Description of some ground states by Puiseux technics

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Abstract

Let (Σ_G^+, σ) be a one-sided transitive subshift of finite type, where the set of symbols is given by a finite spin set S, and the admissible transitions are represented by an irreducible directed graph $G \subset S \times S$. Let $H : \Sigma_G^+ \to \mathbb{R}$ be a locally constant function and, given $\beta > 0$, let $\mu_{\beta H}$ be the Gibbs-equilibrium probability measure associated to the observable $-\beta H$. It is known, by using abstract considerations, that $\{\mu_{\beta H}\}_{\beta > 0}$ converges as $\beta \to +\infty$ to a H-minimimizing probability measure or ground state $\mu_{\min}^H$. For weighted graphs with a small number of vertices, we describe here an algorithm (similar to Puiseux algorithm) that gives the explicit form of $\mu_{\min}^H$.

1 Introduction and main results

The purpose of this article is to present, for specific interaction energy function H, rigourous results on the convergence of Gibbs measures $\{\mu_{\beta H}\}_\beta$ when the temperature $T = \beta^{-1}$ of the system goes to zero. The limit measures thus obtained are called ground states. For most part of the article, the dynamical system is represented by a one-dimensional lattice, or more generally by a transitive subshift of finite type (Σ_G^+, σ), in which some edges may not follow a given edge. The exclusion rule is given by an irreducible finite directed graph $G \subset S \times S$. The set S of vertices of G represents the possible states of the system at each site. We say that the interaction energy function H has long range if it depends on the whole configuration; we say H has short range if it depends only on two adjacent sites.

Our first goal in section 2 is to improve results on the convergence of Gibbs measures for a certain class of long range interaction energy functions H; we use there the language of ergodic optimization theory. More precisely, we prove the

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convergence of Gibbs potentials $\Phi_{\beta H}$ as $\beta \to +\infty$ when the H-minimizing nonwandering set $\Omega(H)$ (see definition 5) admits a unique irreducible component of maximal entropy. Our second aim is to understand the zero-temperature phase diagram for short range interaction energy functions. It is known that [7, 9, 17], for short range interactions, the family of Gibbs measures $\{\mu_{\beta H}\}_\beta$ converges to a unique invariant probability measure called ground state. We present in section 3 the beginning of an algorithm, valid for any weighted directed graph, that describes precisely all possible ground states. We collect all proofs both for general subshift of finite type systems and for weighted directed graphs in sections 4 and 5. We discuss in section 6 the complete phase diagram for all nonsymmetric complete graphs on 3 symbols. We discuss in section 7 the complete phase diagram of ground sates at zero-temperature for the BEG model: a specific model well studied in solid state physics.

We close this introduction by showing on what sort of phase diagram we obtain in the case of the one-dimensional Blume-Emery-Griffits model. The BEG model was initially developed in order to understand the phase transition of mixte systems with two isotopes He^3 and He^4 (see [5]). In particular, it exibits a tricritical point of transition, a first-order and a second-order phase transitions. Our purpose in this introduction is to describe the zero-temperature phase diagram of the one-dimensional BEG model at the level of ground states. A complete discussion on ground states for one-dimensional Ising models can be found in [14]. There are also examples of ground states for more than one dimension (see, for instance, the case of the bidimensional Blume-Capel model in [8]).

We consider a one-dimensional spin system with nearest neighbor interaction given by the hamiltonian

$$H(x) = -J\sum_{\langle i,j\rangle} x_i x_j - K\sum_{\langle i,j\rangle} x_i^2 x_j^2 + \Delta\sum_i x_i^2,$$

where $x_i \in S = \{-1, 0, +1\}$ represents a possible state at the site i.

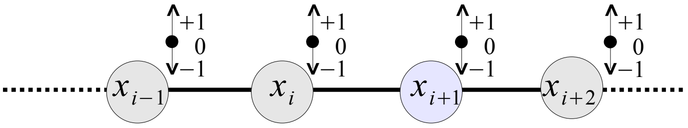


Figure 1: The schematic Blume-Emery-Griffiths model.

For each positive temperature $T = \beta^{-1}$, there exists a unique Gibbs measure, which may be obtained by Ruelle transfer operator method. We first write H in terms of a unique energy function per site H_0, that is, $H = \sum_{i\in\mathbb{Z}} H_0(x_i, x_{i+1})$, where

$$H_0(x,y) = -Jxy - Kx^2y^2 + \frac{\Delta}{2}(x^2+y^2).$$

In BEG model, a site having a state ±1 represents an atom He^4, a site having a state 0 represents He^3. The constant J is supposed to be positive for ferromagnetic systems and negative for antiferromagnetic systems. The constant K takes into account the isotopic interaction, Δ may be interpreted as a chemical potential. An external magnetic field could be added and would give an additional hamiltonian $h\sum_i x_i$. We do not consider this term in this introduction. Even so, we emphasize that the algorithm to be described applies without changes in all these cases, ferromagnetic or antiferromagnetic, with or without external magnetic field.

Ruelle transfer operator method tells us that the Gibbs measure at temperature $T = \beta^{-1}$ is a Markov chain on the finite state space S, defined by an irreducible transition matrix $\big[Q_\beta(x,y)\big]_{x,y\in S}$ and a stationary probability vector $\big[\pi_\beta(x)\big]_{x\in S}$,

$$Q_\beta(x,y) := \frac{\Phi_\beta(y)}{\Phi_\beta(x)} \exp\big[-\beta(H_0(y,x)-F_\beta)\big],\ \pi_\beta(x) := \frac{\Phi^*_\beta(x)\Phi_\beta(x)}{\sum_{y\in S}\Phi^*_\beta(y)\Phi_\beta(y)},$$

where $\exp(-\beta F_\beta)$ denotes the maximal eigenvalue of the transfer operator $\mathcal{L}_\beta$, represented here by a matrix indexed by $S\times S$,

$$\mathcal{L}_\beta = [\mathcal{L}_\beta(x,y)]_{x,y\in S}, \quad \mathcal{L}_\beta(x,y) = \exp(-\beta H_0(x,y)),$$

and $[\Phi_\beta(x)]_{x\in S}$ ($[\Phi^*_\beta(x)]_{x\in S}$) denotes its left (right) eigenvector

$$\sum_{y\in S}\mathcal{L}_\beta(x,y)\Phi^*_\beta(y) = e^{(-\beta F_\beta)}\Phi^*_\beta(x), \quad \sum_{x\in S}\Phi_\beta(x)\mathcal{L}_\beta(x,y) = e^{(-\beta F_\beta)}\Phi_\beta(y),$$

normalized by $\sum_{x\in S}\Phi_\beta(x) = \sum_{x\in S}\Phi^*_\beta(x) = 1$, $\Phi_\beta(x) > 0$, $\Phi^*_\beta(x) > 0$. Notice that in the definition of $Q_\beta(x,y)$, the order of (x,y) has been interchanged in $H_0(y,x)$. Notice also that F_β is also called the free energy in the physics literature.

In BEG model, by numbering the state space $S = \{s_1, s_2, s_3\}$, $s_1 = -1$, $s_2 = 0$ and $s_3 = +1$, and by changing the parameter β to $\epsilon = \exp(-\beta)$, we are left to study a singular perturbation of a one-parameter family of matrices $M_\epsilon = [A(x,y)\epsilon^{a(x,y)}]$, where

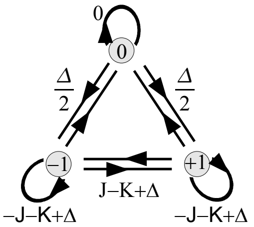
$$A = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix}-J-K+\Delta & \frac{1}{2}\Delta & J-K+\Delta\\ \frac{1}{2}\Delta & 0 & \frac{1}{2}\Delta\\ J-K+\Delta & \frac{1}{2}\Delta & -J-K+\Delta\end{bmatrix}.$$

We summarize the set of possible interactions between two consecutive sites x_i and x_{i+1} by a (directed) graph $G\subset S\times S$ weighted by the principal exponent $a(x,y)$ as explained in figure 2. We also indicate in this figure the mean of a along all simple cycles.

We will show that $\mu_{\beta H}$ converges to a Markov chain μ^H_{min} characterized by an initial law π_∞ and a transition matrix Q_∞ that we describe in figures 3 and 4.

Each region of the plane (J,K) represents the *limit phase* or ground state: each box indicates the initial law, the transition matrix and the beginning of the Puiseux series expansion of the free energy. The three phases of dimension 2 correspond to the case where all exponents 0, $\frac{1}{2}\Delta$, $-J-K+\Delta$, $J-K+\Delta$ and $\frac{1}{3}(J-K+2\Delta)$ are distinct.

For instance, when $J-K+\Delta<0$ and $J<0$, corresponding to the upper left part of the phase diagram, the smallest exponent is $J-K+\Delta$ and the ground



Mean of a along simple cycles:

cycles of order 1	0, $(-J-K+\Delta)$
cycles of order 2	$\frac{1}{2}\Delta$, $(J-K+\Delta)$
cycles of order 3	$\frac{1}{3}(J-K+2\Delta)$

Figure 2: Graph of interactions and minimizing cycles in BEG model.

state μ^H_{min} is equal to the uniform distribution on the intermittent configuration $\cdots -1, +1, -1, +1, \cdots$, more precisely, because we fix an origin, it is equal to a periodic probability measure of period 2:

$$\mu^H_{min} = \frac{1}{2}\delta_{<\cdots+1|-1+1\cdots>} + \frac{1}{2}\delta_{<\cdots-1|+1-1\cdots>}.$$

The ground state is pure and made of atoms with alternate spins ±1. We show, for example, that the initial law π_β, the maximal eigenvalue λ_β and the transition matrix Q_β admit an expansion of the following form

$$\pi_\beta \sim \begin{bmatrix} 1/2 \\ 2e^{-2\beta(-J+K-\Delta/2)} \\ 1/2 \end{bmatrix} \to \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \quad \lambda_\beta = e^{-\beta F_\beta} \sim e^{-\beta(J-K+\Delta)}$$

$$Q_\beta \sim \begin{bmatrix} e^{2\beta J} & 2e^{-2\beta(-J+K-\Delta/2)} & 1 \\ 1/2 & e^{-\beta(-J+K-\Delta)} & 1/2 \\ 1 & 2e^{-2\beta(-J+K-\Delta/2)} & e^{2\beta J} \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}.$$

We notice that, in the region $J-K+\Delta > 0$ and $-J-K+\Delta > 0$, independent of the sign of J, the ground state is pure with only the presence of He^3.

2 A dynamical system approach

We consider a one-sided transitive subshift of finite type (Σ^+_G, σ), where S is a finite set of vertices and $G \subset S \times S$ is an irreducible directed graph representing the admissible transitions from one vertex to another. The main objects are described by a dynamical system and a Hölder observable that will play the role of an interaction energy function,

$$\begin{aligned} &\Sigma^+_G = \big\{x = (x_k)_{k\geq 0} \in S^{\mathbb{N}} \ : \ (x_k, x_{k+1}) \in G, \ \forall\ k \in \mathbb{N}\big\}, \\ &\sigma : \Sigma^+_G \to \Sigma^+_G, \quad \sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots), \\ &H : \Sigma^+_G \to \mathbb{R}. \end{aligned}$$

Here Σ^+_G is the set of configurations compatible with the transitions given by the graph G, and σ is the left translation. We will use the notation $x \overset{G}{\to} y$ to indicate the admissible transition $(x, y) \in G$ between two vertices $x, y \in S$. We denote by C_n any cylinder of length n, that is, a set of configurations $x \in \Sigma^+_G$ whose first n

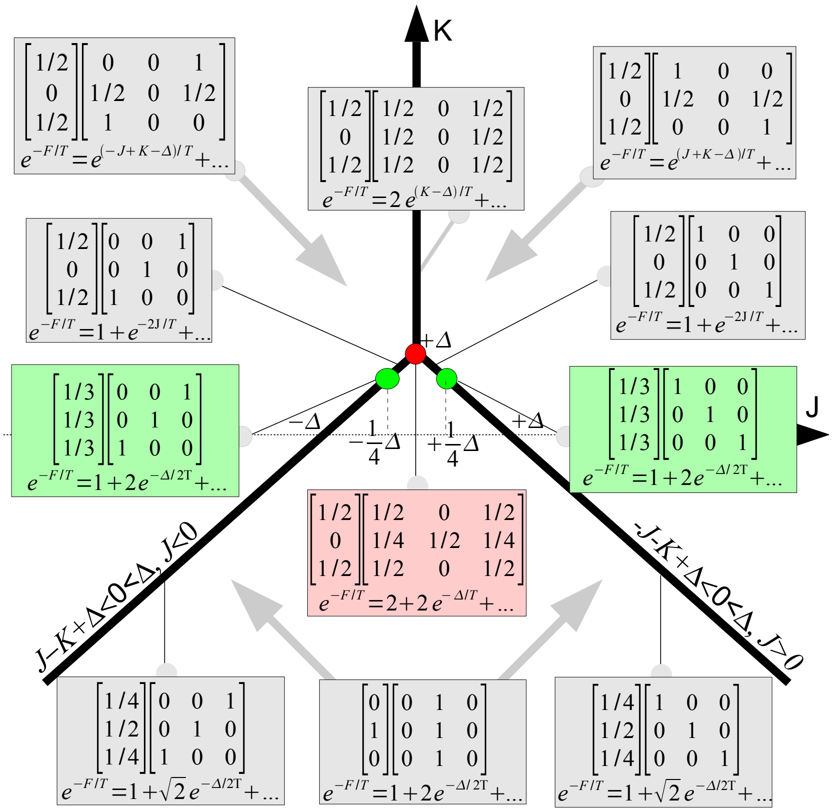


Figure 3: Phase diagram of the BEG model at zero temperature for $\Delta > 0$. The Markov chain structure of the ground state and the Puiseux series expansion of the free energy is shown for each phase.

symbols are prescribed by admissible transitions $i_0 \overset{G}{\to} i_1 \overset{G}{\to} i_2 \overset{G}{\to} \ldots \overset{G}{\to} i_{n-1}$. We use $C_n(x)$ or $[x_0, \ldots, x_{n-1}]$ for a cylinder prescribed by $x_0 \overset{G}{\to} x_1 \overset{G}{\to} \ldots \overset{G}{\to} x_{n-1}$ if x is any allowed configuration. We call $\mathcal{C}_n(G) = \{C_n(x) \ : \ x \in \Sigma_G^+\}$ the set of all cylinders of length n.

The transitivity of (Σ_G^+, σ) (in an equivalent way, the irreducibility of G) guarantees the uniqueness of the Gibbs-equilibrium measure μ_H associated to a Hölder observable $H : \Sigma_G^+ \to \mathbb{R}$. Recall also that Σ_G^+ is a compact metric space equiped with the distance $d(x,y) = 1$ if $x_0 \neq y_0$ and $d(x,y) = (\frac{1}{2})^n$ if $x_0 = y_0, \ldots, x_{n-1} = y_{n-1}$, $x_n \neq y_n$. Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be continuous observables which will play the role of interaction energy functions. Let us recall Ruelle's definition of the pressure of an observable Ψ (which shall be seen as $E + \beta H$).

Definition 1. *Let $\Psi : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We call pressure of Ψ the real value*

$$\mathrm{Pres}(\Psi) := \max\Big\{\mathrm{Ent}(\mu) - \int \Psi \, d\mu \ : \ \mu \in \mathcal{M}(\Sigma_G^+, \sigma)\Big\},$$

where $\mathcal{M}(\Sigma_G^+, \sigma)$ denotes the set of σ-invariant Borel probability measures on Σ_G^+,

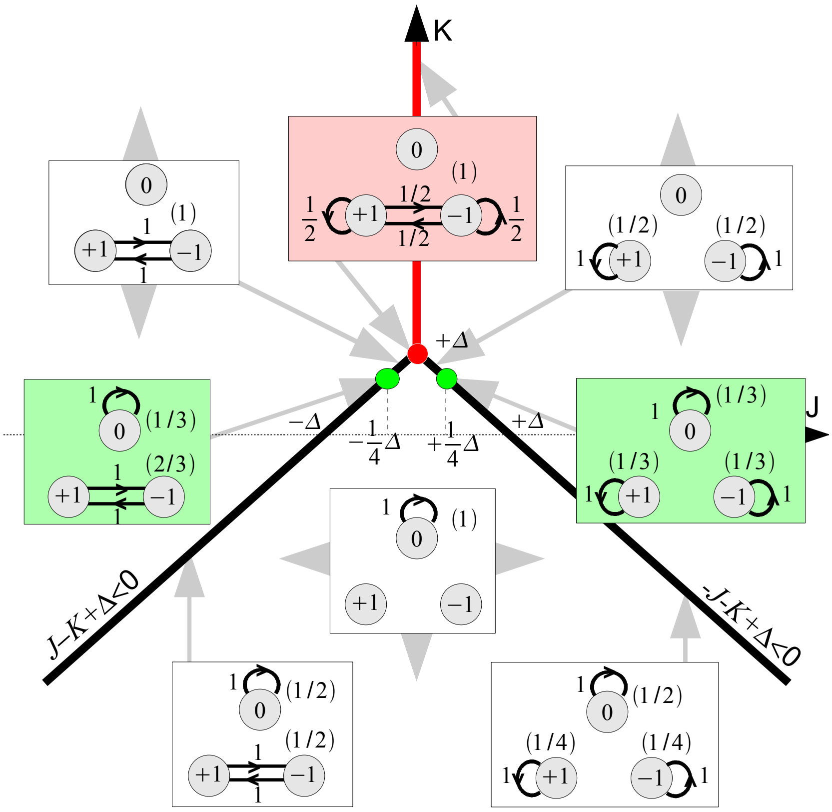


Figure 4: Phase diagram of the BEG model at zero temperature for $\Delta > 0$. Numbers in parenthesis indicate the weigth of each indecomposable (ergodic) Markov chain which contributes to the ground state.

and $\mathrm{Ent}(\mu)$ *denotes the Kolmogorov-Sinai entropy of* σ *with respect to* μ

$$\mathrm{Ent}(\mu) := \lim_{n\to+\infty} \frac{1}{n} \sum_{C_n\in\mathcal{C}_n(G)} -\mu[C_n]\ln\mu[C_n].$$

More generally, for any σ*-invariant Borel probability measure* μ *or* σ*-invariant compact set* Ω*, we call relative pressure with respect to* μ *or* Ω*, respectively,*

$$\mathrm{Pres}(\Psi,\mu) := \mathrm{Ent}(\mu) - \int \Psi\, d\mu,$$

$$\mathrm{Pres}_\Omega(\Psi) := \max\Big\{\mathrm{Pres}(\Psi,\mu)\ :\ \mu\in\mathcal{M}(\Sigma_G^+,\sigma)\ \ \textit{and}\ \ \mathrm{supp}(\mu)\subset\Omega\Big\}.$$

We say that $\mu\in\mathcal{M}(\Sigma_G^+,\sigma)$ *has relative maximal pressure in* Ω *for* Ψ *if*

$$\mathrm{Pres}_\Omega(\Psi) = \mathrm{Pres}(\Psi,\mu) \quad \textit{and} \quad \mathrm{supp}(\mu)\subset\Omega.$$

Definition 2. *We call Gibbs measure associated to* Ψ *a* σ*-invariant Borel probability measure* μ_Ψ *on* Σ_G^+ *satisfying*

$$\mu_\Psi[C_n(x)] \asymp \exp\Big(-\sum_{k=0}^{n-1}\big[\Psi\circ\sigma^k(x) + \mathrm{Pres}(\Psi)\big]\Big), \quad \forall\, x\in\Sigma_G^+, \quad \forall\, n\geq 1.$$

The notation $a_n(x) \asymp b_n(x)$ is a shortcut to $C^{-1}a_n(x) \leq b_n(x) \leq Ca_n(x)$ for some constant $C > 0$ independent of n and x.

It is known that, for any given Hölder observable $\Psi : \Sigma_G^+ \to \mathbb{R}$, there exists a unique Gibbs measure μ_Ψ, which is also the unique σ-invariant Borel probability measure with maximal pressure:

$$\mathrm{Pres}(\Psi) = \mathrm{Pres}(\Psi, \mu_\Psi) > \mathrm{Pres}(\Psi, \mu), \quad \forall\, \mu \in \mathcal{M}(\Sigma_G^+, \sigma) \setminus \{\mu_\Psi\}.$$

For $E, H : \Sigma_G^+ \to \mathbb{R}$ Hölder observables, we are interesting in the convergence (with respect to the weak* topology) of $\mu_{E+\beta H}$ as β tends to $+\infty$, that is, as the temperature $T = \beta^{-1}$ of the system goes to zero.

Question 3. *What are the possible weak* limits of $\mu_{E+\beta H}$ as β tends to $+\infty$? Is there a unique limit ? How can one characterize them in an effective way ?*

We collect in this section several general facts related to these questions. We will show in the next section how to improve these results when H has short range. An immediate observation tells us that all possible weak* limits of $\{\mu_{E+\beta H}\}_\beta$ need to be minimizing:

Definition 4. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We say that a σ-invariant Borel probability measure μ_{min} is H-minimizing if*

$$\int H(x)\, d\mu_{min}(x) = \min\Big\{ \int H(x)\, d\mu(x) \ : \ \mu \in \mathcal{M}(\Sigma_G^+, \sigma)\Big\} =: \bar{H}.$$

The constant $\bar{H} = \int H(x)\, d\mu_{min}(x)$ is called the minimizing ergodic value of H. The set of H-minimizing probability measures is denoted by $\mathcal{M}_{min}(\Sigma_G^+, \sigma, H)$.

It is also easy to prove that the support of all weak* limits of $\{\mu_{E+\beta H}\}_\beta$ need be included into the H-minimizing non-wandering set $\Omega(H)$, whose definition is recalled below.

Definition 5. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. The H-minimizing non-wandering set is defined by*

$$\Omega(H) := \Big\{ x \in \Sigma_G^+ : \forall\, \epsilon > 0,\ \exists\, n \geq 1,\ \exists\, z \in \Sigma_G^+ \ \ s.\ t. \quad d(x,z) < \epsilon,\ d(x, \sigma^n(z)) < \epsilon \text{ and } \Big|\sum_{k=0}^{n-1} [H \circ \sigma^k(z) - \bar{H}]\Big| < \epsilon \Big\}.$$

We remark that $\Omega(H)$ is a compact σ-invariant set.

We now recall some general facts about the set of all possible weak* limits of $\{\mu_{E+\beta H}\}_\beta$. When the temperature is not zero, it is standard to introduce the free energy instead of the Ruelle pressure:

$$F_\beta(E, H) := -\frac{1}{\beta}\mathrm{Pres}(E + \beta H).$$

A weak* limit of a converging subsequence of $\{\mu_{E+\beta H}\}_{\beta \to +\infty}$ is called a *ground state.*

Proposition 6. *[11] Let $H : \Sigma_G^+ \to \mathbb{R}$ be a Hölder observable. A σ-invariant Borel probability measure μ is H-minimizing if, and only if, its support* $\mathrm{supp}(\mu)$ *is included into $\Omega(H)$. Equivalently,*

$$\mathcal{M}_{min}(\Sigma_G^+, \sigma, H) = \{\mu \in \mathcal{M}(\Sigma_G^+, \sigma) \ : \ \mathrm{supp}(\mu) \subset \Omega(H)\}.$$

As we will see in section 3, for a short range energy function H, the existence of several minimizing measures is not unusual. The next proposition states that, by freezing the system, the Gibbs measures accumulate on special minimizing measures. Similar results have been obtained in other contexts (see, for instance, [3] or [15, 18]).

Proposition 7. *[11, 17] Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Then any ground state μ_{min} (any accumulation point of $\{\mu_{E+\beta H}\}_\beta$) is H-minimizing,* $\mathrm{supp}(\mu_{min}) \subset \Omega(H)$ *and μ_{min} has relative maximal pressure in $\Omega(H)$. More precisely, when $\beta \to +\infty$,*

$$\begin{gathered}\mathrm{Pres}(E + \beta H) + \beta \bar{H} \to \mathrm{Pres}_{\Omega(H)}(E), \\ \int H \, d\mu_{E+\beta H} \to \bar{H} = \int H \, d\mu_{min}, \\ \mathrm{Pres}(E, \mu_{E+\beta H}) \to \mathrm{Pres}_{\Omega(H)}(E) = \mathrm{Pres}(E, \mu_{min}).\end{gathered}$$

In particular, the free energy $F_\beta(E, H) := -\frac{1}{\beta} P(E + \beta H)$ converges to the minimizing ergodic value $\bar{H}$. If $\Omega(H)$ supports a unique measure μ_{min} with maximal pressure $P_{\Omega(H)}(E)$, then $\{\mu_{E+\beta H}\}_\beta$ converges to μ_{min}.

The next proposition gives a class of examples where $\{\mu_{E+\beta H}\}_\beta$ converges to a unique minimizing measure.

Proposition 8. *[11] For any $\alpha > 0$, the set of α-Hölder H admitting a unique H-minimizing probability measure is generic in C^α. Thus $\{\mu_{E+\beta H}\}_\beta$ converges to a unique μ_{min} for generic α-Hölder H.*

Gibbs measures have a different functional characterization in terms of the Ruelle transfer operator. They are also called *equilibrium measures.*

Definition 9. *We call Ruelle transfer operator associated to a Hölder observable $\Psi : \Sigma_G^+ \to \mathbb{R}$ the operator $\mathcal{L}_\Psi$ acting on Hölder functions $f : \Sigma_G^+ \to \mathbb{R}$ as follows*

$$\mathcal{L}_\Psi f(x) = \sum_{y \,:\, \sigma(y) = x} e^{-\Psi(y)} f(y), \quad \forall \ x \in \Sigma_G^+,$$

where the summation is taken among all preimages of x by σ.

It is well known that, by extending the standard Perron-Frobenius theory for nonnegative matrices, the Ruelle transfer operator $\mathcal{L}_\Psi$ admits similar “right and left eigenvectors” that we recall in the following proposition.

Proposition 10. *[6, 20, 21] Let $\Psi : \Sigma_G^+ \to \mathbb{R}$ be a Hölder observable. Then there exist a unique left eigenmeasure, or Borel probability measure ν_Ψ on Σ_G^+, a unique normalized right eigenfunction, or positive Hölder function $\Phi_\Psi : \Sigma_G^+ \to \mathbb{R}$, such that*

$$\mathcal{L}_\Psi^* \nu_\Psi = e^{\mathrm{Pres}(\Psi)} \nu_\Psi, \quad \mathcal{L}_\Psi \Phi_\Psi = e^{\mathrm{Pres}(\Psi)} \Phi_\Psi \quad \text{and} \quad \int \Phi_\Psi \, d\nu_\Psi = 1.$$

Moreover, $\mu_\Psi := \Phi_\Psi \nu_\Psi$ is a Gibbs measure and the unique σ-invariant probability that maximizes the pressure for Ψ among all σ-invariant probabilities.

We have seen in proposition 7 that $\{-\frac{1}{\beta} P(E+\beta H)\}_\beta$ converges to $\bar{H}$ and that any weak* limit of $\{\mu_{E+\beta H}\}_\beta$ is H-minimizing. It would be interesting to obtain similar characterizations for limit points of $\{\Phi_{E+\beta H}\}_\beta$ or $\{\nu_{E+\beta H}\}_\beta$. Such a characterization exists for $\{\Phi_{E+\beta H}\}_\beta$: any limit point of $\{-\frac{1}{\beta} \ln \Phi_{E+\beta H}\}_\beta$ is a calibrated sub-action. Let us first recall the definition of a sub-action.

Definition 11. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a continuous observable. We call sub-action with respect to H any continuous function $V : \Sigma_G^+ \to \mathbb{R}$ such that*

$$V \circ \sigma(x) - V(x) \leq H(x) - \bar{H}, \quad \forall\, x \in \Sigma_G^+.$$

We call calibrated sub-action any sub-action V which in addition satisfies

$$V(y) = \min \big\{ V(x) + H(x) - \bar{H} \ : \ x \in \Sigma_G^+, \ \sigma(x) = y \big\}, \quad \forall\, y \in \Sigma_G^+.$$

Similarly to proposition 29 of [11], we obtain easily the following proposition.

Proposition 12. *Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Let $\Phi_{E+\beta H}$ be the right eigenfunction of $\mathcal{L}_{E+\beta H}$, let $\Phi_{E+\beta H} := \exp(-\beta V_{E+\beta H})$. Then $\{V_{E+\beta H}\}_\beta$ is uniformly bounded and has a uniform Hölder norm. Moreover, any accumulation function of $\{V_{E+\beta H}\}_\beta$ is a calibrated sub-action with respect to H.*

If $\Omega(H)$ supports a unique probability measure $\mu_{\min}^H$ with relative maximal pressure $\mathrm{Pres}_{\Omega(H)}(E)$, then $\mu_{E+\beta H} \to \mu_{\min}^H$ although $\mathcal{M}_{\min}(\Sigma_G^+, \sigma, H)$ may not be reduced to a single measure. We nevertheless do not known whether $\{V_{E+\beta H}\}_\beta$ converges or not. We intend to establish the “projective” convergence of $\{V_{E+\beta H}\}_\beta$ in the particular case where $\Omega(H)$ can be split into disjoint irreducible components with a unique component of maximal pressure. The splitting up of $\Omega(H)$ into components uses the following notion of Peierls barrier.

Definition 13. *Let $H : \Sigma_G^+ \to \mathbb{R}$ be a Hölder observable. We call Peierls barrier the function $h(x,y)$ defined on $\Sigma_G^+ \times \Sigma_G^+$ by*

$$h(x,y) := \lim_{\epsilon \to 0} \liminf_{n \to +\infty} S_n^\epsilon(x,y),$$

where

$$S_n^\epsilon(x,y) := \inf \Big\{ \sum_{k=0}^{n-1} (H - \bar{H}) \circ \sigma^k(z) \ : \ d(z,x) < \epsilon \ \text{and} \ d(\sigma^n(z), y) < \epsilon \Big\}.$$

The Peierls barrier may be infinite. If $x \in \Omega(H)$, $h(x,y)$ is finite and Hölder with respect to $y \in \Sigma_G^+$. Notice that $\Omega(H) = \{x \in \Sigma_G^+ \ : \ h(x,x) = 0\}$. Let us recall how the minimizing non-wandering set $\Omega(H)$ can be partitioned into closed invariant sets, which uniquely characterize sub-actions.

Definition 14. *[13] We say that two points x, y of $\Omega(H)$ are equivalent, and we write $x \sim y$, whenever $h(x,y) + h(y,x) = 0$. Equivalent classes are called irreducible components. Irreducible components are σ-invariant and compact.*

We now state the main result of this section.

Theorem 15. *Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Assume that $\Omega(H) = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_r$ admits a finite decomposition into disjoint irreducible components Ω_i and*

$$\mathrm{Pres}_{\Omega(H)}(E) = \mathrm{Pres}_{\Omega_0}(E) > \mathrm{Pres}_{\Omega_1}(E) \geq \ldots \geq \mathrm{Pres}_{\Omega_r}(E).$$

Let $\Phi_{E+\beta H} = \exp(-\beta V_{E+\beta H})$ be the normalized right eigenfunction of the Ruelle transfer operator $\mathcal{L}_{E+\beta H}$. Then uniformly in $y \in \Sigma_G^+$, for any fixed $x_0 \in \Omega_0$,

$$\lim_{\beta \to +\infty} V_{E+\beta H}(y) - V_{E+\beta H}(x_0) = h(x_0, y), \quad \forall \ y \in \Sigma_G^+.$$

Notice that, in the above theorem, $\{\mu_{E+\beta H}\}_\beta$ may not converge to a unique H-minimizing measure. Indeed, any weak* limit has a support in Ω_0 which may contain many minimizing measures. Notice also that the convergence of $\{V_{E+\beta H}\}_\beta$ (as a sequence of functions) depends only on the converge of $\{V_{E+\beta H}(x_0)\}_\beta$ for any fixed $x_0 \in \Omega_0$.

3 A matrix approach to ground state theory

We say that the interaction energy function $H : \Sigma_G^+ \to \mathbb{R}$ has finite range if it only depends on two consecutive symbols $H(x) = H(x_0, x_1)$. By allowing a larger number of vertices in another irreducible finite directed graph G', an energy function of the form $H(x_0, \ldots, x_{d-1})$ can be described by the framework we are going to develop. The main consequence of this strong assumption on the energy function is that the problem of zero-temperature phase diagram is reduced to a problem of singular perturbation of matrices of Puiseux type.

We consider a finite state space S and an irreducible directed graph $G \subset S \times S$ weighted by an energy function $\{\exp[-\beta H(x,y)]\}_{x \overset{G}{\to} y}$, where x, y are particular states in S and $x \overset{G}{\to} y$ denotes an admissible transition given by the graph G. We prefer to introduce a new parameter $\epsilon := \exp(-\beta)$, which goes to zero when β tends to $+\infty$, and a one-parameter family of transfer matrices $[M_\epsilon(x,y)]_{(x,y) \in S \times S}$, adapted to G, defined by

$$\begin{cases} M_\epsilon(x,y) := \exp[-\beta H(x,y)] = \epsilon^{H(x,y)}, & \forall \ (x,y) \in G, \\ M_\epsilon(x,y) := 0, & \forall \ (x,y) \notin G. \end{cases}$$

Notice that M_ϵ is a Perron-Frobenius matrix, that is, a matrix with nonnegative entries. Let $\lambda_\epsilon := \rho_{spec}(M_\epsilon) > 0$ be its spectral radius. Because of the irreducibility of G, λ_ϵ is an eigenvalue of multiplicity 1. Let $[L_\epsilon(x)]_{x\in S}$ and $[R_\epsilon(x)]_{x\in S}$ be the left and right eigenvector of M_ϵ for the eigenvalue λ_ϵ,

$$\sum_{x\in S} L_\epsilon(x)M_\epsilon(x,y) = \lambda_\epsilon L_\epsilon(y), \quad \forall\ y \in S,$$
$$\sum_{y\in S} M_\epsilon(x,y)R_\epsilon(y) = \lambda_\epsilon R_\epsilon(x), \quad \forall\ x \in S,$$

normalized by $\sum_{x\in S} L_\epsilon(x)R_\epsilon(x) = 1$ and $\sum_{x\in S} R_\epsilon(x) = 1$. Notice that $L_\epsilon(x) > 0$ and $R_\epsilon(x) > 0$ for all $x \in S$. Let

$$\pi_\epsilon(x) := L_\epsilon(x)R_\epsilon(x) \quad \text{and} \quad Q_\epsilon(x,y) := M_\epsilon(x,y)\frac{R_\epsilon(y)}{R_\epsilon(x)\lambda_\epsilon}, \quad \forall\ x,y \in S.$$

The Ruelle transfer operator used in the dynamical approach of section 2 is strongly related to a basic eigenvalue problem that we recall in the following remark.

Remark 16. *Assume $H(x) = H(x_0,x_1)$ has short range. Let $\Phi_{\beta H} : \Sigma_G^+ \to \mathbb{R}$ be the right eigenfunction of $\mathcal{L}_{\beta H}$ and $\nu_{\beta H}$ be the left eigenmeasure of $\mathcal{L}_{\beta H}$. Let $\mu_{\beta H}(dx) = \Phi_{\beta H}(x)\nu_{\beta H}(dx)$ be the normalized Gibbs-equilibrium measure associated to βH. Then*

- *i.* $\Phi_{\beta H}(x) = L_\epsilon(x_0)$, $\forall\ x = (x_0,x_1,\dots) \in \Sigma_G^+$.
- *ii.* $\nu_{\beta H}([x_0]) = R_\epsilon(x_0)$, $\forall\ x_0 \in S$.
- *iii.* *$\mu_{\beta H}$ is a Markov chain on Σ_G^+ with initial law π_ϵ and transition matrix Q_ϵ. For any cylinder of size $d+1$, one has*

$$\mu_{\beta H}([x_0,x_1,\dots,x_d]) = L_\epsilon(x_0)\big[\Pi_{i=0}^{d-1}M_\epsilon(x_i,x_{i+1})\big]R_\epsilon(x_d)/\lambda_\epsilon^d.$$

We are interested in describing the possible limits of $\{(\pi_\epsilon,Q_\epsilon)\}_{\epsilon\to0}$ that we also call ground states. In an equivalent way, we want to describe all possible limits of the eigenvalue $\{\lambda_\epsilon\}_{\epsilon\to0}$ and the projective eigenvectors $\{L_\epsilon(x)/L_\epsilon(y)\}_{\epsilon\to0}$ and $\{R_\epsilon(x)/R_\epsilon(y)\}_{\epsilon\to0}$. As in the dynamical system approach, the ground states are localized in a minimizing subgraph similar to the minimizing non-wandering set $\Omega(H)$ recalled in definition 5. We first begin by restricting the class of the one-parameter family of matrices we want to study. We introduce the notion of one-parameter family of Puiseux type in two steps.

Definition 17. *Let $G \subset S\times S$ be a (not necessarily irreducible) directed graph and $\{M_\epsilon\}_{\epsilon>0}$ be a one-parameter family of matrices indexed by S. The graph G is said to be weighted by M_ϵ if $M_\epsilon(x,y) = 0$ whenever $(x,y) \notin G$. The weighted graph (G,M_ϵ) is said to be of exact Puiseux type if there exist a nonnegative matrix $[A(x,y)]_{x,y\in S}$ and an extended real-valued matrix $[a(x,y)]_{x,y\in S}$ such that*

- *i.* $\forall\ (x,y) \notin G$, $A(x,y) = 0$, $a(x,y) = +\infty$ *and* $M_\epsilon(x,y) = 0$.

ii. $\forall\ (x,y) \in G$, $A(x,y) > 0$, $a(x,y) \in \mathbb{R}$ *and*

$$M_\epsilon(x,y) = A(x,y)\epsilon^{a(x,y)} + o(\epsilon^{a(x,y)}).$$

We say shortly $M_\epsilon \sim A\epsilon^a$.

We call G-path of length $n \geq 1$ in S, any sequence $(x_0, \ldots, x_n)$ such that $(x_k, x_{k+1}) \in G$, $\forall\ k = 0, \ldots, n-1$. The support of a G-path $(x_0, \ldots, x_n)$ is the subset $\{(x_k, x_{k+1}) : k = 0, \ldots, n-1\} \subset G$. A cycle of length $n \geq 1$ is a G-path $(x_0, \ldots, x_n)$ in S such that $x_n = x_0$. We call off-diagonal cycle any cycle $(x_0, x_1, \ldots, x_n)$ such that $x_i \neq x_{i+1}$ for all $i = 0, \ldots, n-1$. A simple cycle is a cycle $(x_0, \ldots, x_n)$ such that $x_i \neq x_j$ for all $0 \leq i \neq j < n$. A loop is a cycle (x_0, x_1) of length 1 where $(x_0, x_1) \in G$ and $x_0 = x_1$. We call mean exponent of a cycle the real number $\frac{1}{n}\sum_{i=0}^{n-1} a(x_i, x_{i+1})$.

Definition 18. *Suppose that* (G, M_ϵ) *is an irreducible weighted graph of exact Puiseux type with* $M_\epsilon \sim A\epsilon^a$.

i. We call minimizing mean exponent of (G, M_ϵ) *the real number*

$$\bar{a} := \min\Big\{\frac{1}{n}\sum_{i=0}^{n-1} a(x_i, x_{i+1}) : n \geq 1,\ (x_0, \ldots, x_n) \textit{ is a cycle } \Big\}.$$

We call minimizing cycle any cycle of mean exponent $\bar{a}$.

ii. We call minimizing subgraph the graph $G_{min} \subset S_{min} \times S_{min}$, *where* S_{min} *is the set of states belonging to some minimizing cycle and* G_{min} *is the union of supports of all minimizing cycles.*

iii. We call dominant spectral coefficient of M_ϵ *the spectral radius of* A_{min}

$$\bar{\alpha} := \sup\{|\lambda| \ :\ \lambda \in \operatorname{spec}(A_{min})\} = \rho_{spec}(A_{min}),$$

where $A_{min} = [A(x,y)\mathbb{1}_{G_{min}}(x,y)]_{x,y\in S}$. *Notice that* $\bar{\alpha} > 0$.

Notice that $\bar{a}$ may be obtained by minimizing on the finite set of simple cycles. Although we start with an irreducible graph, G_{min} may not be any more irreducible; G is nevertheless semi-irreducible as explained below.

Definition 19. *A graph* $G \subset S \times S$ *is said to be semi-irreducible if there exist a partition* $S = S_1 \cup \ldots \cup S_d$ *and irreducible subgraphs* $G_i \subset S_i \times S_i$ *such that* $G = G_1 \cup \ldots \cup G_d$. *Note that in* G *there is no transition from* $x_i \in S_i$ *to* $x_j \in S_j$ *for any* $1 \leq i \neq j \leq d$. *The subgraphs* G_i *are called the irreducible components of* G.

Lemma 20. *Let* (G, M_ϵ) *be an irreducible weighted graph of exact Puiseux type. Then the minimizing subgraph* G_{min} *is semi-irreducible.*

In the language of dynamical system, when (G, M_ϵ) is of exact Puiseux type, G_{min} describes the minimizing non-wandering set $\Omega(a)$ introduced in definition 5. More precisely:

Lemma 21. *Let G be an irreducible directed graph and $E, H : \Sigma_G^+ \to \mathbb{R}$ be short range observables. Let $M_\epsilon = A\epsilon^a = [\exp(E(x,y))\epsilon^{H(x,y)}\mathbb{1}_G(x,y)]_{x,y\in S}$. Then (G, M_ϵ) is of exact Puiseux type and satisfies:*

i. The minimizing mean exponent of (G, M_ϵ) is equal to the minimizing ergodic value of H, namely, $\bar{a} = \bar{H}$.

ii. The minimizing non-wandering set $\Omega(H)$ is a subshift of finite type

$$\Omega(H) = \{x \in \Sigma_G^+ : (x_k, x_{k+1}) \in G_{min}, \ \forall\ k \geq 0\} = \Sigma_{G_{min}}^+.$$

iii. The splitting up of $\Omega(H)$ into irreducible components (see definition 14) corresponds to the splitting up of G_{min} into irreducible components $\{G_i\}_{i=1}^d$:

$$\begin{gathered}\Omega(H) = \Omega_1(H) \cup \ldots \cup \Omega_d(H), \ \textit{where} \\ \Omega_i(H) := \{x \in \Sigma_G^+ : (x_k, x_{k+1}) \in G_i, \ \forall\ k \geq 0\}.\end{gathered}$$

iv. The relative pressure of E to $\Omega(H)$ is related to the dominant spectral coefficient of M_ϵ by $\bar{\alpha} = \exp[\mathrm{Pres}_{\Omega(H)}(E)]$.

We now complete the notion of one-parameter family of Puiseux type.

Definition 22. *Let $G \subset S \times S$ be an irreducible directed graph. We call off-diagonal graph the subgraph of G defined by $G^{\mathit{off}} := G\backslash\{(x,x) : x \in S\}$. Notice that G^{off} is again irreducible. If (G, M_ϵ) is a weighted graph, we denote $M_\epsilon^{\mathit{off}}(x,y) := M_\epsilon(x,y)\mathbb{1}_{G^{\mathit{off}}}(x,y)$.*

Definition 23. *Following the definition 17, we say that an irreducible weighted graph (G, M_ϵ) is of general Puiseux type if*

i. The irreducible off-diagonal weighted graph $(G^{\mathit{off}}, M_\epsilon^{\mathit{off}})$ is of exact Puiseux type. Let $\bar{a}_{\mathit{off}}$ be the minimizing mean exponent of $(G^{\mathit{off}}, M_\epsilon^{\mathit{off}})$.

ii. For each $(x,y) \notin G$, $A(x,y) = 0$ and $a(x,y) = +\infty$ (by convention).

iii. For all $x \in S$, $(x,x) \in G$ and one of the two estimates holds

$$\begin{aligned}&M_\epsilon(x,x) = o(\epsilon^{\bar{a}_{\mathit{off}}}) \ \ (\textit{by convention: } A(x,x) = 0, \ a(x,x) = +\infty) \quad \textit{or} \\ &M_\epsilon(x,x) = A(x,x)\epsilon^{a(x,x)} + o(\epsilon^{a(x,x)}), \ A(x,x) > 0, \ a(x,x) \leq \bar{a}_{\mathit{off}}.\end{aligned}$$

Let $G^* := G \setminus \{(x,x) \in G : A(x,x) = 0\}$ and $M_\epsilon^*(x,y) := M_\epsilon(x,y)\mathbb{1}_{G^*}(x,y)$. Notice that G^* is an irreducible directed graph and (G^*, M_ϵ^*) becomes a weighted graph of exact Puiseux type. We call minimizing mean exponent $\bar{a}$ of (G, M_ϵ) the minimizing mean exponent of (G^*, M_ϵ^*). Let G_{min}^* be the minimizing subgraph of G^* and

$$A_{min}^* := [A(x,y)\mathbb{1}_{G_{min}^*}(x,y)]_{x,y\in S}.$$

We call dominant spectral coefficient $\bar{\alpha}$ the spectral radius of A_{min}^*. We call dominant subgraph $\bar{G}$ the subgraph of G defined by the union of all irreducible components of G_{min}^* of dominant spectral coefficient.

Notice that the only difference between the two notions of Puiseux type is that, in the weakest definition, M_ϵ may possess a diagonal term (positive or not) of the form $o(\epsilon^{\bar{a}_{\mathit{off}}})$. We will see soon that that these terms are negligible in the computation of the spectral radius of M_ϵ. Notice also that

$$\bar{a} = \min\{\bar{a}_{\mathit{off}}, a(x,x) : x \in S\}.$$

From lemma 20, the minimizing subgraph G^*_{min} is equal to a disjoint union of irreducible subgraphs: $G^*_{min} = G^*_1 \cup \ldots \cup G^*_d$, where $S_1 \cup \ldots \cup S_d$ is a partition of S^*_{min} and $G^*_i \subset S_i \times S_i$. By just permutating indices, we may consider that the first r subgraphs G^*_i have dominant spectral coefficient $\bar{\alpha}$. In order to do that, we adapt the notation and we say that $\bar{G}_i \subset \bar{S}_i \times \bar{S}_i$ has dominant spectral coefficient if the restricted matrix $A^{ii}_{min} = [A(x,y)\mathbb{1}_{\bar{G}_i}(x,y)]_{x,y\in\bar{S}_i}$ has spectral radius $\bar{\alpha}$.

Main notations 24. *Suppose (G, M_ϵ) is an irreducible weighted graph of general Puiseux type. Let $\bar{G}_1 \subset \bar{S}_1 \times \bar{S}_1, \ldots, \bar{G}_r \subset \bar{S}_r \times \bar{S}_r$, $1 \le r \le d$, be the set of irreducible components of G^*_{min} of dominant spectral coefficient $\bar{\alpha}$. Let $\bar{G} := \bar{G}_1 \cup \ldots \cup \bar{G}_r$ be the dominant subgraph, and $\bar{S} := \bar{S}_1 \cup \ldots \cup \bar{S}_r$ be the set of vertices of $\bar{G}$. Denote $G_0 = G \setminus \bar{G}$ and $S_0 = S \setminus \bar{S}$. We write M_ϵ as a $(r+1)\times(r+1)$ block matrix in the following way*

$$M_\epsilon = \begin{bmatrix} \oplus^r_{i,j=1} M^{ij}_\epsilon & \oplus^r_{i=1} M^{i0}_\epsilon \\ \oplus^r_{j=1} M^{0j}_\epsilon & M^{00}_\epsilon \end{bmatrix},$$

$$M^{00}_\epsilon = [M_\epsilon(x,y)]_{x,y\in S_0}, \quad M^{i0}_\epsilon = [M_\epsilon(x,y)]_{x\in\bar{S}_i, y\in S_0}, \quad M^{0j}_\epsilon = [M_\epsilon(x,y)]_{x\in S_0, y\in\bar{S}_j},$$
$$\textit{and} \quad M^{ij}_\epsilon = [M_\epsilon(x,y)]_{x,y\in\bar{S}_i\times\bar{S}_j}, \ \forall\ 1 \le i,j \le r.$$

We call dominant matrix $\bar{A}$ the diagonal matrix obtained by keeping only the submatrices A^{ii}_{min} with dominant spectral radius

$$\bar{A} := [A(x,y)\mathbb{1}_{\bar{G}}(x,y)]_{x,y\in\bar{S}} = \begin{bmatrix} \bar{A}^{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{A}^{rr} \end{bmatrix},$$
$$\bar{A}^{ii} = [A(x,y)\mathbb{1}_{\bar{G}_i}(x,y)]_{x,y\in\bar{S}_i} = A^{ii}_{min}, \quad \forall\ i = 1,\ldots,r.$$

By convention all matrices $\bar{A}^{ij}$, $1 \le i \ne j \le r$, are equal to 0. Notice that

$$\lambda_\epsilon := \sup\{|\lambda| \,:\, \lambda \in \mathrm{spec}(M_\epsilon)\} = \rho_{spec}(M_\epsilon)$$

is an eigenvalue of mutiplicity 1 and unique on the circle $\{|\lambda| = \lambda_\epsilon\}$. Let L_ϵ and R_ϵ be the left and right eigenvectors of M_ϵ associated to the largest eigenvalue λ_ϵ

$$L_\epsilon = \oplus^r_{i=1} L^i_\epsilon \oplus L^0_\epsilon, \ R_\epsilon = \oplus^r_{i=1} R^i_\epsilon \oplus R^0_\epsilon,$$
$$\sum_{x\in S} L_\epsilon(x)R_\epsilon(x) = 1, \ \textit{and} \ \sum_{x\in S} R_\epsilon(x) = 1,$$

where L_ϵ is a row vector and R_ϵ a column vector. Consider thus

$$\pi_\epsilon(x) := L_\epsilon(x)R_\epsilon(x), \ Q_\epsilon(x,y) := \frac{M_\epsilon(x,y)R_\epsilon(y)}{\lambda_\epsilon R_\epsilon(x)}, \ \textit{and} \ \mu_\epsilon(x,y) := \pi_\epsilon(x)Q_\epsilon(x,y).$$

For each $i = 1, \ldots, r$, $\bar\alpha = \rho_{spec}(\bar A^{ii})$ is an eigenvalue of multiplicity 1 admitting a unique positive left row eigenvector $[\bar L^i(x)]_{x\in\bar S_i}$ and a unique right column eigenvector $[\bar R^i(x)]_{x\in\bar S_i}$ satisfying

$$\bar L^i \bar A^{ii} = \bar\alpha\ \bar L^i,\ \ \bar A^{ii}\bar R^i = \bar\alpha\ \bar R^i,$$
$$\sum_{x\in\bar S_i} \bar L^i(x)\bar R^i(x) = 1,\ \ and\ \sum_{x\in\bar S_i}\bar R^i(x) = 1.$$

Let $\bar\pi^i$, $\bar Q^{ii}$ and $\bar\mu_i$ be defined on $\bar G_i$ as follows

$$\bar\pi^i(x) := \bar L^i(x)\bar R^i(x),\ \bar Q^{ii}(x,y) := \frac{\bar A^{ii}(x,y)\bar R^i(y)}{\bar\alpha\bar R^i(x)},\ \bar\mu_i(x,y) := \bar\pi^i(x)\bar Q^{ii}(x,y).$$

We extend $\bar\mu_i$ on $G\setminus\bar G_i$ by 0.

In the language of dynamical system, the main known result in this setting is recalled in the following theorem.

Theorem 25. *[7, 17, 9] Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be short range observables defined on a transitive subshift of finite type Σ_G^+ given by an irreducible directed graph G. Let $\mu_{E+\beta H}$ be the Gibbs measure associated to $E+\beta H$. For $\epsilon = e^{-\beta}$, consider $M_\epsilon = [A(x,y)\epsilon^{a(x,y)}]_{x,y\in S}$ the transfer matrix, where*

$$\begin{cases} a(x,y) = H(x,y) & and\ \ A(x,y) = e^{E(x,y)}, & \forall\ (x,y)\in G, \\ a(x,y) = +\infty & and\ \ A(x,y) = 0, & \forall\ (x,y)\notin G. \end{cases}$$

We recall that $\mu_{E+\beta H}$ weights each cylinder $[x_0,\ldots,x_n]\in\mathcal{C}_{n+1}(G)$ as

$$\mu_{E+\beta H}([x_0,\ldots,x_n]) = L_\epsilon(x_0)\Big[\prod_{k=0}^{n-1} M_\epsilon(x_k,x_{k+1})\Big]R_\epsilon(x_n)/\lambda_\epsilon^n.$$

Let $\bar G_1,\ldots,\bar G_r$ be the dominant irreducible components of G_{min}. Let $\bar\mu_i$ be the Gibbs measure associated to E restricted to $\Sigma_{\bar G_i}^+$,

$$\bar\mu_i([x_0,\ldots,x_n]) = \bar L^i(x_0)\Big[\prod_{k=0}^{n-1}\bar A^{ii}(x_k,x_{k+1})\Big]\bar R^i(x_n)/\bar\alpha^n,\ \ \forall\,[x_0,\ldots,x_n]\in\mathcal{C}_{n+1}(\bar G_i).$$

Then, the family $\{\mu_{E+\beta H}\}_\beta$ converges to

$$\mu_{min}^{E,H} := \lim_{\beta\to+\infty}\mu_{E+\beta H} = \sum_{i=1}^r c_i^{E,H}\bar\mu_i,$$

where $c_i^{E,H} = \mu_{min}^{E,H}(\bar G_i) \geq 0$ and $\sum_{i=1}^r c_i^{E,H} = 1$.

The existence of the limit in theorem 25 is the main point and was proved by Brémont in [7] using semi-algebraic technics. Leplaideur in [17] gave a dynamical proof and has identified the limit as a barycenter of minimizing measure of maximal pressure. Akian, Bapat and Gaubert (see [1, 2]) using min-plus methods have obtained similar results. Chazottes, Gambaudo and Ugalde in [9] gave a more algorithmic proof. Nekhoroshev has obtained [19] the convergence to a ground state for generic one-dimensional spin systems with nearest neighbors interaction. Chazottes and Hochman in [10] showed a counter-example for the convergence of Gibbs measures associated to a long range interaction.

We intend to partially extend theorem 25 to the case of irreducible weighted graphs (G, M_ϵ) of general Puiseux type. We explain the first two steps of an algorithm based on Puiseux-series expansions. These two steps are enough to describe the limits $\lim_{\epsilon\to0}\pi_\epsilon = \pi_{min}$ and $\lim_{\epsilon\to0}Q_\epsilon = Q_{min}$ for matrices of small dimension. The main difficulty is to identify which irreducible components of G^*_{min} support μ_{min}. The first step consists in writting M_ϵ in a normal form; this step makes use of the notion of correctors (equivalent to the notion of sub-actions introduced in definition 11). The second step consists in aggregating all the states in the same irreducible component, obtaining thus a new weighted graph with a lower dimension.

Definition 26. *Suppose that (G, M_ϵ) is a weighted graph of general Puiseux type, $M_\epsilon \sim A\epsilon^a$, G^*_{min} is the minimizing subgraph of G^*, and $\bar a$ is the minimizing mean exponent of (G, M_ϵ). We call corrector any function $v : S \to \mathbb{R}$ such that*

$$a(x,y) \geq v(y) - v(x) + \bar a, \quad \forall\ (x,y) \in G^*.$$

The corrector is said to be backward or forward calibrated if

$$\begin{aligned} v(y) + \bar a &= \min\nolimits_{x:(x,y)\in G^*}\{v(x) + a(x,y)\}, \quad \forall\ y \in S \quad \textit{(backward)},\\ v(x) - \bar a &= \max\nolimits_{y:(x,y)\in G^*}\{v(y) - a(x,y)\}, \quad \forall\ x \in S \quad \textit{(forward)}.\end{aligned}$$

It is said to be separating if

$$\begin{aligned} a(x,y) &= v(y) - v(x) + \bar a, \quad \forall\ (x,y) \in G^*_{min},\\ a(x,y) &> v(y) - v(x) + \bar a, \quad \forall\ (x,y) \in G^* \setminus G^*_{min}.\end{aligned}$$

It is easy to show that separating correctors exists. We just want to make clear that this notion is a key part to understand the singular perturbations of Perron matrices.

Lemma 27. *The notations being given in definition 26, there exist (not necessarily unique) backward or forward calibrated correctors. There exist (not necessarily unique) separating correctors. The difference of two correctors is constant on each irreducible component.*

The first step of the algorithm is described below.

Algorithm 28 (I. Reduction to a normal form). *Let (G, M_ϵ) be an irreducible weighted graph of general Puiseux type, $M_\epsilon \sim A\epsilon^a$. From main notations 24, recall the partition of S into dominant and non dominant indices: $S = \cup_{i=1}^r \bar S_i \cup S_0$. For $v : S \to \mathbb{R}$ a separating corrector, denote $\Delta_\epsilon(v) := \mathrm{diag}[\epsilon^{v(x)} : x \in S]$ and $\tilde a(x,y) := a(x,y) + v(x) - v(y) - \bar a \geq 0$ for all $(x,y) \in G^*$. Then*

- $\tilde{M}_\epsilon := \Delta_\epsilon(v) M_\epsilon \Delta_\epsilon(v)^{-1}\epsilon^{-\bar{a}} = A^*_{min} + \tilde{N}_\epsilon$ *and* $\tilde{N}_\epsilon = o(1)$*;*
- $A^*_{\min} = \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix}$*, where* $\bar{A} := \mathrm{diag}[\bar{A}^{ii} : i = 1, \ldots, r]$ *is the diagonal matrix of dominant matrices* $\bar{A}^{ii}$*, and* D *is a nonnegative matrix indexed by* S_0 *such that* $\rho_{spec}(D) < \rho_{spec}(\bar{A}^{11}) = \ldots = \rho_{spec}(\bar{A}^{rr})$*;*
- $(G^{\mathit{off}}, \tilde{M}_\epsilon^{\mathit{off}})$ *is an irreducible weighted graph of exact Puiseux type;*
- $\forall\,(x,y) \in G^{\mathit{off}}$*,* $\tilde{M}_\epsilon(x,y) \sim A(x,y)\epsilon^{\tilde{a}(x,y)}$*,* $A(x,y) > 0$*,* $\tilde{a}(x,y) \geq 0$*.*

Wa say that $(G, \tilde{M}_\epsilon)$ *is a normal form of* (G, M_ϵ)*. Let* $\tilde{L}_\epsilon$ *and* $\tilde{R}_\epsilon$ *denote the left and right eigenvectors of* $\tilde{M}_\epsilon$ *for* $\tilde{\lambda}_\epsilon := \rho_{spec}(\tilde{M}_\epsilon)$*. Then* $\tilde{\lambda}_\epsilon = \lambda_\epsilon \epsilon^{-\bar{a}}$ *and*

$$\tilde{L}_\epsilon(x) = \epsilon^{-v(x)} L_\epsilon(x) \quad \text{and} \quad \tilde{R}_\epsilon(x) = \epsilon^{v(x)} R_\epsilon(x), \quad \forall\, x \in S.$$

The following proposition extends proposition 7 in the sense that we admit a more general form of transfer matrix.

Proposition 29. *Let* (G, M_ϵ) *be an irreducible weighted graph of general Puiseux type. Then*

i. $\lambda_\epsilon \sim \bar{\alpha}\epsilon^{\bar{a}}$*;*

ii. $\mu_\epsilon(x,y) \to 0$ *for all* $(x,y) \notin \bar{G}$*,* $\pi_\epsilon(x) \to 0$ *for all* $x \in S_0$*;*

iii. any accumulation measure $\bar{\mu}$ *of* $(\mu_\epsilon)_{\epsilon>0}$ *is of the form* $\bar{\mu} = \sum_{i=1}^r \bar{\mu}(\bar{G}_i)\bar{\mu}_i$*.*

We recover the fact that, if G^*_{min} admits a unique irreducible component of dominant spectral coefficient ($r = 1$), then $\mu_\epsilon \to \bar{\mu}_1$, $\pi_\epsilon(x) \to \bar{\pi}^1(x)$ for all $x \in \bar{S}_1$ and $\pi_\epsilon(x) \to 0$ elsewhere.

The second step of the algorithm is an operation of aggregation.

Algorithm 30 (II. Reduction to an aggregated form). *Let* (G, M_ϵ) *be an irreducible weighted graph of general Puiseux type. Assume that* $(G, \tilde{M}_\epsilon)$ *is a normal form of* (G, M_ϵ)*. We write*

$$\tilde{M}_\epsilon = \begin{bmatrix} \oplus_{i,j=1}^r \tilde{M}_\epsilon^{ij} & \oplus_{i=1}^r \tilde{M}_\epsilon^{i0} \\ \oplus_{j=1}^r \tilde{M}_\epsilon^{0j} & \tilde{M}_\epsilon^{00} \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix} + \tilde{N}_\epsilon.$$

(Notice that $\bar{A}(x,y) = A(x,y)\mathbb{1}_{\tilde{a}(x,y)=0}$ *for all* $x, y \in \bar{S} = \bar{S}_1 \cup \ldots \cup \bar{S}_r$*.) The right eigenvector* $\tilde{R}_\epsilon$ *is solution of the system*

$$\begin{cases} \sum_{j=1}^r \tilde{M}_\epsilon^{ij}\tilde{R}_\epsilon^j + \tilde{M}_\epsilon^{i0}\tilde{R}_\epsilon^0 &= \tilde{\lambda}_\epsilon \tilde{R}_\epsilon^i, \quad \forall\, i = 1, \ldots, r, \\ \sum_{j=1}^r \tilde{M}_\epsilon^{0j}\tilde{R}_\epsilon^j + \tilde{M}_\epsilon^{00}\tilde{R}_\epsilon^0 &= \tilde{\lambda}_\epsilon \tilde{R}_\epsilon^0. \end{cases}$$

As $\rho_{spec}(\tilde{M}_\epsilon^{00}) \to \rho_{spec}(D) < \bar{\alpha} \sim \tilde{\lambda}_\epsilon$*,* $\tilde{R}_\epsilon^0$ *can be written linearly with respect to* $\tilde{R}_\epsilon^i$*. We thus obtain*

$$\sum_{j=1}^r \Big(\tilde{M}_\epsilon^{ij} + \tilde{M}_\epsilon^{i0}(\tilde{\lambda}_\epsilon - \tilde{M}_\epsilon^{00})^{-1}\tilde{M}_\epsilon^{0j}\Big)\tilde{R}_\epsilon^j = \tilde{\lambda}_\epsilon \tilde{R}_\epsilon^i.$$

We take the scalar product of each equation by the left eigenvector $\bar L^i$. We extract the dominant term $\bar A$ and obtain a new weighted graph $(G^{(1)}, M_\epsilon^{(1)})$ indexed by $S^{(1)} := \{1, \dots, r\}$ defined in the following way. For $i \neq j$, let $\mathcal{P}(i,j)$ denote the set of G-admissible paths $\underline{x} := (x_0, \dots, x_n)$ such that $n \geq 1$, $x_0 \in \bar S_i$, $x_1, \dots, x_{n-1} \in S_0$ and $x_n \in \bar S_j$. Then

- *for all $i \neq j$, $(i,j) \in G^{(1)}$ if, and only if, $\mathcal{P}(i,j) \neq \emptyset$;*
- *for all $i = 1, \dots, r$, $(i,i) \in G^{(1)}$ (by convention);*
- $M_\epsilon^{(1)}(i,j) = \bar L^i\Big(\tilde N_\epsilon^{ij} + \tilde M_\epsilon^{i0}(\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}\tilde M_\epsilon^{0j}\Big)\frac{\tilde R_\epsilon^j}{\bar L^j \tilde R_\epsilon^j}$.

The new eigenvalue problem is related to the previous one by

$$\sum_{j=1}^{r} M_\epsilon^{(1)}(i,j)R_\epsilon^{(1)}(j) = (\tilde\lambda_\epsilon - \bar\alpha)R_\epsilon^{(1)}(i), \quad R_\epsilon^{(1)}(i) = \bar L^i \tilde R_\epsilon^i, \quad \forall\ i = 1, \dots, r.$$

We say that $(G^{(1)}, M_\epsilon^{(1)})$ is an aggregated form of (G, M_ϵ). Note that $\sum_{i=1}^r R_\epsilon^{(1)}(i)$ may not be equal to 1.

Proposition 31. *Let (G, M_ϵ) be an irreducible weighted graph of general Puiseux type. Let $(G^{(1)}, M_\epsilon^{(1)})$ be its aggregated form defined by the separating corrector $v : S \to \mathbb{R}$. If $\tilde a(x,y) = a(x,y) + v(x) - v(y) - \bar a$ for all $(x,y) \in G^*$ and $\underline{x} = (x_0, \dots, x_n)$ belongs to $\mathcal{P}(i,j)$, denote $\tilde a(\underline{x}) := \sum_{i=0}^{n-1} \tilde a(x_i, x_{i+1})$. Then*

i. $(G^{(1)\mathit{off}}, M_\epsilon^{(1)\mathit{off}})$ is an irreducible weighted graph of exact Puiseux type, with $M_\epsilon^{(1)\mathit{off}} \sim A^{(1)}\epsilon^{a^{(1)}}$, where, for all $(i,j) \in G^{(1)\mathit{off}}$,

$$a^{(1)}(i,j) := \min\big\{\tilde a(\underline{x}) \ :\ \underline{x} \in \mathcal{P}(i,j)\big\} \quad \text{and}$$

$$A^{(1)}(i,j) := \sum_{\substack{\underline{x}=(x_0,\dots,x_n)\in\mathcal{P}(i,j) \\ \tilde a(\underline{x}) = a^{(1)}(i,j)}} \frac{\bar L^i(x_0)\Pi_{k=0}^{n-1}A(x_k, x_{k+1})\bar R^j(x_n)}{\bar\alpha^{n(\underline{x})-1}};$$

ii. for all $i = 1, \dots, r$ and $x, y \in \bar S_i$,

$$\frac{L_\epsilon^i(x)}{L_\epsilon^i(y)} \sim \frac{\epsilon^{v(x)}}{\epsilon^{v(y)}}\frac{\bar L^i(x)}{\bar L^i(y)}, \quad \text{and} \quad \frac{R_\epsilon^i(x)}{R_\epsilon^i(y)} \sim \frac{\epsilon^{-v(x)}}{\epsilon^{-v(y)}}\frac{\bar R^i(x)}{\bar R^i(y)};$$

iii. for all $i \neq j \in \{1, \dots, r\}$ and $x \in \bar S_i$,

$$Q_\epsilon(x,y) \to 0,\ \forall\ y \in \bar S_j \cup S_0, \qquad Q_\epsilon(x,y) \to \bar Q^{ii}(x,y),\ \forall\ y \in \bar S_i.$$

Notice that no estimate is given in the previous proposition for the quotients $R_\epsilon^i(x)/R_\epsilon^j(y)$ if $x \in \bar S_i$ and $y \in \bar S_j$.

Algorithm 32 (**III. Induction**). *Assume by induction one can prove*

$$\frac{R_\epsilon^{(1)}(i)}{R_\epsilon^{(1)}(j)} \sim \gamma^{(1)}(i,j)\epsilon^{c^{(1)}(i,j)}, \quad \forall\ i=1,\dots,r,$$

for some real coefficients $\gamma^{(1)}(i,j) = \gamma^{(1)}(j,i)^{-1} > 0$ *and* $c^{(1)}(i,j) = -c^{(1)}(j,i)$. *Notice that proposition 31.ii easily implies*

$$\frac{\tilde R_\epsilon^i(x)}{R_\epsilon^{(1)}(i)} \sim \frac{\bar R^i(x)}{\bar L^i \bar R^i} = \bar R^i(x), \quad \forall\ i=1,\dots,r, \quad \forall\ x\in \bar S_i.$$

Let G' *be the graph containing either* (x,x) *for* $x\in S_0$ *or* (x_0,x_n) *if* $(x_0,\dots,x_n)$ *is a path of* $G\cap(S_0\times S_0)$ *such that* $D(x_k,x_k+1)>0$. *Let* $M'_\epsilon = (\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}$. *Then* (G',M'_ϵ) *is a weighted graph of exact Puiseux type (see lemma 48). It follows that*

$$\frac{\tilde R_\epsilon^0(x)}{R_\epsilon^{(1)}(1)} = \sum_{j=1}^r (\tilde\lambda_\epsilon - \tilde M_\epsilon^{00})^{-1}\tilde M_\epsilon^{0j}\frac{\tilde R_\epsilon^j}{R_\epsilon^{(1)}(1)}(x) \sim \gamma^{(1)}(x)\epsilon^{c^{(1)}(x)}$$

for some coefficients $\gamma^{(1)}(x)>0$ *and* $c^{(1)}(x)\in\mathbb{R}$. *One thus may obtain*

$$\frac{R_\epsilon(x)}{R_\epsilon(y)} \sim \gamma(x,y)\epsilon^{c(x,y)}, \quad \forall\ x,y\in S,$$

for some real coefficients $\gamma(x,y) = \gamma(y,x)^{-1} > 0$ *and* $c(x,y) = -c(y,x)$. *The normalization* $\sum_{x\in S} R_\epsilon(x) = 1$ *then implies*

$$R_\epsilon(x) = \frac{1}{\sum_{y\in S}\frac{R_\epsilon(y)}{R_\epsilon(x)}} \sim \rho(x)\epsilon^{r(x)}, \quad \forall\ x\in S, \quad \textit{with}$$

$$\rho(x) := \Big(\sum_{y=\arg\max c(x,y)} \gamma(y,x)\Big)^{-1} \quad \textit{and} \quad r(x) := \max_{y\in S} c(x,y).$$

Similar equivalences can be written for $L_\epsilon(x)$ *and* $Q_\epsilon(x,y)$. *In particular, the limits* $\lim_{\epsilon\to 0}\pi_\epsilon(x)$ *and* $\lim_{\epsilon\to0} Q_\epsilon(x,y)$ *exist for all* $x,y\in S$.

4 Proofs of results stated in section 2

We begin by proving the results of section 2 for a transitive subshift of finite type (Σ_G^+,σ) defined by an irreducible directed graph G on a finite state space S. Let $E,H:\Sigma_G^+\to\mathbb{R}$ be two Hölder functions. Proposition 7 has been noticed many times as in [11, 17]. We nevertheless give the proof of this proposition in order to point out the following inequalities.

Lemma 33. *For any* $\beta>0$, $\mathrm{Pres}_{\Omega(H)}(E) \le \mathrm{Pres}(E+\beta H) + \beta\bar H \le \mathrm{Pres}(E)$. *If* $\mu_{E+\beta H}$ *is the Gibbs-equilibrium measure of* $E+\beta H$, *then*

$$0\le \beta\Big(\int H\,d\mu_{E+\beta H} - \bar H\Big) \le \mathrm{Pres}(E) - \mathrm{Pres}_{\Omega(H)}(E), \quad \textit{and}$$

$$\mathrm{Pres}_{\Omega(H)}(E) \le \mathrm{Ent}\ (\mu_{E+\beta H}) - \int E\,d\mu_{E+\beta H}.$$

Proof. On the one hand, if $\mu_{\min}$ is any H-minimizing probability with relative maximal pressure in $\Omega(H)$, then

$$\begin{aligned}\operatorname{Pres}_{\Omega(H)}(E) - \beta\bar{H} = \operatorname{Ent}(\mu_{\min}) - \int E\,d\mu - \beta\bar{H} = \\ = \operatorname{Ent}(\mu_{\min}) - \int (E+\beta H)\,d\mu_{\min} \le \operatorname{Pres}(E+\beta H).\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{Pres}(E+\beta H) &= \operatorname{Ent}(\mu_{E+\beta H}) - \int (E+\beta H)\,d\mu_{E+\beta H}, \quad \text{either} \\ &\le \operatorname{Ent}(\mu_{E+\beta H}) - \int E\,d\mu_{E+\beta H} - \beta\bar{H}, \quad \text{or} \\ &\le \operatorname{Pres}(E) - \beta\int H\,d\mu_{E+\beta H} \le \operatorname{Pres}(E) - \beta\bar{H}.\end{aligned}$$

□

Proof of proposition 7. We first remark

$$0 \le \int H\,d\mu_{E+\beta H} - \bar{H} \le \frac{1}{\beta}[\operatorname{Pres}(E) - \operatorname{Pres}_{\Omega(H)}(E)]$$

implies that $\{\int H\,d\mu_{E+\beta H}\}_\beta$ converges to $\bar{H}$ as $\beta \to +\infty$ and that any weak* limit of $\{\mu_{E+\beta H}\}_\beta$ is actually minimizing for H. Let μ_∞ be a weak* accumulation probability. We next observe that the upper semi-continuity of the entropy map $\beta \mapsto \operatorname{Ent}(\mu_{E+\beta H})$ implies

$$\begin{aligned}\operatorname{Pres}_{\Omega(H)}(E) &\ge \operatorname{Ent}(\mu_\infty) - \int E\,d\mu_\infty \\ &\ge \limsup_{\beta\to+\infty}\Big(\operatorname{Ent}(\mu_{E+\beta H}) - \int E\,d\mu_{E+\beta H}\Big) \ge \operatorname{Pres}_{\Omega(H)}(E).\end{aligned}$$

All inequalities in the previous estimate are therefore equalities and lim sup should be understood as a limit. □

The rest of this part is now devoted to the proof of theorem 15. We first give some complements on the Peierls barrier. As usual, define the Birkhoff sum of an observable $\Psi : \Sigma_G^+ \to \mathbb{R}$ as

$$S_n\Psi(x) = \sum_{k=0}^{n-1} \Psi\circ\sigma^k(x), \quad \forall\, x \in \Sigma_G^+.$$

Lemma 34. *Let $h(x,y)$ be the Peierls barrier introduced in definition 13.*

i. The function $h : \Sigma_G^+ \times \Sigma_G^+ \to \mathbb{R}\cup\{+\infty\}$ is lower semi-continuous.

ii. If $V : \Sigma_G^+ \to \mathbb{R}$ is a continuous sub-action, $V(y) - V(x) \le h(x,y)$.

iii. *For any* $x \in \Omega(H)$, $h(x,\cdot) : \Sigma_G^+ \to \mathbb{R}$ *is Hölder (and finite).*

iv. *For any* $x, y, z \in \Sigma_G^+$, $h(x,z) \leq h(x,y) + h(y,z)$.

v. *For any* $y \in \Sigma_G^+$, $h(\cdot, y) : \Sigma_G^+ \to \mathbb{R} \cup \{+\infty\}$ *is a coboundary of* $H - \bar{H}$,

$$(H - \bar{H})(x) + h(\sigma(x), y) = h(x,y), \quad \forall\ x, y \in \Sigma_G^+.$$

vi. *For any* $x \in \Sigma_G^+$, $\sigma^n(x) \in \Omega(H) \Rightarrow h(x, \sigma^n(x)) = S_n(H - \bar{H})(x)$.

Proof. Items i, ii, iii and iv are well known and have been discussed, for instance, in [11, 12, 13].

Item v. Suppose $\epsilon \in (0,1)$. If z' is close to $\sigma(x)$, $d(z', \sigma(x)) < \epsilon$, one can find z close to x, $d(z,x) < \epsilon/2$, such that $\sigma(z) = z'$. Hence, if $\mathrm{osc}_1(H,\eta) := \sup\big\{H(x) - H(y)\ :\ d(x,y) \leq \eta\big\}$, then

$$S_{n+1}^{\epsilon/2}(x,y) \leq (H - \bar{H})(x) + S_n^{\epsilon}(\sigma(x), y) + \mathrm{osc}_1(H, \epsilon/2).$$

Conversely, if $d(z,x) < \epsilon$, then $d(\sigma(z), \sigma(x)) < 2\epsilon$. Therefore

$$S_{n+1}^{\epsilon}(x,y) \geq (H - \bar{H})(x) + S_n^{2\epsilon}(\sigma(x), y) - \mathrm{osc}_1(H, \epsilon).$$

Item v is proved by taking $\liminf_{n \to +\infty}$ first and $\lim_{\epsilon \to 0}$ afterwards.

Item vi. From the previous item, we have by induction

$$S_n(H - \bar{H})(x) + h(\sigma^n(x), y) = h(x,y).$$

If $y = \sigma^n(x) \in \Omega(H)$, then $h(y,y) = 0$ and item vi is proved. □

From now on the minimizing non-wandering set $\Omega(H)$ can be decomposed into a disjoint union of irreducible components $\Omega(H) = \Omega_0 \cup \ldots \cup \Omega_r$ (see definition 14). Each Ω_i is necessarily closed and invariant. We fixed once for all $x_i^* \in \Omega_i$. We recall that $\Omega_i = \{x \in \Sigma_G^+\ :\ h(x, x_i^*) + h(x_i^*, x) = 0\}$ and that, for any $i \neq j$, $h(x_i^*, x_j^*) + h(x_j^*, x_i^*) > 0$.

Lemma 35. *Assume* $\Omega(H) = \Omega_0 \cup \ldots \cup \Omega_r$ *is a disjoint union of irreducible components. Let* $V : \Sigma_G^+ \to \mathbb{R}$ *be any continuous sub-action. Then*

i. *The quantities* $\bar{h}_V(i,j) := h(x_i^*, x_j^*) - V(x_j^*) + V(x_i^*)$ *are nonnegative and independent of the choice of* $x_i^* \in \Omega_i$.

ii. $\bar{h}_V(i,i) = 0$ *for all* $i = 0, 1, \ldots, r$.

iii. *If* $\bar{h}_V(0,j) = 0$ *for all* $j = 1, \ldots, r$ *and* V *is a calibrated sub-action, then* $V(y) - V(x) = h(x,y)$ *for all* $x \in \Omega_0$ *and* $y \in \Sigma_G^+$, *that is,* V *is unique provided* $V(x_0)$ *is known for some* $x_0 \in \Omega_0$.

Proof. Item *i*. Let $h_V(x,y) := h(x,y) - V(y) + V(x) \geq 0$ for all $x, y \in \Sigma_G^+$. Hence, $x \sim y$ if, and only if, $h_V(x,y) + h_V(y,x) = 0$ if, and only if, $h_V(x,y) = h_V(y,x) = 0$. Suppose $x, x', y, y' \in \Omega(H)$ satisfy $x \sim x'$ and $y \sim y'$. Because of lemma 34.*iv*,

$$h_V(x,y) \leq h_V(x,x') + h_V(x',y) = h_V(x',y).$$

Equivalently $h_V(x',y) \leq h_V(x,y)$ and thus $h_V(x',y) = h_V(x,y)$. For the same reason, $h_V(x',y) = h_V(x',y')$. We just have proved $h_V(x,y) = h_V(x',y')$.

Item *ii*. It is immediate from the definition of $\bar{h}_V$.

Item *iii*. From [4, 12], calibrated sub-actions have the following characterization $V(y) = \min\{V(x) + h(x,y) \ : \ x \in \Omega(H)\}$ for all $y \in \Sigma_G^+$. Then, for any fixed $x_0 \in \Omega_0$, on the one hand,

$$\begin{aligned} V(y) &= \min_{j=0,\ldots,r} \min_{x \in \Omega_j} [V(x) + h(x,y)] \\ &\geq \min_{j=0,\ldots,r} \min_{x \in \Omega_j} [V(x) - V(x_0) + h(x,x_0)] + V(x_0) + h(x_0,y) \\ &= V(x_0) + h(x_0,y). \end{aligned}$$

On the other hand, because V is a sub-action, $h(x_0,y) \geq V(y) - V(x_0)$. We have proved that $V(y) = V(x) + h(x,y)$ for all $x \in \Omega_0$ and $y \in \Sigma_G^+$. □

Let $\Phi_{E+\beta H} = \exp(-\beta V_{E+\beta H})$ and $\nu_{E+\beta h}$ be, respectively, the eigenfunction and the eigenmeasure of the Ruelle transfer operator $\mathcal{L}_{E+\beta H}$, normalized by $\int \Phi_{E+\beta H} \, d\nu_{E+\beta H} = 1$. We know that $\{V_{E+\beta H}\}_\beta$ has uniform sup-norm and uniform Hölder norm. Let V_∞ be any accumulation point in the C^0 topology. Proposition 12 tells us that V_∞ is calibrated. We assume that $\mathrm{Pres}_{\Omega_0}(E) > \mathrm{Pres}_{\Omega_1 \cup \ldots \cup \Omega_r}(E)$. We want to prove that $V_\infty(y) - V_\infty(x) = h(x,y)$ for any $x \in \Omega_0$ and $y \in \Sigma_G^+$, which will show that, for any fixed $x_0 \in \Omega_0$,

$$V_{E+\beta H}(y) - V_{E+\beta H}(x_0) \to V_\infty(y) - V_\infty(x_0), \quad \text{uniformly in } y \in \Sigma_G^+.$$

That convergence will indeed follow from lemma 35.*iii* and the next lemma.

Lemma 36. *Let $V : \Sigma_G^+ \to \mathbb{R}$ be any sub-action and $\bar{h}_V(i,j)$ be defined as in lemma 35. Assume, for any $j = 1, \ldots, r$, there exists $i = 0, 1, \ldots, r$, $i \neq j$, such that $\bar{h}_V(i,j) = 0$. Then $\bar{h}_V(0,j) = 0$ for all $j = 1, \ldots, r$.*

Proof. Assume by contradiction that $\bar{h}_V(0,j_1) > 0$ for some $j_1 = 1, \ldots, r$. Define $J := \{j = 1, \ldots, r \ : \ \bar{h}_V(0,j) > 0\}$. Notice that if $j_1 \in J$ and $\bar{h}_V(j_2,j_1) = 0$ for some $j_2 = 0, 1, \ldots, r$, $j_2 \neq j_1$, then necessarily $j_2 \neq 0$ and $j_2 \in J$. By hypothesis, one can therefore construct a sequence $j_1, j_2, \ldots \in J$ such that

$$\ldots = \bar{h}_V(j_3,j_2) = \bar{h}_V(j_2,j_1) = 0 \quad \text{and} \quad j_{k+1} \neq j_k.$$

Because the number of irreducible components is finite, there exist two distinct indices $s < t$ such that $\bar{h}_V(j_t,j_{t-1}) = \ldots = \bar{h}_V(j_{s+1},j_s) = 0$ and $j_s = j_t$. We obtain, for instance, $\bar{h}_V(j_s,j_{s+1}) = 0 = \bar{h}_V(j_{s+1},j_s)$, which is in contradiction with $\Omega_{j_{s+1}} \neq \Omega_{j_s}$. □

In order to be able to apply the initial assumption of lemma 36, we fix from now on $j = 1, \ldots, r$, $\tilde{\Omega} = \Omega_j$ and $\bar{\Omega} = \cup_{i \neq j} \Omega_i$. Clearly, $\bar{\Omega}$ and $\tilde{\Omega}$ are disjoint closed invariant sets and $\mathrm{Pres}_{\bar{\Omega}}(E) > \mathrm{Pres}_{\tilde{\Omega}}(E)$. We want to show that

$$\min\{h(x,y) - V_\infty(y) + V_\infty(x) \ : \ x \in \bar{\Omega} \text{ and } y \in \tilde{\Omega}\} = 0.$$

We begin by introducing some notations.

Notations 37. *Let $V : \Sigma_G^+ \to \mathbb{R}$ be any Hölder sub-action. Consider the function*

$$h_V(x,y) := h(x,y) - V(y) + V(x) \geq 0, \quad \forall \ x, y \in \Sigma_G^+,$$

which is the Peierls barrier of the observable $H_V := H - \bar{H} - V \circ \sigma + V \geq 0$. Assume that $\Omega(H) = \bar{\Omega} \cup \tilde{\Omega}$ is a disjoint union of two closed σ-invariant sets with $\tilde{\Omega}$ irreducible. For $\epsilon > 0$, denote

$$K_V(\tilde{\Omega}, \epsilon) := \{x \in \Sigma_G^+ \ : \ \exists \ y \in \tilde{\Omega} \ \ s.\ t. \ \ h_V(x,y) \leq \epsilon\}.$$

We will need to approximate $\mathrm{Pres}_{\tilde{\Omega}}(E)$ by the pressure of E restricted to transitive subshifts of finite type $\tilde{\Sigma}_d \supset \tilde{\Omega}$ which decrease to $\tilde{\Omega}$. In order to introduce them, the following notion will be useful.

Definition 38. *A closed σ-invariant set $\tilde{\Omega} \subset \Sigma_G^+$ is said to be quasi-transitive if, for any $x, y \in \tilde{\Omega}$, for any $\epsilon > 0$, there exist $z \in \Sigma_G^+$ and an integer $n \geq 0$ such that*

$$d(z,x) < \epsilon, \quad d(\sigma^n(z), y) < \epsilon \quad \textit{and} \quad d(\sigma^k(z), \tilde{\Omega}) < \epsilon, \quad \forall \ k = 0, 1, \ldots, n.$$

Lemma 39. *Any isolated irreducible component $\tilde{\Omega}$ of $\Omega(H)$ (there exists an open set $\tilde{U}$ containing $\tilde{\Omega}$ such that $\tilde{U} \cap \Omega(H) = \tilde{\Omega}$) is quasi-transitive.*

Proof. Let V be any Hölder separating sub-action, namely, a Hölder sub-action such that $H_V^{-1}(0) = \Omega(H)$ (for details, see [13]). For $\epsilon > 0$, let U_ϵ and $\tilde{U}_\epsilon$ be neighborhoods of size ϵ of $\Omega(H)$ and $\tilde{\Omega}$, respectively. Assume ϵ is sufficiently small enough so that if $z \in \tilde{U}_\epsilon$ and $k \geq 1$ is the first time such that $\sigma^{k-1}(z) \in \tilde{U}_\epsilon$ and $\sigma^k(z) \notin \tilde{U}_\epsilon$, then $\sigma^k(z) \notin U_\epsilon$. Let $\eta > 0$ sufficiently small enough so that $\{z \in \Sigma_G^+ : H_V(z) < \eta\} \subset U_\epsilon$. Since $\tilde{\Omega}$ is irreducible, given $x, y \in \tilde{\Omega}$, there exist infinitely many positive integers n and points $z_n \in \Sigma_G^+$ such that

$$d(z_n, x) < \epsilon, \quad d(\sigma^n(z_n), y) < \epsilon \quad \text{and} \quad S_n H_V(z_n) < \eta.$$

Since $z_n \in \tilde{U}_\epsilon$ and $H_V \circ \sigma^k(z_n) < \eta$, then $\sigma^k(z_n) \in \tilde{U}_\epsilon$, $\forall \ k = 0, 1, \ldots, n$. □

Lemma 40. *Let $\tilde{\Omega}$ be a quasi-transitive closed σ-invariant set. Let $\tilde{U}_d$ be the union of all cylinders $B = [x_0, x_1, \ldots, x_{d-1}]$ of length d such that $B \cap \tilde{\Omega} \neq \emptyset$. Consider $\tilde{\Sigma}_d = \{x \in \Sigma_G^+ : \sigma^n(x) \in \tilde{U}_d, \ \forall \ n \geq 0\} \supset \tilde{\Omega}$. Then*

i. $(\tilde{\Sigma}_d, \sigma)$ is bi-Hölder conjugate to a transitive subshift of finite type.

ii. There exists a constant $\tilde{C}_d > 0$ such that

$$\tilde{C}_d^{-1} \leq \sum_{\substack{x \in \tilde{\Sigma}_d \\ \sigma^n(x) = y}} \exp[-S_n(E + \mathrm{Pres}_{\tilde{\Sigma}_d}(E))(x)] \leq \tilde{C}_d, \quad \forall \ y \in \tilde{\Sigma}_d, \ \forall \ n \geq 0.$$

iii. $\lim_{d\to+\infty}\operatorname{Pres}_{\tilde\Sigma_d}(E) = \operatorname{Pres}_{\tilde\Omega}(E)$.

Proof. Item *i*. Let $\tilde S(d)$ be the set of cylinders $[x_0,\ldots,x_{d-1}]$ which have a non-empty intersection with $\tilde\Omega$. Let $\tilde G(d)\subset \tilde S(d)\times\tilde S(d)$ be the graph defined by the transitions

$$[x_0,\ldots,x_{d-1}] \xrightarrow{\tilde G(d)} [x'_1,\ldots,x'_d] \ \Leftrightarrow\ (x_1,\ldots,x_{d-1}) = (x'_1,\ldots,x'_{d-1}) \text{ and } x_{d-1}\overset{G}{\to} x'_d.$$

Let $\Sigma^+_{\tilde G(d)}$ be the subshift of finite given by the graph $\tilde G(d)$. Thus $\Sigma^+_{\tilde G(d)}$ is transitive since $\tilde\Omega$ is quasi-transitive and $\Sigma^+_{\tilde G(d)}$ is bi-Hölder conjugate to $\tilde\Sigma_d$ by the conjugacy $\{[x_0^n,\ldots,x_{d-1}^n]\}_{n\ge0}\mapsto\{x_0^n\}_{n\ge0}$.

Item *ii*. This estimate is true for any transitive subshift of finite type, being invariant under topological conjugacy.

Item *iii*. Since $\tilde\Omega\subset\tilde\Sigma_d$, we have on the one hand $\operatorname{Pres}_{\tilde\Omega}(E)\le\operatorname{Pres}_{\tilde\Sigma_d}(E)$. On the other hand, if $\tilde\mu_d$ denotes the equilibrium measure associated to the observable $E:\tilde\Sigma_d\to\mathbb{R}$ and $\tilde\mu_\infty$ denotes an accumulation point of $\{\tilde\mu_d\}_{d\to+\infty}$, then $\operatorname{supp}(\tilde\mu_\infty)\subset\tilde\Omega$ and

$$\begin{aligned}\limsup_{d\to+\infty}\operatorname{Pres}_{\tilde\Sigma_d}(E) &= \limsup_{d\to+\infty}\Big(\operatorname{Ent}(\tilde\mu_d)-\int E\,d\tilde\mu_d\Big)\\ &\le \operatorname{Ent}(\tilde\mu_\infty)-\int E\,d\tilde\mu_\infty\le\operatorname{Pres}_{\tilde\Omega}(E).\end{aligned}$$

We have proved that $\operatorname{Pres}_{\tilde\Sigma_d}(E)\to\operatorname{Pres}_{\tilde\Omega}(E)$. □

Lemma 41. *Consider the decomposition* $\Omega(H)=\bar\Omega\cup\tilde\Omega$ *as in notations 37. For a Hölder sub-action* $V:\Sigma^+_G\to\mathbb{R}$, *assume* $\min\{h_V(x,y): x\in\bar\Omega$ *and* $y\in\tilde\Omega\}>\epsilon>0$. *Then*

i. $K_V(\tilde\Omega,\epsilon)$ *is closed, invariant and disjoint from* $\bar\Omega$. *Moreover,*

$$S_nH_V(x)\le\epsilon,\quad\forall\ x\in K_V(\tilde\Omega,\epsilon),\ \forall\ n\ge0.$$

ii. If $\tilde U\supset\tilde\Omega$ *is open and disjoint from* $\bar\Omega$, *then*

$$\sup_{x\in K_V(\tilde\Omega,\epsilon),\ n\ge1}\operatorname{card}\{j=0,1,\ldots,n-1\ :\ \sigma^j(x)\notin\tilde U\}<+\infty.$$

(Every orbit of $K_V(\tilde\Omega,\epsilon)$ *stays most of the time in* $\tilde U$.*)*

iii. If $\tilde C(n):=\sup\Big\{\sum_{x\in K_V(\tilde\Omega,\epsilon),\ \sigma^n(x)=y}\exp[-S_n(E+\operatorname{Pres}_{\tilde\Omega}(E))(x)] : y\in\tilde\Omega\Big\}$ *for every* $n\ge1$, *then* $\limsup_{n\to+\infty}\frac1n\ln\tilde C(n)\le0$.

Proof. For simplicity, denote $\tilde K=K_V(\tilde\Omega,\epsilon)$.

Item *i*. Since $h(x,y)$ is lower semi-continuous and $\tilde\Omega$ is compact, we deduce that $\tilde K$ is closed. From lemma 34.*v*, we have

$$h_V(\sigma(x),y)\le H_V(x)+h_V(\sigma(x),y)=h_V(x,y),\quad\forall\ x,y\in\Sigma^+_G.$$

In particular, $h_V(x,y) \leq \epsilon \Rightarrow h_V(\sigma(x),y) \leq \epsilon$, which shows that $\tilde{K}$ is invariant. Iterating this last formula, we also obtain

$$S_n H_V(x) \leq S_n H_V(x) + h_V(\sigma^n(x),y) \leq h_V(x,y), \quad \forall\ x,y \in \Sigma_G^+.$$

Hence, $S_n H_V(x)$ is uniformly bounded on $n \geq 0$ and $x \in \tilde{K}$.

Item *ii*. Suppose by contradiction there exist a sequence of points $\{x_n\}_{n\geq 1}$ of $\tilde{K}$ such that

$$\mathrm{card}\{j=0,1,\ldots,n : \sigma^j(x_n) \notin \tilde{U}\} \to +\infty.$$

Let $\eta_0 > \eta_1 > \cdots$ be a sequence of positive real numbers decreasing to 0. Let $\{B_i(\eta_0)\}_i$ be a finite cover of $\tilde{K}\backslash\tilde{U}$ by balls of radius η_0. One of these balls contains infinitely many points of $\{\sigma^j(x_n) : j=0,1,\ldots,n,\ n\geq 1\}$. More precisely, there exist a subsequence $\{x_{k_0(n)}\}_{n\geq 1}$ (with $k_0 : \mathbb{N} \to \mathbb{N}$ increasing) and a ball B_{i_0} of radius η_0 such that

$$\mathrm{card}\{j=0,1,\ldots,k_0(n) : \sigma^j(x_{k_0(n)}) \in B_{i_0}\} \to +\infty.$$

By covering B_{i_0} by balls $\{B_i(\eta_1)\}_i$ of radius η_1, one can extract a second subsequence $\{x_{k_0\circ k_1(n)}\}_{n\geq 1}$ (with $k_1 : \mathbb{N} \to \mathbb{N}$ increasing) and choose one of these balls B_{i_1} so that

$$\mathrm{card}\{j=0,1,\ldots,k_0\circ k_1(n) : \sigma^j(x_{k_0\circ k_1(n)}) \in B_{i_1}\} \to +\infty.$$

We continue by induction. Let $k^j(n) = k_0 \circ \ldots \circ k_j(n)$ and z be an accumulation point of $\{B_{i_j}\}_{j\geq 0}$. Let

$$0 = s_0^j < s_1^j < \ldots < s_{r^j(n)-1}^j < s_{r^j(n)}^j = k^j(n)$$

be the successive times $\{s_l^j\}_{l=1}^{r^j(n)-1}$ such that $\sigma^{s_l^j}(x_{k^j(n)}) \in B_{i_j}$. By construction $r^j(n) \to +\infty$. Notice that

$$\sum_{l=0}^{r^j(n)-1} S_{(s_{l+1}^j - s_l^j)} H_V \circ \sigma^{s_l^j}(x_{k^j(n)}) = S_{k^j(n)} H_V(x_{k^j(n)}) \leq \epsilon.$$

Therefore, for infinitely many indices j, one can consider $z_j := \sigma^{s_l^j}(x_{k^j(n)})$ and $n_j := s_{l+1}^j - s_l^j$ for some $l = 1,\ldots,r^j(n)-1$ in such a way that $S_{n_j}H_V(z_j) \to 0$. As $z_j, \sigma^{n_j}(z_j), z \in B_{i_j}$ and $\mathrm{diam}(B_{i_j}) \to 0$, we have proved that $z \in \Omega(H) = \bar{\Omega} \cup \tilde{\Omega}$. Since $z \in \tilde{K} \setminus \tilde{U}$ and $\tilde{K} \setminus \tilde{U}$ is disjoint from $\bar{\Omega}$ and $\tilde{\Omega}$, we obtain a contradiction.

Item *iii*. Let $S(d)$ be the set of non-empty cylinders of Σ_G^+ of size d and $G(d) \subset S(d) \times S(d)$ be the graph whose transitions are given by

$$[x_0,\ldots,x_{d-1}] \overset{G(d)}{\longrightarrow} [x_1',\ldots,x_d'] \ \Leftrightarrow\ (x_1,\ldots,x_{d-1}) = (x_1',\ldots,x_{d-1}') \text{ and } x_{d-1} \overset{G}{\to} x_d'.$$

Denote the oscillation of the Birkhoff sums of E by

$$\mathrm{osc}_n(E) := \sup_{\gamma,x,y}\{S_nE(|\gamma x\rangle) - S_nE(|\gamma y\rangle) :$$
$$\gamma = v_{-n}\ldots v_{-2}v_{-1},\ v_{-1} \overset{G(d)}{\longrightarrow} x \text{ and } v_{-1} \overset{G(d)}{\longrightarrow} y\},$$

where $|\gamma x\rangle$ is the concatenation of a finite $G(d)$-admissible path $\gamma = v_{-n} \dots v_{-2}v_{-1}$ in $S(d)$ and a point x in Σ_G^+, and $v_{-1} \overset{G(d)}{\longrightarrow} x$ just denotes $v_{-1} \overset{G(d)}{\longrightarrow} [x_0, \dots, x_{d-1}]$. Hence, if $v_{-i} = [v_{-i}^0, \dots, v_{-i}^{d-1}] \in S(d)$, $i = 1, \dots, n$, then

$$|\gamma x\rangle := (v_{-n}^0, \dots, v_{-1}^0, x_0, x_1, \dots) \in \Sigma_G^+.$$

More generally, if $\gamma = v_{-n} \dots v_{-1}$ and $\gamma' = v'_{-n'} \dots v'_{-1}$ are $G(d)$-admissible paths of length n and n', we say that γ can be concatenated to γ' if $v_{-1} \overset{G(d)}{\longrightarrow} v'_{-n'}$. Write then $\gamma\gamma' = v_{-n} \dots v_{-1}v'_{-n'} \dots v'_{-1}$.

As in the proof of lemma 40.i, we also consider $\tilde{S}(d)$ the set of vertices $[x_0, \dots, x_{d-1}] \in S(d)$ such that $[x_0, \dots, x_{d-1}] \cap \tilde{\Omega} \neq \emptyset$ and the subgraph $\tilde{G}(d) = G(d) \cap \tilde{S}(d) \times \tilde{S}(d)$. We choose once for all a finite set $\tilde{\Gamma}_d$ of $\tilde{G}(d)$-admissible paths which connects all vertices of $\tilde{S}(d)$ to all vertices of $\tilde{S}(d)$. Given $y \in \tilde{\Omega}$, each inverse branch of order n of y can be written as $x = |\gamma y\rangle$, where $\gamma = v_{-n} \dots v_{-1}$ is a $G(d)$-admissible path and $v_{-1} \overset{G(d)}{\longrightarrow} v_0 := [y_0, \dots, y_{d-1}]$. We partition γ into sub-paths so that alternatively γ_{2i} is a path in $\tilde{S}(d)$ and γ_{2i+1} is a path in $S(d) \setminus \tilde{S}(d)$. More precisely, we consider $\gamma = \gamma_r \dots \gamma_1\gamma_0$ as concatenation of paths γ_i of length n_i (possibly $n_0 = 0$ if $v_{-1} \notin \tilde{S}(d)$ and γ_0 is the empty path) in such a way that

$$\begin{aligned}
&\gamma_0 = v_{-(n_0)} \dots v_{-(1)} \quad \text{is a path in } \tilde{S}(d),\\
&\gamma_1 = v_{-(n_0+n_1)} \dots v_{-(n_0+1)} \quad \text{is a path in } S(d) \setminus \tilde{S}(d),\\
&\gamma_2 = v_{-(-n_0+n_1+n_2)} \dots v_{-(n_0+n_1+1)} \quad \text{is a path in } \tilde{S}(d), \quad \textit{et cetera.}
\end{aligned}$$

We associate to each such an inverse branch γ a new path $\tilde{\gamma}$ in $\tilde{S}(d)$ of the form $\tilde{\gamma} = \tilde{\gamma}_r \dots \tilde{\gamma}_0$, given by the concatenation of paths $\tilde{\gamma}_i$ of length $\tilde{n}_i$ such that $\tilde{\gamma}_{2i} = \gamma_{2i}$ and each sub-path γ_{2i+1} outside $\tilde{S}(d)$ has been replaced by a sub-path $\tilde{\gamma}_{2i+1} = \tilde{v}_{-(\tilde{n}_0+\dots+\tilde{n}_{2i+1})} \dots \tilde{v}_{-(\tilde{n}_0+\dots+\tilde{n}_{2i}+1)}$ in $\tilde{S}(d)$ chosen in $\tilde{\Gamma}_d$ so that

$$\tilde{v}_{-(\tilde{n}_0+\dots+\tilde{n}_{2i}+1)} \overset{\tilde{G}(d)}{\longrightarrow} \tilde{v}_{-(\tilde{n}_0+\dots+\tilde{n}_{2i})} \text{ and } \tilde{v}_{-(\tilde{n}_0+\dots+\tilde{n}_{2i+1}+1)} \overset{\tilde{G}(d)}{\longrightarrow} \tilde{v}_{-(\tilde{n}_0+\dots+\tilde{n}_{2i+1})}.$$

Let $\tilde{n} = \tilde{n}_0 + \tilde{n}_1 \dots + \tilde{n}_r$ be the length of the path $\tilde{\gamma}$. Denote $x_i = |\gamma_i\gamma_{i-1} \dots \gamma_0 y\rangle$ and $\tilde{x}_i = |\tilde{\gamma}_i\tilde{\gamma}_{i-1} \dots \tilde{\gamma}_0 y\rangle$. We want to compare

$$S_n E(|\gamma y\rangle) = \sum_{i=0}^{r} S_{n_i} E(x_i) \quad \text{and} \quad S_{\tilde{n}} E(|\tilde{\gamma} y\rangle) = \sum_{i=0}^{r} S_{\tilde{n}_i} E(\tilde{x}_i).$$

Either γ_i corresponds to a path outside $\tilde{S}(d)$, then

$$S_{n_i} E(x_i) \geq S_{\tilde{n}_i} E(\tilde{x}_i) - (n_i + \tilde{n}_i)\|E\|_\infty,$$

or γ_i corresponds to a path inside $\tilde{S}(d)$, then $\tilde{\gamma}_i = \gamma_i$, $\tilde{x}_i$ and x_i have the same symbols during a period $n_i = \tilde{n}_i$,

$$S_{n_i} E(x_i) \geq S_{\tilde{n}_i} E(\tilde{x}_i) - \operatorname{osc}_{n_i}(E).$$

Let $\tilde{L}_d$ be the maximal length of paths in $\tilde{\Gamma}_d$. Then

$$S_nE(|\gamma y\rangle) \geq S_{\tilde{n}}E(|\tilde{\gamma} y\rangle) - \sum_{i \text{ odd}} n_i(1+\tilde{L}_d)\|E\|_\infty - \sum_{i \text{ even}} \sup\nolimits_n \operatorname{osc}_n(E).$$

Since $\operatorname{card}\{i : \ i \text{ even}\} \leq \operatorname{card}\{i : \ i \text{ odd}\} + 1 \leq 2\sum_{i \text{ odd}} n_i$, we obtain

$$S_nE(|\gamma y\rangle) \geq S_{\tilde{n}}E(|\tilde{\gamma} y\rangle) - [(1+\tilde{L}_d)\|E\|_\infty + 2\sup\nolimits_n \operatorname{osc}_n(E)] \sum_{i \text{ odd}} n_i.$$

We assume from now on that the inverse branch $x = |\gamma y\rangle$ belongs to $\tilde{K}$. From item *ii*, we know that $\sum_{i \text{ odd}} n_i \leq \tilde{N}_d$ is bounded by a constant independent of x and n which only depends on the neighborhood of $\tilde{\Omega}$, $\tilde{U}_d = \cup\{C : C \in \tilde{S}(d)\}$ for d sufficiently large enough. Notice that

$$\sum_{i \text{ odd}} \tilde{n}_i \leq \sum_{i \text{ odd}} \tilde{L}_d \leq \sum_{i \text{ odd}} n_i\tilde{L}_d \leq \tilde{N}_d\tilde{L}_d.$$

We obtain in particular $\tilde{n} = \sum_{i=0}^r \tilde{n}_i \in [n-\tilde{N}_d, n+\tilde{N}_d\tilde{L}_d]$.

In the previous construction, we associate to each inverse branch $x = |\gamma y\rangle \in \tilde{K}$ of length n of y a new inverse branch $\tilde{x} = |\tilde{\gamma} y\rangle$ of length $\tilde{n}$ for the subshift of finite type $(\tilde{\Sigma}_d, \sigma)$ as defined in lemma 40. Since the association $x \mapsto \tilde{x}$ is not injective, we want to bound from above the cardinal of each fiber. Hence, if $\tilde{\gamma}$ has lenght $\tilde{n} \geq 3\tilde{N}_d$, fix a partition $\tilde{I}_r \cup \ldots \cup \tilde{I}_0$ of $\{-\tilde{n}, \ldots, -1\}$ into $r+1$ disjoint consecutive intervals, with $r \in \{1, \ldots, 3\tilde{N}_d\}$, in order to determine a decomposition $\tilde{\gamma} = \tilde{\gamma}_r \ldots \tilde{\gamma}_0$ such that $\tilde{\gamma}_i$ has lenght $\operatorname{card}(\tilde{I}_i)$. The possible $\gamma = \gamma_r \ldots \gamma_0$ associated to $\tilde{\gamma} = \tilde{\gamma}_r \ldots \tilde{\gamma}_0$ must have lenght $n \in [\tilde{n} - \tilde{N}_d\tilde{L}_d, \tilde{n} + \tilde{N}_d]$ and each γ_{2i+1} has lenght at most $\tilde{N}_d$. The cardinal of each fiber is thus bound from above by

$$[\tilde{N}_d(\tilde{L}_d+1)+1]\Big(\sum_{k=1}^{\tilde{N}_d}(\operatorname{card}(S))^k\Big)^{\tilde{N}_d} \sum_{r=1}^{3\tilde{N}_d} \binom{\tilde{n}}{r} \leq \tilde{C}_d' \ n^{3\tilde{N}_d},$$

for some constant $\tilde{C}_d'$ depending only on d. Let

$$\tilde{C}_d'' := \tilde{C}_d' \exp[((1+\tilde{L}_d)\|E\|_\infty + 2\sup\nolimits_n \operatorname{osc}_n(E))\tilde{N}_d].$$

Then

$$\sum_{x\in\tilde{K}, \ \sigma^n(x)=y} \exp[-S_nE(x)] \leq \tilde{C}_d'' \ n^{3\tilde{N}_d} \sum_{\tilde{n}=n-\tilde{N}_d}^{n+\tilde{N}_d\tilde{L}_d} \ \sum_{\tilde{x}\in\tilde{\Sigma}_d, \ \sigma^{\tilde{n}}(\tilde{x})=y} \exp[-S_{\tilde{n}}E(\tilde{x})].$$

Denote $\tilde{C}_d''' := \tilde{C}_d''[\tilde{N}_d(\tilde{L}_d+1)+1]\tilde{C}_d \exp[\tilde{N}_d\tilde{L}_d \operatorname{Pres}_{\tilde{\Sigma}_d}(E)]$, where $\tilde{C}_d$ is the positive constant given by lemma 40.*ii*. Therefore, we get

$$\sum_{x\in\tilde{K}, \ \sigma^n(x)=y} \exp[-S_nE(x)] \leq \tilde{C}_d''' \ n^{3\tilde{N}_d} \ \exp[n\operatorname{Pres}_{\tilde{\Sigma}_d}(E)].$$

Since $\mathrm{Pres}_{\tilde{\Sigma}_d}(E) \to \mathrm{Pres}_{\tilde{\Omega}}(E)$, we finally obtain

$$\limsup_{n\to+\infty} \frac{1}{n} \ln \Big(\sup \Big\{ \sum_{x\in\tilde{K},\ \sigma^n(x)=y} \exp[-S_n(E + \mathrm{Pres}_{\tilde{\Omega}}(E))(x)] : y \in \tilde{\Omega} \Big\} \Big) \le 0.$$

□

In order to prove theorem 15, we summarize in the following proposition the main technical result, which consists in relating the pressure of disjoint parts of the minimizing non-wandering set $\Omega(H)$ and the levels of the Peierls barrier $h(x, y)$ between these parts.

Proposition 42. *Let $E, H : \Sigma_G^+ \to \mathbb{R}$ be Hölder observables. Assume $\Omega(H)$ can be written as a disjoint union $\Omega(H) = \bar{\Omega} \cup \tilde{\Omega}$ of two closed invariant sets. Assume $\tilde{\Omega}$ is irreducible. Let V_∞ be any accumulation point (in the C^0 topology) of $\{V_{E+\beta H}\}_{\beta\to+\infty}$ where $\Phi_{E+\beta H} = \exp(-V_{E+\beta H})$ is the right eigenfunction of the Ruelle operator $\mathcal{L}_{E+\beta H}$ normalized by $\int \Phi_{E+\beta H}\, d\nu_{E+\beta H} = 1$. Then*

$$\mathrm{Pres}_{\bar{\Omega}}(E) > \mathrm{Pres}_{\tilde{\Omega}}(E) \Longrightarrow \min_{x\in\bar{\Omega},\ y\in\tilde{\Omega}} h(x, y) - V_\infty(y) + V_\infty(x) = 0$$

Proof. By contradiction, we suppose that

$$\min_{x\in\bar{\Omega},\ y\in\tilde{\Omega}} h_{V_\infty}(x, y) > \epsilon > 0.$$

Let $\tilde{K} = K_V(\tilde{\Omega}, \epsilon)$ as in notation 37. We consider $\Phi_{E+\beta H}$ as an eigenfunction of $\mathcal{L}^n_{E+\beta H}$ for some $n = n(\beta)$ that will be chosen later. Given $y \in \tilde{\Omega}$, we thus have

$$1 = \sum_{x\in\Sigma_G^+,\ \sigma^n(x)=y} \exp[-\beta S_n(H - \bar{H} - V_{E+\beta H}\circ\sigma + V_{E+\beta H})(x)]$$
$$\exp[-S_nE(x)] \exp[-n(\mathrm{Pres}(E + \beta H) + \beta\bar{H})].$$

We split this sum into two parts

$$I' = \sum_{x\in\Sigma_G^+\backslash\tilde{K},\ \sigma^n(x)=y} \ldots, \quad I'' = \sum_{x\in\tilde{K},\ \sigma^n(x)=y} \ldots .$$

We choose β large enough so that $\|V_{E+\beta H} - V_\infty\|_\infty < \frac{1}{4}\eta$, with $\eta < \epsilon$ to be determined. From lemma 34.*vi*, we have $S_nH_{V_\infty}(x) = h_{V_\infty}(x, y)$, which yields

$$S_n(H - \bar{H} - V_{E+\beta H}\circ\sigma + V_{E+\beta H})(x) \ge h_{V_\infty}(x, y) - 2\|V_{E+\beta H} - V_\infty\|_\infty.$$

We recall from lemma 33 the following inequalities

$$\mathrm{Pres}_{\Omega(H)}(E) \le \mathrm{Pres}(E + \beta H) + \beta\bar{H} \le \mathrm{Pres}(E).$$

We also recall how to compute the pressure using a counting argument on inverse branches ($C = \exp[2\|V_E\|_\infty]$)

$$C^{-1}\exp[n\mathrm{Pres}(E)] \le \sum_{x\in\Sigma_G^+,\ \sigma^n(x)=y} \exp[-S_nE(x)] \le C\exp[n\mathrm{Pres}(E)].$$

Therefore, the first part can be bounded from above in the following way

$$\begin{aligned} I' &\leq \sum_{x\in\Sigma_G^+\setminus\tilde{K},\ \sigma^n(x)=y} \exp[-\beta\frac{\epsilon}{2}]\exp[-S_nE(x)]\exp[-n\mathrm{Pres}_{\Omega(H)}(E)], \\ &\leq C\exp[-\beta\frac{\epsilon}{2}]\exp[n(\mathrm{Pres}(E)-\mathrm{Pres}_{\Omega(H)}(E))]. \end{aligned}$$

The second part is bounded from above using the estimate of lemma 41.*iii*

$$\begin{aligned} I'' &\leq \sum_{x\in\tilde{K},\ \sigma^n(x)=y} \exp[\beta\frac{\eta}{2}]\exp[-S_nE(x)]\exp[-n\mathrm{Pres}_{\Omega(H)}(E)], \\ &\leq \tilde{C}(n)\exp[\beta\frac{\eta}{2}]\exp[n(\mathrm{Pres}_{\tilde{\Omega}}(E)-\mathrm{Pres}_{\bar{\Omega}}(E))]. \end{aligned}$$

We now choose η and $n=n(\beta)$ so that

$$\begin{aligned} -\beta\frac{\epsilon}{2}+n(\mathrm{Pres}(E)-\mathrm{Pres}_{\Omega(H)}(E)) &< -n\frac{\eta}{2}, \\ \beta\frac{\eta}{2}-n(\mathrm{Pres}_{\bar{\Omega}}(E)-\mathrm{Pres}_{\tilde{\Omega}}(E)) &< -n\frac{\eta}{2}, \end{aligned}$$

that is, $\eta/2 < \mathrm{Pres}_{\bar{\Omega}}(E)-\mathrm{Pres}_{\tilde{\Omega}}(E)$ and

$$\frac{\eta/2}{\mathrm{Pres}_{\bar{\Omega}}(E)-\mathrm{Pres}_{\tilde{\Omega}}(E)-\eta/2} < \frac{n}{\beta} < \frac{\epsilon/2}{\mathrm{Pres}(E)-\mathrm{Pres}_{\Omega(H)}(E)+\eta/2}.$$

We thus have obtained, for a subsequence $n\to+\infty$,

$$1 = I'+I'' \leq (C+\tilde{C}(n))\exp[-n\frac{\eta}{2}] \to 0,$$

which is clearly a contradiction. □

Proof of Theorem 15. As before, we fix an accumulation point V_∞ of the sequence $\{V_{E+\beta H}\}_{\beta\to+\infty}$. Let $\Omega(H)=\Omega_0\cup\ldots\Omega_r$ be a disjoint union of irreducible components. By hypothesis, $\mathrm{Pres}_{\Omega_0}(E) > \mathrm{Pres}_{\Omega_1\cup\ldots\Omega_r}(E)$. For $j=1,\ldots,r$, denote $\bar{\Omega}=\cup_{i\neq j}\Omega_i$ and $\tilde{\Omega}=\Omega_j$. Since $\mathrm{Pres}_{\bar{\Omega}}(E) > \mathrm{Pres}_{\tilde{\Omega}}(E)$, proposition 42 implies $\bar{h}_{V_\infty}(i,j)=0$ for some $i\neq j$. Lemma 36 shows that $\bar{h}_{V_\infty}(0,j)=0$ for all $j=1,\ldots,r$. Since V_∞ is calibrated, lemma 35.*iii* implies finally

$$h(x_0,y)=V_\infty(y)-V_\infty(x_0), \quad \forall\ x_0\in\Omega_0,\ \forall\ y\in\Sigma_G^+.$$

If $x_0\in\Omega_0$ is fixed, the sequence $\{V_{E+\beta H}(\cdot)-V_{E+\beta H}(x_0)\}_{\beta\to+\infty}$ has a unique accumulation point $h(x_0,\cdot)$ and therefore converges. □

5 Proofs of results stated in section 3

We study in this section the algorithmic aspects of singular perturbations of Perron matrices of Puiseux type. We start with a weighted irreducible graph (G,M_ϵ) of (general) Puiseux type (recall definition 23) and we write formally $M_\epsilon \sim A\epsilon^a$.

The first step of the algorithm consists in conjugating M_ϵ by a diagonal matrix $\mathrm{diag}[\epsilon^{v(x)} : x \in S]$ so that all entries in $S \times S \setminus G^*_{min}$ are negligeable with respect to $\epsilon^{\bar a}$. The construction of the corrector $v(x)$ is performed in two steps: $v(x)$ is a calibrated corrector in the first step and separating in the second one. A Peierls barrier $h_a(x,y)$ between two vertices is introduced as in definition 13.

Definition 43. *Let $G \subset S \times S$ be an irreducible graph and $a : G \to \mathbb{R}$ be a weigth on each edge. The Peierls barrier (associated to a) between two vertices $x, y \in S$ is defined by*

$$h_a(x,y) := \min \Big\{ \sum_{k=0}^{n-1} (a(x_k, x_{k+1}) - \bar a) : n \geq 1, \\ (x_0, \ldots, x_n) \textit{ is a } G\textit{-admissible path}, \ x_0 = x \textit{ and } x_n = y \Big\}.$$

Notice that it is enough to minimize on simple path: thanks to the choice of the constant $\bar a$, each cycle $(x_0, \ldots, x_n)$ satisfies $\sum_{k=0}^{n-1}(a(x_k, x_{k+1}) - \bar a) \geq 0$ and may be eliminated from the sum.

We summarize several properties of $h_a(x,y)$. Item *vi* of the following lemma gives the definition of the irreducible components of G_{min} and proves lemma 20.

Lemma 44. *Suppose (G, M_ϵ) is an irreducible graph of exact Puiseux type, with $M_\epsilon \sim A\epsilon^a$, and $h_a(x,y)$ is the Peierls barrier associated to $a : G \to \mathbb{R}$. Then*

i. $\forall\ (x_0, \ldots, x_n)$ *G-admissible path,* $h_a(x_0, x_n) \leq \sum_{k=0}^{n-1}(a(x_k, x_{k+1}) - \bar a)$.

ii. $\forall\ x, y, z \in S$, $h_a(x,z) \leq h_a(x,y) + h_a(y,z)$.

iii. $\forall\ x \in S$, $h_a(x,x) \geq 0$.

iv. $\forall\ x \in S$, $h_a(x,x) = 0 \Leftrightarrow x \in S_{min}$.

v. A cycle has a support in G_{min} if, and only if, it is minimizing.

vi. G_{min} is semi-irreducible and its irreducible components are given by the equivalence classes of the relation

$$\forall\ x, y \in S_{min},\ x \sim_a y \Leftrightarrow h_a(x,y) + h_a(y,x) = 0 \\ \Leftrightarrow x \textit{ and } y \textit{ belong to the same minimizing cycle.}$$

Proof. Items *i*, *ii*, *iii* and *iv* are obvious from the definition of h_a.

Item *v*. By the definition of G_{min}, the support of all minimizing cycle is included in G_{min}. Conversely, let $(x_0, \ldots, x_n)$ be a cycle of G_{min}. Each (x_k, x_{k+1}) is the initial segment of a minimizing cycle $(z_0^k, \ldots, z_{p_k}^k)$ with $p_k \geq 2$, $z_0^k = x_k$ and $z_1^k = x_{k+1}$. The union of the supports of these minimizing cycles can be written as a union of the supports of two (*a priori* not minimizing) cycles $(x_0, x_1, \ldots, x_n)$ and

$$(y_0, \ldots, y_{q_n}) = (z_1^{n-1}, \ldots, z_{p_{n-1}}^{n-1}, z_1^{n-2}, \ldots, z_{p_{n-2}}^{n-2}, \ldots, z_1^0, \ldots, z_{p_0}^0)$$

of length $q_n = p_0 + \dots p_{n-1} - n$. Since

$$0 = \sum_{k=0}^{n-1} \sum_{i=0}^{p_k - 1} (a(z_i^k, z_{i+1}^k) - \bar{a}) = \sum_{k=0}^{n-1} (a(x_k, x_{k+1}) - \bar{a}) + \sum_{k=0}^{q_n - 1} (a(y_k, y_{k+1}) - \bar{a}),$$

both cycles $(x_0, \dots, x_n)$ and $(y_0, \dots, y_{q_n})$ are indeed minimizing.

Item *vi*. Consider the relation on S_{min}: $x \sim_a y$ if, and only if, x and y belong the support of the same minimizing cycle of length ≥ 1. Since the union of two minimizing cycles with a common point is again a minimizing cycle, the previous relation is an equivalence relation. If $x \sim_a y$, then there exists a minimizing cycle $(x_0, \dots, x_n)$ such that $x = x_0$ and $y = x_i$ for some $0 < i < n$. Therefore,

$$0 \leq h_a(x,x) \leq h_a(x,y) + h_a(y,x) \leq \sum_{k=0}^{i-1} (a(x_k, x_{k+1}) - \bar{a}) + \sum_{k=i}^{n-1} (a(x_k, x_{k+1}) - \bar{a}) = 0,$$

and $h_a(x,y) + h_a(y,x) = 0$. Conversely, suppose $h_a(x,y) + h_a(y,x) = 0$. So each minimum $h_a(x,y)$ or $h_a(y,x)$ is reached by a G-admissible path $(x_0, \dots, x_i)$ or $(x_i, \dots, x_n)$, with $x_0 = x$, $x_i = y$ and $x_n = y$. Then $(x_0, \dots, x_n)$ is a minimizing cycle containing both x and y. □

In the framework of a dynamical system where the weighted graph (G, M_ϵ) is given by $M_\epsilon(x,y) = \exp(E(x,y)) \epsilon^{H(x,y)} \mathbb{1}_G(x,y)$ for two short range potentials $E, H : \Sigma_G^+ \to \mathbb{R}$, we show that the two notions of minimizing non-wandering set $\Omega(H)$ and minimizing subgraphs coincide. Let $a(x,y) = H(x,y)$ if $(x,y) \in G$ and $a(x,y) = +\infty$ otherwise.

Proof of Lemma 21. Item *i*. Let $x = (x_0, x_1, \dots) \in \Sigma_G^+$. Since G is irreducible, there is a G-admissible path joining x_n to x_0, $(x_0^n, x_1^n, \dots, x_{p_n}^n)$ of length p_n at most the cardinal of S. Then $(y_0, \dots, y_{n+p_n}) = (x_0, \dots, x_{n-1}, x_0^n, \dots, x_{p_n}^n)$ is a cycle and

$$\bar{H} = \inf_{x \in \Sigma_G^+} \liminf_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} H \circ \sigma^k(x) = \inf_{x \in \Sigma_G^+} \liminf_{n \to +\infty} \frac{1}{n + p_n} \sum_{k=0}^{n+p_n-1} a(y_k, y_{k+1}) \geq \bar{a}.$$

The converse $\bar{H} \leq \bar{a}$ is obtained by taking a periodic point $x = (x_0, \dots, x_n)^\infty$ with $(x_0, \dots, x_n)$ a minimizing cycle.

Item *ii*. Let $h(x,y) = \lim_{\epsilon \to 0} \liminf_{n \to +\infty} S_n^\epsilon(x,y)$ be the Peierls barrier introduced in definition 13. We first show that $h(x,y) \geq h_a(x_0, y_0)$ for any $x, y \in \Sigma_G^+$. Indeed, for ϵ sufficiently small, for any $z = (z_0, z_1, \dots) \in \Sigma_G^+$ satisfying $d(x,z) < \epsilon$ and $d(\sigma^n(z), y) < \epsilon$, we have $z_0 = x$ and $z_n = y_0$ and therefore $S_n^\epsilon(x,y) \geq h_a(x,y)$. Let $x = (x_0, x_1, \dots) \in \Omega(H)$. Since $0 = h(x,x) \geq h_a(x_0, x_0) \geq 0$, $x_0 \in S_{min}$. Hence $\sigma^n(x) \in \Omega(H)$ implies $x_n \in S_{min}$ for any n. Moreover,

$$\begin{aligned} 0 = h(x,x) &= (H - \bar{H})(x) + h(\sigma(x), x) \\ &\geq (a(x_0, x_1) - \bar{a}) + h_a(x_1, x_0) \geq h_a(x_0, x_1) + h_a(x_1, x_0) \geq 0. \end{aligned}$$

In particular, $(a(x_0,x_1)-\bar{a})+h_a(x_1,x_0)=0$. By choosing a path $(y_1,\ldots,y_n)$ joining x_1 to x_0 which realizes the minimum in $h_a(x_1,x_0)$, we obtain a minimizing cycle $(x_0,x_1,y_2,\ldots,y_n)$. We have just proved $(x_0,x_1)\in G_{min}$ and more generally $(x_k,x_{k+1})\in G_{min}$. Thus, $\Omega(H)\subset\Sigma^+_{G_{min}}$. Conversely, suppose $x\in\Sigma^+_{G_{min}}$. Let $n\geq 1$ and $k=0,\ldots,n-1$. Then any (x_k,x_{k+1}) is the beginning of a minimizing cycle $(x_0^k,x_1^k,\ldots,x_{p_k}^k)$ with $p_k\geq 2$. Consider z_n the periodic point of period $q_n=p_0+\ldots+p_{n-1}+n$ given by

$$z_n=(x_0,\ldots,x_{n-1},x_1^{n-1},\ldots,x_{p_{n-1}-1}^{n-1},x_1^{n-2},\ldots,x_{p_{n-2}-1}^{n-2},\ldots,x_1^0,\ldots,x_{p_0}^0)^\infty.$$

Then $d(z_n,x)\to 0$ when $n\to+\infty$ and $\sum_{k=0}^{q_n-1}(a(z_k,z_{k+1})-\bar{a})=0$. We have proved that $x\in\Omega(H)$.

Item *iii*. We first show that, if $x=(x_0,x_1,\ldots),y=(y_0,y_1,\ldots)\in\Omega(H)$, then $x\sim y$ if, and only if, $x_0\sim_a y_0$. Indeed, on the one hand,

$$x\sim y\Leftrightarrow h(x,y)+h(y,x)=0\Rightarrow h_a(x_0,y_0)+h_a(y_0,x_0)=0\Leftrightarrow x_0\sim_a y_0.$$

On the other hand, suppose $x_0\sim_a y_0$. Since $(x_k,x_{k+1}),(y_k,y_{k+1})\in G_{min}$ for all $k=0,\ldots,p-1$, by transitivity we have that $x_p\sim_a y_0$ and $y_p\sim_a x_0$. For infinitely many m and n, one can find a G_{min}-cycle of length $q=2p+m+n$ containing both $(x_0,\ldots,x_{p-1})$ and $(y_0,\ldots,y_{p-1})$ of the following form

$$(x_0,\ldots,x_{p-1},z_p,\ldots,z_{p+m-1},y_0,\ldots,y_{p-1},z_{2p+m},\ldots,z_{2p+m+n}).$$

Let $z\in\Sigma^+_{G_{min}}$ be the corresponding periodic point. For any $\epsilon>0$, if p is large enough, for infinitely many m and n, one has

$$d(z,x)<\epsilon,\quad d(\sigma^{p+m}(z),y)<\epsilon,\quad d(\sigma^{2p+m+n}(z),x)<\epsilon,$$
$$S^\epsilon_{p+m}(x,y)+S^\epsilon_{p+n}(y,x)\leq\sum_{k=0}^{2p+m+n-1}(H-\bar{H})\circ\sigma^k(z)=0.$$

By taking $\liminf$ when $m\to\infty$ and $n\to\infty$ first and $\lim$ when $\epsilon\to 0$, one obtains $h(x,y)+h(y,x)=0$, that is, $x\sim y$. Since G_{min} is equal to the disjoint union of irreducible components $G_i\subset S_i\times S_i$ with no transition from S_i to S_j when $i\neq j$, $\Omega(H)=\Sigma^+_{G_{min}}$ is equal to the disjoint union of $\Omega_i(H)=\Sigma^+_{G_i}$. The equivalence between $x\sim y$ and $x_0\sim_a y_0$ shows that $\Omega_1(H),\ldots,\Omega_d(H)$ are the irreducible components of $\Omega(H)$.

Item *iv*. The pressure of E restricted to $\Omega(H)$ is equal to the maximum of the pressure of E restricted on each $\Omega_i(H)$. It is well known (see, for instance, [20]) that the two notions of spectral radius α_i of the matrix $A^{ii}_{\min}=[e^{E(x,y)}\mathbb{1}_{G_i}(x,y)]_{x,y\in S_i}$ and the pressure of E restricted to $\Sigma^+_{G_i}$ coincide: $\alpha_i=\exp[\mathrm{Pres}_{\Omega_i(H)}(E)]$ and $\bar{\alpha}=\max_{1\leq i\leq d}\alpha_i=\exp[\mathrm{Pres}_{\Omega(H)}(E)]$. □

The first step of the algorithm consists in finding a normal form for M_ϵ. This step is done using a diagonal matrix $\mathrm{diag}[\epsilon^{v(x)}:x\in S]$ where $v:S\to\mathbb{R}$ is a separating corrector. We prove the existence of such a corrector.

Proof of Lemma 27. Given $z^* \in S_{min}$, consider

$$u(x) := h_a(z^*, x), \quad \forall\, x \in S,$$

where h_a is the Peierls barrier associated to a introduced in definition 43. Items i and ii of lemma 44 and the fact that the Peierls barrier between two vertices is realized by a G-admissible path easily show that u is a backward calibrated corrector. Let $G_1 \subset S_1 \times S_1, \ldots, G_d \subset S_d \times S_d$ be the irreducible components of the minimizing subgraph $G_{min} \subset S_{min} \times S_{min}$. Denote $S_0 = S \setminus (S_1 \cup \ldots \cup S_d)$. We consider then

$$\tilde{a}(x, y) := a(x, y) - u(y) + u(x) - \bar{a} \geq 0, \quad \forall\, x, y \in S.$$

Notice that the mean of $\tilde{a}$ on any minimizing cycle is zero and therefore $\tilde{a}(x, y) = 0$ whenever $(x, y) \in G_{min}$. We introduce a new directed graph. The set of vertices $\tilde{S}$ is made of classes of two kinds: a class $[x]$ reduced to one point for all $x \in S_0$ and d classes $[G_1] \ldots [G_d]$ where all vertices in each G_i are identified into one vertex. For any $x \in S$, we note by $[x]$ the class containing x. Let $\tilde{G} \subset \tilde{S} \times \tilde{S}$ be the graph whose transitions are defined as follows

$$[x] \overset{\tilde{G}}{\to} [y] \iff [x] \neq [y] \quad \text{and} \quad \min\{\tilde{a}(x', y') \,:\, x' \in [x],\, y' \in [y]\} = 0.$$

The main observation is that there is no cycle in $\tilde{G}$ and we can define a decreasing "height" function $\eta : S \to [0, \epsilon]$ as small as we want so that η is constant on each class $[x]$ and

$$[x] \overset{\tilde{G}}{\to} [y] \iff \eta(x) > \eta(y), \quad \forall\, x, y \in S.$$

We claim that, for ϵ small enough,

$$v(x) := u(x) + \eta(x), \quad \forall\, x \in S$$

is a separating corrector for $a(x, y)$ or equivalently $\eta(x)$ is a separating corrector for $\tilde{a}(x, y)$. Indeed, on the one hand, if $(x, y) \in G_{min}$, x and y belong to the same irreducible component of G, $\eta(x) = \eta(y)$ and $\tilde{a}(x, y) = 0 = \eta(y) - \eta(x)$. On the other hand, if $(x, y) \in G \setminus G_{min}$, we discuss two cases. In the first case, $([x], [y])$ is not an edge of $\tilde{G}$. This implies $\tilde{a}(x, y) > 0$ since $(x, y) \notin G_{min}$. We choose then $\epsilon > 0$ such that $\tilde{a}(x, y) > \eta(y) - \eta(x)$. In the second case, $([x], [y])$ is an edge of $\tilde{G}$. Since η is decreasing along the edges, $\tilde{a}(x, y) \geq 0 > \eta(y) - \eta(x)$ independently of ϵ. As S is finite, the number of constraints on ϵ is finite. □

In order to prove proposition 29, we recall some notions of entropy and pressure for graphs weighted by Perron matrices.

Definition 45. *Let $G \subset S \times S$ be a directed graph weighted by a Perron matrix $[M(x, y)]_{x,y \in S}$. We call transshipment any a probability measure $\mu(x, y)$ on G such that $\pi(y) := \sum_{x \in S} \mu(x, y) = \sum_{x \in S} \mu(y, x)$, for all $y \in S$. The entropy of a transshipment μ is given by*

$$\mathrm{Ent}(\mu) := \sum_{(x,y) \in G} -\mu(x, y) \ln \frac{\mu(x, y)}{\pi(x)}.$$

We say the transshipment μ *is supported by* M *if* $M(x,y)=0$ *implies* $\mu(x,y)=0$. *In this case, the pressure of* M *with respect to* μ *is given by*

$$\mathrm{Pres}(M,\mu) := \mathrm{Ent}(\mu) + \sum_{(x,y)\in G} \mu(x,y)\ln M(x,y).$$

We recall that, if G *is irreducible and* $\lambda = \rho_{spec}(M)$, *then* $\mathrm{Pres}(M,\mu) \le \ln\lambda$ *for any transshipment* μ *supported by* M, *with equality if, and only if,* $\mu(x,y) = L(x)M(x,y)R(y)/\lambda$, *where* $[L(x)]_{x\in S}$ *and* $[R(x)]_{x\in S}$ *are the left and right eigenvectors of* M *for the eigenvalue* λ.

We shall also use a known result on the perturbation of the spectrum of matrices. See Kato's monography [16] for more elaborate statements.

Lemma 46. *For any matrix* $M \in \mathrm{Mat}(n,\mathbb{C})$, *for any* $\epsilon > 0$, *there exists* $\eta > 0$ *such that, if* $H \in \mathrm{Mat}(n,\mathbb{C})$ *and* $\|H\| < \eta$, *then* $\mathrm{spec}(M+H) \subset \mathrm{spec}(M) + B_\epsilon$, *where* B_ϵ *denotes the disk of radius* ϵ *centered at* 0. *In particular,* $M \mapsto \rho_{spec}(M)$ *is continuous on* $\mathrm{Mat}(n,\mathbb{C})$.

Proof of proposition 29. Notice that it is enough to assume M_ϵ is written in a normal form

$$M_\epsilon = \hat{M} + N_\epsilon,\ \hat{M} = \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix},\ \bar{A} = \mathrm{diag}[\bar{A}^{11},\dots,\bar{A}^{rr}],\ \bar{\alpha} = \rho_{\mathrm{spec}}(\bar{A}^{ii}),$$

where $\bar{A}^{ii}$ is nonnegative irreducible, D is nonnegative with $\rho_{\mathrm{spec}}(D) < \bar{\alpha}$, and $N_\epsilon = o(1)$. We also assume M_ϵ is nonnegative by changing if necessary M_ϵ to $M_\epsilon - \eta_\epsilon \mathrm{Id}$ where $\eta_\epsilon = 0 \wedge \min\{M_\epsilon(x,x) : x \in S\}$. Notice that L_ϵ and R_ϵ do not change and that $\eta_\epsilon = o(1)$.

Let thus $\hat{G}$ be the subgraph of G defined by $(x,y) \in \hat{G} \Leftrightarrow \bar{A}(x,y) > 0$ or $D(x,y) > 0$. Let $\hat{M}_\epsilon(x,y) = M_\epsilon(x,y)$ if $(x,y) \in \hat{G}$, $\hat{M}_\epsilon(x,y) = M_\epsilon^{1/2}(x,y)$ if $(x,y) \in G \setminus \hat{G}$. On the one hand, we remark that

$$\begin{aligned}\ln\lambda_\epsilon = \mathrm{Pres}(M_\epsilon,\mu_\epsilon) = \mathrm{Pres}(\hat{M}_\epsilon,\mu_\epsilon) + \sum_{(x,y)\in G\setminus\hat{G}} \mu_\epsilon(x,y)\ln M_\epsilon^{1/2}(x,y) \le \\ \le \ln\rho_{\mathrm{spec}}(\hat{M}_\epsilon) + \sum_{(x,y)\in G\setminus\hat{G}} \mu_\epsilon(x,y)\ln M_\epsilon^{1/2}(x,y) \le \ln\rho_{\mathrm{spec}}(\hat{M}_\epsilon).\end{aligned}$$

Consider now $\bar{G}_1$ (an irreducible component of $G^*_{\min}$ of dominant spectral coeficient $\bar{\alpha}$) weighted by $\hat{M}_\epsilon^{11}(x,y) = M_\epsilon(x,y)\mathbb{1}_{\bar{G}_1}(x,y)$. Let $\hat{\mu}_\epsilon^1$ be the transshipment defined on $\bar{G}_1$ by

$$\hat{\mu}_\epsilon^1(x,y) = \hat{L}_\epsilon^1(x)\hat{M}_\epsilon^{11}(x,y)\hat{R}_\epsilon^1(y)/\rho_{\mathrm{spec}}(\hat{M}_\epsilon^{11}),$$

and extended by 0 on $G \setminus \bar{G}_1$. Then, on the other hand, one has

$$\ln\lambda_\epsilon \ge \mathrm{Pres}(\hat{M}_\epsilon^{11},\hat{\mu}_\epsilon^1) = \ln\rho_{\mathrm{spec}}(\hat{M}_\epsilon^{11}).$$

Lemma 46 tells us that $\rho_{\mathrm{spec}}(\hat{M}_\epsilon) \sim \rho_{\mathrm{spec}}(\hat{M}_\epsilon^{11}) \sim \bar{\alpha}$. Hence, the two previous inequalities show that $\lambda_\epsilon \sim \bar{\alpha}$ (item *i*), as well as $\mu_\epsilon(x,y) \to 0$ whenever $(x,y) \notin \hat{G}$. They also show that any accumulation point $\bar{\mu}$ of $(\mu_\epsilon)_{\epsilon>0}$ has maximal pressure

$$\ln \bar{\alpha} = \lim_{\epsilon \to 0} \ln \lambda_\epsilon \leq \lim_{\epsilon \to 0} \Big[\mathrm{Ent}(\mu_\epsilon) + \sum_{(x,y)\in \hat{G}} \mu_\epsilon(x,y) \ln M_\epsilon(x,y)\Big] = \mathrm{Pres}(\hat{M}, \bar{\mu}) \leq \ln \bar{\alpha}.$$

(The first inequality comes from the fact that $\ln M_\epsilon(x,y) < 0$ if $(x,y) \in G \setminus \hat{G}$. Notice also that $\bar{\mu}$ has support on $\hat{G}$.) For $\bar{G}$ the dominant subgraph, let $\bar{\mu}_{\bar{G}}$ and $\bar{\mu}_{\hat{G}\setminus\bar{G}}$ be the induced transshipments on $\bar{G}$ and $\hat{G} \setminus \bar{G}$, respectively. Since

$$\ln \bar{\alpha} = \mathrm{Pres}(\hat{M}, \bar{\mu}) = \bar{\mu}(\bar{G})\mathrm{Pres}(\bar{A}, \bar{\mu}_{\bar{G}}) + \bar{\mu}(\hat{G} \setminus \bar{G})\mathrm{Pres}(D, \bar{\mu}_{\hat{G}\setminus\bar{G}}),$$

we obtain $\bar{\mu}(\hat{G} \setminus \bar{G}) = 0$, that is, $\mu_\epsilon(x,y) \to 0$ whenever $(x,y) \notin \bar{G}$ (item *ii*).

Consider $\bar{\pi}^i(x) = \sum_{y\in\bar{S}_i} \bar{\mu}(x,y)/\bar{\mu}(\bar{G}_i)$ for any $x \in \bar{S}_i$. Let $\bar{\mu}_i$ be the induced transshipment on $\bar{G}_i$, $\bar{\mu}_i(x,y) = \bar{\mu}(x,y)/\bar{\mu}(\bar{G}_i)$ whenever $\bar{\mu}(\bar{G}_i) \neq 0$. The main remark is the following coboundary property

$$\sum_{x\in\bar{S}_i} \bar{\mu}_i(x,y) = \sum_{x\in\bar{S}_i} \bar{\mu}_i(y,x),\ \forall\, y \in \bar{S}_i \ \Rightarrow \sum_{(x,y)\in\bar{S}_i\times\bar{S}_i} \bar{\mu}_i(x,y) \ln\Big(\frac{\bar{R}^i(y)}{\bar{R}^i(x)}\Big) = 0.$$

Then $\ln \bar{\alpha} = \sum_{i=1}^r \bar{\mu}(\bar{G}_i)\mathrm{Pres}(\bar{A}^{ii}, \bar{\mu}_i)$ and

$$\mathrm{Pres}(\bar{A}^{ii}, \bar{\mu}_i) = \sum_{\substack{x\in\bar{S}_i \\ \bar{\pi}^i(x)\neq 0}} \bar{\pi}^i(x) \sum_{y\in\bar{S}_i} \frac{\bar{\mu}_i(x,y)}{\bar{\pi}^i(x)} \ln\Big(\frac{\bar{A}^{ii}(x,y)\bar{R}^i(y)/\bar{R}^i(x)}{\bar{\mu}_i(x,y)/\bar{\pi}_i(x)}\Big).$$

Each sum over $y \in \bar{S}_i$ is bounded from above by

$$\ln\Big(\sum_{y\in\bar{S}_i} \bar{A}^{ii}(x,y)\bar{R}^i(y)/\bar{R}^i(x)\Big) = \ln \bar{\alpha},$$

with equality if, and only if, $\bar{\mu}_i(x,y)/\bar{\pi}^i(x) = \bar{A}^{ii}(x,y)\bar{R}^i(y)/(\bar{\alpha}\bar{R}^i(x))$, $\forall\, y \in \bar{S}_i$. We thus have proved (whether or not $\bar{\pi}_i(x) = 0$)

$$\frac{\bar{\pi}^i(x)}{\bar{R}^i(x)} \bar{A}^{ii}(x,y) = \bar{\alpha}\frac{\bar{\mu}_i(x,y)}{\bar{R}^i(y)}, \quad \forall\, x,y \in \bar{S}_i.$$

By summing over x, using the fact that $\bar{\mu}_i$ is a transshipment, we obtain that $[\bar{\pi}^i(x)/\bar{R}^i(x)]_{x\in\bar{S}_i}$ is a left eigenvector of $\bar{A}^{ii}$ for the eigenvalue $\bar{\alpha}$. In particular, if $\bar{\pi}^i(x) \neq 0$ for some $x \in \bar{S}_i$, $\bar{\pi}^i(y) \neq 0$ for all $y \in \bar{S}_i$ and

$$\bar{\pi}^i(y) = \bar{L}^i(y)\bar{R}^i(y), \quad \bar{\mu}_i(x,y) = \bar{L}^i(x)\bar{A}^{ii}(x,y)\bar{R}^i(y)/\bar{\alpha}.$$

(Item *iii* is proved.) □

Before proving proposition 31, we give some complements to the theory of series of equivalences.

Lemma 47. *Let* $(A_n)_{n\geq 0}$ *be a sequence of positive numbers and* $(A_n(\epsilon))_{n\geq 0}$ *be a sequence of functions. We assume that* $A_n = O(\delta^n)$ *for some* $\delta \in (0,1)$ *and* $\big(A_n(\epsilon)/A_n\big)^{1/n} \to 1$ *as* $\epsilon \to 0$ *uniformly in* $n \geq 0$. *Then*

$$\sum_{n\geq 0} A_n(\epsilon) \sim \sum_{n\geq 0} A_n.$$

Proof. Denote $h_n(\epsilon) := \big(A_n(\epsilon)/A_n\big)^{1/n} - 1$. Let $\eta \in (0,1)$ be small enough so that $\delta(1+\eta) < 1$. Fix a constant $C > 0$ such that $A_n \leq C\delta^n$, for all $n \geq 0$. Choose a positive integer N large enough so that

$$(1-\eta)\sum_{n\geq N} A_n < \eta \sum_{n\geq 0} A_n \quad \text{and} \quad C\sum_{n\geq N} \delta^n(1+\eta)^n < \eta\sum_{n\geq 0} A_n.$$

For ϵ small enough, one has $(1-\eta)\sum_{n=0}^{N-1} A_n \leq \sum_{n=0}^{N-1} A_n(\epsilon) \leq (1+\eta)\sum_{n=0}^{N-1} A_n$, as well as $h_n(\epsilon) < \eta$ uniformly in n, which in particular yields

$$\sum_{n\geq N} A_n(\epsilon) < \sum_{n\geq N} A_n(1+\eta)^n \leq C\sum_{n\geq N} \delta^n(1+\eta)^n.$$

Considering all these inequalities, for all ϵ small enough, we obtain that

$$(1-2\eta)\sum_{n\geq 0} A_n < \sum_{n\geq 0} A_n(\epsilon) < (1+2\eta)\sum_{n\geq 0} A_n.$$

□

In the following lemma, we extend the notion of weighted graph (G, M_ϵ) of general Puiseux type to the case in which G is not irreducible and we show that the resolvant is of exact Puiseux type.

Lemma 48. *Let* (G, M_ϵ) *be a (not necessarily irreducible) weighted graph. Assume* $M_\epsilon = D + N_\epsilon$, *where* D *is nonnegative,* $\rho_{spec}(D) < 1$, $N_\epsilon = o(1)$. *Suppose* (G, M_ϵ) *is of general Puiseux type in the following sense:*

$$M_\epsilon(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin G, \\ A_\epsilon(x,y)\epsilon^{a(x,y)} & \text{if } (x,y) \in G \text{ and } x \neq y, \\ A_\epsilon(x,y) & \text{if } (x,x) \in G,\ x = y \text{ and } D(x,x) > 0, \\ o(1) & \text{if } (x,x) \in G,\ x = y \text{ and } D(x,x) = 0, \end{cases}$$

where $A_\epsilon(x,y) \sim A(x,y) > 0$ *and* $a(x,y) \geq 0$ *in the second and third cases, and by convention* $A(x,y) = 0$ *and* $a(x,y) = +\infty$ *in the other cases. Let* $\mathcal{P}(x,y)$ *be the set of* G*-admissible paths* $\underline{x} = (x_0, \ldots, x_n)$ *of length* $n \geq 1$ *such that* $x_0 = x$ *and* $x_n = y$. *Consider the directed graph*

$$G' = \big\{(x,x) \,:\, x \in S\big\} \cup \big\{(x,y) \in S \times S \,:\, \mathcal{P}(x,y) \neq \emptyset\big\}$$

and define $M'_\epsilon := (Id - M_\epsilon)^{-1}$. *Then* (G', M'_ϵ) *is a weighted graph of exact Puiseux type. More precisely,*

$$M'_\epsilon(x,y) = 0 \Leftrightarrow (x,y) \notin G' \quad \text{and} \quad M'_\epsilon(x,y) \sim A'(x,y)\epsilon^{a'(x,y)} \Leftrightarrow (x,y) \in G',$$

with $a'(x,y) = \begin{cases} 0 & \textit{if } x=y \\ \min\{a(\underline{x}) \,:\, \underline{x} \in \mathcal{P}(x,y)\} & \textit{if } x \neq y \end{cases}, \quad \forall\ (x,y) \in G'$,
and $A'(x,y) = \mathbb{1}_{(x=y)} + \sum_{\underline{x} \in \mathcal{P}(x,y) \,:\, a(\underline{x}) = a'(x,y)} \Pi_{i=0}^{n(\underline{x})-1} A(x_i,x_{i+1}),\ \forall\ (x,y) \in G'$,

where $n(\underline{x})$ *is the lenght of the path* $\underline{x} \in \mathcal{P}(x,y)$ *and* $a(\underline{x}) := \sum_{i=0}^{n(\underline{x})-1} a(x_i,x_{i+1})$. *(By convention* $A'(x,y)=0$ *and* $a'(x,y)=+\infty$ *for all* $(x,y) \notin G'$.*)*

Proof. Part 1. We first assume that (G, M_ϵ) is of exact Puiseux type,

$$M_\epsilon(x,y) = \begin{cases} 0 & \forall\ (x,y) \notin G, \\ A_\epsilon(x,y)\epsilon^{a(x,y)} & \forall\ (x,y) \in G, \end{cases}$$

where $A_\epsilon(x,y) \sim A(x,y) > 0$ and $a(x,y) \geq 0$ if $(x,y) \in G$, $A(x,y) = 0$ and $a(x,y) = +\infty$ if $(x,y) \notin G$. Note that $D(x,y) > 0$ if, and only if, $a(x,y) = 0$. Since $\rho_{\text{spec}}(M_\epsilon)$ converges to $\rho_{\text{spec}}(D) < 1$, $(\text{Id} - M_\epsilon)$ is invertible and

$$M'_\epsilon(x,y) = \sum_{n\geq 0} M_\epsilon^n(x,y) = \mathbb{1}_{(x=y)} + \sum_{\underline{x} \in \mathcal{P}(x,y)} \Pi_{i=0}^{n(\underline{x})-1} M_\epsilon(x_i,x_{i+1}).$$

Since M_ϵ is a nonnegative matrix, M'_ϵ is nonnegative too. Moreover,

$$M'_\epsilon(x,y) = 0 \iff x \neq y \ \text{ and } \ \mathcal{P}(x,y) = \emptyset \iff (x,y) \notin G'.$$

For $(x,y) \in G'$, let $\mathcal{P}(x,y,k)$ be the subset of paths $\underline{x} \in \mathcal{P}(x,y)$ such that

$$k = \text{card}\{i = 0, \ldots, n(\underline{x}) - 1 \,:\, a(x_i,x_{i+1}) > 0\}.$$

If $\underline{x} \in \mathcal{P}(x,y,k)$ and $k \geq 1$, then $a(\underline{x})$ takes a finite number of distinct values $a_{k,l}$,

$$0 < k a_{min} \leq a_{k,1} < a_{k,2} < \cdots < a_{k,p_k} \leq k a_{max},$$

with $a_{min} := \min\{a(x,y) : a(x,y) > 0\}$ and $a_{max} := \max\{a(x,y) : a(x,y) < +\infty\}$. Notice that the set of exponents $\{a_{k,l} \,:\, k \geq 1,\ 1 \leq l \leq p_k\}$ is finite on each bounded interval. Let $\mathcal{P}(x,y,k,l)$ be the subset of paths $\underline{x} \in \mathcal{P}(x,y,k)$ such that $a(\underline{x}) = a_{k,l}$. By developing all products M_ϵ^n, one obtains

$$\begin{aligned} M'_\epsilon(x,y) = \mathbb{1}_{(x=y)} &+ \sum_{\underline{x} \in \mathcal{P}(x,y,0)} \Pi_{i=0}^{n(\underline{x})-1} A_\epsilon(x_i,x_{i+1}) \\ &+ \sum_{k\geq 1} \sum_{l=1}^{p_k} \Big(\sum_{\underline{x} \in \mathcal{P}(x,y,k,l)} \Pi_{i=0}^{n(\underline{x})-1} A_\epsilon(x_i,x_{i+1}) \Big) \epsilon^{a_{k,l}}. \end{aligned}$$

Let $\mathcal{P}(x,y,0,0) := \mathcal{P}(x,y,0)$ by convention and $\mathcal{P}_n(x,y,k,l)$ be the set of paths $\underline{x} \in \mathcal{P}(x,y,k,l)$ of length $n(\underline{x}) = n$. Denote

$$A_{n,k,l}(\epsilon) := \sum_{\underline{x} \in \mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A_\epsilon(x_i,x_{i+1}), \quad A_{n,k,l} := \sum_{\underline{x} \in \mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A(x_i,x_{i+1}).$$

We use lemma 47 to show that $\sum_{n\geq 1} A_{n,k,l}(\epsilon) \sim \sum_{n\geq 1} A_{n,k,l}$ (one only considers terms (n,k,l) such that $\mathcal{P}_n(x,y,k,l) \neq \emptyset$). Since $\rho_{\text{spec}}(D) < 1$, there exists a positive matrix $[\tilde{D}(x,y)]_{x,y\in S}$ such that

$$\rho_{\text{spec}}(\tilde{D}) < 1 \quad \text{and} \quad \tilde{D}(x,y) > D(x,y), \quad \forall\, x,y \in S.$$

Since $A(x,y) = D(x,y)$ whenever $D(x,y) > 0$, one obtains

$$A_{n,k,l} \leq \sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} \tilde{D}(x_i, x_{i+1}) \Big(\frac{\max A}{\min \tilde{D}}\Big)^k \leq \tilde{D}^n(x,y) \Big(\frac{\max A}{\min \tilde{D}}\Big)^k.$$

Choose $\tilde{\delta}$ such that $\rho_{\text{spec}}(\tilde{D}) < \tilde{\delta} < 1$. Then $\tilde{D}^n(x,y) = O(\tilde{\delta}^n)$, and in particular $A_{n,k,l} = O(\tilde{\delta}^n)$. Given $\eta \in (0,1)$, for ϵ small enough,

$$(1-\eta)A(x,y) < A_\epsilon(x,y) < (1+\eta)A(x,y), \quad \forall\, (x,y) \in G.$$

For all non empty set $\mathcal{P}_n(x,y,k,l)$,

$$(1-\eta)^n < \frac{\sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A_\epsilon(x_i,x_{i+1})}{\sum_{\underline{x}\in\mathcal{P}_n(x,y,k,l)} \Pi_{i=0}^{n-1} A(x_i,x_{i+1})} < (1+\eta)^n.$$

We have thus obtained $\big(A_{n,k,l}(\epsilon)/A_{n,k,l}\big)^{1/n} \to 1$ uniformly in n.

We now show that the rest of the series

$$R_K(\epsilon) := \sum_{k\geq K} \sum_{l=1}^{p_k} \Big(\sum_{n\geq 1} A_{n,k,l}(\epsilon)\Big) \epsilon^{a_{k,l}}$$

is negligible with respect to the first non zero term $(\sum_{n\geq 1} A_{n,k,l})\epsilon^{a_{k,l}}$. More precisely, we show that, for any $\mathsf{a} > 0$, there exists $K \geq 1$ such that $R_K(\epsilon) = o(\epsilon^{\mathsf{a}})$ as $\epsilon \to 0$. Indeed, let d be the dimension of the matrix M_ϵ, then $p_k \leq d^{2k}$ and

$$R_K(\epsilon) \leq \sum_{k\geq K} \Big(\sum_{n\geq 1} \|\tilde{D}^n\|\Big) \Big(d^2 \frac{\max A}{\min \tilde{D}} \epsilon^{a_{min}}\Big)^k \leq C_K \epsilon^{K a_{min}} = o(\epsilon^{\mathsf{a}})$$

as soon as $\mathsf{a} < K a_{min}$.

Therefore, $M'_\epsilon(x,y) \sim A'(x,y)\epsilon^{a'(x,y)}$ for all $(x,y) \in G'$.

Part 2. We now assume that (G, M_ϵ) is of general Puiseux type as described in the statement. We first notice that M'_ϵ admits a different expression

$$M'_\epsilon = \frac{1}{2}\Big(\text{Id} - \frac{\text{Id} + M_\epsilon}{2}\Big)^{-1} \quad \text{where} \quad \frac{\text{Id} + M_\epsilon}{2} = \frac{\text{Id} + D}{2} + \frac{N_\epsilon}{2},$$

with $\rho_{\text{spec}}(\frac{1}{2}(\text{Id} + D)) < 1$ and $\frac{1}{2}N_\epsilon = o(1)$. Since $(G, \frac{1}{2}(\text{Id} + M_\epsilon))$ is of exact Puiseux type, one obtains from part 1 that (G', M'_ϵ) is of exact Puiseux type.

We now want to determine a' and A' in this case. Let Δ_ϵ be the diagonal matrix built from the principal diagonal of N_ϵ. Hence,

$$N_\epsilon = \Delta_\epsilon + \tilde{N}_\epsilon, \quad \tilde{N}_\epsilon(x,x) = 0, \ \forall\, x \in S.$$

Let $\tilde{M}_\epsilon := D + \tilde{N}_\epsilon$, $\tilde{G} := G \setminus \{(x,x) \,:\, D(x,x) = 0\}$. Then $(\tilde{G}, \tilde{M}_\epsilon)$ is of exact Puiseux type. Moreover,

$$M'_\epsilon = (\mathrm{Id} - \tilde{M}_\epsilon - \Delta_\epsilon)^{-1} = (\mathrm{Id} - \tilde{M}'_\epsilon \Delta_\epsilon)^{-1} \tilde{M}'_\epsilon = \sum_{n \geq 0} (\tilde{M}'_\epsilon \Delta_\epsilon)^n \tilde{M}'_\epsilon,$$

where $\tilde{M}'_\epsilon := (\mathrm{Id} - \tilde{M}_\epsilon)^{-1}$. From part 1, we know that $(\tilde{G}', \tilde{M}'_\epsilon)$ is of exact Puiseux type. Let a' and A' be defined as in part 1 by using $(\tilde{G}, \tilde{M}_\epsilon)$. Then

$$\tilde{M}'_\epsilon(x,y) = A'_\epsilon(x,y) \epsilon^{a'(x,y)}, \quad \forall\, x, y \in S,$$

with $A'_\epsilon(x,y) \sim A'(x,y) > 0$ if $(x,y) \in \tilde{G}'$ and $A'_\epsilon(x,y) = 0$ if $(x,y) \notin \tilde{G}'$. Since G and $\tilde{G}$ have the same off-diagonal entries, $G' = \tilde{G}'$. We show by induction there exist matrices $(B_{n,\epsilon})_{n \geq 1}$ such that

$$\begin{cases} (\tilde{M}'_\epsilon \Delta_\epsilon)^n \tilde{M}'_\epsilon(x,y) = B_{n,\epsilon}(x,y) \epsilon^{a'(x,y)} & \forall\, (x,y) \in G' \\ (\tilde{M}'_\epsilon \Delta_\epsilon)^n \tilde{M}'_\epsilon(x,y) = 0 = B_{n,\epsilon}(x,y) & \forall\, (x,y) \notin G' \end{cases} \quad \text{and}$$

$$\lim_{\epsilon \to 0} \big(B_{n,\epsilon}(x,y)\big)^{1/n} = 0, \quad \text{uniformly in } n \geq 1.$$

Since $(\tilde{M}'_\epsilon \Delta_\epsilon)^{n+1} \tilde{M}'_\epsilon = (\tilde{M}'_\epsilon \Delta_\epsilon)^n \tilde{M}'_\epsilon \Delta_\epsilon \tilde{M}'_\epsilon$, for all $x, y \in S$ one has

$$\begin{aligned} (\tilde{M}'_\epsilon \Delta_\epsilon)^{n+1} \tilde{M}'_\epsilon(x,y) &= \sum_{z \in S} (\tilde{M}'_\epsilon \Delta_\epsilon)^n \tilde{M}'_\epsilon(x,z) \Delta_\epsilon(z,z) \tilde{M}'_\epsilon(z,y) \\ &= \sum_{z \in S} B_{n,\epsilon}(x,z) \Delta_\epsilon(z,z) A'_\epsilon(z,y) \epsilon^{a'(x,z) + a'(z,y)}. \end{aligned}$$

If $(x,y) \notin G'$, then $(x,z) \notin G'$ or $(z,y) \notin G'$ and the above sum is null. Thus by convention $B_{n+1,\epsilon}(x,y) = 0$. If $(x,y) \in G'$ and $z \in S$ is such that $(x,z) \in G'$ and $(z,y) \in G'$, then $a'(x,y) \leq a'(x,z) + a'(z,y)$. Let

$$B_{n+1,\epsilon}(x,y) := \sum_{z \in S} B_{n,\epsilon}(x,z) \Delta_\epsilon(z,z) A'_\epsilon(z,y) \epsilon^{a'(x,z) + a'(z,y) - a'(x,y)}.$$

By taking the supremum in $x, y \in S$, we obtain

$$\sup_{x,y} \Big(B_{n+1,\epsilon}(x,y)\Big) \leq \sup_{x,y} \Big(B_{n,\epsilon}(x,y)\Big) \sup_{x,y} \Big(d \Delta_\epsilon(x,y) A'(x,y)\Big).$$

As $\Delta_\epsilon = o(1)$, we have proved that $(B_{n,\epsilon}(x,y))^{1/n} \to 0$ uniformly in n. Besides,

$$M'_\epsilon(x,y) = A'_\epsilon(x,y) \epsilon^{a'(x,y)} \Big[1 + \sum_{n \geq 1} \frac{B_{n,\epsilon}(x,y)}{A'_\epsilon(x,y)}\Big] \sim A'(x,y) \epsilon^{a'(x,y)} \quad \text{for all } (x,y) \in G',$$

and $M'_\epsilon(x,y) = 0$ for all $(x,y) \notin G'$. □

Proof of proposition 31. Notice that it is enough to assume (G, M_ϵ) is reduced to a normal form and $M_\epsilon = \tilde{M}_\epsilon$ is nonnegative (by possibly substracting $\eta_\epsilon \mathrm{Id}$, where $\eta_\epsilon := 0 \wedge \min\{M_\epsilon(x,x) \,:\, x \in S\}$ is negligible with respect to λ_ϵ). We prove item i at the end.

Item *ii.* We only prove the equivalence $\tilde{R}^i_\epsilon(x)/\tilde{R}^i_\epsilon(y) \sim \bar{R}^i(x)/\bar{R}^i(y)$. We consider the vector space indexed by $\bar{S}_i$. The vectors are supposed to be column vectors. Let us consider the projector onto $\bar{R}^i$ defined by

$$V \mapsto (\bar{L}^i V)\bar{R}^i, \quad (V \text{ is a column vector}),$$

or as a (square) matrix $\bar{R}^i\bar{L}^i$. Notice that the kernel $\{V : \bar{L}^i V = 0\}$ is invariant by $\bar{A}^{ii}$. The complementary projector is denoted $\bar{P}^{ii} := \mathrm{Id} - \bar{R}^i\bar{L}^i$. We then obtain a decomposition of $\bar{A}^{ii}$

$$\bar{A}^{ii} = \bar{\alpha}\bar{R}^i\bar{L}^i + \bar{D}^{ii} \quad \text{or} \quad \bar{D}^{ii} = \bar{P}^{ii}\bar{A}^{ii} = \bar{A}^{ii}\bar{P}^{ii}.$$

Since $\bar{A}^{ii}$ is irreducible, $\bar{\alpha}$ has multiplicity 1 and $\rho_{\text{spec}}(\bar{D}^{ii}) < \bar{\alpha}$. By multiplying by $\bar{P}^{ii}$ the equation

$$\sum_{j=1}^{r}\Big(\tilde{M}^{ij}_\epsilon + \tilde{M}^{i0}_\epsilon(\tilde{\lambda}_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}\tilde{M}^{0j}_\epsilon\Big)\tilde{R}^j_\epsilon = \tilde{\lambda}_\epsilon\tilde{R}^i_\epsilon,$$

one obtains

$$\sum_{j=1}^{r}(\tilde{\lambda}_\epsilon - \bar{D}^{ii})^{-1}\bar{P}^{ii}\Big(\tilde{N}^{ij}_\epsilon + \tilde{M}^{i0}_\epsilon(\tilde{\lambda}_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}\tilde{M}^{0j}_\epsilon\Big)\tilde{R}^j_\epsilon = \bar{P}^{ii}\tilde{R}^i_\epsilon.$$

(We use the fact that $\tilde{N}^{ij}_\epsilon = \tilde{M}^{ij}_\epsilon$ when $i \neq j$ and that $\bar{A}^{ii}\bar{P}^{ii} = \bar{D}^{ii}\bar{P}^{ii}$.) We first claim that $\tilde{R}^i_\epsilon/\bar{L}^i\tilde{R}^i_\epsilon$ is bounded, or equivalently that $\tilde{R}^i_\epsilon(x)/\tilde{R}^i_\epsilon(y)$ is bounded for all $x, y \in \bar{S}_i$. Notice that all following terms are nonnegative

$$\tilde{M}^{ij}_\epsilon(x,y) \geq 0 \quad \text{or} \quad \tilde{M}^{i0}_\epsilon(\tilde{\lambda}_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}\tilde{M}^{0j}_\epsilon(x,y) \geq 0.$$

(The second inequality follows from lemma 48.) By the irreducibility of $\bar{A}^{ii}$, if $(x_0, \ldots, x_n)$ is a path joining x to y such that $\bar{A}^{ii}(x_k, x_{k+1}) > 0$, then

$$\frac{\tilde{R}^i_\epsilon(x_0)}{\tilde{R}^i_\epsilon(x_n)} \geq \frac{\Pi_{k=0}^{n-1}\tilde{M}^{ii}_\epsilon(x_k, x_{k+1})}{\tilde{\lambda}^n_\epsilon} \sim \frac{\Pi_{k=0}^{n-1}\bar{A}^{ii}(x_k, x_{k+1})}{\bar{\alpha}^n} > 0.$$

By reversing x and y, we prove the claim. We now claim that all following terms are negligible

$$\frac{\tilde{N}^{ij}_\epsilon\tilde{R}^j_\epsilon}{\bar{L}^i\tilde{R}^i_\epsilon} = o(1) \quad \text{or} \quad \frac{\tilde{M}^{i0}_\epsilon(\tilde{\lambda}_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}\tilde{M}^{0j}_\epsilon\tilde{R}^j_\epsilon}{\bar{L}^i\tilde{R}^i_\epsilon} = o(1).$$

Notice that these terms are nonnegative, except perhaps $\tilde{\mu}^i_\epsilon := \tilde{N}^{ii}_\epsilon\tilde{R}^i_\epsilon/\bar{L}^i\tilde{R}^i_\epsilon$ which is negligible because of the first claim. We conclude by observing that all terms on the left hand side of the following equality are nonnegative and that the right hand side is negligible

$$\sum_{j=1}^{r}\frac{\bar{L}^i\big(\tilde{N}^{ij}_\epsilon\delta_{(i\neq j)} + \tilde{M}^{i0}_\epsilon(\tilde{\lambda}_\epsilon - \tilde{M}^{00}_\epsilon)^{-1}\tilde{M}^{0j}_\epsilon\big)\tilde{R}^j_\epsilon}{\bar{L}^i\tilde{R}^i_\epsilon} = \tilde{\lambda}_\epsilon - \bar{\alpha} - \tilde{\mu}^i_\epsilon = o(1).$$

Therefeore, we have proved that $\dfrac{\tilde R^i_\epsilon}{\bar L^i \tilde R^i_\epsilon} - \bar R^i = \dfrac{\bar P^{ii} \tilde R^i_\epsilon}{\bar L^i \tilde R^i_\epsilon} = o(1)$.

Item *iii*. Let $i, j \in \{1, \dots, r\}$, $x \in \bar S_i$ and $y \in S$. We have already proved in the first part that

$$\begin{aligned}
\frac{\tilde M^{ii}_\epsilon(x,y)\tilde R^i_\epsilon(y)}{\tilde\lambda_\epsilon \tilde R^i_\epsilon(x)} \sim \frac{\bar A^{ii}(x,y)\bar R^i(y)}{\bar\alpha \bar R^i(x)} &= \bar Q^{ii}(x,y), \quad \forall\ x,y \in \bar S_i,\\
\frac{\tilde M^{ij}_\epsilon(x,y)\tilde R^j_\epsilon(y)}{\tilde\lambda_\epsilon \tilde R^i_\epsilon(x)} &= o(1), \quad \forall\ x \in \bar S_i,\ \forall\ y \in \bar S_j,\ i \neq j,\\
\frac{\tilde M^{i0}_\epsilon(x,y)\tilde R^0_\epsilon(y)}{\tilde\lambda_\epsilon \tilde R^i_\epsilon(x)} = \frac{\tilde M^{i0}_\epsilon(x,y)\big(\sum_{j=1}^r (\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon \tilde R^j_\epsilon\big)(y)}{\tilde\lambda_\epsilon \tilde R^i_\epsilon(x)} &= o(1).
\end{aligned}$$

(In the two last estimates, we use the fact that the sum over y in each case is negligible.) We then obtain

$$Q_\epsilon(x,y) = \frac{\tilde M_\epsilon(x,y)\tilde R_\epsilon(y)}{\tilde\lambda_\epsilon \tilde R_\epsilon(x)} \to \begin{cases} \bar Q^{ii}(x,y), & \forall\ x,y \in \bar S_i,\\ 0, & \forall\ x \in \bar S_i,\ \forall\ y \in \bar S_j \cup S_0,\ i \neq j. \end{cases}$$

Item *i*. Let $i \neq j$, then $M^{(1)}_\epsilon(i,j) = \bar L^i \Big(\tilde M^{ij}_\epsilon + \tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon\Big)\frac{\tilde R^j_\epsilon}{\bar L^j \tilde R^j_\epsilon}$. We want to show that

$$M^{(1)}_\epsilon(i,j) = \begin{cases} 0 & \forall\ (i,j) \notin G^{(1)},\\ A^{(1)}_\epsilon(i,j)\epsilon^{a^{(1)}(i,j)} & \forall\ (i,j) \in G^{(1)},\ i \neq j, \end{cases}$$

where $A^{(1)}_\epsilon(i,j) = 0$ in the first case and $A^{(1)}(i,j) \sim A^{(1)}(i,j) > 0$ in the second one. From item *ii*, we know that $\tilde R^j_\epsilon / \bar L^j \tilde R^j_\epsilon \sim \bar R^j$. Since $\bar L^i$ and $\bar R^j$ have positive coefficients, it is enought to find equivalences to $\tilde M^{ij}_\epsilon(x,y)$ and $\tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon(x,y)$ when $x \in \bar S_i$ and $y \in \bar S_j$. From lemma 48, we know that $(\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}$ is of exact Puiseux type on the graph containing either $\{(x_1,x_1) : x_1 \in S_0\}$ or $\{(x_1,x_{n-1}) : (x_1,\dots,x_{n-1})$ is a path of $\tilde G \cap S_0 \times S_0\}$, where $\tilde G$ is obtained from G by substracting all (x,x) such that $D(x,x) = 0$. We write $(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}(x,y) \sim \bar\alpha^{-1}A'(x,y)\epsilon^{a'(x,y)}$. Therefore, for $x \in \bar S_i$ and $y \in \bar S_j$, one has $\tilde M^{ij}_\epsilon(x,y) \sim A(x,y)\epsilon^{a(x,y)}$ and

$$\tilde M^{i0}_\epsilon(\tilde\lambda_\epsilon - \tilde M^{00}_\epsilon)^{-1}\tilde M^{0j}_\epsilon(x,y) \sim \sum_{z,w \in S_0} A(x,z)\frac{A'(z,w)}{\bar\alpha}A(w,y)\epsilon^{a(x,z)+a'(z,w)+a(w,y)}.$$

One can see the previous estimate as a sum over paths $\underline{x}$ of two kinds. Either there exists a G-admissible path $\underline{x} = (x,z,y)$ (for $z = w$), or there exists a G-admissible path $\underline{x} = (x_0,\dots,x_n)$ of lenght $n \geq 3$, with $x_0 = x$, $x_1 = z$, $x_{n-1} = w$, $x_n = y$, such that the intermediate path $(x_1,\dots,x_{n-1})$ is $(\tilde G \cap S_0 \times S_0)$-admissible and realizes the minimun in the definition of $a'(z,w)$. Each one of these terms is of the form

$$\Big[\Pi_{k=0}^{n-1}A(x_k,x_{k+1})/\bar\alpha^{n-1}\Big]\epsilon^{\sum_{k=0}^{n-1}a(x_k,x_{k+1})}.$$

The dominant term is obtained by minimizing $a(\underline{x})$ over $\underline{x}$. □

6 Complete classification for 3-states spin systems

We consider in this section a full weighted graph of exact Puiseux type on 3 states. More precisely, for $S = \{1,2,3\}$, we consider $G = S \times S$ weighted by

$$M_\epsilon(x,y) = \exp[-\beta H(x,y)] = \epsilon^{H(x,y)}, \quad \epsilon = e^{-\beta}, \quad \forall\, x,y \in S.$$

We assume (by substracting $\bar{H}$) that H has been normalized: $\bar{H} = 0$. We are interested in describing the unique ground state μ^H_{min} obtained (recall the notations of section 3) as a limit of

$$\big(\pi_\epsilon(x), Q_\epsilon(x,y)\big) = \Big(L_\epsilon(x)R_\epsilon(x), \frac{M_\epsilon(x,y)R_\epsilon(y)}{\lambda_\epsilon R_\epsilon(x)}\Big)$$

when $\epsilon \to 0$. As it will be clear from the computation, the limit depends from the possibility to expand each quotient $R_\epsilon(x)/R_\epsilon(y)$ and $L_\epsilon(x)/L_\epsilon(y)$ into a Puiseux series of an *a priori* arbitrarily large precision. The algorithm is based on the dimension of the matrix M_ϵ. We will obtain a finite set of possible ground states μ^H_{min} and the zero-temperature phase diagram describes all the domains in the parameter space $\{H(x,y) \,:\, x,y \in S\}$ having a fixed ground state μ^H_{min}. The dimension of this parameter space is *a priori* 9; we will reduce it to 2 in the following discussion. We describe each domain according to the number of irreducible components of the minimizing subgraph. We use algorithm 28 to conjugate M_ϵ to a simpler matrix $M'_\epsilon = \Delta_\epsilon M_\epsilon \Delta_\epsilon^{-1}$, which (by possibly permuting $\{1,2,3\}$) takes one of the following form.

i. A unique dominant irreducible component.

   – When the dominant spectral radius $\bar{\alpha}$ is equal to 1, $G_{min} = \bar{G}$ is irreducible and there are three possibilities corresponding to $\bar{S} = \{1,2,3\}$, $\bar{S} = \{1,2\}$ and $\bar{S} = \{1\}$,

$$M'_\epsilon = \begin{bmatrix} \epsilon^a & 1 & \epsilon^{b'} \\ \epsilon^{c'} & \epsilon^b & 1 \\ 1 & \epsilon^{a'} & \epsilon^c \end{bmatrix}, \ M''_\epsilon = \begin{bmatrix} \epsilon^a & 1 & \epsilon^d \\ 1 & \epsilon^b & \epsilon^e \\ \epsilon^{d'} & \epsilon^{e'} & \epsilon^c \end{bmatrix}, \ M'''_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^c \\ \epsilon^{a'} & \epsilon^b & \epsilon^d \\ \epsilon^{c'} & \epsilon^{d'} & \epsilon^e \end{bmatrix}.$$

   (Notice that all coefficients $a, a', b, \ldots$ are positive.)

   – When $\bar{\alpha} > 1$, $\bar{G} = G_{min}$ is obtained by replacing in the previous M'_ϵ any (but at least one) $a, a', b, \ldots$ by 0, and in M''_ϵ one of the two coefficients a and/or b by 0 and leaving $c, c', d, \ldots$ positive. When $\bar{\alpha} > 1$, $\bar{G} \subset G_{min}$ with two irreducible components is obtained by replacing a and/or b in M''_ϵ by 0 and c by 0. Notice that we obtain a finite liste of possible $\bar{\alpha}$.

ii. Two irreducible components with equal dominant spectral radius:

$$\bar{\alpha} = 1, \quad M'_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & \epsilon^c & 1 \\ \epsilon^{b'} & 1 & \epsilon^d \end{bmatrix}, \quad \text{or} \quad M''_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & \epsilon^d \end{bmatrix}.$$

iii. Three irreducible components with dominant spectral radius 1:

$$\bar{\alpha} = 1, \quad M'_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}.$$

In order to simplify notations, we introduce the following convention

$$a\#b = 1 \quad \text{if} \quad a \neq b, \quad a\#b = 2 \quad \text{if} \quad a = b.$$

In the case of one irreducible component with dominant spectral coefficient ($r = 1$), $\pi_\epsilon(x) \to 0$ for all $x \in S \setminus \bar{S}$ and $\pi_\epsilon(x) \to \bar{\pi}^1(x)$ for all $x \in \bar{S}$. For instance, for M'_ϵ, M''_ϵ and M'''_ϵ, respectively, π_ϵ converges to $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, $[\frac{1}{2}, \frac{1}{2}, 0]$ and $[1, 0, 0]$. We now treat in detail the two remaining cases *ii* and *iii*.

6.1 Two irreducible components. Part I

We first consider the matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & \epsilon^c & 1 \\ \epsilon^{b'} & 1 & \epsilon^d \end{bmatrix}, \quad a, a', b, b', c, d > 0.$$

We already know that $\lambda_\epsilon \sim 1$, $R_\epsilon(2) \sim R_\epsilon(3)$ and $L_\epsilon(2) \sim L_\epsilon(3)$. We collapse the two components 2 and 3 and obtain for the right eigenvector

$$M^{(1)}_\epsilon = \begin{bmatrix} 0 & (\epsilon^a R_2 + \epsilon^b R_3)/(R_2 + R_3) \\ \epsilon^{a'} + \epsilon^{b'} & (\epsilon^c R_2 + \epsilon^d R_3)/(R_2 + R_3) \end{bmatrix} \sim \begin{bmatrix} 0 & \frac{a\#b}{2}\epsilon^{a\wedge b} \\ a'\#b'\epsilon^{a'\wedge b'} & \frac{c\#d}{2}\epsilon^{c\wedge d} \end{bmatrix}.$$

Note that $M^{(1)}_\epsilon$ is of exact Puiseux type. Let r and ρ be the minimizing mean exponent and the dominant spectral radius of $M^{(1)}_\epsilon$. Then $\lambda^{(1)}_\epsilon = \lambda_\epsilon - 1 \sim \rho\epsilon^r$,

$$r = \min\Big(c \wedge d, \frac{a \wedge b + a' \wedge b'}{2}\Big), \ \frac{R_1}{R_3} \sim \frac{a\#b}{\rho}\epsilon^{a\wedge b - r}, \ \frac{L_1}{L_3} \sim \frac{a'\#b'}{\rho}\epsilon^{a'\wedge b' - r}.$$

We thus obtain a complete formula for the transition matrix

$$Q_\epsilon \sim \begin{bmatrix} 1 & \frac{\rho}{a\#b}\epsilon^{a - a\wedge b + r} & \frac{\rho}{a\#b}\epsilon^{b - a\wedge b + r} \\ \frac{a\#b}{\rho}\epsilon^{a' + a\wedge b - r} & \epsilon^c & 1 \\ \frac{a\#b}{\rho}\epsilon^{b' + a\wedge b - r} & 1 & \epsilon^d \end{bmatrix} \to Q^H_{min} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and for the ground state

$$\frac{\pi_\epsilon(2)}{\pi_\epsilon(3)} \sim 1 \quad \text{and} \quad \frac{\pi_\epsilon(1)}{\pi_\epsilon(3)} \sim \frac{(a\#b)(a'\#b')}{\rho^2}\epsilon^{a\wedge b + a'\wedge b' - 2r}.$$

We are left to discuss the value of ρ according to the choice of the exponents contributing in the definition of r. We recall that ρ is the largest eigenvalue of the dominant matrix $A^{(1)}_{min}$.

6.1.1 Case $c \wedge d < (a \wedge b + a' \wedge b')/2$:

In this case, $r = c \wedge d$,

$$A^{(1)}_{min} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{c\#d}{2} \end{bmatrix}, \ \rho = \frac{c \wedge d}{2}, \ \lambda_\epsilon = 1 + \frac{c \wedge d}{2} \epsilon^{c \wedge d} + \ldots, \ \mu^H_{min} = \big[0, \tfrac{1}{2}, \tfrac{1}{2}\big].$$

6.1.2 Case $c \wedge d > (a \wedge b + a' \wedge b')/2$:

In this case, $r = \frac{1}{2}(a \wedge b + a' \wedge b')$,

$$A^{(1)}_{min} = \begin{bmatrix} 0 & \frac{a\#b}{2} \\ a'\#b' & 0 \end{bmatrix}, \ \rho = \sqrt{\frac{(a\#b)(a'\#b')}{2}}, \ \mu^H_{min} = \big[\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\big].$$

6.1.3 Case $c \wedge d = (a \wedge b + a' \wedge b')/2$:

In this case, $r = c \wedge d$,

$$A^H_{min} = \begin{bmatrix} 0 & \frac{a \wedge b}{2} \\ a' \wedge b' & \frac{c \wedge d}{2} \end{bmatrix}, \quad \rho = \frac{c\#d}{4} \Big[1 + \sqrt{1 + 8 \frac{(a\#b)(a'\#b')}{(c\#d)^2}} \ \Big]$$

and the ground state is proportional to

$$\mu^H_{min} \propto \begin{bmatrix} 16(a\#b)(a'\#b')/(c\#d)^2 \\ \Big[1 + \sqrt{1 + 8(a\#b)(a'\#b')/(c\#d)^2}\Big]^2 \\ \Big[1 + \sqrt{1 + 8(a\#b)(a'\#b')/(c\#d)^2}\Big]^2 \end{bmatrix}$$

or $\mu^H_{min}(1) = \dfrac{4(a\#b)(a'\#b')/(c\#d)^2}{1 + 8(a\#b)(a'\#b')/(c\#d)^2 + \sqrt{1 + 8(a\#b)(a'\#b')/(c\#d)^2}}.$

We summarize the discussion in figure 5.

6.2 Two irreducible components. Part II

We consider now the matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & \epsilon^d \end{bmatrix}, \quad a, a', b, b', c, c', d > 0.$$

Let $[L_\epsilon(x)]_{x=1,2,3}$ and $[R_\epsilon(x)]_{x=1,2,3}$ be the left and right eigenvector for the largest eigenvalue λ_ϵ. We eliminate the negligible variable $x = 3$ by substituting $L_\epsilon(3)$ or $R_\epsilon(3)$ in the first two equations. We substract the dominant term 1 of λ_ϵ and obtain

$$L^{(1)}_\epsilon M^{(1)}_\epsilon = \lambda^{(1)}_\epsilon L^{(1)}_\epsilon, \quad M^{(1)}_\epsilon R^{(1)}_\epsilon = \lambda^{(1)}_\epsilon R^{(1)}_\epsilon.$$

We summarize the discussion in figure 6.

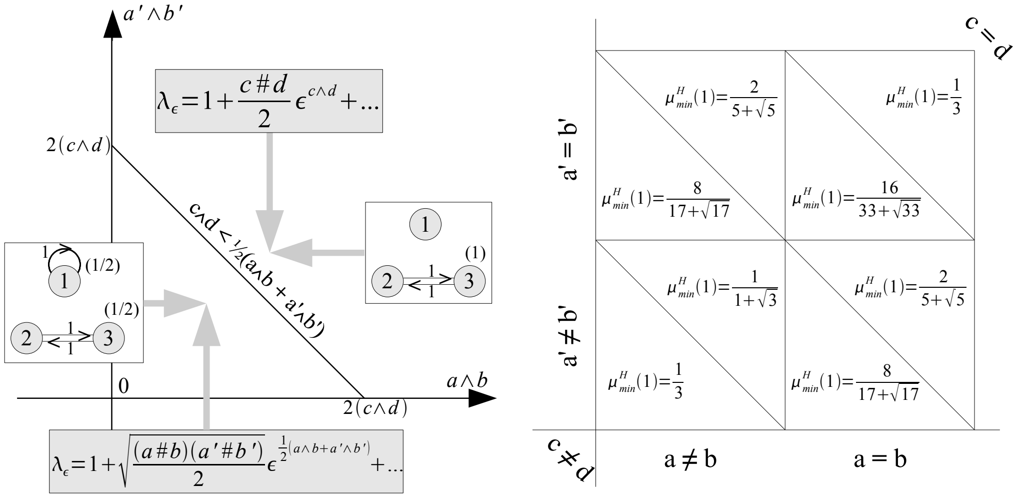


Figure 5: Phase diagram for a 3×3 matrix with two irreducible components: part I. In the left diagram, numbers in parenthesis indicate the weight of each irreducible components. In the right diagram, the value of $\mu^H_{min}(1)$ is shown for the case $c \wedge d = \frac{1}{2}(a \wedge b + a' \wedge b')$.

6.3 Three irreducible components

We consider the matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}, \quad a, a', b, b', c, c' > 0.$$

We know from propositions 29 and 31 that $\lambda_\epsilon \sim 1$ and $Q_\epsilon \to \text{Id}$. We want to show that $[\pi_\epsilon(x)]_{x=1,2,3} = [L_\epsilon(x)R_\epsilon(x)]_{x=1,2,3}$ converges to some raw vector $[\mu^H_{min}(x)]_{x=1,2,3}$ identified to the ground state as a barycenter of 3 Dirac masses:

$$\mu^H_{min} = \mu^H_{min}(1)\delta_{<1^\infty>} + \mu^H_{min}(2)\delta_{<2^\infty>} + \mu^H_{min}(3)\delta_{<3^\infty>}.$$

Thanks to the special form of the matrix, the steps of algorithm 30 are immediate: $M^{(1)}_\epsilon = M_\epsilon - \text{Id}$, $\lambda^{(1)}_\epsilon = \lambda_\epsilon - 1$, $L^{(1)}_\epsilon = L_\epsilon$ and $R^{(1)}_\epsilon = R_\epsilon$. We want to apply again algorithm 30 by reducing $M^{(1)}_\epsilon$ to a normal form as in algorithm 28. We call $\bar{a}^{(1)}$ the minimizing mean exponent of $M^{(1)}_\epsilon$ and $A^{(1)}_{min}$ the matrix associated to the graph of minimizing cycles. Notice that $A^{(1)}_{min}$ admits a unique irreducible component. Let $v : S \to \mathbb{R}$ be a separating corrector and $\tilde{M}_\epsilon := \Delta_\epsilon(v)M^{(1)}_\epsilon\Delta_\epsilon(v)^{-1}\epsilon^{-\bar{a}^{(1)}} = A^{(1)}_{min} + \tilde{N}_\epsilon$. Denote $\tilde{L}_\epsilon(x) = \epsilon^{-v(x)}L^{(1)}_\epsilon(x)$ and $\tilde{R}_\epsilon(x) = \epsilon^{v(x)}R^{(1)}_\epsilon(x)$. Proposition 31 tells us that

$$\frac{\tilde{L}_\epsilon(x)}{\tilde{L}_\epsilon(y)} \sim \frac{\bar{L}(x)}{\bar{L}(y)}, \quad \frac{\tilde{R}_\epsilon(x)}{\tilde{R}_\epsilon(y)} \sim \frac{\bar{R}(x)}{\bar{R}(y)}, \ \forall\, x, y \in \bar{S}, \quad \text{and} \quad \tilde{L}_\epsilon(x)\tilde{R}_\epsilon(x) \to 0, \ \forall\, x \in S_0,$$

where $\bar{L}$ and $\bar{R}$ are the left and right eigenvectors of the dominant matrix $\bar{A}$.

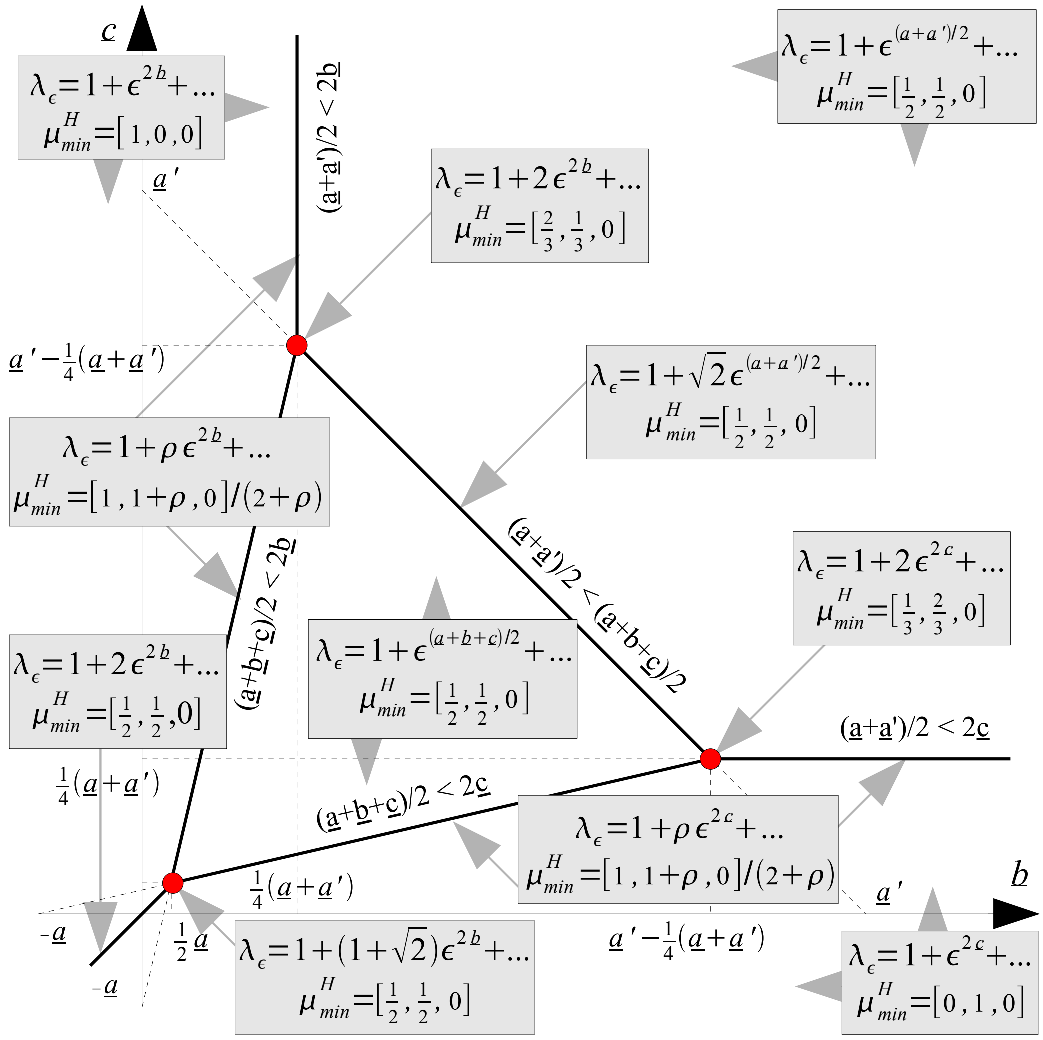


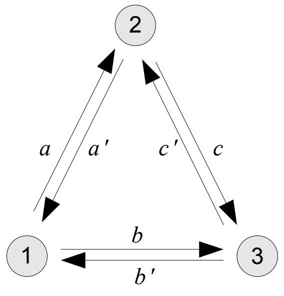
Figure 6: Phase diagram for a 3×3 matrix with two irreducible components: part II. We assume $\underline{a} < \underline{a}'$. The ground state is a barycenter of the periodic measures δ_1 and δ_2.

In order to simplify the phase transition diagram, we change the coefficients:

$$\underline{a} := \frac{1}{2}(a+a'), \quad \underline{b} := \frac{1}{2}(b+b'),$$
$$\underline{c} := c + \frac{1}{2}(b'-b) + \frac{1}{2}(a-a'), \quad \underline{c}' := c' + \frac{1}{2}(b-b') + \frac{1}{2}(a'-a).$$

Then $\dfrac{c+c'}{2} = \dfrac{\underline{c}+\underline{c}'}{2}$, $\dfrac{a+b'+c}{3} = \dfrac{\underline{a}+\underline{b}+\underline{c}}{3}$, $\dfrac{a'+b+c'}{3} = \dfrac{\underline{a}+\underline{b}+\underline{c}'}{3}$.

We now discuss the different phases according to the coincidence set of multiple order of minimizing cycles. We discuss only the case $\underline{c} < \underline{c}'$. The purely symmetric case $a = a'$, $b = b'$, $c = c'$ is done in section 7. We show in figure 7 the location of all possible minimizing cycles.



Mean along the cycles of order 2 and 3:

cycles of order 2	cycles of order 3
$\frac{1}{2}(a+a')$	$\frac{1}{3}(a+b'+c)$
$\frac{1}{2}(b+b')$	$\frac{1}{3}(a'+b+c')$
$\frac{1}{2}(c+c')$	

Figure 7: Graph of interactions and minimizing cycles of $M_\epsilon - \mathrm{Id}$.

6.3.1 Case $\underline{a} < \min\{\underline{b}, \frac{1}{2}(\underline{c}+\underline{c}'), \frac{1}{3}(\underline{a}+\underline{b}+\underline{c})\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{\underline{a}}, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{2}, \frac{1}{2}, 0].$$

6.3.2 Case $\underline{b} < \min\{\underline{a}, \frac{1}{2}(\underline{c}+\underline{c}'), \frac{1}{3}(\underline{a}+\underline{b}+\underline{c})\}$:

$$\bar{a}^{(1)} = \underline{b}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{\underline{b}}, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{2}, 0, \frac{1}{2}].$$

6.3.3 Case $\frac{1}{2}(\underline{c}+\underline{c}') < \min\{\underline{a}, \underline{b}, \frac{1}{3}(\underline{a}+\underline{b}+\underline{c})\}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c}+\underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{(\underline{c}+\underline{c}')/2}, \quad \text{and} \quad \mu^H_{min} = [0, \frac{1}{2}, \frac{1}{2}].$$

6.3.4 Case $\frac{1}{3}(\underline{a}+\underline{b}+\underline{c}) < \min\{\underline{a}, \underline{b}, \frac{1}{2}(\underline{c}+\underline{c}')\}$:

$$\bar{a}^{(1)} = \frac{1}{3}(\underline{a}+\underline{b}+\underline{c}), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \epsilon^{(\underline{a}+\underline{b}+\underline{c})/3},$$

$$\tilde{L}_\epsilon \propto [1,1,1], \quad \tilde{R}_\epsilon \propto [1,1,1]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}].$$

Notice that the reverse cycle $1 \to 3 \to 2 \to 1$ is negligible against the cycle $1 \to 2 \to 3 \to 1$ since its exponent is higher.

6.3.5 Case $\underline{a} = \frac{1}{3}(\underline{a}+\underline{b}+\underline{c}) < \min\{\underline{b}, \frac{1}{2}(\underline{c}+\underline{c}')\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \kappa\epsilon^{\underline{a}},$$

$$\tilde{L}_\epsilon \propto [\kappa^2, \kappa, 1], \quad \tilde{R}_\epsilon \propto [\kappa, \kappa^2, 1]^T, \quad \text{and} \quad \mu^H_{min} = [1+\kappa, 1+\kappa, 1]/(3+2\kappa),$$

where κ is the largest eigenvalue of $A^{(1)}_{min}$ and satisfies $\kappa^3 - \kappa - 1 = 0$.

6.3.6 Case $\underline{b} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \min\{\underline{a}, \frac{1}{2}(\underline{c} + \underline{c}')\}$:

$$\bar{a}^{(1)} = \underline{b}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \kappa \epsilon^{\underline{b}},$$

$$\tilde{L}_\epsilon \propto [\kappa, 1, \kappa^2], \quad \tilde{R}_\epsilon \propto [\kappa^2, 1, \kappa]^T, \quad \text{and} \quad \mu^H_{min} = [1+\kappa, 1, 1+\kappa]/(3+2\kappa).$$

($A^{(1)}_{min}$ admits the same characteristic polynomial as before.)

6.3.7 Case $\frac{1}{2}(\underline{c} + \underline{c}') = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \min\{\underline{a}, \underline{b}\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \kappa \epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, \kappa^2, \kappa], \quad \tilde{R}_\epsilon \propto [1, \kappa, \kappa^2]^T, \quad \text{and} \quad \mu^H_{min} = [1, 1+\kappa, 1+\kappa]/(3+2\kappa).$$

6.3.8 Case $\underline{a} = \underline{b} < \min\{\frac{1}{2}(\underline{c} + \underline{c}'), \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})\}$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \sqrt{2} \epsilon^{\underline{a}},$$

$$\tilde{L}_\epsilon \propto [\sqrt{2}, 1, 1], \quad \tilde{R}_\epsilon \propto [\sqrt{2}, 1, 1]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{2}, \frac{1}{4}, \frac{1}{4}].$$

6.3.9 Case $\underline{a} = \frac{1}{2}(\underline{c} + \underline{c}') < \min\{\underline{b}, \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})\}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \sqrt{2} \epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, \sqrt{2}, 1], \quad \tilde{R}_\epsilon \propto [1, \sqrt{2}, 1]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}].$$

6.3.10 Case $\underline{b} = \frac{1}{2}(\underline{c} + \underline{c}') < \min\{\underline{a}, \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})\}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \sqrt{2} \epsilon^{(\underline{c}+\underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, 1, \sqrt{2}], \quad \tilde{R}_\epsilon \propto [1, 1, \sqrt{2}]^T, \quad \text{and} \quad \mu^H_{min} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}].$$

6.3.11 Case $\underline{a} = \underline{b} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \frac{1}{2}(\underline{c} + \underline{c}')$:

$$\bar{a}^{(1)} = \underline{a}, \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \rho \epsilon^{\underline{a}},$$

$$\tilde{L}_\epsilon \propto [\rho, 1, \rho], \quad \tilde{R}_\epsilon \propto [\rho, \rho, 1]^T, \quad \text{and} \quad \mu^H_{min} = [\rho, 1, 1]/(2 + \rho),$$

where ρ is the positive root of $\rho^3 - 2\rho - 1 = (\rho + 1)(\rho^2 - \rho - 1) = 0$.

6.3.12 Case $\underline{a} = \frac{1}{2}(\underline{c} + \underline{c}') = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \underline{b}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \rho \epsilon^{(\underline{c} + \underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [\rho, \rho, 1], \quad \tilde{R}_\epsilon \propto [1, \rho, \rho]^T, \quad \text{and} \quad \mu^H_{min} = [1, \rho, 1]/(2 + \rho).$$

6.3.13 Case $\underline{b} = \frac{1}{2}(\underline{c} + \underline{c}') = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) < \underline{a}$:

$$\bar{a}^{(1)} = \frac{1}{2}(\underline{c} + \underline{c}'), \quad A^{(1)}_{min} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \lambda^{(1)}_\epsilon \sim \rho \epsilon^{(\underline{c} + \underline{c}')/2},$$

$$\tilde{L}_\epsilon \propto [1, \rho, \rho], \quad \tilde{R}_\epsilon \propto [\rho, 1, \rho]^T, \quad \text{and} \quad \mu^H_{min} = [1, 1, \rho]/(2 + \rho).$$

We summarize the preceding discussion in the figure 8.

7 Zero-temperature phase diagram for BEG model

We give in this section a complete description of the zero-temperature phase diagram for the Blume-Emery-Griffiths model. We apply the algorithm proposed in section 3 to $S = \{-, 0, +\}$, $G = S \times S$ and $M_\epsilon(x, y) = \epsilon^{H_0(x,y)}$ for all $x, y \in S$, where

$$H_0 = \begin{bmatrix} -J - K + \Delta & \frac{1}{2}\Delta & J - K + \Delta \\ \frac{1}{2}\Delta & 0 & \frac{1}{2}\Delta \\ J - K + \Delta & \frac{1}{2}\Delta & -J - K + \Delta \end{bmatrix}.$$

We discuss the different cases according to the choice of the parameters wich contribute to the minimizing mean exponent $\bar{a}$. In all cases, we have

$$M_\epsilon = \begin{bmatrix} \epsilon^a & \epsilon^b & \epsilon^c \\ \epsilon^b & 1 & \epsilon^b \\ \epsilon^c & \epsilon^b & \epsilon^a \end{bmatrix}, \quad \pi_\epsilon = \begin{bmatrix} L_\epsilon(-)R_\epsilon(-) \\ L_\epsilon(0)R_\epsilon(0) \\ L_\epsilon(+)R_\epsilon(+) \end{bmatrix}, \quad Q_\epsilon = \begin{bmatrix} \frac{\epsilon^a R_\epsilon(-)}{\lambda_\epsilon R_\epsilon(-)} & \frac{\epsilon^b R_\epsilon(0)}{\lambda_\epsilon R_\epsilon(-)} & \frac{\epsilon^c R_\epsilon(+)}{\lambda_\epsilon R_\epsilon(-)} \\ \frac{\epsilon^b R_\epsilon(-)}{\lambda_\epsilon R_\epsilon(0)} & \frac{R_\epsilon(0)}{\lambda_\epsilon R_\epsilon(0)} & \frac{\epsilon^b R_\epsilon(+)}{\lambda_\epsilon R_\epsilon(0)} \\ \frac{\epsilon^c R_\epsilon(-)}{\lambda_\epsilon R_\epsilon(+)} & \frac{\epsilon^b R_\epsilon(0)}{\lambda_\epsilon R_\epsilon(+)} & \frac{\epsilon^a R_\epsilon(+)}{\lambda_\epsilon R_\epsilon(+)} \end{bmatrix},$$

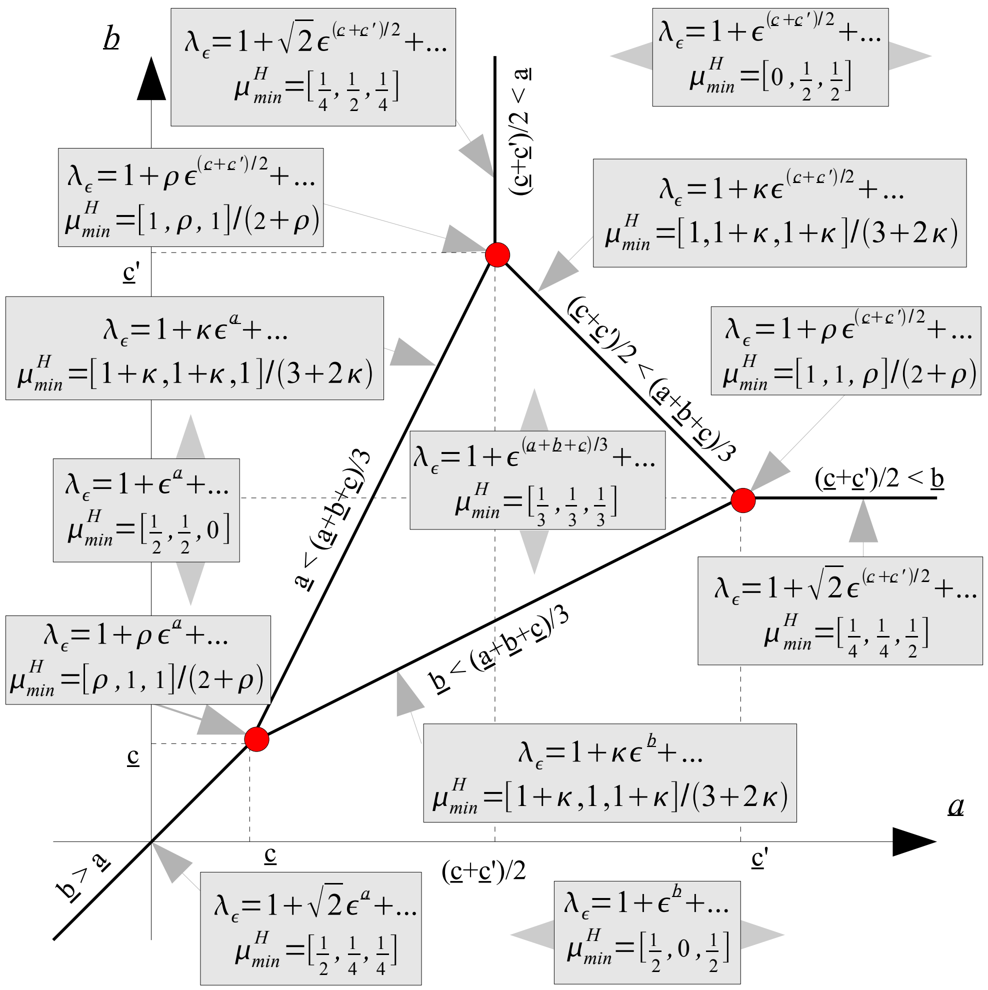


Figure 8: Phase diagram for a 3×3 matrix with three irreducible components. We assume $\underline{c} < \underline{c}'$. The ground state is a barycenter of the three periodic measures $\delta_{<1^\infty>}$, $\delta_{<2^\infty>}$ and $\delta_{<3^\infty>}$. The constants ρ and κ are solutions of $\rho^2-\rho-1=0$ and $\kappa^3-\kappa-1=0$. The exact values of these constants are $\rho=\frac{1}{2}(1+\sqrt{5})$ and $\kappa=\sqrt[3]{\frac{1}{2}(1-\sqrt{23/27})}+\sqrt[3]{\frac{1}{2}(1+\sqrt{23/27})}$.

normalized by $\sum_{x\in S}L_\epsilon(x)R_\epsilon(x)=1$ and $\sum_{x\in S}R_\epsilon(x)=1$. Because of the symmetry of M_ϵ, $L_\epsilon=R_\epsilon$ and $\pi_\epsilon(x)=R^2_\epsilon(x)/\sum_x R^2_\epsilon(x)$. We also simplify the computation by noticing that $R_\epsilon(-)=R_\epsilon(+)$. We recall that G_{min} is the minimizing subgraph and $\bar\alpha$ is the dominant spectral coefficient. We only present the details of the computations for $\Delta>0$, the other situations being analogous.

7.1 Case $J-K+\Delta<0$, $J<0$:

Case: $c<\min(0,a,b)$. We know that

$$\bar a=c,\quad A_{min}=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\quad \bar\alpha=1,\quad \lambda_\epsilon\sim\epsilon^c$$

and G_{min} has one irreducible component $(-) \rightleftharpoons (+)$. We aggregate the two components $(\pm)$ by adding $R_\epsilon(\pm) := R_\epsilon(-) + R_\epsilon(+)$ and eliminate the negligible term $R_\epsilon(0)$. The new singular eigenvalue problem obtained in algorithm 30, $M_\epsilon^{(1)} R_\epsilon^{(1)} = \lambda_\epsilon^{(1)} R_\epsilon^{(1)}$, is actually reduced to a unique equation with unique unknown $R_\epsilon^{(1)} := R_\epsilon(\pm)$. More precisely,

$$\begin{cases} (\epsilon^a + \epsilon^c) R_\epsilon(\pm) + 2\epsilon^b R_\epsilon(0) & = & \lambda_\epsilon R_\epsilon(\pm), \\ \epsilon^b R_\epsilon(\pm) + R_\epsilon(0) & = & \lambda_\epsilon R_\epsilon(0), \end{cases}$$

$$R_\epsilon(0) = \frac{\epsilon^b}{\lambda_\epsilon - 1} R_\epsilon(\pm) \ll R_\epsilon(\pm), \quad \lambda_\epsilon^{(1)} := \lambda_\epsilon - \epsilon^c = \epsilon^a + \frac{2\epsilon^{2b}}{\lambda_\epsilon - 1},$$

which yields

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-c} \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \epsilon^{-J+K-\Delta/2} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(-J+K-\Delta/2)} \\ 1/2 \end{bmatrix},$$

$$Q_\epsilon \sim \begin{bmatrix} \epsilon^{a-c} & 2\epsilon^{2(b-c)} & 1 \\ 1/2 & \epsilon^{-c} & 1/2 \\ 1 & 2\epsilon^{2(b-c)} & \epsilon^{a-c} \end{bmatrix} = \begin{bmatrix} \epsilon^{-2J} & 2\epsilon^{2(-J+K-\Delta/2)} & 1 \\ 1/2 & \epsilon^{-J+K-\Delta} & 1/2 \\ 1 & 2\epsilon^{2(-J+K-\Delta/2)} & \epsilon^{-2J} \end{bmatrix}.$$

7.2 Case $-J - K + \Delta < 0$, $J > 0$:

Case: $a < \min(0, b, c)$. G_{min} has two irreducible components with identical spectral coefficient, $(-) \leftrightarrow (-)$ and $(+) \leftrightarrow (+)$, and as before $R_\epsilon(0) \ll R_\epsilon(-) = R_\epsilon(+)$. We thus obtain

$$\bar{a} = a, \quad A_{min} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim \epsilon^a,$$

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-a} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(b-a)} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1 & 2\epsilon^{2(b-a)} & \epsilon^{c-a} \\ 1/2 & \epsilon^{-a} & 1/2 \\ \epsilon^{c-a} & 2\epsilon^{2(b-a)} & 1 \end{bmatrix}.$$

7.3 Case $-J - K + \Delta > 0$, $J - K + \Delta > 0$:

Case: $0 < \min(a, b, c)$. G_{min} has one irreducible component $(0) \leftrightarrow (0)$, $\bar{\alpha} = 1$ and $R_\epsilon(-) = R_\epsilon(+) \ll R_\epsilon(0)$. We obtain

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim 1,$$

$$R_\epsilon \sim \begin{bmatrix} \epsilon^b \\ 1 \\ \epsilon^b \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} \epsilon^{2b} \\ 1 \\ \epsilon^{2b} \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^a & 1 & \epsilon^c \\ \epsilon^{2b} & 1 & \epsilon^{2b} \\ \epsilon^c & 1 & \epsilon^a \end{bmatrix}.$$

As in case 7.1, we eliminate the negligible term $R_\epsilon(\pm)$ and get a new graph $G^{(1)}$ reduced to a singleton

$$R_\epsilon(\pm) = \frac{2\epsilon^b}{\lambda_\epsilon - (\epsilon^a + \epsilon^c)} R_\epsilon(0) \quad \text{and} \quad \lambda_\epsilon - 1 = \frac{2\epsilon^{2b}}{\lambda_\epsilon - (\epsilon^a + \epsilon^c)} \sim 2\epsilon^{2b}.$$

7.4 Case $J - K + \Delta = 0$, $J < 0$:

Case: $c = 0 < \min(a, b)$. We know that

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim 1,$$

and G_{min} has two irreducible components $(-) \rightleftharpoons (+)$ and $(0) \leftrightarrow (0)$. No state $x \in S$ is *a priori* negligible. We then aggregate the states $(\pm)$ by adding $R_\epsilon(\pm) := R_\epsilon(+) + R_\epsilon(-)$ and obtain a new eigenvalue problem $M_\epsilon^{(1)} R_\epsilon^{(1)} = \lambda_\epsilon^{(1)} R_\epsilon^{(1)}$, where

$$M_\epsilon^{(1)} := \begin{bmatrix} \epsilon^a & 2\epsilon^b \\ \epsilon^b & 0 \end{bmatrix}, \quad R_\epsilon^{(1)} := \begin{bmatrix} R_\epsilon(\pm) \\ R_\epsilon(0) \end{bmatrix} \quad \text{and} \quad \lambda_\epsilon^{(1)} := \lambda_\epsilon - 1.$$

We then have to discuss three subcases.

7.4.1 Subcase $J < -\frac{1}{4}\Delta < 0$:

Subcase: $b < a$. The minimizing subgraph $G_{min}^{(1)}$ has one irreducible component $(\pm) \rightleftharpoons (0)$ with minimizing mean exponent $\bar{a}^{(1)} = b$ and dominant spectral coefficient $\bar{\alpha}^{(1)} = \sqrt{2}$. We obtain

$$\lambda_\epsilon^{(1)} \sim \sqrt{2}\epsilon^b, \quad R_\epsilon^{(1)} \propto \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \quad R_\epsilon \propto \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and}$$

$$\lambda_\epsilon = 1 + \sqrt{2}\epsilon^b + \dots, \quad \pi_\epsilon \sim \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^a & \sqrt{2}\epsilon^b & 1 \\ \epsilon^b/\sqrt{2} & 1 & \epsilon^b/\sqrt{2} \\ 1 & \sqrt{2}\epsilon^b & \epsilon^a \end{bmatrix}.$$

7.4.2 Subcase $J = -\frac{1}{4}\Delta < 0$:

Subcase: $a = b$. $G_{min}^{(1)}$ has one irreducible component $(\pm) \leftrightarrow (\pm) \rightleftharpoons (0)$ with dominant spectral coefficient $\bar{\alpha}^{(1)} = 2$ (the spectral radius of $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$), and the right eigenvector $R_\epsilon^{(1)}$ is proportional to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We obtain $\lambda_\epsilon^{(1)} \sim 2\epsilon^b$ and

$$\lambda_\epsilon = 1 + 2\epsilon^b + \dots, \quad R_\epsilon \sim \pi_\epsilon \sim \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^b & \epsilon^b & 1 \\ \epsilon^b & 1 & \epsilon^b \\ 1 & \epsilon^b & \epsilon^b \end{bmatrix}.$$

7.4.3 Subcase $-\frac{1}{4}\Delta < J < 0$:

Subcase: $a < b$. The minimizing subgraph $G_{min}^{(1)}$ has one irreducible component $(\pm) \leftrightarrow (\pm)$ with dominant spectral coefficient $\bar{\alpha}^{(1)} = 1$. We obtain therefore

$\lambda_\epsilon^{(1)} \sim \epsilon^a$, $R_\epsilon(0) = \epsilon^{b-a} R_\epsilon(\pm) \ll R_\epsilon(\pm)$, $\lambda_\epsilon = 1 + \epsilon^a + \ldots$ and

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-a} \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(b-a)} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} \epsilon^a & 2\epsilon^{2b-a} & 1 \\ \epsilon^a/2 & 1 & \epsilon^a/2 \\ 1 & 2\epsilon^{2b-a} & \epsilon^a \end{bmatrix}.$$

7.5 Case $-J - K + \Delta = 0$, $J > 0$:

Case: $a = 0 < \min(b, c)$. One then has

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 1, \quad \lambda_\epsilon \sim 1.$$

The minimizing subgraph G_{min} has three irreducible components $(-) \leftrightarrow (-)$, $(0) \leftrightarrow (0)$ and $(+) \leftrightarrow (+)$. Once again we simplify the proof by noticing that $R_\epsilon(-) = R_\epsilon(+)$, but it is so far not clear which state dominates. The reduction to an aggregated form consists in simply eliminating the first term of λ_ϵ in the Puiseux series:

$$M_\epsilon^{(1)} = M_\epsilon - \mathrm{Id}, \quad M_\epsilon^{(1)} R_\epsilon^{(1)} = \lambda_\epsilon^{(1)} R_\epsilon^{(2)}, \quad R_\epsilon^{(1)} = R_\epsilon, \quad \lambda_\epsilon^{(1)} = \lambda_\epsilon - 1.$$

The new graph $G^{(1)}$ has possible minimizing mean exponents $\bar{a}^{(1)} = b$ or c. Let $\bar{\alpha}^{(1)}$ be the associated dominant spectral coefficient. We discuss three subcases.

7.5.1 Subcase $0 < \frac{1}{4}\Delta < J$:

Subcase: $b < c$. $G_{min}^{(1)}$ has one irreducible component $(-) \rightleftharpoons (0) \rightleftharpoons (+)$ with $\bar{a}^{(1)} = b$. Moreover,

$$A_{min}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{\alpha}^{(1)} = \sqrt{2}, \quad R_\epsilon^{(1)} \propto \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

Then $\lambda_\epsilon = 1 + \sqrt{2}\epsilon^b + \ldots$ and

$$R_\epsilon \propto \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1 & \sqrt{2}\epsilon^b & \epsilon^c \\ \epsilon^b/\sqrt{2} & 1 & \epsilon^b/\sqrt{2} \\ \epsilon^c & \sqrt{2}\epsilon^b & 1 \end{bmatrix}.$$

7.5.2 Subcase $0 < \frac{1}{4}\Delta = J$:

Subcase: $c = b$. The subgraph $G_{min}^{(1)}$ has one irreducible component $(-) \rightleftharpoons (0) \rightleftharpoons (+) \rightleftharpoons (-)$ and

$$A_{min}^{(1)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \bar{\alpha}^{(1)} = 2, \quad R_\epsilon^{(1)} \propto \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We thus obtain

$$\lambda_\epsilon = 1 + 2\epsilon^b + \ldots, \qquad R_\epsilon \sim \pi_\epsilon \sim \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \qquad Q_\epsilon \sim \begin{bmatrix} 1 & \epsilon^b & \epsilon^b \\ \epsilon^b & 1 & \epsilon^b \\ \epsilon^b & \epsilon^b & 1 \end{bmatrix}.$$

7.5.3 Subcase $0 < J < \frac{1}{4}\Delta$:

Subcase: $c < b$. $G_{min}^{(1)}$ has one irreducible component $(-) \rightleftharpoons (+)$ with minimizing mean exponent $\bar{a}^{(1)} = c$ and $\bar{\alpha}^{(1)} = 1$. We aggregate the states $(\pm)$, $R_\epsilon^{(1)}(\pm) := R_\epsilon^{(1)}(-) + R_\epsilon^{(1)}(+)$, and eliminate $R_\epsilon^{(1)}(0) \ll R_\epsilon^{(1)}(\pm)$ to obtain a third graph (reduced to a singleton)

$$\begin{cases} \epsilon^c R_\epsilon^{(1)}(\pm) + 2\epsilon^b R_\epsilon^{(1)}(0) & = & \lambda_\epsilon^{(1)} R_\epsilon^{(1)}(\pm), \\ \epsilon^b R_\epsilon^{(1)}(\pm) & = & \lambda_\epsilon^{(1)} R_\epsilon^{(1)}(0), \end{cases} \qquad \epsilon^c + \frac{2\epsilon^{2b}}{\lambda_\epsilon^{(1)}} = \lambda_\epsilon^{(1)}.$$

We get $\lambda_\epsilon = 1 + \epsilon^c + \ldots$ and

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-c} \\ 1/2 \end{bmatrix}, \qquad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2(b-c)} \\ 1/2 \end{bmatrix}, \qquad Q_\epsilon \sim \begin{bmatrix} 1 & 2\epsilon^{2b-c} & \epsilon^c \\ \epsilon^c/2 & 1 & \epsilon^c/2 \\ \epsilon^c & 2\epsilon^{2b-a} & 1 \end{bmatrix}.$$

7.6 Case $J = 0 < \Delta < K$:

Case: $a = c < \min(0, b)$. One has

$$\bar{a} = a, \quad A_{min} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 2, \quad \lambda_\epsilon \sim 2\epsilon^a.$$

G_{min} has one irreducible component $(-) \leftrightarrow (-) \rightleftharpoons (+) \leftrightarrow (+)$ with minimizing mean exponent $\bar{a} = a$ and dominant spectral coefficient $\bar{\alpha} = 2$. We again aggregate the states $(\pm)$, $R_\epsilon(\pm) := R_\epsilon(-) + R_\epsilon(+)$, and eliminate $R_\epsilon(0) \ll R_\epsilon(\pm)$ in order to introduce a new singular eigenvalue problem

$$\begin{cases} 2\epsilon^a R_\epsilon(\pm) + 2\epsilon_\epsilon^R(0) & = & \lambda_\epsilon R_\epsilon(\pm), \\ \epsilon^b R_\epsilon(\pm) + R_\epsilon(0) & = & \lambda_\epsilon R_\epsilon(0), \end{cases} \qquad 2\epsilon^a + \frac{2\epsilon^{2b}}{\lambda_\epsilon - 1} = \lambda_\epsilon.$$

We thus obtain

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{b-a}/2 \\ 1/2 \end{bmatrix}, \qquad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^{2(b-a)}/2 \\ 1/2 \end{bmatrix}, \qquad Q_\epsilon \sim \begin{bmatrix} 1/2 & \epsilon^{2(b-a)}/2 & 1/2 \\ 1/2 & \epsilon^{-a}/2 & 1/2 \\ 1/2 & \epsilon^{2(b-a)} & 1/2 \end{bmatrix}.$$

7.7 Case $J = 0 < \Delta = K$:

Case: $a = c = 0 < b$. We have

$$\bar{a} = 0, \quad A_{min} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{\alpha} = 2, \quad \lambda_\epsilon \sim 2.$$

G_{min} has two irreducible components with spectral coefficients equal to 1 and 2, whose graphs are $(0) \leftrightarrow (0)$ and $(-) \leftrightarrow (-) \rightleftharpoons (+) \leftrightarrow (+)$, respectively. We aggregate $(\pm)$ into a unique state $R_\epsilon(\pm) := R_\epsilon(-) + R_\epsilon(+)$ and obtain

$$\left\{ \begin{array}{lcl} 2R_\epsilon(\pm) + 2\epsilon^b R_\epsilon(0) & = & \lambda_\epsilon R_\epsilon(\pm), \\ \epsilon^b R_\epsilon(\pm) + R_\epsilon(0) & = & \lambda_\epsilon R_\epsilon(0), \end{array} \right. \qquad 2 + \frac{2\epsilon^{2b}}{\lambda_\epsilon - 1} = \lambda_\epsilon.$$

We thus get $\lambda_\epsilon = 2 + 2\epsilon^{2b} + \dots$ and

$$R_\epsilon \sim \begin{bmatrix} 1/2 \\ \epsilon^b \\ 1/2 \end{bmatrix}, \quad \pi_\epsilon \sim \begin{bmatrix} 1/2 \\ 2\epsilon^{2b} \\ 1/2 \end{bmatrix}, \quad Q_\epsilon \sim \begin{bmatrix} 1/2 & \epsilon^{2b} & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & \epsilon^{2b} & 1/2 \end{bmatrix}.$$

We recall that the previous discussion is summarized in figures 3 and 4.

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