Conley's Spectral Sequence via the Sweeping Algorithm

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Abstract

In this article we consider a spectral sequence (E^r, d^r) associated to a filtered Morse-Conley chain complex (C, Δ), where Δ is a connection matrix. The underlying motivation is to understand connection matrices under continuation. We show how the spectral sequence is completely determined by a family of connection matrices. This family is obtained by a sweeping algorithm for Δ over fields $\mathbb{F}$ as well as over $\mathbb{Z}$. This algorithm constructs a sequence of similar matrices $\Delta^0 = \Delta, \Delta^1, \ldots$, where each matrix is related to the others via a change-of-basis matrix. Each matrix Δ^r over $\mathbb{F}$ (resp., over $\mathbb{Z}$) determines the vector space (resp., $\mathbb{Z}$-module) E^r and the differential d^r. We also prove the integrality of the final matrix Δ^R produced by the sweeping algorithm over $\mathbb{Z}$ which is quite surprising, mainly because the intermediate matrices in the process may not have this property. Several other properties of the change-of-basis matrices as well as the intermediate matrices Δ^r are obtained. The sweeping algorithm and the computation of the spectral sequence (E^r, d^r) are implemented in the software *Mathematica*®.

Keywords: Connection matrix, spectral sequences, sweeping algorithm, Conley index, integer programming, computer implementation.

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1 Introduction

In this article, we consider M an n-dimensional compact Riemannian manifold, $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ a Morse decomposition of M and a filtered Conley chain complex C with finest filtration $\{F_p\}$[1]. A Morse decomposition of M is a collection $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ of mutually disjoint compact invariant subsets of M such that that if $\gamma \in M \setminus \cup_{p=1}^m M_p$, then there exist $p' < p$ with $\omega(\gamma) \in M_{p'}$ and $\omega^*(\gamma) \in M_p$. In other words, $\mathcal{D}(M)$ contains the recurrent behavior of the flow. A subset of M which belongs to some Morse decomposition is called a *Morse set.* In our case, each Morse set, M_p, is a nondegenerate singularity of the gradient flow φ of a Morse function $f : M \to \mathbb{R}$.

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[1] A filtration $F = \{F_p\}$ on a chain complex C is a sequence of subcomplexes F_pC for all integers p such that $F_pC \subset F_{p+1}C$.

As in [CdRS] we consider a Morse chain complex with connection matrix Δ. Given a Morse decomposition $\mathcal{D}(M)$ with m Morse sets, a connection matrix is a $m \times m$ matrix whose entries are homomorphisms between the homological Conley indices associated to the Morse sets (see [Fr1], [Fr2] and [Mo]). The nonzero entries of a connection matrix register the existence of connecting orbits in φ.

We make use of the algebraic-topological tool called *spectral sequence* in the setting described above. Our goal is to explain how a connection matrix Δ determines the spectral sequence, i.e, how it determines the spaces E^r and how it induces the differentials d^r. By considering a connection matrix over a field $\mathbb{F}$ it is possible to obtain a sweeping algorithm which characterizes the convergence process of the spectral sequence. To achieve this we use this algorithm to sweep the connection matrix.

This process was first described in [CdRS] for connection matrices over $\mathbb{Z}$. In this paper we prove that the sweeping algorithm holds for a connection matrix over $\mathbb{F}$. This algorithm consists of changing the basis of a connection matrix, $\Delta^r = (M^{r-1})^{-1}\Delta^{r-1}M^{r-1}$, as the spectral sequence (E^r, d^r) associated to a Morse-Conley chain complex unfolds. The sweeping algorithm preserves the upper triangular structure as well as the nilpotency of Δ^r, throughout the process producing a sequence of connection matrices over $\mathbb{F}$. However, this is not necessarily true over $\mathbb{Z}$. In fact, fractional entries show up in several of the computational examples. It is therefore surprising that the final matrix Δ^R in the sweeping algorithm over $\mathbb{Z}$ is always integral, and, thus, a connection matrix. This is the subject of Section 4. Several other properties of the change-of-basis matrices as well as of the intermediate matrices Δ^r are obtained.

A major interest in the Conley index theory is to understand flows and connection matrices under continuation. Our main motivation for characterizing properties of the intermediate matrices is to better understand the continuation behavior associated to the initial matrix Δ.

Both versions of the sweeping algorithm are implemented using the program *Mathematica*®.

In [CdRS], we treated the case where the chain complex C was a $\mathbb{Z}$-module generated by the singularities and graded by their indices, i.e.,

$$C_k = \bigoplus_{x \in \mathrm{crit}_k f} \mathbb{Z}\langle x \rangle$$

where $\mathrm{crit}_k(f)$ is the set of index k critical points of f. In this case, the connection matrix $\Delta : C \to C$ associated to $\mathcal{D}(M)$ is defined as the differential of the graded Morse chain complex $C = \mathbb{Z}\langle \mathrm{crit} f \rangle$, i.e., determined by the maps $\Delta_k : C_k \to C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \mathrm{crit}_{k-1} f} n(x,y)\langle y \rangle,$$

where $n(x, y)$ is the intersection number of x and y. The intersection number is defined for nondegenerate singularities x and y of indices k and $k-1$ respectively, since the set of connecting orbits is finite. By orienting the unstable and stable manifolds respectively, the intersection number $n(x, y)$ is the number of connecting orbits counted with orientation. In order to count orbits with orientation, choose a regular value c of f with $f(y) < c < f(x)$ and $n(x, y)$ is the intersection number of the spheres $S^{k-1} = W^u(x) \cap f^{-1}(c)$ and $S^{n-k} = W^s(y) \cap f^{-1}(c)$. For more details see [Sa1] and [R3].

When we have $\mathbb{F} = \mathbb{Z}_2$, C is the $\mathbb{Z}_2$ vector space

$$C = \mathbb{Z}_2\langle \mathrm{crit} f \rangle$$

and the connection matrix $\Delta : C \to C$ associated to $\mathcal{D}(M)$ is the differential of the graded Morse chain complex C, i.e., it is determined by the maps $\Delta_k : C_k \to C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \mathrm{crit}_{k-1} f} a(x,y)\langle y \rangle,$$

where $a(x,y)$ is the number of connecting orbits counted mod 2 for nondegenerate singularities x and y of indices k and $k-1$ respectively. We recall that Δ is an upper triangular nilpotent matrix.

Without loss of generality, we may assume that the columns of the matrix Δ are ordered with respect to k. The property we need to ensure is that the map Δ_k is filtration preserving. Hence, the columns of Δ may be partitioned into subsets J_0, J_1, J_2, $\dots$, such that J_s are the columns associated with index-k critical points, for some k. This implies that the matrix Δ is block upper triangular, as illustrated in Figure 1 below, that is, if $\Delta_{ij} \neq 0$ then $i \in J_{s-1}$ and $j \in J_s$, for some s. The entries with row indices in J_{s-1} and column indices in J_s constitute the s-th block B_s. The entries in B_s determine the map Δ_k, for some k. There is however a subtlety regarding notation. We use $\Delta_{k\ ij}$ as an "enhanced" synonym to Δ_{ij}, in the sense that it refers to the same entry, but carries the additional information that this entry belongs to the block in the column set associated with index-k critical points. Notice however, that the subscript s of column set J_s usually does not coincide with the index of the critical points associated with the s-th block.

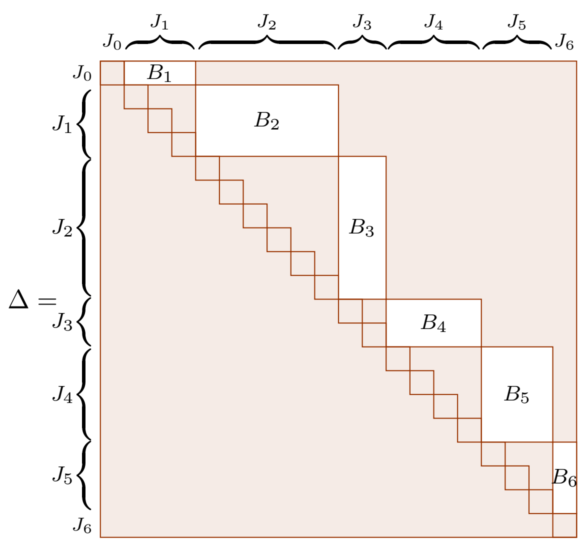


Figure 1: Connection matrix with 6 blocks.

We denote this filtered graded Morse chain complex by

$$(C, \Delta) = (\mathbb{F}\langle \mathrm{crit} f \rangle, \Delta).$$

We will use the notation of the boundary operator ∂ and its matrix Δ interchangeably.

Note that the r-th diagonal of Δ has entries $\Delta_{p+1-r,p+1}$, which are related to the connections between unstable and stable manifolds of M_{p+1} and M_{p+1-r}, for $p \in \{r, \dots, m-1\}$. Clearly, if column $(p+1)$ intersects the submatrix Δ_k, then M_{p+1} and M_{p+1-r} are respectively singularities of Morse index k and $k-1$, which we denote by h_k and h_{k-1}. These singularities are in filtrations $F_p \setminus F_{p-1}$ and $F_{p-r} \setminus F_{p-r-1}$, respectively. In summary, the r-th diagonal, when intersected with Δ_k, is registering information of numerically consecutive singularities of Morse indices k and $k-1$. We will use the same notation to indicate an elementary chain of C, that is, the elementary chain h_k^{p+1} is associated to the column $(p+1) \in J_k$, which corresponds to the singularity of Morse index k in filtration $F_p \setminus F_{p-1}$[2].

The notation h_k^s indicates the elementary k-chain associated to the column s of Δ.

Given a chain complex (C, ∂) endowed with an increasing filtration F^pC, such that $\partial(F^pC) \subset F^pC$ (and we assume here $F^{-1}C = 0$), the associated spectral sequence is (a generally infinite) sequence of chain complexes (E^r, d^r) (see [D] and [Sp]). Roughly, each stage contains information about longer and longer parts of the differential: the differential d^0 in the complex at the first stage is the part of ∂ which does not decrease filtration, d^1 concerns the part of ∂ which reduces filtration by no more than 1, and so on. Moreover, $H(E^r, d^r) = E^{r+1}$.

A *bigraded module* E^r over a principal ideal domain R is an indexed collection of R-modules $E^r_{p,q}$, for every pair of integers p and q. In this article, we work with $R = \mathbb{F}$ and hence the bigraded modules E^r are actually vector spaces over $\mathbb{F}$. A *differential* d^r *of bidegree* $(-r, r-1)$ is a collection of homomorphisms $d^r : E_{p,q} \to E_{p-r,q+r-1}$ for all p and q, such that $d^r \circ d^r = 0$. The *homology module* $H(E^r)$ is the bigraded module

$$H_{p,q}(E^r) = \frac{\mathrm{Ker} d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\mathrm{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

A *spectral sequence* $\{E^r, d^r\}$, $r \geq 0$, is a sequence of chain complexes where each chain complex E^r is the homology module of the previous one, i.e.,

- E^r is bigraded module, d^r is a differential with bidegree $(-r, r-1)$ in E^r;
- For $r \geq 0$ there exists an isomorphism $H(E^r) \approx E^{r+1}$.

In general, we will omit reference to q in this section since its role will be important only when considering more general Morse sets of a Morse decomposition. In our case, when the Morse set is a singularity of index k, the only q such that $E^r_{p,q}$ is nonzero is $q = k - p$. Hence, it is understood that E^r_p is in fact $E^r_{p,k-p}$.

For a filtered graded chain complex (C, ∂) we can define a spectral sequence

$$E^r_p = Z^r_p / (Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1})$$

where,

$$Z^r_p = \{c \in F_pC \mid \partial c \in F_{p-r}C\}.$$

Hence, the module Z^r_p consists of chains in F_pC with boundary in $F_{p-r}C$. This makes it natural to look at chains associated to the columns of the connection matrix to the left of and including the column $(p+1)$. This guarantees that any linear combination of chains respects the filtration. Furthermore, since the boundary of the chains must be in $F_{p-r}C$ we must consider columns or linear combinations that respect the filtration and that have the property that the entries in rows $i > (p - r + 1)$ are all zeroes. Hence, a significant entry in the connection matrix is the element on the r-th diagonal on the row $(p - r + 1)$ and the column $(p + 1)$.

However, as r increases, the $\mathbb{F}$-modules E^r_p change generators. In practice, the generators of the complex C mentioned above are very specific: singularities in the Morse case. The domain of d^r, E^r, is a certain quotient of a subgroup of C. Elements in this domain are represented by elements of C whose appropriate classes are in the kernels of all previous differentials $d^s, s < r$. Finding a system that span E^r in terms of the original basis of C is, in practice, a non-trivial matter but it can be obtained as a product of the sweeping algorithm.

[2] Note that the numbering on the columns are shifted by one with respect to the subindex p of the filtration F_p.

2 Sweeping Algorithm on Connection Matrices

The sweeping algorithm for constructing the spectral sequence (E^r, d^r) associated with a connection matrix Δ was introduced in [CdRS]. The spaces E^r are bigraded modules over a principal ideal domain R, assumed, in that work, to be $\mathbb{Z}$. The algorithm developed therein is repeated below in condensed form, for completeness. The notation adopted regarding matrices is introduced in Table 1.

$A_{i\bullet}$	i-th row of matrix A
$A_{\bullet j}$	j-th column of matrix A
$A_{I\bullet}$	submatrix of A with entries[3] a_{ij} such that $i \in I$
A_{IJ}	submatrix of A with entries a_{ij} such that $i \in I$ and $j \in J$, where I (resp., J) is a nonempty subset of the set of row indices (resp., column indices)
A^ℓ	ℓ-th matrix in a sequence, nonnegative superscripts **do not** denote exponents
$(A^\ell)^{-1}$	the inverse of matrix A^ℓ

Table 1: Notation adopted for (sub)matrices.

Given a $m \times m$ connection matrix Δ, the sweeping algorithm constructs a family of matrices Δ^r, for $r = 0, \ldots, m-2$, recursively, where $\Delta^0 = \Delta$. At each iteration, the entries of Δ^r are obtained from Δ^0 by performing a change of basis over R, that is, $\Delta^r = (P^{r-1})^{-1}\Delta^0 P^{r-1}$ and marking the r-th diagonal to the right of and parallel to the main diagonal, or r-th diagonal for short. Thus, the main diagonal is the 0-th diagonal. The construction of Δ^r is completed only after the markup of the entries along the r-th diagonal. That is, each matrix Δ^r comprises two kinds of information: numerical (the values of the entries themselves) and qualitative (the marks assigned to specific entries). The change of basis is determined by certain entries in Δ^r which we will refer to as *change-of-basis pivots*. These, on the other hand, depend on the previous classification of certain non-null entries of Δ^{r-1} as *primary pivots*. Primary pivot marks are permanent, i.e., matrix Δ^r inherits all the primary pivots of Δ^{r-1}. On the other hand, if the non-null entry of Δ^r is a change-of-basis pivot, then the corresponding entry in Δ^{r+1} is zero and unmarked. In the illustrations of the algorithm, primary pivot entries will be encircled, whereas change-of-basis pivots will be encased in boxes.

The sweeping algorithm is used to determine the Conley spectral sequence (E^r, d^r) associated to the Morse complex (C, Δ) and the finest filtration $\{F_p\}$. As previously stated, we will assume the singularities to be ordered with respect to the filtration. It simplifies the notation and implies that the nilpotent upper triangular connection matrix Δ has a block structure. That is, the set of column indices $\{1, \ldots, n\}$ may be partitioned into consecutive nonempty subsets $J_0, J_1, \ldots, J_b$ determined by the indices present in the chain complex as follows. Suppose the distinct indices present in the chain complex C, in ascending order, are $0 = g_0, g_1, \ldots, g_b$. Then $|J_s|$ is the number of critical points of index g_s, for $s = 0, \ldots, b$. If $g_s = k$, then the first and last column of J_s are denoted by f_k and ℓ_k, respectively. Thus the column partition implies a block partition of Δ, with the s-th block constituted by entries Δ_{ij} with $(i, j) \in J_{s-1} \times J_s$. The s-th block is non-null only if the indices g_{s-1} and g_s are consecutive integers. If $g_s = k$, the columns in J_s are associated with the elementary chains

[3] When there is no danger of ambiguity, the comma between row and column indices is omitted.

$\{h_k^{f_k}, h_k^{f_k+1}, \ldots, h_k^{\ell_k}\}$.

Figure 1 illustrates the block structure of a connection matrix Δ. This structure may correspond to several different sets of indices. It is compatible with, for example, one index-0 critical point ($g_0 = 0$), three index-2 critical points ($g_1 = 2$), six index-3 critical points ($g_2 = 3$), two index-5 critical points ($g_3 = 5$), four index-6 critical points ($g_4 = 6$), three index-7 critical points ($g_5 = 7$) and one index-9 critical point ($g_6 = 9$). If that were the case, blocks B_1, B_3 and B_6 must be null. Furthermore, we would have $f_0 = \ell_0 = 1$, $f_2 = 2$, $\ell_2 = 4$, $f_3 = 5$, $\ell_3 = 10$, $f_5 = 11$, $\ell_5 = 12$, $f_6 = 13$, $\ell_6 = 16$, $f_7 = 17$, $\ell_7 = 19$, $f_9 = \ell_9 = 20$.

Sweeping Algorithm over $\mathbb{Z}$

Input: nilpotent $m \times m$ upper triangular matrix Δ and column partition $J_0, J_1, \ldots, J_b$.

Initialization Step:

- $r = 0$
- $\Delta^r = \Delta$
- $P^r = I$ ($m \times m$ identity matrix)

Iterative Step: (Repeated until all diagonals parallel and to the right of the main diagonal have been swept)

Matrix Δ update

$r \leftarrow r + 1$

$\Delta^r = (P^{r-1})^{-1} \Delta^0 P^{r-1}$

Markup

Sweep entries of Δ^r in the r-th diagonal:

If $\Delta^r_{i,i+r} \neq 0$ **and** $\Delta^r_{\bullet,i+r}$ does not contain a primary pivot

Then If $\Delta^r_{i\bullet}$ contains a primary pivot

Then mark $\Delta^r_{\bullet,i+r}$ as a change-of-basis pivot

Else mark $\Delta^r_{i,i+r}$ as a primary pivot

Matrix P update

$P^r \leftarrow P^{r-1}$

For each change-of-basis pivot $\Delta^r_{k\,ij}$ update the j-th column of P^r as follows

Let $I = \{i, \ldots, \ell_{k-1}\}$, $J = \{f_k, \ldots, j\}$, $A = \Delta^0_{IJ}$, $c = |J|$

Let $x^* \in \mathbb{Z}^c$ be the optimal solution to

$$\begin{array}{lrcl} \min & x_c & & \\ \text{subject to} & Ax & = & 0 \\ & x_c & \geq & 1 \\ & x & \in & \mathbb{Z}^c \end{array}$$

$P^r_{Jj} \leftarrow x^*$

Final Step:

Matrix Δ update

$r \leftarrow r + 1$

$\Delta^r = (P^{r-1})^{-1} \Delta^0 P^{r-1}$

The construction of P^r is designed to guarantee that all entries below and including a change-of-basis entry are zeroed out in Δ^{r+1}. This and other properties of the algorithm over $\mathbb{Z}$ will be further explored in Section 4. It turns out that this step can be considerably simplified when one works with fields, using the fact that field elements have multiplicative inverses. In this case, one does not need to work with the original matrix Δ, or Δ^0, but can consider Δ^{r-1} directly. Since the basis change will be from Δ^{r-1} to Δ^r, we assign a different notation to the change-of-basis matrix. The algorithm is thus altered.

Sweeping Algorithm over $\mathbb{F}$

Input: nilpotent $m \times m$ upper triangular matrix Δ and column partition $J_0, J_1, \ldots, J_b$.

Initialization Step:

$r = 0$

$\Delta^r = \Delta$

$M^r = I$ ($m \times m$ identity matrix)

Iterative Step: (Repeated until all diagonals parallel and to the right of the main diagonal have been swept)

Matrix Δ update

$r \leftarrow r + 1$

$\Delta^r = (M^{r-1})^{-1}\Delta^{r-1}M^{r-1}$

Markup

Sweep entries of Δ^r in the r-th diagonal:

If $\Delta^r_{i,i+r} \neq 0$ **and** $\Delta^r_{\bullet,i+r}$ does not contain a primary pivot

Then If $\Delta^r_{i\bullet}$ contains a primary pivot

Then mark $\Delta^r_{\bullet,i+r}$ as a change-of-basis pivot

Else mark $\Delta^r_{i,i+r}$ as a primary pivot

Matrix M construction

$M^r \leftarrow I$

For each change-of-basis pivot Δ^r_{ij} change the j-th column of M^r as follows

Let p be such that Δ^r_{ip} is a primary pivot

$M^r_{pj} \leftarrow -\Delta^r_{ij}/\Delta^r_{ip}$

Final Step:

Matrix Δ update

$r \leftarrow r + 1$

$\Delta^r = (M^{r-1})^{-1}\Delta^{r-1}M^{r-1}$

The following proposition embodies the main properties of the family $\{\Delta^0, \Delta^1, \Delta^2, \ldots\}$ regarding the pattern of certain zero entries therein. Loosely speaking, it establishes that entries marked as primary pivots in an iteration remain non-null as the algorithm progresses and that entries below primary pivots and change-of-basis pivots are always null. It is necessary to show that the algorithm is well defined, that is, the division operation in the matrix M construction step is valid.

Proposition 2.1 *Let $\{\Delta^0, \Delta^1, \ldots\}$ be the sequence of matrices produced by the sweeping algorithm over $\mathbb{F}$. Then* *(i) all matrices inherit the block structure of Δ^0;*

(ii) the non-null entries of Δ^r strictly below the r-th diagonal are either primary pivots or are above a primary pivot;

(iii) primary pivot entries of Δ^r are non-null.

Proof: The statements are trivially true for Δ^0. Assume by induction that they are true for Δ^r. Consider the sweeping of the r-th diagonal of Δ^r. By the markup rules, if an entry on the r-th diagonal is marked as a primary pivot, then it must be non-null. Furthermore, there are no primary pivots below it. But since these entries lie strictly below the r-th diagonal, by the induction hypothesis, they must be null. Now suppose $\Delta^r_{i,i+r}$ is a change-of-basis pivot. Then, by the markup rules, $\Delta^r_{i,i+r} \neq 0$ and there is a primary pivot on the same row, say Δ^r_{ip}. Due to the order in which the entries are swept, this primary pivot, marked at an earlier iteration, must lie on a lower diagonal, and thus to the left of the change-of-basis pivot, so $p < i + r$. By the induction hypothesis, $\Delta^r_{ip} \neq 0$. We conclude that the change-of-basis matrix M^r is well defined.

The matrix M^r has unit diagonal and, for each column j such that $\Delta^r_{\bullet j}$ has a change-of-basis pivot, has precisely another (off-diagonal) non-null entry. In particular, if $\Delta^r_{ij} = \Delta^r_{i,i+r}$ is a change-of-basis pivot and $\Delta^r_{i,p}$ is the primary pivot on row i, then $M^r_{pj} = -\Delta^r_{ij}/\Delta^r_{pj} \neq 0$ is in the upper triangular part of the matrix, since $p < j$. Furthermore, this entry is in the triangular region above the diagonal and below the block containing the change-of-basis pivot. Figure 2 gives a close-up of this region of M^r, showing in gray the position of the relevant change-of-basis pivot, the primary pivot and the block containing them in Δ^r.

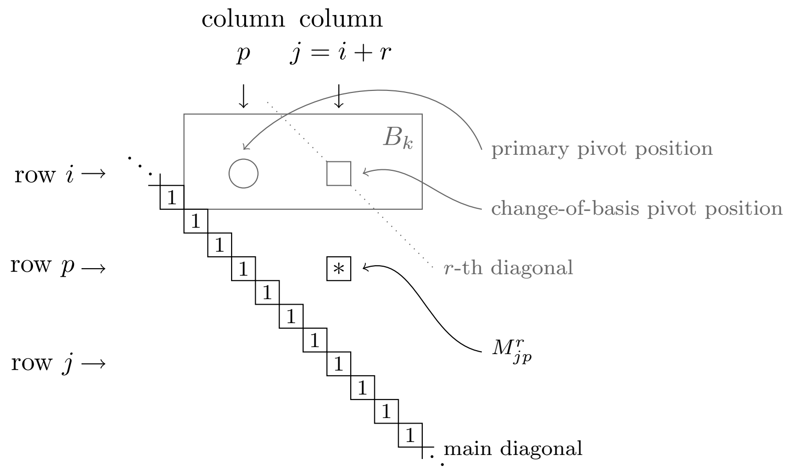


Figure 2: Relative position of non-null entries of M^r.

Thus the post- multiplication of Δ^r by M^r zeroes out the change-of-basis pivots, since the appropriate multiple of the column containing the primary pivot is added to the column containing the change-of-basis pivot:

$$(\Delta^r M^r)_{\bullet j} = \begin{cases} \Delta^r_{\bullet j}, & \text{if column } j \text{ of } \Delta^r \text{ does not contain a change-of-basis pivot,} \\ \Delta^r_{\bullet j} - \frac{\Delta^r_{j-r,j}}{\Delta^r_{j-r,p}} \Delta^r_{\bullet p}, & \text{otherwise, and } \Delta^r_{j-r,p} \text{ is a primary pivot.} \end{cases}$$

Furthermore, by the induction hypothesis, the entries below primary pivots strictly below the r-th diagonal are null, so this addition does not introduce non-null entries below the r-th diagonal of $\Delta^r M^r$ on the column containing the change-of-basis pivot. Additionally, it does not affect the block structure, since both columns $\Delta^r_{\bullet p}$ and $\Delta^r_{\bullet j}$ belong to the same block. Finally, if an entry on the r-th diagonal is marked as a primary pivot, then, by the markup rules, it is non-null and has no primary pivots below it. Hence, by the induction hypothesis, the entries below the primary pivot positions on the r-th diagonal are null, and the entries in the primary pivot positions are non-null. This means that the non-null entries of $\Delta^r M^r$ strictly below the $(r+1)$-th diagonal lie either on or above primary pivot positions and this matrix also inherits the block structure of Δ^0.

It remains to see what happens with the pre-multiplication by $(M^r)^{-1}$. It is easy to see that this inverse is obtained from M^r by reversing the sign of the off-diagonal entries, so it has the same nonzero pattern as M^r. Since we're performing a pre-multiplication, it is more convenient to think in terms of the rows of $(M^r)^{-1}$. If Δ^r_{ij} is a change-of-basis pivot, and the primary pivot on row i is located on column p, then $(M^r)^{-1}_{pj} = \Delta^r_{ij}/\Delta^r_{ip}$, where $i < p < j$, and p and j belong to the same subset of the column partition, say J_k. Thus, if row p of $(M^r)^{-1}$ has two nonzero entries, then column p of Δ^r contains a primary pivot. Pre-multiplication of $\Delta^r M^r$ by $(M^r)^{-1}$ will add to row p of $\Delta^r M^r$ a multiple of row $j > p$. Because we're adding to row p a multiple of a row below it, in the same block, this operation won't disrupt the zero patterns nor the block structure already established for $\Delta^r M^r$. Hence the non-null entries of Δ^{r+1} strictly below the $(r+1)$-th diagonal are either primary pivots or lie above a primary pivot, and, by induction, the statement is true for all matrices in the sequence. ■

Given a non-null entry Δ^r_{ij} such that $j \in J_s$, with $g_s = k$, then the i-th row is associated to a $(k-1)$-chain, whereas the j-th column is associated to a k-chain. This association is made explicit by the notation $\Delta^r_{k\ ij}$. Notice, however, that these might not be elementary chains any longer. The elementary chain associated to column j of Δ_k may have been replaced by a linear combination of the elementary chains $h_k^{f_k}, \ldots, h_k^j$ at some previous iteration.

If $\Delta^r_{k\ ij}$ is a change-of-basis pivot, then there is a column, namely, the p-th column, associated to a k-chain such that $\Delta^r_{k\ ip}$ is a primary pivot. Then we have to perform a change of basis on Δ^r by adding to the j-th column of Δ^r the p-th column of Δ^r multiplied by $(-\Delta^r_{k\ ip})^{-1}\Delta^r_{k\ ij}$, in order to zero out the entry $\Delta^r_{k\ ij}$, without introducing nonzero entries in $\Delta^r_{k\ sj}$ for $s > i$. Once this is done, we obtain a k-chain associated to the j-th column of Δ^{r+1}. It is a linear combination over $\mathbb{F}$ of the p-th column and the j-th column of Δ^r such that ${\Delta^{r+1}}_{k\ ij} = 0$. It is also a linear combination of h_k columns of Δ on and to the left of the j-th column. Hence, the j-th column of Δ^r is an h_k column and it corresponds to a linear combination over $\mathbb{F}$

$$\sigma_k^{j,r} = \sum_{s=f_k}^{j} c_s^{j,r} h_k^s$$

of h_k columns of Δ, recall that f_k-th column is the first column in Δ associated to a k-chain. The notation of $\sigma_k^{j,r}$ indicates the Morse index k and the j-th column of Δ^r. Note that $c_j^{j,r} = 1$.

It follows that the j-th column of Δ^{r+1} is an h_k column given by

$$\sigma_k^{j,r+1} = \underbrace{\sum_{s=f_k}^{j} c_s^{j,r} h_k^s}_{\sigma_k^{j,r}} + q_p \underbrace{\sum_{s=f_k}^{t} c_s^{p,r} h_k^s}_{\sigma_k^{p,r}} = c_{f_k}^{j,r+1} h_k^{f_k} + c_{f_k+1}^{j,r+1} h_k^{f_k+1} + \cdots + c_{j-1}^{j,r+1} h_k^{j-1} + c_j^{j,r+1} h_k^j, \tag{1}$$

where $q_p = (-{\Delta^r}_{k\ ip})^{-1}{\Delta^r}_{k\ ij}$ and $c_j^{j,r+1} = 1$.

Figure 3 illustrates the markup process at the r-th iteration. Primary pivots are encircled and change-of-basis pivots are encased in boxes. The figure shows part of the block associated with index k, as the r-th diagonal is swept.

It is clear that the first column of any Δ_k cannot undergo any change of basis since there is no column, and thus no primary pivots, to its left.

Once the above procedure is done for all change-of-basis pivots of the r-th diagonal of Δ^r we can define a change-of-basis matrix M^r, and let $\Delta^{r+1} = (M^r)^{-1}\Delta^r M^r$. Equivalently, $\Delta^{r+1} = (P^r)^{-1}\Delta^0 P^r$ where $P^r = M^0 M^1 \ldots M^r$.

Therefore, the matrix Δ^{r+1} has numerical values determined by the change of basis over $\mathbb{F}$ of Δ^r. In particular, all the change-of-basis pivots on the r-th diagonal Δ^r are zero in Δ^{r+1}. See Figure 3.

$$
\begin{array}{c}
\begin{array}{cccccc}
\sigma_k^{s,r} & \sigma_k^{s+1,r} & \sigma_k^{s+2,r} & \sigma_k^{s+3,r} & \sigma_k^{s+4,r} & \sigma_k^{s+5,r}
\end{array}
\\
\begin{array}{c}
\sigma_{k-1}^{s,r} \\ \sigma_{k-1}^{s+1,r} \\ \sigma_{k-1}^{s+2,r} \\ \sigma_{k-1}^{s+3,r} \\ \sigma_{k-1}^{s+4,r}
\end{array}
\left(
\begin{array}{cccccc}
1 & \boxed{1} & 1 & & & \\
0 & 0 & 1 & 1 & & \\
0 & 0 & 1 & \boxed{1} & 1 & \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right)
\begin{array}{c}
r+1 \\ r
\end{array}
\end{array}
$$

Figure 3: Diagonals r and $r+1$.

Remark 2.2 *Note that the change of bases we perform when we consider the connection matrix with entries in $\mathbb{Z}$ are more complicated, see [CdRS]. We add a linear combination over $\mathbb{Q}$ of all the h_k columns s of Δ^r with $f_k \leq s < j$, where f_k is the first column associated with index k, to the positive integer multiple of the j-th column of Δ^r, in order to zero out the entry $\Delta^r_{k\ ij}$ without introducing nonzero entries in $\Delta^r_{k\ sj}$ for $s > i$. Moreover, this integer multiple of the j-th column of Δ^r, which we will denote by u, has to be the minimal positive integer with this property. The resulting linear combination should be of the form $\beta^{f_k} h_k^{f_k} + \cdots + \beta^{j-1} h_k^{j-1} + \beta^j h_k^j$ where β^s are integers for $s = f_k, \ldots, j$.*

The integer u is called leading coefficient *of the change of basis. Note that it is the minimal leading coefficient of a change of basis. Once this is done, we obtain a k-chain associated to the j-th column of Δ^{r+1}. It is a linear combination over $\mathbb{Q}$ of the s-th h_k columns $f_k \leq s < j$ of Δ^r plus an integer multiple u of the j-th column of Δ^r such that $\Delta^{r+1}_{k\ ij} = 0$. It is also an integer linear combination of h_k columns of Δ on and to the left of the j-th column. In this case, the j-th column of Δ^{r+1} is*

$$\sigma_k^{j,r+1} = u \underbrace{\sum_{s=f_k}^{j} c_s^{j,r} h_k^s}_{\sigma_k^{j,r}} + q_{j-1} \underbrace{\sum_{s=f_k}^{j-1} c_s^{j-1,r} h_k^s}_{\sigma_k^{j-1,r}} + \cdots + q_{f_k+1} \underbrace{(c_{f_k}^{f_k+1,r} h_k^{f_k} + c_{f_k+1}^{f_k+1,r} h_k^{f_k+1})}_{\sigma_k^{f_k+1,r}} + q_{f_k} \underbrace{c_{f_k}^{f_k,r} h_k^{f_k}}_{\sigma_k^{f_k,r}}, \tag{2}$$

or, equivalently,

$$\begin{aligned}(uc_{f_k}^{j,r} + q_{j-1} c_{f_k}^{j-1,r} + \cdots + q_{f_k} c_{f_k}^{f_k,r}) h_k^{f_k} + (uc_{f_k+1}^{j,r} + q_{j-1} c_{f_k+1}^{j-1,r} + \cdots + q_{f_k+1} c_{f_k+1}^{f_k+1,r}) h_k^{f_k+1} + \cdots \\ \cdots + (uc_{j-1}^{j,r} + q_{j-1} c_{j-1}^{j-1,r}) h_k^{j-1} + uc_j^{j,r} h_k^j\end{aligned} \tag{3}$$

with $c_{f_k}^{f_k,r} = 1$ and

$$c_{f_k}^{j,r+1} = uc_{f_k}^{j,r} + q_{j-1} c_{f_k}^{j-1,r} + \cdots + q_{f_k} c_{f_k}^{f_k,r} \in \mathbb{Z} \tag{4}$$

$$c_{f_k+1}^{j,r+1} = uc_{f_k+1}^{j,r} + q_{j-1} c_{f_k+1}^{j-1,r} + \cdots + q_{f_k+1} c_{f_k+1}^{f_k+1,r} \in \mathbb{Z} \tag{5}$$

$$\vdots$$

$$c_{j-1}^{j,r+1} = uc_{j-1}^{j,r} + q_{j-1} c_{j-1}^{j-1,r} \in \mathbb{Z} \tag{6}$$

$$c_j^{j,r+1} = uc_j^{j,r} \in \mathbb{Z}. \tag{7}$$

Note that if the primary pivot of the i-th row is on the t-th column then the rational number q_t is nonzero in $q_t \sum_{s=f_k}^{t} c_s^{t,r} h_k^s$ *and such that*

$$\Delta_{k\ ij}^{r+1} = u\Delta_{k\ ij}^{r} + q_t \Delta_{k\ it}^{r} = 0.$$

Since $u \geq 1$ is unique, the coefficient q_t is uniquely defined.

3 Conley Spectral sequence

In this section we present some results which indicate how the sweeping algorithm produces Conley's spectral sequence. Those were established in [CdRS] for connection matrices over $\mathbb{Z}$. In what follows we present a version for connection matrices over $\mathbb{F}$.

We will start describing basic properties of the Δ^r's produced by the sweeping algorithm which are to be used in the proof of the main theorems. More specifically our attention will be directed towards characterizing properties associated with the primary and change-of-basis pivots which are essential in determining the spectral sequence.

It is easy to see that all Δ^r's are upper triangular and nilpotent since they are recursively obtained from the initial connection matrix Δ by change of bases over $\mathbb{F}$.

Note that, as in [CdRS], if the entry $\Delta_{k\ p-r+1,p+1}^r$ has been identified by the sweeping algorithm as a primary pivot or a change-of-basis pivot then $\Delta_{k\ s,p+1}^r = 0$ for all $s > p - r + 1$.

Proposition 3.1 asserts that we cannot have more than one primary pivot in a fixed row or column. Moreover, if there is a primary pivot in row i then there is no primary pivot in column i.

Proposition 3.1 *Let $\{\Delta^r\}$ be the resulting family of matrices produced by the sweeping algorithm applied to a connection matrix Δ. Given any two primary pivots $\Delta_{k\ ij}^r$ and $\Delta_{k\ ms}^r$ we have that $\{i, j\} \cap \{m, s\} = \emptyset$.*

The proof is completely analogous to the proof for connection matrices over $\mathbb{Z}$ given in [CdRS]. Note that this proposition is also a particular case of Proposition 4.3.

In order to simplify notation, reference to the index k in the matrix Δ_k^r will be omitted whenever it is not necessary.

3.1 The Spaces E_p^r of the Spectral Sequence

The spaces E_p^r are determined when we apply the sweeping algorithm to the matrix Δ. The primary and change-of-basis pivots of Δ^r produced by the sweeping algorithm play an important role in determining the generators of Z_p^r.

Recall that

$$E_p^r = Z_p^r / (Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1})$$

where,

$$Z_p^r = \{c \in F_p C \mid \partial c \in F_{p-r} C\}.$$

Each h_k column of the connection matrix Δ represents connections of an elementary chain h_k of C_k to an elementary chain h_{k-1} of C_{k-1}.

The space $Z^r_{p,k-p} = \{c \in F_pC_k; \partial c \in F_{p-r}C_{k-1}\}$ is generated by k-chains contained in F_p with boundaries in F_{p-r}. This corresponds in the matrix Δ to all the h_k columns to the left of the column $(p+1)$ or linear combinations of these h_k columns, such that their boundaries (nonzero entries) are above the row $(p-r+1)$ [4].

The index k singularity in $F_p \setminus F_{p-1}$ corresponds to the k chain associated to the column $(p+1)$ of Δ. Hence we denote this singularity by h_k^{p+1}.

Proposition 3.2 is an important result since it establishes a formula for $Z^r_{p,k-p}$ using the chains $\sigma_k^{p,r}$ determined in the sweeping algorithm.

Proposition 3.2 $Z^r_{p,k-p} = \mathbb{F}[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \ldots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}]$ *where* f_k *is the first column in* Δ *associated to a* k*-chain and* $\mu^{j,\zeta} = 0$ *whenever the primary pivot of the* j*-th column is below the row* $(p-r+1)$ *and* $\mu^{j,\zeta} = 1$ *otherwise.*

Proof: Note that the $\sigma_k^{p+1-\xi,r-\xi}$ is associated to column $(p+1-\xi)$ of the matrix Δ^ξ. By definition, $\mu^{p+1-\xi,r-\xi} = 1$ if and only if the primary pivot on column $(p+1-\xi)$ is above row $(p+1-\xi)-(r-\xi) = p-r+1$. It is easy to verify that chains associated to columns with primary pivots below row $(p-r+1)$ do not correspond to generators of $Z^r_{p,k-p}$. Consider a k-chain $\sigma_k^{p+1-\xi,r-\xi}$, with $\xi \in \{0, \ldots, p+1-\kappa\}$, associated to column $(p+1-\xi)$ of $\Delta^{r-\xi}$ such that the primary pivot of column $(p+1-\xi)$ of $\Delta^{r-\xi}$ is above row $(p-r+1)$. For the latter primary pivots we show that $\sigma_k^{p+1-\xi,r-\xi}$ is a k-chain which corresponds to a generator of Z^r_p. It is easy to see that $\sigma_k^{p+1-\xi,r-\xi}$ is in F_pC_k for $\xi \geq 0$. Furthermore, the step $(r-\xi)$ in the sweeping method has zeroed out all change-of-basis pivots below the diagonal $(r-\xi)$. In other words, all nonzero entries of column $(p+1-\xi)$ of $\Delta^{r-\xi}$ are above row $(p+1-\xi)-(r-\xi) = (p-r+1)$. Hence, the boundary of $\sigma_k^{p+1-\xi,r-\xi}$ is in $F_{p-r}C_{k-1}$.

We now show that any element in Z^r_p is a linear integer combination of $\mu^{p+1-\xi,r-\xi}\sigma_k^{p+1-\xi,r-\xi}$ for $\xi = 0, \ldots, p+1-\kappa$. This is done by multiple induction in p and r.

- Consider F_{f_k-1}, where f_k is the first column of Δ associated to a k-chain. Let ξ be such that the boundary of $h_k^{f_k}$ is in $F_{f_k-1-\xi}C_{k-1}$.

  1. $Z^r_{f_k-1}$ is generated by k-chain in $F_{f_k-1}C_k$ with boundaries in $F_{f_k-1-r}C_{k-1}$. Note that there exists only one chain $h_k^{f_k}$ in $F_{f_k-1}C_k$. Hence
     (a) If $\xi < r$ then $\partial h_k^{f_k} \notin F_{f_k-1-r}C_{k-1}$. Thus, $Z^r_{f_k-1} = 0$
     (b) If $\xi > r$ than $\partial h_k^{f_k} \in F_{f_k-1-r}C_{k-1}$. Thus, $Z^r_{f_k-1} = \mathbb{F}[h_k^{f_k}]$
  2. On the other hand, $\sigma_k^{f_k,r}$ is a k-chain associated to column f_k of Δ^r. Since there is no change of basis caused by the sweeping method that affects the first column of Δ_k, $\sigma_k^{f_k,r} = h_k^{f_k}$. Furthermore, $\mu^{f_k,r} = 1$ if and only if the boundary of $h_k^{f_k} = \sigma_k^{f_k,r}$ is above the r-th diagonal. Hence
     (a) If $\xi < r$ then $\mu^{f_k,r} = 0$. Thus $\mathbb{F}[\mu^{f_k,r}\sigma_k^{f_k,r}] = 0$
     (b) If $\xi > r$ then $\mu^{f_k,r} = 1$. Thus $\mathbb{F}[\mu^{f_k,r}\sigma_k^{f_k,r}] = \mathbb{F}[\sigma_k^{f_k,r}] = \mathbb{F}[h_k^{f_k}]$.

  Hence $Z^r_{f_k-1} = \mathbb{F}[\mu^{f_k,r}\sigma_k^{f_k,r}]$.

[4] The expressions "above the row" and "to the left of the column" shall include the row or column in question, whereas the expressions "below the row" and "to the right of the column" shall not include the row or column in question.

- Let diagonal ξ_1 be the first in Δ that intersects Δ_k. All the columns of Δ corresponding to the chains $h_k^{p+1}, \ldots, h_k^{f_k}$ have nonzero entries above the diagonal ξ_1, thus, above row $(p - \xi_1 + 1)$ of Δ.

  1. By definition $Z_p^{\xi_1}$ is generated by k-chains contained in F_pC_k with boundary in $F_{p-\xi_1}C_{k-1}$. Since the columns of Δ associated to the chains $h_k^{p+1}, \ldots, h_k^{f_k}$ have nonzero entries above row $(p - \xi_1 + 1)$, this implies that the boundaries are in $F_{p-\xi_1}C_{k-1}$, i.e.,
  $$Z_p^{\xi_1} = \mathbb{F}[h_k^{p+1}, \ldots, h_k^{f_k}].$$
  2. Since nonzero entries in the columns of Δ associated to the chains $h_k^{p+1}, \ldots, h_k^{f_k}$ are all above the diagonal ξ_1, then $\sigma_k^{j,\xi_1} = h_k^j$, $j = f_k, \ldots, p+1$ and $\mu^{j,\xi_1} = 1$, $j = f_k, \ldots, p+1$. Hence,
  $$\mathbb{F}[\mu^{p+1,\xi_1}\sigma_k^{p+1,\xi_1}, \ldots, \mu^{f_k,\xi_1-p-1+f_k}\sigma_k^{f_k,\xi_1-p-1+f_k}] = \mathbb{F}[h_k^{p+1}, \ldots, h_k^{f_k}].$$

  Therefore, $Z_p^{\xi_1} = \mathbb{F}[\mu^{p+1,\xi_1}\sigma_k^{p+1,\xi_1}, \ldots, \mu^{f_k,\xi_1-p-1+f_k}\sigma_k^{f_k,\xi_1-p-1+f_k}]$.

- We assume that the generators of Z_{p-1}^{r-1} correspond to k-chains associated to $\sigma_k^{p+1-\xi,r-\xi}$, $\xi = 1, \ldots, p+1-f_k$ whenever the primary pivot of column $(p+1-\xi)$ is above row $(p-r+1)$. If the primary pivot of column $(p+1)$ is below row $(p-r+1)$ then $Z_p^r = Z_{p-1}^{r-1}$ and it is the case when $\mu^{p+1,r} = 0$. Suppose now that the primary pivot of column $(p+1)$ is above row $(p-r+1)$. Let $b_{f_k}, \ldots, b_{p+1} \in \mathbb{F}$ and $\mathfrak{h}_k = b^{p+1}h_k^{p+1} + \cdots + b^{f_k}h_k^{f_k}$ be a k-chain corresponding to an element of $Z_{p,k-p}^r$. We know that $\mathfrak{h}_k$ is in F_p and its boundary is above row $(p-r+1)$. If $b^{p+1} = 0$ then $\mathfrak{h}_k \in Z_{p-1}^{r-1}$ and the result follows by the induction hypothesis. Suppose $b^{p+1} \neq 0$.

  Thus we can rewrite $\mathfrak{h}_k$ as
  $$\mathfrak{h}_k = b^{p+1}\sigma_k^{p+1,r} + (b^p - b^{p+1}c_p^{p+1,r})h_k^p + \cdots + (b^{f_k} - b^{p+1}c_{f_k}^{p+1,r})h_k^{f_k}.$$

  Note that $\mathfrak{h}_k - b_{p+1}\sigma_k^{p+1,r} = (b^p - b^{p+1}c_p^{p+1,r})h_k^p + \cdots + (b^{f_k} - b^{p+1}c_{f_k}^{p+1,r})h_k^{f_k} \in F_{p-1}$. Moreover, since $\mathfrak{h}_k$ and $\sigma_k^{p+1,r}$ have their boundaries above row $(p-r+1)$, then the boundary of $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r}$ is above row $(p-r+1)$. Hence $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} \in Z_{p-1}^{r-1}$. By the induction hypothesis we have that $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} = a_p\mu^{p,r-1}\sigma_k^{p,r-1} + \cdots + a_{f_k}\mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}$ i.e,
  $$\mathfrak{h}_k = b^{p+1}\sigma_k^{p+1,r} + a_p\mu^{p,r-1}\sigma_k^{p,r-1} + \cdots + a_{f_k}\mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}.$$

■

Note that sometimes the index j in Δ^j is negative and in this case we adopt the following convention: when $j < 0$ we have $\Delta^j = \Delta$.

The next lemma establishes a formula which is used in Theorem 3.4.

Lemma 3.3 *Suppose that* $\partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} \nsubseteq Z_{p-1,k-(p-1)}^{r-1}$. *Then*
$$Z_{p-1,k-(p-1)}^{r-1} + \partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} = Z_p^r$$

Proof: Since $\partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} \nsubseteq Z_{p-1,k-(p-1)}^{r-1}$ then $Z_{p-1,k-(p-1)}^{r-1} + \partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1}$ is a subspace of
$$Z_{p,k-p}^r = \mathbb{F}[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \ldots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}]$$

but it is not a subspace of

$$Z^{r-1}_{p-1,k-(p-1)} = \mathbb{F}[\mu^{p,r-1}\sigma_k^{p,r-1}, \mu^{p-1,r-2}\sigma_k^{p-1,r-2}, \ldots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}].$$

Then $\mu^{p+1,r} = 1$ and $Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = Z^r_p$.

■

When we have a chain complex and a connection matrix over $\mathbb{Z}$ this formula is harder to be obtained since it detects torsion in the spectral sequence. In this case, when $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$ then

$$Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = \mathbb{Z}[\ell\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \ldots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}]$$

where

$$\ell = gcd\{\mu^{r+p,r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p}, \ldots, \mu^{f_{k+1},f_{k+1}-p-1}c^{p+1,f_{k+1}-p-1}_{p+1}\Delta^{f_{k+1}-p-1}_{p+1,f_{k+1}}\}/c^{p+1,r}_{p+1}$$

f_k is the first column associated to a k-chain and f_{k+1} is the first column associated to a $(k+1)$-chain as it can be seen with more detail in [CdRS].

Theorem 3.4 *The matrix Δ^r obtained from the sweeping algorithm applied to Δ determines E^r_p.*

Proof: We have to prove that

$$E^r_{p,k-p} = \frac{Z^r_{p,k-p}}{Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}}$$

is either zero or a finitely generated space whose generator corresponds to a k-chain associated to column $(p+1)$ of Δ^r. The entry $\Delta^r_{p-r+1,p+1}$ is on the r-th diagonal and plays a crucial role in determining the generators of $E^r_{p,k-p}$. Since $\Delta^r_{p-r+1,p+1}$ is a nonzero entry on the r-th diagonal, it can be either a primary pivot, a change-of-basis pivot or it is in a column above a primary pivot. A zero entry can be in a column above a primary pivot or all entries below it are also zero. The proof is a consequence of formulas obtained in Proposition 3.2 and Lemma 3.3 considering each one of the possibilities for $\Delta^r_{p-r+1,p+1}$.

1. Suppose the entry $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot. Then $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$. Therefore, the chain associated to column $(p+1)$ in Δ^r corresponds to a generator of $Z^r_{p,k-p}$. By the sweeping method this chain is a linear combination over $\mathbb{F}$ of the h_k columns of Δ to the left of column $(p+1)$ such that the coefficient of column $(p+1)$ is 1. This chain is $\sigma_k^{p+1,r}$ and since the coefficient of column $(p+1)$ is nonzero, $\sigma_k^{p+1,r}$ is not contained in the generators of $Z^{r-1}_{p-1,k-(p-1)}$.

   Claim 1: If $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot then $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.

   The generators of $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ must correspond to $(k+1)$-chains associated to h_{k+1} columns with the property that their boundaries are above row $(p+1)$ and consequently all entries below row $(p+1)$ are zero. Hence the entries of these h_{k+1} columns on row $(p+1)$ must, by the sweeping method, either be a primary pivot or a zero entry. See figure 4.

   By Proposition 3.1, row $(p+1)$ cannot contain a primary pivot since we have assumed that column $(p+1)$ has a primary pivot. Therefore, the entries of these h_{k+1} columns in row $(p+1)$ must be zeroes. It follows that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ does not contain in its set of generators the generator $\sigma_k^{p+1,r}$. The claim follows.

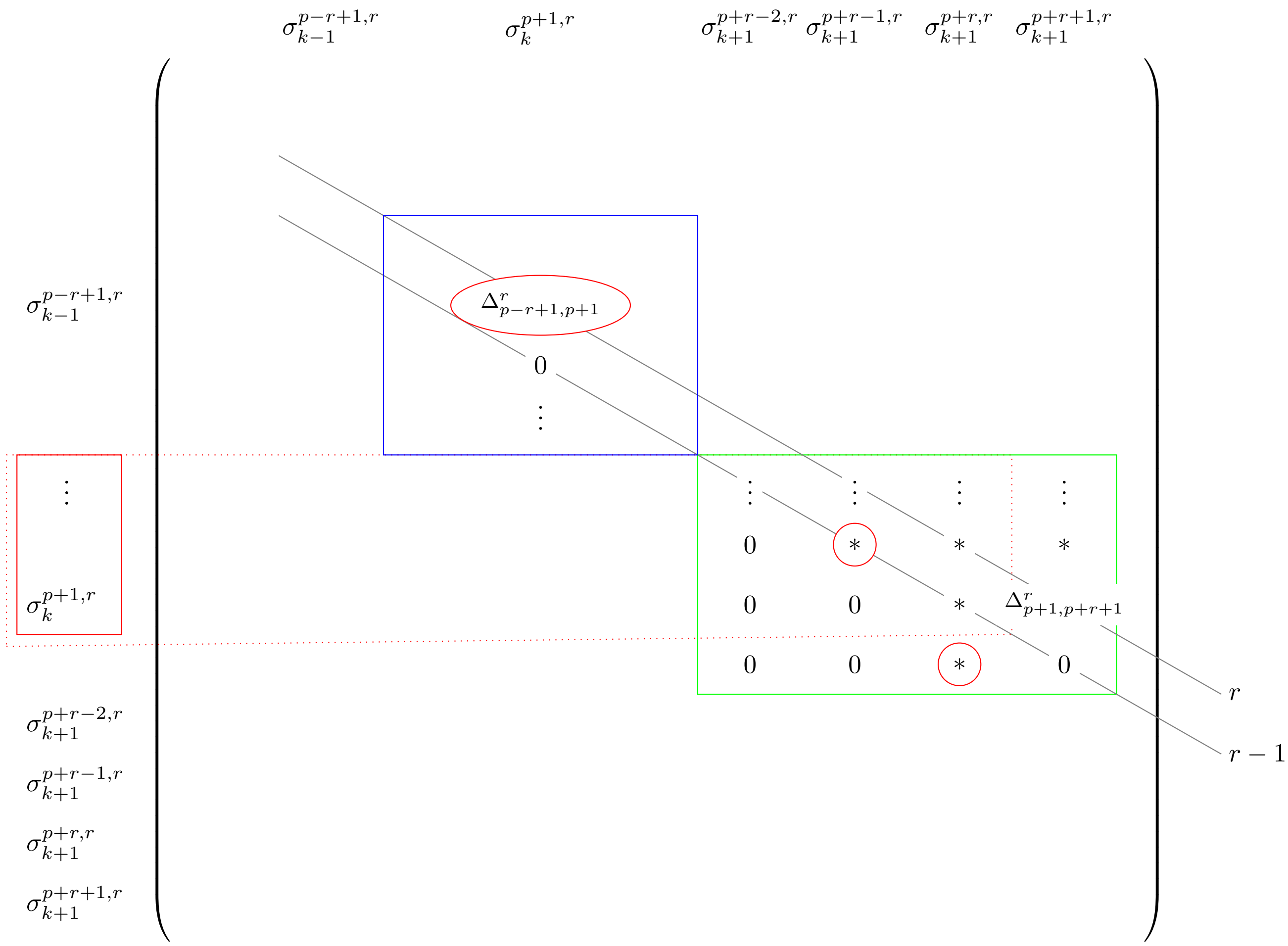


Figure 4: $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.

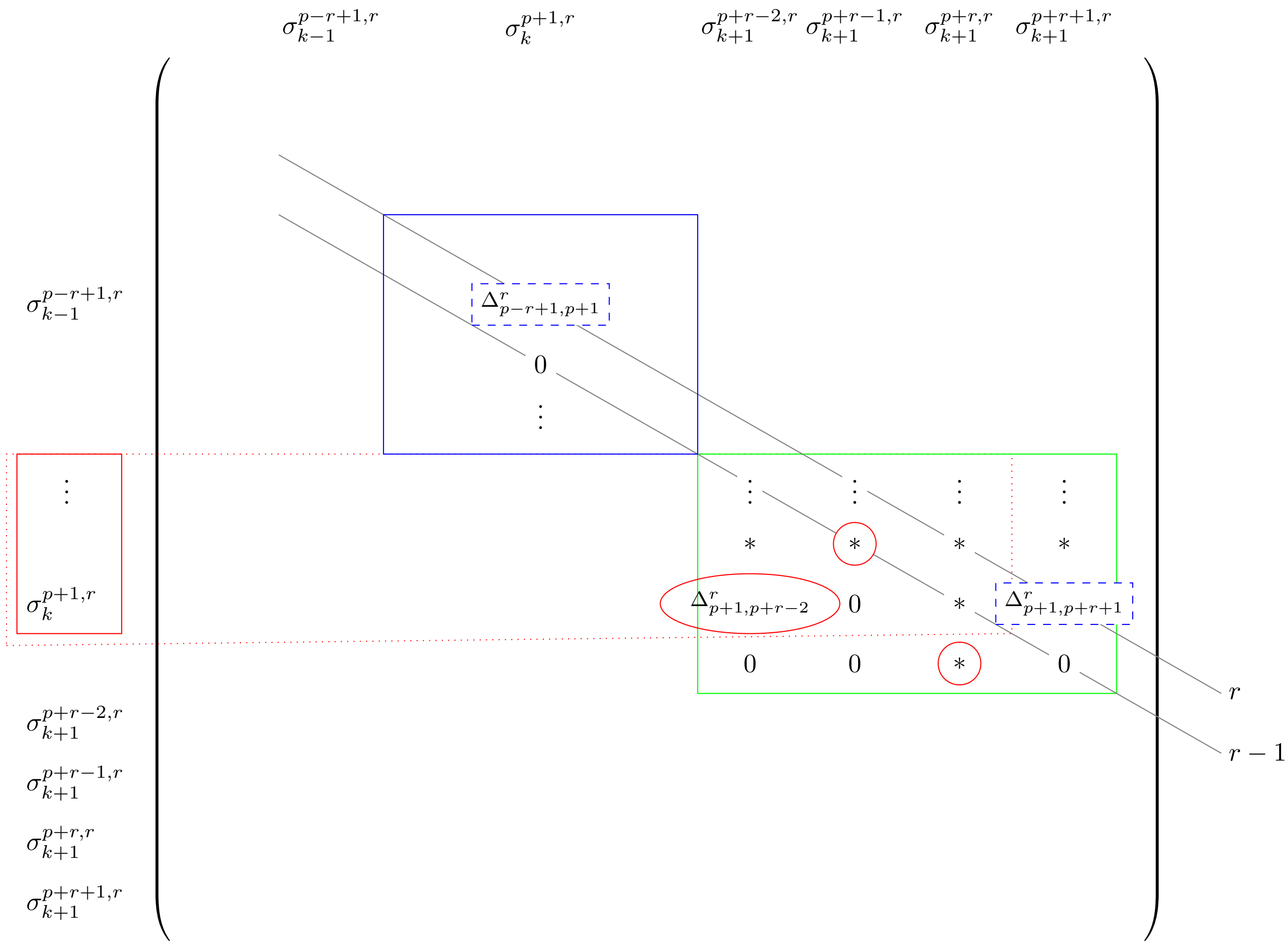


Figure 5: $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$.

By Proposition 3.2 we have that $E^r_{p,k-p} = \mathbb{F}[\sigma_k^{p+1,r}]$.

2. If the entry $\Delta^r_{p-r+1,p+1}$ is identified by the sweeping method as a change-of-basis pivot then the sweeping method guarantees that $\Delta^{r+1}_{p-r+1,p+1} = 0$. Furthermore, $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ and, like in the previous case, the generator $\sigma_k^{p+1,r}$ corresponding to the k-chain associated to column $(p+1)$ in Δ^r is a generator of $Z^r_{p,k-p}$.

   Thus we have to analyze row $(p+1)$. There are two possibilities:

   (a) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above row p.

   In this case, as before, by Proposition 3.2 $E^r_{p,k-p} = \mathbb{F}[\sigma_k^{p+1,r}]$.

   (b) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, there exist elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a nonzero entry in row $(p+1)$ which is necessarily a primary pivot. See figure 5.

   By Lemma 3.3 and Proposition 3.2 $E^r_{p,k-p} = 0$.

3. If the entry $\Delta^r_{p-r+1,p+1}$ is nonzero, but is not a primary pivot nor a change-of-basis pivot then it must be an entry above a primary pivot. In other words, there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. It follows that $\sigma_k^{p+1,r}$ is not in $Z^r_{p,k-p}$. Thus, $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and hence $E^r_{p,k-p} = 0$.

4. If the entry $\Delta^r_{p-r+1,p+1}$ is a zero entry we have the following possibilities:

   (a) There is a primary pivot below $\Delta^r_{p-r+1,p+1}$ i.e, there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. In this case the generator $\sigma_k^{p+1,r}$ corresponding to the k-chain associated to column $(p+1)$ is not a generator of Z^r_p and hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$. It follows that $E^r_{p,k-p} = 0$.

   (b) $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$. In this case, the generator $\sigma_k^{p+1,r}$ corresponding to the k-chain associated to column $(p+1)$ in Δ^r is a generator of $Z^r_{p,k-p}$. Thus we must analyze row $(p+1)$. We have the following possibilities:

      i. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above row p.

      In this case, as before, by Proposition 3.2 $E^r_{p,k-p} = \mathbb{F}[\sigma_k^{p+1,r}]$.

      ii. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, there exist elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a nonzero entry in row $(p+1)$. By Proposition 3.2 and Lemma 3.3 $E^r_{p,k-p} = 0$.

5. The entry $\Delta^r_{p-r+1,p+1}$ is not in Δ^r_k. This includes the case where $p-r+1 < 0$, i.e, $\Delta^r_{p-r+1,p+1}$ is not on the matrix Δ^r.

   The analyzes of E^r_p is very similar to the previous one, i.e, we have two possibilities:

   (a) There is a primary pivot in column $(p+1)$ in a diagonal $\overline{r} < r$. In this case the generator corresponding to the k-chain associated to column $(p+1)$, $\sigma_k^{p+1,r}$ is not a generator of $Z^r_{p,k-p}$. Hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and $E^r_{p,k-p} = 0$.

   (b) All the entries in Δ^r in column $(p+1)$ in diagonals lower than r are zero, i.e, the generator corresponding to the k-chain associated to column $(p+1)$, $\sigma_k^{p+1,r}$ in Δ^r is a generator of $Z^r_{p,k-p}$. Then we have to analyze row $(p+1)$.

      i. If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$ then, by Proposition 3.2, $E^r_{p,k-p} = \mathbb{F}[\sigma_k^{p+1,r}]$.

      ii. If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$ then, by Proposition 3.2 and Lemma 3.3, $E^r_{p,k-p} = 0$.

■

3.2 The Differentials of the Spectral Sequence

We will describe how the sweeping algorithm applied to Δ induces the differentials $d^r_p : E^r_p \to E^r_{p-r}$ in the spectral sequence. The results obtained in the case of a complex and a connection matrix over $\mathbb{Z}$ can be seen in [CdRS]. We will denote by f_k the first column of a connection matrix associated to a k-chain and by f_{k+1} the first column associated to a $(k+1)$-chain.

Lemma 3.5 *Let $E^r_p = \mathbb{F}[\sigma_k^{p+1,r}]$ and suppose that $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeroes below it. Then*

1. *If $\Delta^r_{p+1,p+r+1}$ is a primary pivot, $E^{r+1}_{p,k-p} = 0$.*

2. *If $\Delta^r_{p+1,p+r+1}$ is a zero entry with a column of zeroes below it, $E^{r+1}_{p,k-p} = \mathbb{F}[\sigma_k^{p+1,r+1}]$.*

Proof: Since $\Delta^r_{p-r+1,p+1}$ is zero with a column of zero entries below it then $\sigma_k^{p+1,r+1} \in Z^{r+1}_{p,k-p}$ and hence $Z^r_{p-1,k-(p-1)} \subsetneq Z^{r+1}_{p,k-p}$. Moreover, since $E^r_p = \mathbb{F}[\sigma_k^{p+1,r}]$ then $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$. But the difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$

and $\partial Z^r_{p+r,(k+1)-(p+r)}$ is that the last one includes the boundary of column $(p+r+1)$. The element in column $(p+r+1)$ and row $(p+1)$ is $\Delta^r_{p+1,p+r+1}$.

If $\Delta^r_{p+1,p+r+1}$ is a primary pivot then $\partial Z^r_{p+r,(k+1)-(p+r)} \nsubseteq Z^r_{p-1,k-(p-1)}$ and $E^{r+1}_{p,k-p} = 0$.

If $\Delta^r_{p+1,p+r+1} = 0$ then $\partial Z^r_{p+r,(k+1)-(p+r)} \subseteq Z^r_{p-1,k-(p-1)}$ and, $E^{r+1}_p = \mathbb{F}[\sigma^{p+1,r}_k]$.

■

Theorem 3.6 *If E^r_p and E^r_{p-r} are both nonzero, then the map $d^r_p : E^r_p \to E^r_{p-r}$ is induced by δ^r_p, i.e, multiplication by the entry $\Delta^r_{p-r+1,p+1}$ whenever it is either a primary pivot or a zero with a column of zero entries below it.*

Proof: Suppose that E^r_p and E^r_{p-r} are both nonzero. We must show in each of the following cases that

$$\frac{\mathrm{Ker}\,\delta^r_p}{\mathrm{Im}\,\delta^r_{p+r}} = E^{r+1}_p.$$

Since we want E^r_p nonzero, it follows from Theorem 3.4, that we must consider three cases for the entry $\Delta^r_{p-r+1,p+1}$: primary pivot, change-of-basis pivot and zero with a column of zeroes below it. However, if $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot then there exists a primary pivot in row $(p-r+1)$ on a diagonal below the r-th diagonal and hence $E^r_{p-r} = 0$. Hence, whenever E^r_p and E^r_{p-r} are both nonzero, the entry $\Delta^r_{p-r+1,p+1}$ in Δ^r is either a primary pivot or a zero with a column of zero entries below it.

1. Suppose $\Delta^r_{p-r+1,p+1}$ is a primary pivot.

   In this case $E^r_p = \mathbb{F}[\sigma^{p+1,r}_k]$ and $E^r_{p-r} = \mathbb{F}[\sigma^{p-r+1,r}_{k-1}]$. We have the following sequence:

$$\cdots \longleftarrow \mathbb{F}[\sigma^{p-r+1,r}_{k-1}] \xleftarrow{\delta^r_p} \mathbb{F}[\sigma^{p+1,r}_k] \xleftarrow{\delta^r_{p+r}} E^r_{p+r} \longleftarrow \cdots \tag{8}$$

   Since $\delta^r_p : \mathbb{F}[\sigma^{p+1,r}_k] \to \mathbb{F}[\sigma^{p-r+1,r}_{k-1}]$ is multiplication by $\Delta^r_{p-r+1,p+1} \neq 0$ then $\mathrm{Ker}\,\delta^r_p = 0$. Hence $\dfrac{\mathrm{Ker}\,\delta^r_p}{\mathrm{Im}\,\delta^r_{p+r}} = 0$.

   On the other hand, since $\Delta^r_{p-r+1,p+1}$ is a primary pivot then $\sigma^{p+1,r+1}_k = \sigma^{p+1,r}_k \notin Z^{r+1}_p$. Thus $Z^{r+1}_p = Z^r_{p-1}$ and $E^{r+1}_p = 0$.

2. Suppose $\Delta^r_{p-r+1,p+1} = 0$ with a column of zeroes below it. In this case $\mathrm{Ker}\,\delta^r_p = E^r_p$ and $\sigma^{p+1,r}_k = \sigma^{p+1,r+1}_k$.

   (a) If $\Delta^r_{p+1,p+r+1}$ is an entry above a primary pivot then we have $E^r_{p+r} = 0$ and hence $\mathrm{Im}\,\delta^r_{p+r} = 0$. Thus,

$$\frac{\mathrm{Ker}\,\delta^r_p}{\mathrm{Im}\,\delta^r_{p+r}} = E^r_p.$$

   On the other hand, since $\mu^{p+r+1,r} = 0$ then $E^{r+1}_p = E^r_p$.

   (b) If $\Delta^r_{p+1,p+r+1} = 0$ with a column of zero entries below it then $\mathrm{Im}\,\delta^r_{p+r} = 0$ and

$$\frac{\mathrm{Ker}\,\delta^r_p}{\mathrm{Im}\,\delta^r_{p+r}} = E^r_p.$$

   On the other hand, it follows from Lemma 3.5 that $E^{r+1}_p = E^r_p$.

   (c) If $\Delta^r_{p+1,p+r+1}$ is a primary pivot then $E^r_p = \mathbb{F}[\sigma^{p+1,r}_k]$ and $E^r_{p+r} = \mathbb{F}[\sigma^{p+r+1,r}_k]$.

$$\cdots \longleftarrow E^r_{p-r} \xleftarrow{\delta^r_p} \mathbb{F}[\sigma^{p+1,r}_k] \xleftarrow{\delta^r_{p+r}} \mathbb{F}[\sigma^{p+r+1,r}_{k+1}] \longleftarrow \cdots \tag{9}$$

Therefore,

$$\frac{\operatorname{Ker}\delta_p^r}{\operatorname{Im}\delta_{p+r}^r} = \frac{\mathbb{F}[\sigma_k^{p+1,r}]}{\mathbb{F}[\sigma_k^{p+1,r}]} = 0.$$

On the other hand, since $\Delta_{p+1,p+r+1}^r$ is a primary pivot, by Lemma 3.5 $E_{p,k-p}^{r+1} = 0$.

We have seen that in all cases

$$\frac{\operatorname{Ker} d_p^r}{\operatorname{Im} d_{p+r}^r} = E_{p,k-p}^{r+1} = \frac{\operatorname{Ker}\delta_p^r}{\operatorname{Im}\delta_{p+r}^r}.$$

■

These results are also true for connection matrices over $\mathbb{Z}$, see [CdRS], however the proofs are intrinsically more difficult due to the presence of torsion in the homology.

4 Properties of the sweeping algorithm over $\mathbb{Z}$

Computational experiments with the sweeping algorithm over $\mathbb{Z}$, significantly expanded with the aid of its computer implementation and of the random generator of matrices Δ, described in Section 5.4, led to the conjecture that the final matrix produced by the sweeping algorithm over $\mathbb{Z}$ is integral. The very nature of the sweeping algorithm made it reasonable to expect the occurrence of fractional numbers in the Δ^r matrices, which was quickly confirmed. On the other hand, it was somewhat puzzling to witness the consistent disappearance of these fractional numbers at the end of the algorithm. Beyond its immediate relevance, there is the fact that this result opens up the door to interpretations of the final matrix, and instigates investigation into the information it might carry. In this section, we conduct an algebraic study of the sequence of matrices produced by the algorithm, establishing properties that eventually lead to the desired result.

The algorithm included in Section 2 is in an equivalent, albeit slightly different, format from the one described in [CdRS]. The difference lies in the matrix P update step. We begin by establishing that the algorithm is well defined. This is a consequence of the following proposition, analogous to the one presented for the sweeping algorithm over $\mathbb{F}$.

Proposition 4.1 *Let $\{\Delta^0, \Delta^1, \ldots\}$ and $\{P^0, P^1, \ldots\}$ be the sequence of connection and change-of-basis matrices, respectively, produced by the sweeping algorithm over $\mathbb{Z}$. Then the following are true regarding these matrices:*

(i) All matrices $\{\Delta^0, \Delta^1, \ldots\}$ inherit the block structure of Δ^0.

(ii) The non-null entries of $\Delta^0 P^{r-1}$ strictly below the r-th diagonal are located on or above a primary pivot position.

(iii) The non-null entries of Δ^r strictly below the r-th diagonal are either primary pivots (always non-null) or are above a primary pivot.

(iv) The change-of-basis matrices P^r have the following block upper triangular structure: it is block diagonal, block k has entries with indices in $J_k \times J_k$, for $k = 1, \ldots, b-1$, and each block is upper triangular and invertible.

Proof: The matrices Δ^0 and $P^0 = I$ trivially satisfy the proposition. The proof is by induction.

Suppose by induction that (i)–(iv) are true for Δ^r and P^{r-1}. Given a change-of-basis pivot $\Delta_{k\; i,i+r}^r = \Delta_{k\; ij}^r$ with $j \in J_s$ such that $g_s = k$, let $A = \Delta_{IJ}^0$, where $I = \{i, \ldots, \ell_{k-1}\}$ and $J = \{f_k, \ldots, j\}$, see illustration in Figure 6. Let $c = |J|$. Consider the corresponding entries in Δ^r:

$$\Delta_{IJ}^r = (P^{r-1})_{I\bullet}^{-1} \Delta^0 P_{\bullet J}^{r-1}.$$

Given that the inverse of an upper triangular matrix is also upper triangular, the inverse of P^{r-1} shares its block upper triangular structure. Thus, we may rewrite the previous equality as

$$\Delta^r_{IJ} = (P^{r-1})^{-1}_{II} \Delta^0_{IJ} P^{r-1}_{JJ}.$$

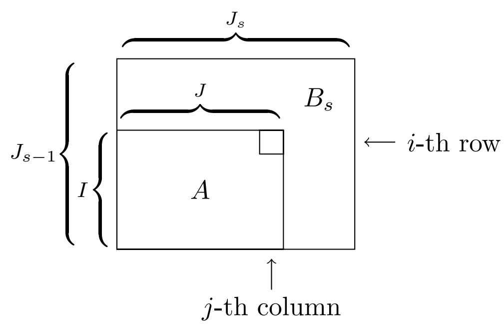


Figure 6: Submatrix A determined by a change-of-basis pivot in position (i, j).

By the markup rules, there is $p \in J_s$ such that Δ^r_{ip} is a primary pivot. Then, by the induction hypothesis, $\Delta^r_{i'p} = 0$, for $i' > i$. Thus, the $c \times 1$ column vector y given by

$$y_{j'} = \begin{cases} \Delta^r_{ip}, & \text{if } j' + \ell_{k-1} = j, \\ -\Delta^r_{ij}, & \text{if } j' + \ell_{k-1} = p, \\ 0, & \text{otherwise,} \end{cases}$$

is a rational solution to the linear system

$$\Delta^r_{IJ}\, y = 0,$$

since P^{r-1} and Δ^0 are integral. This implies that the rational vector $\bar{x} = P^{r-1}_{JJ}\, y$ satisfies

$$A\bar{x} = 0.$$

Furthermore, since P^{r-1} is block upper triangular and invertible, $\bar{x}_c = P^{r-1}_{jj} y_c \neq 0$. Therefore, an appropriate multiple of $\bar{x}$ solves the integer linear program of the change-of-basis update step:

$$\begin{array}{lrcl} \min & x_c & & \\ \text{s.t.} & Ax & = & 0 \\ & x_c & \geq & 1 \\ & x & \in & \mathbb{Z}^c. \end{array}$$

The above integer program is thus feasible, bounded by construction (since $x_c \geq 1$) and the data defining it is integral. It is a well known consequence, see [NW], that it has an optimal solution x^*. Hence the matrix P update step is well defined. Notice that $x^*_c = u$, the leading coefficient associated with the change-of-basis pivot Δ^r_{ij}.

The components of the optimal solution x^* will replace entries of P^{r-1} in positions $(f_k, j), \ldots, (j, j)$. Thus the change-of-basis matrix P^r is integral, has the same block upper triangular structure, and is invertible, since the diagonal entries that may have changed (one for each change-of-basis pivot) were replaced with positive (since $x^*_c \geq 1$) entries. Condition (iv) is verified for P^r. The matrix update rule implies that condition (i) is satisfied.

By construction of P^r, only columns containing change-of-basis pivots detected on the r-th diagonal change from $\Delta^0 P^{r-1}$ to $\Delta^0 P^r$. In particular, the construction of P^r implies that entries in positions (i, j), $\ldots$, (ℓ_{k-1}, j) are zeroed out, for $j \in J_k$ such that $\Delta^r_{i,i+r} = \Delta^r_{ij}$ is a change-of-basis pivot. Thus, columns containing primary pivots, either on or below the r-th diagonal, do not change. Hence, by induction and the markup rules, $\Delta^0 P^r$ satisfies (ii).

Matrix Δ^{r+1} is obtained pre-multiplying $\Delta^0 P^r$ by $(P^r)^{-1}$. The fact that $(P^r)^{-1}$ has the same structure as P^r implies that, in the pre-multiplication, a row $i \in J_s$ of $\Delta^0 P^r$ will be replaced by a linear combination of rows i, $i+1$, $\ldots$, ℓ_k, with the coefficient of row i being non-null. Thus, since we have already established that $\Delta^0 P^r$ satisfies (ii), it follows that $(P^r)^{-1}\Delta^0 P^r$ satisfies (iii). ■

The following corollary is a consequence of Proposition 4.1 and the sweeping algorithm over $\mathbb{Z}$. The markup rules imply the uniqueness of the primary pivot below a nonzero entry, since each column may have at most one primary pivot. The relative positions of the rectangular blocks of Δ^R, the last matrix produced by the sweeping algorithm over $\mathbb{Z}$, and the triangular blocks of P^{R-1} are illustrated in Figure 7. Notice that, since the columns j, for $j \in J_1$, are null, block T_1^r is an identity matrix of order J_1, for all r.

Corollary 4.2 *Let Δ^R be the last matrix produced by the sweeping algorithm over $\mathbb{Z}$. Then*

(i) The primary pivot entries are non-null and each non-null entry is located above a unique primary pivot.

(ii) The s-th block of Δ^R is given by

$$B_s^R = (T_{s-1}^{R-1})^{-1} B_s T_s^{R-1}, \quad \text{for } 1 \le k \le b, \tag{10}$$

where $B_s = \Delta_{J_{s-1}J_s}$.

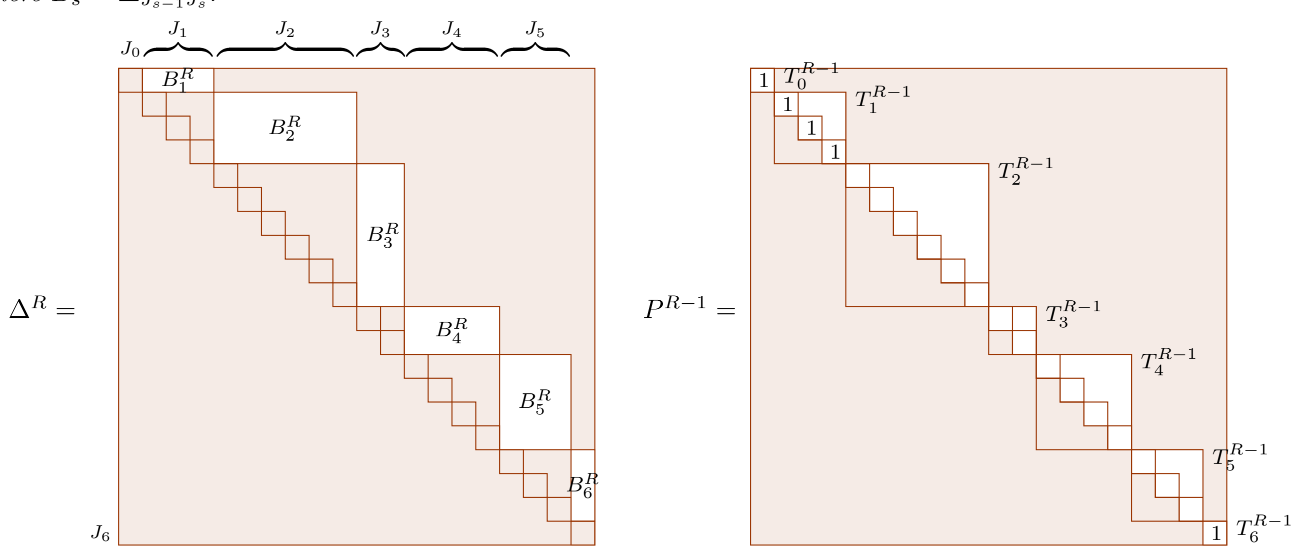


Figure 7: Block structures of Δ^R and P^{R-1}.

The sweeping algorithm leads to a complementary relation between a column $j \in J_s$ of Δ^R containing a primary pivot and its j-th row, established in the following proposition and illustrated in Figure 8. Notice that this proposition is valid for both versions of the algorithm, over $\mathbb{F}$ and over $\mathbb{Z}$, since it follows from the zero pattern established in Proposition 2.1 for $\mathbb{F}$ and in Proposition 4.1 for $\mathbb{Z}$.

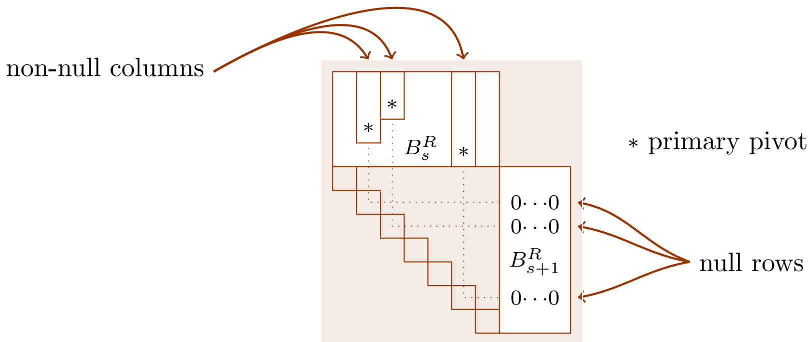


Figure 8: Complementarity relation between columns of B_s^R and rows of B_{s+1}^R.

Proposition 4.3 *Let Δ^R be the last matrix produced by the sweeping algorithm over $\mathbb{Z}$. If the j-th column of Δ^R is non-null, then its j-th row is null, or, equivalently,*

$$\Delta^R_{\bullet j}\,\Delta^R_{j\bullet} = 0, \quad \textit{for all } j. \tag{11}$$

Proof: Equation (11) is trivial when $\Delta^R_{\bullet j}$ is a zero column. So suppose $\Delta^R_{\bullet j} \neq 0$. By the inherited block structure established in Proposition 4.1, there exists s such that $j \in J_s$. By Proposition 4.1, the non-null columns of Δ^R are precisely the columns containing primary pivots. Label the primary pivots of block s in increasing order of row index: if $\Delta^R_{i_1 j_1}, \ldots, \Delta^R_{i_a j_a}$ are the primary pivots in block s, then $i_1 < i_2 < \cdots < i_a$. Thus, $j_1, j_2, \ldots, j_a$ are the non-null columns of the s-th block of Δ^R. Furthermore, $\Delta^R_{i_a j_a}$ is the unique nonzero entry of row $\Delta^R_{i_a\bullet}$, row i_{a-1} has a nonzero entry in column j_{a-1} and may have another one in column j_a, and so on.

The fact that Δ^R is nilpotent implies that

$$0 = \Delta^R_{i_a\bullet}\Delta^R_{\bullet j'} = \Delta^R_{i_a j_a}\Delta^R_{j_a j'},$$

for fixed arbitrary j'. Using the fact that the primary pivot entry is non-null, we conclude that $\Delta^R_{j_a j'} = 0$, for all j'. Repeating the argument for i_{a-1} and using the fact that the j_a-th row of Δ^R is null, we establish that its j_{a-1}-th row is null. The nullity of rows $j_{a-2}, \ldots, j_1$ of Δ^R follow analogously. ∎

In the proof of the main result, we need the following modified version of Proposition 4.3.

Corollary 4.4 *Let Δ^R be the last matrix produced by the sweeping algorithm over $\mathbb{Z}$. Let $k = g_s$. Then*

$$B_s(T_s^{R-1})_{\bullet j}((T_s^{R-1})^{-1}B_{s+1})_{j\bullet} = 0, \tag{12}$$

for $j + \ell_{k-1} \in J_s$, and $s \in \{1, \ldots, b\}$.

Proof: Let $\bar{\jmath} = j + \ell_{k-1} \in J_s$, for $s \in \{1, \ldots, b\}$. Then, equations (10) and (11) imply

$$\begin{aligned} 0 &= \Delta^R_{J_{s-1}\bar{\jmath}}\,\Delta^R_{\bar{\jmath}J_{s+1}} \\ &= ((T_{s-1}^{R-1})^{-1}B_sT_s^{R-1})_{\bullet j}\;((T_s^{R-1})^{-1}B_{s+1}T_{s+1}^{R-1})_{j\bullet} \\ &= (T_{s-1}^{R-1})^{-1}B_s(T_s^{R-1})_{\bullet j}\;((T_s^{R-1})^{-1})_{j\bullet}B_{s+1}T_{s+1}^{R-1}. \end{aligned} \tag{13}$$

Pre-multiplying (13) by T_{s-1}^{R-1} and post-multiplying by $(T_{s+1}^{R-1})^{-1}$ gives (12). ∎

We may now prove the integrality result.

Theorem 4.5 *The last matrix Δ^R produced by the sweeping algorithm over $\mathbb{Z}$ is integral.*

Proof: By Corollary 4.2, it suffices to show that the rectangular blocks B_s^R are integral. By the integrality of P^{R-1} and (10), it is sufficient to show that $(T_s^{R-1})^{-1}v = \alpha$ is integral, where v is a fixed arbitrary column of B_{s+1}, for $s \in \{0, \dots, b-1\}$. Or, equivalently, we need to show that the unique solution of the linear system

$$T_s^{R-1}\alpha = v \tag{14}$$

is integral. Notice that we need only consider $s \in \{1, \dots, b-1\}$, since $T_0^r = I$, for all r.

Now T_s^{R-1} is upper triangular, which means that system (14) is ready to be solved by back substitution. This is the basic tool behind the constructive proof that follows. The values of the components of α are calculated in reverse order, from the last to the first one. As soon as a component is calculated, its value is inserted in the system, producing a system with one less variable. Of course we need induction to ascertain that this iterative procedure will work with arbitrarily sized systems.

Let $q = |J_s|$, the order of T_s^{R-1}. Let $p = |J_{s-1}|$, so block B_s is $p \times q$. Notice that row and column indices of a block are shifted with respect to the indices in the whole matrix. If, for instance, $g_s = k$, then $(B_s)_{ij} = \Delta_{i+\ell_{k-2}, j+\ell_{k-1}}$. Integrality of α_q, the last component of α, will result from the rules for the construction of the change-of-basis matrix and the fact that $B_s B_{s+1} = 0$, a consequence of the nilpotency of Δ. Replacing the value obtained for α_q in the system (14) amounts to removing the last column of the coefficient matrix and updating the right-hand-side. In order to proceed we need to show how to "update" the other matrices involved in the integrality argument.

We introduce some notation to simplify the induction argument. Let $A^1 = B_s = \Delta_{J_{s-1}J_s}$, $v^1 = v = (B_{s+1})_{\bullet a}$ be a fixed arbitrary column of block $s+1$, and $U^1 = T_s^{R-1}$. In the new notation, system (14) becomes

$$U^1\alpha = v^1. \tag{15}$$

Instead of working with a sequence of shrinking systems, we construe the back substitution procedure as generating a sequence of equally sized linear systems equivalent to (15), but increasingly easy to solve. As the i-th system $U^i\alpha = v^i$ is considered, the last $i-1$ components of α have already been computed. The relevant matrices have the following characteristics:

(a) The first $q-i+1$ columns of U^i coincide with the first $q-i+1$ columns of U^1, while the last $i-1$ columns of U^i are equal to the last $i-1$ columns of a $q \times q$ identity matrix. Notice that this makes U^i upper triangular and invertible.

(b) The last $i-1$ columns of matrix A^i, the update of matrix $A^1 = B_k$, coincide with the last $i-1$ columns of matrix $B_s T_s^{R-1}$, while the first $q-i+1$ columns of A^i retain the original values.

(c) The last $i-1$ components of v^i contain the (integral) values already obtained for the last $i-1$ components of α, that is $(T_s^{R-1})^{-1}_{q\bullet}v, \dots, (T_s^{R-1})^{-1}_{q-i+2,\bullet}v$. The first $q-i+1$ components of v^i have been updated to take into account the fact that the variables $\alpha_q, \dots, \alpha_{q-i+2}$ have been eliminated from the first $q-i+1$ equations. They are integral, as well.

(d) $A^i v^i = A^i_{\bullet,\{1,\dots,q-i+1\}} v^i_{\{1,\dots,q-i+1\}} = 0$.

At the end of the back substitution, $U^{q+1} = I$, $A^{q+1} = B_s T_s^{R-1}$ and $\alpha = v^{q+1} = (T_s^{R-1})^{-1}v$. Matrix equations will be presented that produce sequences of matrices with the prescribed characteristics.

We start the induction proof by calculating α_q. Since U^1 is upper triangular, we have

$$\alpha_q = \frac{v_q^1}{U_{qq}^1}. \tag{16}$$

There are two possibilities for the last column of the k-th block: at some point of the sweeping algorihtm it contained a change-of-basis pivot or not. In the latter case, this last column was not altered during the sweeping algorithm and, since the change-of-basis matrix is initialized with the identity matrix, $U^1_{\bullet q} = e_q$, so that $\alpha_q = v^1_q/1$ and thus clearly integral, since v, a fixed arbitrary column of B_{s+1}, is integral.

Now suppose the last column of the s-th block contained at least one change-of-basis pivot during the sweeping algorithm. By the rules for the change-of-basis matrix update, each time a change-of-basis pivot is marked in a column of a block, say block s, the corresponding column in the triangular block of the change-of-basis matrix, located just below block s, is updated. The old column is superseded by the vector obtained in the solution of the minimization problem constructed in the update step. Thus only the last change-of-basis pivot occuring in that column need be considered. Letting $g_s = k$, the position of this last change-of-basis pivot in the connection matrix is $(\tilde{\imath}, \ell_k)$. The corresponding position in A^1 is $(i, q) = (\tilde{\imath} - \ell_{k-2}, \ell_k - \ell_{k-1})$. Then $x = U^1_{\bullet q}$ is the optimal solution to the optimization problem

$$\begin{array}{lrcl} \min & x_q & & \\ \text{subject to} & A^1_{I\bullet}x & = & 0 \\ & x_q & \geq & 1 \\ & x & \in & \mathbb{Z}^q, \end{array} \tag{17}$$

where $I = \{i, \ldots, p\}$. If U^1_{qq} is a divisor of v^1_q, then (16) implies α_q is integral. Suppose not. The nilpotency of Δ implies that v^1 belongs to the null space of A^1, which means $\pm v^1$ are integral solutions to the system $A^1_{I\bullet}x = 0$. Therefore, if U^1_{qq} is not a divisor of v^1_q, the optimality of $U^1_{\bullet q}$ implies $U^1_{qq} < |v^1_q|$. Consider the following linear combination, with integral coefficients, of v^1 and $U^1_{\bullet q}$:

$$w = \operatorname{sgn}(v^1_q)\, v^1 - \left\lfloor \frac{|v^1_q|}{U^1_{qq}} \right\rfloor U^1_{\bullet q}.$$

By construction, w satisfies the constraints of (17), and $w_q < U^1_{qq}$, contradicting the optimality of $U^1_{\bullet q}$. Thus U^1_{qq} is indeed a divisor of v^1_q and α_q is integral.

Now let

$$D^1 = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & U^1_{\bullet q} \\ & & 1 & \\ \hline 0 & \cdots & 0 & \end{array}\right).$$

The fact that U^1 is upper triangular and invertible implies that D^1 satisfies these same properties. Letting $U^2 = (D^1)^{-1}U^1$, a straightforward calculation gives

$$U^2 = (D^1)^{-1}U^1 = \left(\begin{array}{c|c|c|c} U^1_{\bullet 1} & \cdots & U^1_{\bullet, q-1} & \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array} \end{array}\right),$$

so U^2 satisfies (a). Letting

$$A^2 = A^1 D^1,$$

we see that A^2 satisfies (b). Pre-multiplying (15) by $(D^1)^{-1}$, we obtain the equivalent system

$$U^2\alpha = (D^1)^{-1}U^1\alpha = (D^1)^{-1}v^1 = v^2. \tag{18}$$

Notice that, by construction, v^2 belongs to the null space of A^2, since

$$A^2v^2 = A^1D^1(D^1)^{-1}v^1 = 0. \tag{19}$$

Since $U^2_{\bullet q} = e_q$, we conclude that

$$v^2_q = \frac{v^1_q}{U^1_{qq}} \quad \text{and} \quad v^2_j = v^1_j - v^2_q\, U^1_{jq}, \text{ for } j = 1, \dots, q-1,$$

which implies that v^2 is integral and satisfies (c).

Finally, (12) implies

$$A^2_{\bullet q}v^2_q = B_s(T^{R-1}_s)_{\bullet q}\,((T^{R-1}_s)^{-1}B_{s+1})_{qa} = 0. \tag{20}$$

Therefore, using (19) and (20), we have

$$0 = A^2v^2 = A^2_{\bullet,\{1,\dots,q-1\}}v^2_{\{1,\dots,q-1\}} + A^2_{\bullet q}v^2_q = A^2_{\bullet,\{1,\dots,q-1\}}v^2_{\{1,\dots,q-1\}},$$

which means (d) is also satisfied.

Now assume by induction that the linear system $U^i\alpha = v^i$ is equivalent to (15), that A^i, v^i and U^i satisfy (a), (b), (c) and (d).

Again there are two possibilities for column $\ell_k - i + 1$ of the s-th block: it contained a change-of-basis pivot or not. The latter is the simplest case, as before, since then $U^i_{\bullet,q-i+1} = U^1_{\bullet,q-i+1} = e_{q-i+1}$ and $\alpha_{q-i+1} = v^i_{q-i+1}/U^i_{q-i+1,q-i+1} = v^i_{q-i+1}/U^1_{q-i+1,q-i+1} = v^i_{q-i+1}/1 \in \mathbb{Z}$, by the induction hypotheses. On the other hand, assuming there was a change-of-basis pivot in column $\ell_{k-1} - i + 1$, the column vector $x = U^i_{\{1,\dots,q-i+1\},q-i+1} = U^1_{\{1,\dots,q-i+1\},q-i+1}$ is the optimal solution to

$$\begin{array}{rrcl} \min & x_{q-i+1} & & \\ \text{s.t.} & A^i_{I,\{1,\dots,q-i+1\}}x & = & 0 \\ & x_{q-i+1} & \geq & 1 \\ & x & \in & \mathbb{Z}^{q-i+1}, \end{array} \tag{21}$$

where I is some nonempty subset of the rows of A^i. Then, if $U^i_{q-i+1,q-i+1}$ is not a divisor of v^i_{q-i+1}, optimality of $U^i_{\{1,\dots,q-i+1\},q-i+1}$ implies $U^i_{q-i+1,q-i+1} < |v^i_{q-i+1}|$, since $v^i_{\{1,\dots,q-i+1\}}$ is feasible for (21).

But then, the linear combination with integral coefficients

$$w = \operatorname{sgn}(v^i_{q-i+1})\, v^i_{\{1,\dots,q-i+1\}} - \left\lfloor \frac{|v^i_{q-i+1}|}{U^i_{q-i+1,q-i+1}} \right\rfloor U^i_{\{1,\dots,q-i+1\},q-i+1}$$

is also feasible for (21) and satisfies $w_{q-i+1} < U^i_{q-i+1,q-i+1}$, contradicting optimality of the latter.

Therefore $U^i_{q-i+1,q-i+1}$ must divide v^i_{q-i+1} and

$$\alpha_{q-i+1} = \frac{v^i_{q-i+1}}{U^i_{q-i+1,q-i+1}} \in \mathbb{Z}.$$

It remains to show how to update the matrices and vector such that the new set satisfies (a) through (d). Let

$$D^i = \left(\begin{array}{ccc|c|ccc} 1 & & & & 0 & \cdots & 0 \\ & \ddots & & & & \ddots & \\ & & 1 & U^i_{\bullet,q-i+1} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & & 1 & & \\ & \ddots & & (q-i+1)\text{-th} & & \ddots & \\ 0 & \cdots & 0 & \text{column} & & & 1 \end{array}\right), \qquad U^{i+1} = (D^i)^{-1}U^i, \qquad A^{i+1} = A^iD^i,$$

and

$$v^{i+1} = (D^i)^{-1}v^i.$$

Induction hypotheses and direct computation show that U^{i+1}, A^{i+1} and v^{i+1} satisfy (a) through (d). Therefore, these properties also hold for U^{q+1}, A^{q+1} and v^{q+1}, which implies the desired integrality result. ■

5 Comments on the Implementation of the Sweeping Algorithm

5.1 Sweeping Algorithm over $\mathbb{Z}$

The computer implementations of the sweeping algorithms over $\mathbb{Z}$ and $\mathbb{Z}_2$ are available for download at [Me]. Also available is a set of numerical experiments with the algorithms. The algorithms were implemented in *Mathematica*® 6.0.0.

The implementation of the sweeping algorithm over $\mathbb{Z}$ offers the choice of two methods for the matrix P update step. The first method invokes *Mathematica*'s built-in `Minimize` command to solve the integer linear program constructed in the update step. In the second method, the solution of this optimization program is coded from scratch, using the concept of the integer normal form of a matrix. This choice is indicated via the optional Boolean parameter `Nf`. The other optional parameter is `Erp`. It should be set to `True` if one wants the computation of the sequence of differentials (E^r, d^r). Thus, for instance, after defining a matrix `A` and a column partition `partition`, the command

```
SpectralSequenceConstruction[A, partition]
```

calculates the sequence $\{\Delta^0 = \mathtt{A}, \Delta^1, \ldots,\}$ using the built-in *Mathematica* command to solve the minimization in the matrix P update step. On the other hand,

```
SpectralSequenceConstruction[A, partition, Nf -> True]
```

indicates the choice of the second method for the matrix P update step. To obtain the computation of the sequence (E^r, d^r) one should use

```
SpectralSequenceConstruction[A, partition, Erp -> True]
```

or

```
SpectralSequenceConstruction[A, partition, Nf -> True, Erp -> True].
```

The definition of the input matrix can be done entry by entry, but since it usually has a great number of null entries, it is easier to construct a null matrix of the desired order and then assign the non-null entries separetely. The matrix shown in Figure 9 (a), for instance, was built using the commands

```
A = Table[0, {10}, {10}];
A[[2,6]] = 7;
A[[2,7]] = -5;
A[[3,4]] = 5;
A[[3,5]] = 3;
A[[3,6]] = 2;
A[[4,8]] = 1;
A[[4,9]] = -3;
A[[5,8]] = -5;
A[[5,9]] = 5;
A[[6,8]] = 5;
A[[7,8]] = 7;
```

Its column partition is defined with

```
partition = {1, 0, 2, 4, 2, 0, 1};
```

meaning there is one critical point of index 0, none of index 1, two of index 2, four of index 3, 2 of index 4, none of index 5 and one of index 6.

The execution of the function `SpectralSequenceConstruction` causes the printout of the sequence of Δ's, as well as the expressions of the k-chains σ_k's associated with the columns thereof, as illustrated in Figures 9 (a) and (b). Primary pivot entries are indicated by means of a yellow background and darker edge. The change-of-basis pivots have blue background and dashed edges. Null entries are left blank. Only blocks with non-null entries are highlighted. The diagonal being swept is indicated with a thin gray line.

$$
\Delta^0 = \begin{array}{c|cccccccccc|}
 & h_0^1 & h_2^2 & h_2^3 & h_3^4 & h_3^5 & h_3^6 & h_3^7 & h_4^8 & h_4^9 & h_6^{10} \\
\hline
h_0^1 & & & & & & & & & & \\
h_2^2 & & & & 0 & 0 & 7 & -5 & & & \\
h_2^3 & & & & 5 & 3 & 2 & 0 & & & \\
h_3^4 & & & & & & & & 1 & -3 & \\
h_3^5 & & & & & & & & -5 & 5 & \\
h_3^6 & & & & & & & & 5 & 0 & \\
h_3^7 & & & & & & & & 7 & 0 & \\
h_4^8 & & & & & & & & & & \\
h_4^9 & & & & & & & & & & \\
h_6^{10} & & & & & & & & & & \\
\hline
\end{array}
$$

(a) Initial matrix

$$
\Delta^3 = \begin{array}{c|cccccccccc|}
 & \sigma_0^{1,3} & \sigma_2^{2,3} & \sigma_2^{3,3} & \sigma_3^{4,3} & \sigma_3^{5,3} & \sigma_3^{6,3} & \sigma_3^{7,3} & \sigma_4^{8,3} & \sigma_4^{9,3} & \sigma_6^{10,3} \\
\hline
\sigma_0^{1,3} = h_0^1 & & & & & & & & & & \\
\sigma_2^{2,3} = h_2^2 & & & & 0 & 0 & 7 & -5 & & & \\
\sigma_2^{3,3} = h_2^3 & & & & \boxed{5} & 0 & 2 & 0 & & & \\
\sigma_3^{4,3} = h_3^4 & & & & & & & & -2 & 0 & \\
\sigma_3^{5,3} = 5\,h_3^5 - 3\,h_3^4 & & & & & & & & -1 & 1 & \\
\sigma_3^{6,3} = h_3^6 & & & & & & & & 5 & 0 & \\
\sigma_3^{7,3} = h_3^7 & & & & & & & & \boxed{7} & 0 & \\
\sigma_4^{8,3} = h_4^8 & & & & & & & & & & \\
\sigma_4^{9,3} = h_4^9 & & & & & & & & & & \\
\sigma_6^{10,3} = h_6^{10} & & & & & & & & & & \\
\hline
\end{array}
$$

(b) Matrix after 3-rd diagonal is swept

Figure 9: Sample output of the routine `SpectralSequenceConstruction`.

The function `SpectralSequenceConstruction` returns the number of diagonals that are swept. Hence the command

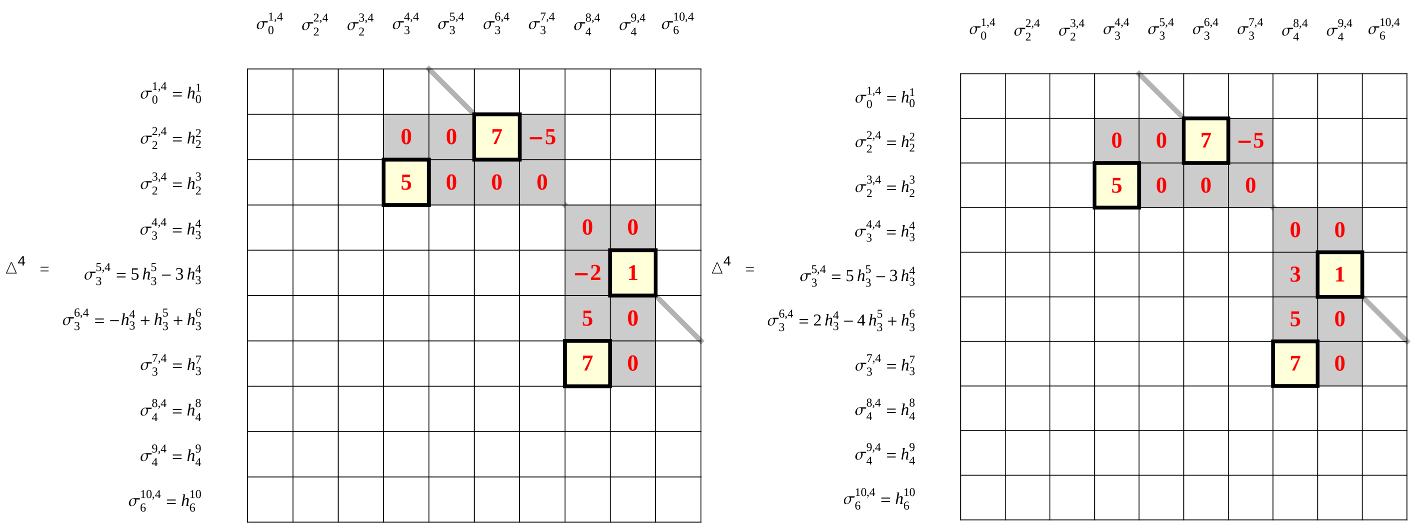
```
n = SpectralSequenceConstruction[A, partition, Nf -> True]
```

executes the function `SpectralSequenceConstruction` assigns to `n` the number of diagonals. This is useful if one wishes to examine the several change-of-basis matrices constructed during the algorithm, which can be done with the command

```
Table[changeofbasismatrix[i] // MatrixForm, i, 1, n]
```

since the function `SpectralSequenceConstruction` assigns to `changeofbasismatrix[i]` the value of P^i.

The choice of methods for the matrix P update step highlights the fact that, although the optimum value of the minimization problem in this step is unique, the optimal solution is not. Figure 10 shows the matrices Δ^4 obtained from Δ^3 shown in Figure 9 (b), using both choices. Notice that using *Mathematica*'s built-in `Minimize` function $\sigma_2^{6,4} = -h_2^4 + h_2^5 + h_2^6$, whereas using the routine based on the normal form $\sigma_2^{6,4} = 2h_2^4 - 4h_2^5 + h_2^6$. Although the entries in Δ_2^4 are the same in both cases, notice that entries in Δ_3^4 are not.



(a) Δ^4 obtained using built-in `Minimize` (b) Δ^4 obtained using code based on normal form

Figure 10: Continuation of instance illustrated in Figure 9.

The computation of the differential (E^r, d^r) involves a similar optimization problem. Proposition 4.1 and Theorem 4.4 of [CdRS] imply that

$$E^r_{p,k-p} = \frac{Z^r_{p,k-p}}{Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}}, \tag{22}$$

where

$$Z^r_{p,k-p} = \mathbb{Z}[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \ldots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}]$$

and the remaining terms have analogous expressions.

As the sweeping algorithm proceeds, we may express the various σ's as integer linear combinations of the elementary h's. Figures 11 and 12 shows a typical situation involving the computation of E_5^4 and E_6^4. By construction of (22), the matrix containing the coefficients of the generators in $Z^r_{p,k-p}$ has one column more than the matrix containing the coefficients of the generators of $Z^{r-1}_{p-1,k-(p-1)}$. Let A be the matrix containing the coefficients of the generators in $Z^{r-1}_{p-1,k-(p-1)}$ followed by the coefficients of the generators in $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$. Let v be the vector containing the coefficients in the expression of

$\mu^{p+1,r}\sigma_k^{p+1,r}$ as a linear combination of h's. Consider the minimization problem

$$\begin{aligned} \min \quad & \alpha \\ \text{subject to} \quad & Ax = \alpha v \\ & \alpha \geq 1 \\ & x_i, \alpha \in \mathbb{Z}, \quad \forall\, i. \end{aligned} \tag{23}$$

The the quotient space is: (i) $\mathbb{Z}[\mu^{p+1,r}\sigma_k^{p+1,r}]$ if problem (23) is infeasible, that is, it is not possible to express a positive integer multiple of v as an integer linear combination of the columns of A, (ii) $\{0\}$ if the optimum value of (23) is 1, and (iii) $\mathbb{Z}_{\alpha^*}[\mu^{p+1,r}\sigma_k^{p+1,r}]$ if the optimal value of (23) is $\alpha^* > 1$. In the example shown in Figure 11, it is easy to conclude that $E_5^4 = \mathbb{Z}[\mu^{6,4}\sigma_3^{6,4}] = \mathbb{Z}[-h_3^4 + h_3^5 + h_3^6]$ and $E_6^4 = \mathbb{Z}_7[h_3^7]$, since $7\mu^{7,4}\sigma_3^{7,4} = \partial\mu^{8,1}\sigma_4^{8,1} + \partial\mu^{9,2}\sigma_4^{9,2} + 2\mu^{4,1}\sigma_3^{4,1} - 5\mu^{6,3}\sigma_3^{6,3}$, and this is clearly the smallest possible multiple of $\mu^{7,4}\sigma_3^{7,4}$ that can be expressed as an integer linear combination of the other generators in the denominator.

E_5^4

Numerator:

	$\mu^{4,2}\ \sigma_3^{4,2}$	$\mu^{5,3}\ \sigma_3^{5,3}$	$\mu^{6,4}\ \sigma_3^{6,4}$
h_3^4	0	-3	-1
h_3^5	0	5	1
h_3^6	0	0	1
h_3^7	0	0	0

Denominator:

1st term of denominator:

	$\mu^{4,2}\ \sigma_3^{4,2}$	$\mu^{5,3}\ \sigma_3^{5,3}$
h_3^4	0	-3
h_3^5	0	5
h_3^6	0	0
h_3^7	0	0

2nd term of denominador:

	$\partial\mu^{8,2}\ \sigma_4^{8,2}$	$\partial\mu^{9,3}\ \sigma_4^{9,3}$
h_3^4	0	-3
h_3^5	0	5
h_3^6	0	0
h_3^7	0	0

Figure 11: Matrices involved in the computation of E_5^4, for Δ shown in Figure 9.

E_6^4

Numerator:

	$\mu^{4,1}\ \sigma_3^{4,1}$	$\mu^{5,2}\ \sigma_3^{5,2}$	$\mu^{6,3}\ \sigma_3^{6,3}$	$\mu^{7,4}\ \sigma_3^{7,4}$
h_3^4	1	0	0	0
h_3^5	0	1	0	0
h_3^6	0	0	1	0
h_3^7	0	0	0	1

Denominator:

1st term of denominator:

	$\mu^{4,1}\ \sigma_3^{4,1}$	$\mu^{5,2}\ \sigma_3^{5,2}$	$\mu^{6,3}\ \sigma_3^{6,3}$
h_3^4	1	0	0
h_3^5	0	1	0
h_3^6	0	0	1
h_3^7	0	0	0

2nd term of denominador:

	$\partial\mu^{8,1}\ \sigma_4^{8,1}$	$\partial\mu^{9,2}\ \sigma_4^{9,2}$
h_3^4	1	-3
h_3^5	-5	5
h_3^6	5	0
h_3^7	7	0

Figure 12: Matrices involved in the computation of E_6^4, for Δ shown in Figure 9.

5.2 Sweeping Algorithm over $\mathbb{Z}_2$

The implementation of the sweeping over $\mathbb{Z}_2$ is considerably simpler. The pertinent command in this case is

`BinarySpectralSequenceConstruction[A, partition]`

where `A` and `partition` have the same meaning as before. Examples may be easily constructed by noting that, if A is a integer nilpotent matrix, then $A \mod 2$ is a (binary) nilpotent matrix with respect to the field $\mathbb{Z}_2$. Figure 13 illustrates the program's output in this case.

5.3 Numerical Experiments

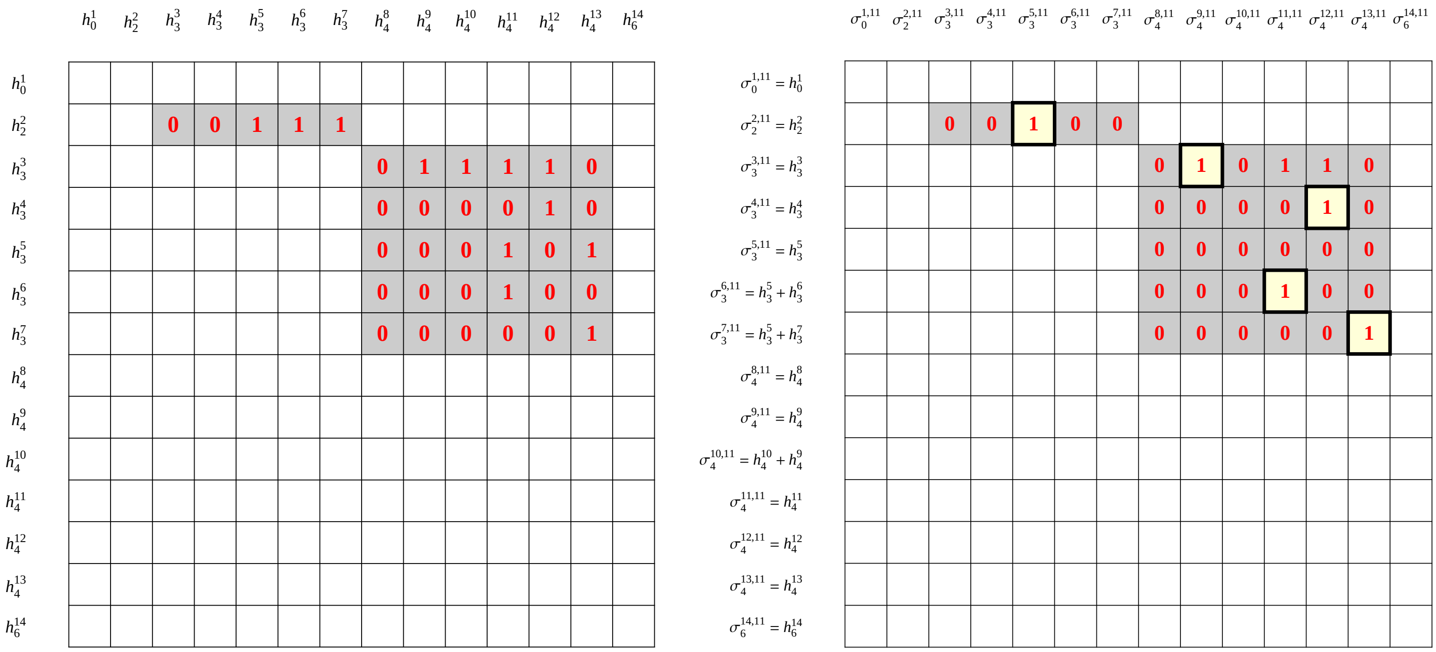
Several numerical experiments are available for download at [Me]. There are two versions of the notebook with the numerical experiments:

`SweepingAlgorithm_ExecutedNumericalExperiments.nb`

and

`SweepingAlgorithm_NumericalExperiments.nb`

Both contain the commands for generating 15 integral connection matrices and 8 binary connection matrices and the commands for running the appropriate version of the sweeping algorithm, but only the former contains the corresponding output as well. Any notebook may be perused with the freely available *Mathematica* player from Wolfram, see description at `http://www.wolfram.com/products/player/`. The full *Mathematica* program is needed to execute the commands in a notebook.



(a) Δ^0 (b) Δ^{11} (final matrix)

Figure 13: Example of the sweeping algorithm over $\mathbb{Z}_2$.

5.4 Random Connection Matrix Generator

The routine for generating random integral connection matrices is invoked with the command

RandomConnectionMatrixGenerator[n];

where `n` is the desired order of the matrix. It accepts three optional parameters:

(i) `seed`: the integral number to be used as a seed for *Mathematica*'s built-in pseudorandom number generator. Although it is not necessary to supply a value for this option, it is a good policy to do so, in order to be able to replicate the outcome.

(ii) `numberofblocks`: the desired number of blocks of Δ.

(iii) `range`: an integer value used to regulate the amplitude of the entries in the resulting connection matrix Δ.

The matrix shown in Figure 13 (a) is, modulo 2, the matrix obtained with the command

```
RandomConnectionMatrixGenerator[14, seed -> 15, numberofblocks -> 2, range -> 4]
```

6 Conclusion

The importance of having a computational tool such as the one presented here cannot be overly emphasized. It opens possibilities of investigating problems which relate topology and dynamics via experiments on a large class of connection matrices.

For instance, there are many questions which have yet to be answered, such as the ones presented in [CdRS], concerning the existence of minimal paths, where time reversal occurs only for connecting orbits associated to primary pivots in the sweeping algorithm.

With this computational tool at hand, we will also be able to investigate the appearance of torsion in the spectral sequence. Our goal is to search for properties in the connection matrix which either make this torsion disappear, or permit it to remain in the stabilization of the unfolding of the spectral sequence.

Also, one of our main results in this paper, namely the integrality of the last matrix in the sweeping algorithm over $\mathbb{Z}$, raises the question of whether this procedure can be related to a continuation as in [Fr3] of a flow associated to the initial connection matrix. Some examples indicate this might be true.

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