordinate transformation and multiplication of the vector field by a formal power series with no zeros near 0, (3.1) can be normalized to a Hamiltonian vector field, with H given in the statement of Theorem 3.2 and X_H given by

$$\begin{cases} \dot{x}_1 & = & -\alpha x_2 - a x_2\Delta_1 - c x_2\Delta_1^2 \\ \dot{x}_2 & = & -\alpha x_1 - a x_1\Delta_1 - c x_1\Delta_1^2 \\ \dot{y}_1 & = & -\beta y_2 - b y_2\Delta_2 \\ \dot{y}_2 & = & \beta y_1 + b y_1\Delta_2, \end{cases} \tag{4.5}$$

where $a = \pm 1$, $b = \pm$ and $c \in \mathbb{R}$.

We proceed order by order. We start multiplying the 3-jet j^3X of X by the function

$$h_1 = 1 - \frac{b_{1,0}}{\beta}A - \frac{a_{0,1}}{\alpha}B$$

yielding

$$\begin{aligned} j^3(h_1 \cdot X) &= (-\alpha x_2 - x_2\tilde{a}_{1,0}A)\frac{\partial}{\partial x_1} + (\alpha x_1 + x_1\tilde{a}_{1,0}A)\frac{\partial}{\partial x_2} \\ &+ \left(-\beta y_2 - y_2\tilde{b}_{0,1}B\right)\frac{\partial}{\partial y_1} + \left(\beta y_1 + y_1\tilde{b}_{0,1}B\right)\frac{\partial}{\partial y_1} \end{aligned}$$

where $\tilde{a}_{1,0} = \left(\frac{\beta a_{1,0} - b_{1,0}\alpha}{\beta}\right)$ and $\tilde{b}_{0,1} = \left(\frac{\alpha b_{0,1} - a_{0,1}\beta}{\alpha}\right)$.

Moreover, due to the genericity conditions (3.2), we can scale $\tilde{a}_{1,0}$ and $\tilde{b}_{0,1}$ to ± 1. We thus may normalize j^3X to
(4.6)

$$(-\alpha x_2 - x_2\varepsilon_1 A)\frac{\partial}{\partial x_1}+(\alpha x_1 + x_1\varepsilon_1 A)\frac{\partial}{\partial x_2}+(-\beta y_2 - y_2\varepsilon_2 B)\frac{\partial}{\partial y_1}+(\beta y_1 + y_1\varepsilon_2 B)\frac{\partial}{\partial y_1}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. We note that the signs of ε_1 and ε_2 are independent.

Having normalized j^3X to (4.6), next we consider the normalization of the homogenous terms of degree 5. Let $X^{(5)}$ denote these terms, as obtained after the previous normalization step. We may write

$$\begin{aligned} X^{(5)} &= -x_2\left(a_{2,0}A^2 + a_{1,1}AB + a_{0,2}B^2\right)\frac{\partial}{\partial x_1} + x_1\left(a_{2,0}A^2 + a_{1,1}AB + a_{0,2}B^2\right)\frac{\partial}{\partial x_2} \\ &- y_2\left(b_{2,0}A^2 + b_{1,1}AB + b_{0,2}B^2\right)\frac{\partial}{\partial y_1} + y_1\left(b_{2,0}A^2 + b_{1,1}AB + b_{0,2}B^2\right)\frac{\partial}{\partial y_1} \end{aligned}$$

We now multiply the vector field X by a function $h_2 = 1+\theta_1 A^2+\theta_2 AB+\theta_3 B^2$, where θ's are parameters that will be specified further below.

The third jet of the resulting vector field $h_2 \cdot X$ is the same of X, and the terms of $h_2 \cdot X$ of homogeneous degree 5 are

$$\begin{aligned} \tilde{X}^{(5)} &= -x_2\left((a_{2,0}+\theta_1)A^2 + (a_{1,1}+\theta_2)AB + (a_{0,2}+\theta_3)B^2\right)\frac{\partial}{\partial x_1} \\ &+ x_1\left((a_{2,0}+\theta_1)A^2 + (a_{1,1}+\theta_2)AB + (a_{0,2}+\theta_3)B^2\right)\frac{\partial}{\partial x_2} \\ &- y_2\left((b_{2,0}+\theta_1)A^2 + (b_{1,1}+\theta_2)AB + (b_{0,2}+\theta_3)B^2\right)\frac{\partial}{\partial y_1} \\ &+ y_1\left((b_{2,0}+\theta_1)A^2 + (b_{1,1}+\theta_2)AB + (b_{0,2}+\theta_3)B^2\right)\frac{\partial}{\partial y_1} \end{aligned}$$

We have to pass $\tilde{X}^{(5)}$ to

$$X_H^{(5)} = -cx_2A^2\frac{\partial}{\partial x_1} - cx_1A^2\frac{\partial}{\partial x_2},$$

so we we perform a change of coordinates generated by

$$\begin{aligned} &Y_2 = x_1(\gamma_{1,0}A + \gamma_{0,1}B)\tfrac{\partial}{\partial x_1} + x_2(\gamma_{1,0}A + \gamma_{0,1}B)\tfrac{\partial}{\partial x_2} \\ &y_1(\delta_{1,0}A + \delta_{0,1}B)\tfrac{\partial}{\partial y_1} + y_2(\delta_{1,0}A + \delta_{0,1}B)\tfrac{\partial}{\partial y_2} \end{aligned}$$

where γ's and δ's are real parameters. Now, just like in the previous case, we have to solve $[\tilde{X}^{(3)}, Y] = X^{(5)} - X_H^{(5)}$. One can check that a solution of this system is given by

$$\begin{aligned} \theta_1 &= -b_{2,0} \\ \theta_2 &= \frac{1}{6}\frac{-2\gamma_{1,0}a_{1,0} + 2\gamma_{1,0}b_{1,0}\alpha - b_{2,0}\beta + a_{2,0}\beta}{\beta} \\ \theta_3 &= -a_{0,2} \\ \delta_{1,0} &= -\frac{1}{2}\frac{\alpha(2\gamma_{0,1}b_{1,0}\alpha + a_{1,1}\beta - 2\gamma_{0,1}a_{1,0}\beta - b_{1,1}\beta)}{\beta(b_{0,1}\alpha - a_{0,1}\beta)} \\ \delta_{0,1} &= -\frac{1}{2}\frac{\alpha(-b_{0,2} + a_{0,2})}{b_{0,1}\alpha - a_{0,1}\beta} \end{aligned},$$

with $\gamma_{1,0}, \gamma_{0,1}$ free variables.

Let us now fix the (Hamiltonian) polynomial vector field j^5X as obtained by the above normalization procedure. We proceed to show that all higher order terms can be eliminated. In order to normalize $X^{(2k+1)}$, $k \geq 3$, we first multiply the vector field by a function of the form

$$h_k = 1 + \sum_{j=0}^{k} \theta_{k-j,j} A^{(k-j)} B^j, \tag{4.7}$$

with parameters $\theta_{k-j,j}$ and subsequently perform a change of coordinates, generated by a vector field of the form

$$Y_k = \sum_{j=0}^{k-1} A^{k-1-j} B^j (\alpha_{k-1-j,j} z_1 \frac{\partial}{\partial z_1} + \beta_{k-1-j,j} z_2 \frac{\partial}{\partial z_2}) \tag{4.8}$$

where α's and β's are parameters. Then $(Y_k)_*(h_k X)$ takes the form

$$\begin{aligned} &iz_1 \left(1 + \varepsilon_1 A + cA^2 + \sum_{j=0}^{k-1} (a_{k-j,j}\theta_{k-j,j} - 2\varepsilon_1 \alpha_{k-1-j,j}) A^{k-j} B^j + (a_{0,k} + \theta_{0,k}) B^k \right) \frac{\partial}{\partial z_1} \\ &+ iz_2 \left(\lambda + \varepsilon_2 B + \sum_{j=1}^{k} (b_{k-j,j} + \lambda\theta_{k-j,j} - 2\varepsilon_1 \beta_{k-1-j,j-1}) A^{k-j} B^j + (b_{k,0} + \lambda\theta_{k,0}) A^k \right) \frac{\partial}{\partial z_2} \end{aligned}.$$

The elimination of all terms is equivalent to the solvability, with respect to the parameters $\theta_{i,j}$, $\alpha_{i,j}$ and $\beta_{i,j}$, of the following system of equations:

$$\begin{aligned} a_{k-j,j} + \theta_{k-j,j} - 2\varepsilon_1 \alpha_{k-1-j,j} &= 0, \quad j = 0, \ldots, k-1 \\ b_{k-j,j} + \lambda\theta_{k-j,j} - 2\varepsilon_2 \beta_{k-j,j-1} &= 0, \quad j = 1, \ldots, k, \\ a_{0,k} + \theta_{0,k} &= 0, \\ b_{k,0} + \lambda\theta_{k,0} &= 0, \end{aligned}$$

where $a_{i,j}, b_{i,j} \in \mathbb{R}$ are constants whose values depend on the details of $X^{(2k+1)}$. Due to the upper triangular form of this system of the equations, its solvability is evident.

We remark that we do not need the coefficients β's (in the same way we did not used the coefficients γ's in the normalization of 5-jet). The explanation for that is easy: while the reparametrization function h_k eliminate the monomials in the first two coordinates, the change of coordinate eliminate the monomials in the last two equations. □

4.2. **Non-semi-simple** $1:1$ **resonance.** Without loss of generality, we assume that the eigenvalues of the vector fields are equal to $\pm i$. We start from the reversible Belitskii normal form (which has its nonlinear terms commuting with the transpose of the linear part)

$$\begin{aligned} X = \; & (-x_2 + y_1 - x_2(a_1 A + a_2 v) + \cdots) \tfrac{\partial}{\partial x_1} + (x_1 + y_2 + x_1(a_1 A + a_2 v) + \cdots) \tfrac{\partial}{\partial x_2} \\ & + (-y_2 - y_2(a_1 A + a_2 v) + x_1(a_3 v + a_4 A) + \cdots) \tfrac{\partial}{\partial y_1} \\ & + (y_1 + y_1(a_1 A + a_2 v) + x_2(a_3 v + a_4 A) + \cdots) \tfrac{\partial}{\partial y_2}, \end{aligned}$$

where the dots denote the higher order terms and the a's denote parameters. The 3-jet of the vector field is Hamiltonian with respect to the standard symplectic form if and only if $a_3 = -2a_1$, with Hamiltonian function $H_4 = (y_1^2 + y_2^2)/2 + v + a_1 Av + a_2 v^2/2 - a_4 A^2/4$.

The condition $a_3 = -2a_1$ can be achieved by a re-scaling of time. After multiplication of X by $f = 1 + \alpha A$ where α is a parameter, we get:

$$
\begin{aligned}
(4.9) \qquad (fX) = \; & \Big(-x_2 + y_1 - x_2((a_1+\alpha)A + a_2 v) + \underline{y_1(\alpha A)} \cdots \Big) \tfrac{\partial}{\partial x_1} \\
& + \Big(x_1 + y_2 + x_1((a_1+\alpha)A + a_2 v) + \underline{y_2(\alpha A)} + \cdots \Big) \tfrac{\partial}{\partial x_2} \\
& + (-y_2 - y_2((a_1+\alpha)A + a_2 v) + x_1(a_3 v + a_4 A) + \cdots) \tfrac{\partial}{\partial y_1} \\
& + (y_1 + y_1((a_1+\alpha)A + a_2 v) + x_2(a_3 v + a_4 A) + \cdots) \tfrac{\partial}{\partial y_2}
\end{aligned}
$$

Using in turn another coordinate transformation to the Belitskii normal form, the underlined terms of (4.9) may be eliminated. Moreover, the remaining terms of (4.9) are not affected because they belong to the complement of the image of the homological operator; the change of coordinate

$$
Id + \begin{pmatrix} x_1((1/2)\alpha\Delta_2) + y_2(-\alpha v) \\ x_2((1/2)\alpha\Delta_2) - y_1(-\alpha v) \\ 0 \\ 0 \end{pmatrix}
$$

realize this. So the desired Hamiltonian form is obtained if $\alpha \in \mathbb{R}$ is chosen such that $a_3 = -2(a_1 + \alpha)$.

Starting with the $(2k-3)$-jet in Hamiltonian form, with $k \geq 2$, we proceed to normalize the $(2k-1)$-jet

$$
\begin{aligned}
X^{(2k-1)} = \; & \Big(-x_2 \textstyle\sum_{j=1}^k b_j A^{k-j} v^{j-1}\Big) \tfrac{\partial}{\partial x_1} + \Big(x_1 \textstyle\sum_{j=1}^k b_j A^{k-j} v^{j-1}\Big) \tfrac{\partial}{\partial x_2} + \\
& \Big(-y_2 \textstyle\sum_{j=1}^k b_j A^{k-j} v^{j-1} + x_1 \textstyle\sum_{j=1}^k c_j A^{k-} v^{j-1}\Big) \tfrac{\partial}{\partial y_1} + \\
& \Big(y_1 \textstyle\sum_{j=1}^k b_j A^{k-j} v^{j-1} + x_2 \textstyle\sum_{j=1}^k c_j A^{k-j} v^{j-1}\Big) \tfrac{\partial}{\partial y_2}
\end{aligned}
$$

as follows. We first multiply by a function f of the form $f = 1 + \sum_{j=1}^{k-1} \theta_j A^{k-j} v^{j-1}$ to obtain

$$
\begin{aligned}
\tilde{X}^{(2k-1)} = \; & \Big(-x_2 \textstyle\sum_{j=1}^k (b_j + \theta_j) A^{k-j} v^{j-1} + \underline{y_1 \textstyle\sum_{j=1}^{k-1} \theta_j A^{k-j} v^{j-1}}\Big) \tfrac{\partial}{\partial x_1} + \\
& \Big(x_1 \textstyle\sum_{j=1}^k (b_j + \theta_j) A^{k-j} v^{j-1} + \underline{y_2 \textstyle\sum_{j=1}^{k-1} \theta_j A^{k-j} v^{j-1}}\Big) \tfrac{\partial}{\partial x_2} + \\
& \Big(-y_2 \textstyle\sum_{j=1}^k (b_j + \theta_j) A^{k-j} v^{j-1} + x_1 \textstyle\sum_{j=1}^k c_j A^{k-} v^{j-1}\Big) \tfrac{\partial}{\partial y_1} + \\
& \Big(y_1 \textstyle\sum_{j=1}^k (b_j + \theta_j) A^{k-j} v^{j-1} + x_2 \textstyle\sum_{j=1}^k c_j A^{k-j} v^{j-1}\Big) \tfrac{\partial}{\partial y_2}
\end{aligned} .
$$

As above the underlined terms can be killed, without changing the other terms, by the change of coordinate

$$
Id + \begin{pmatrix} x_1 f(v,A,B) - y_2 g(v,A) \\ x_2 f(v,A,B) + y_1 g(v,A) \\ 0 \\ 0 \end{pmatrix},
$$

with $g(v,A) = \sum_{j=0}^{k-2} \theta_{k-j-1} v^{k-j-1} A^j$ and $f(v,A,B) = B \sum_{j=0}^{k-2} \theta_{1+j} A^{k-j} v^j$.

So and we may choose $\theta_j,\ j = 1, \ldots, k$ such that

$$
jc_{j+1} = -2(k-j)(b_j + \theta_j), \quad j = 1, \ldots, k.
$$

$\square$.

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