

Range Control Charts Revisited: Simpler Tippett-like Formulae, It's Practical Implementation and the Study of False Alarm

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Abstract

This paper presents simpler alternative formulae and procedures of implementation to deal with the relative range statistic (moments, distribution and quantiles) used in the construction of range control charts for process dispersion monitoring. The Tippett's integral formulae for sample relative range moments and distribution are revisited and simpler new alternative expressions are proposed together with an easy computational implementation procedure based on it's relation with the Tukey maximum studentized range statistic.

Also, such proposed methods are applied in the assessment of the range chart performance considering false alarm comparison between exact control limits charts versus normal approximated versions (3-sigma Shewhart procedure), which show the serious drawbacks of such misplaced control limits. These much simple tools introduced here, we believe, will permit the presentation of R control charts more transparently and without unrealistic normal approximations or blind dependence on tables, avoiding the serious limitations of such "ad hoc" practice.

KEY WORDS: Exact Control Limits, False Alarm, R Control Charts, Relative Range, Tippett's integrals, Tukey Statistic.

1 INTRODUCTION

It can be said that the birth of modern statistical process control took place when Walter A. Shewhart developed the concept of a control chart based on the monitoring of process mean level (\bar{X} chart) and process dispersion (R or S charts). However, it was Leonard Tippett that studied in detail the sample Range Statistic R and it's relative version $W = R/\sigma$ (moment formulae and sampling distribution), providing the necessary statistical background for the R chart, which has been the most used tool for monitoring process variability since then. For original references, see Shewhart, W.A.(1926) and Tippett, L.H.R.(1925); for actual corresponding references see, for instance, Montgomery, D.C.(2008) and David, H.A.(1981).

Although (for typical small samples at least, say $n \leq 8$, as usual) there is no great loss of statistical efficiency in using the sample range R (instead of the sample standard deviation S) for monitoring or estimating process variability, however, there are certainly two very

important drawbacks with the Shewhart-Tippett procedures (which are used until nowadays) for monitoring variability. And how to overcome them is the object of the present paper.

The two drawbacks are the following:

1. The “lack of simplicity” of the Tippett’s integral formulae for moments and distribution of $W = R/\sigma$ ($\mathbb{E}(W)$, $V(W)$, $F(w)$, $f(w)$, presented at section 2) is evident. It’s too much formulae, some expressions are not very simple, most formulae deductions are not straight, there are some shortcomings with their implementation (tables, software), etc. Some of these implementation shortcomings (W distribution) are vital, since the tables available (for instance, Pearson, E.S. and Hartley, H.O., 1943; Harter, H.L., 1960; Harter, H.L. and Balakrishnan, N., 1997) were not designed specifically for quality control applications (where $\alpha = 0.0027$ or 0.0020 and not 0.05 or 0.01 as usual in the tables).
2. The inadequacy of approximating the R distribution by normal (the 3-sigma limits proposed by Shewhart and used by it’s followers until nowadays) is also evident. The domain of R (positive) and the shape (skew) are totally different in a normal distribution, what causes the “problem of misplace control limits” (see Alwan, L.C. and Roberts, H.V., 1995).

In fact, we can consider the second drawback just as a consequence of the first; if the distribution of W was not easy to implement at that time (in the second decade of the XX century), a normal approximation was an easier or practical alternative found by Shewhart.

The two main consequences of these drawbacks are:

1. the use of misplaced control limits in the R control chart (with significant inflation of false alarm probability and difficulty in the lower limit specification), which is recognized by some authors (for instance, Alwan, L.C. and Roberts, H.V.,1995; Alwan, L.,2000, and others), neglected by others, and no definitive simple solution presented; this point will be discussed in more detail at sections 3 and 4 of this paper.
2. the lack of transparency or “ad hoc” way in the presentation of range control charts in the main textbooks about the subject (see instance, Montgomery, D.C., 2008; Ryan, T.P., 1989; Devor, R.E. et al,2007; Grant, E.L. and Leavenworth, R.S.,1996,; Kenett, R. and Zachs, S.,1998 and others); this occurs perhaps because the Tippett’s theory is not simple enough. Some other authors try to present some theoretical background for the R chart (see Burr, I.W.,1974; Alwan, L.,2000; Costa, A.F.B. et al,2004), but without full success.

In order to overcome these difficulties, we reformulate drastically the Tippett’s formulae, reducing the number and complexity of them and giving a simple way of implementation, exploring a simple relationship that exists between the statistics relative range and Tukey maximum studentized range for multiple comparison tests. For details about this last statistic, see Tukey, J.W.,1953, or Copenhaver, M.D. and Holland, B.,1988.

The organization of the paper is the following. After this introduction, we revisit

and analyse the Tippett's formulae at section 2. In section 3 it's presented the new simpler formulation and in section 4 it's presented one application or illustration involving a false alarm analysis. Final discussions and conclusions are presented at section 5, followed by the references.

2 Tippett's Formulae Revisited

2.1 Basic Notation and Background

Let's consider the following basic notation, definitions and background results about order statistics and moments:

(i) **notation:** X_1, X_2, \dots, X_n are iid (independent and identically distributed) random variables with distribution function $F(x)$, forming a (simple) random sample of the continuous characteristic X , and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ are the corresponding order statistics. In practice, the important case is when $X \sim \text{Normal}$, when $F(x) = \Phi(x)$, but moderate departures from normality are acceptable in this context (Burr, I.W., 1967; Brown, R.A., 1974).

(ii) **definitions:** $R = X_{(n)} - X_{(1)}$ is the sample range or amplitude, and, for $r = 1, 2, \dots, n-1$, $S_r = X_{(r+1)} - X_{(r)}$ are the "spacings", such that $R = \sum_{r=1}^{n-1} S_r$. Also, $W = R/\sigma$ is the sample relative range, where σ is the standard deviation of X . The mean and standard deviation of W will be represented by $d_2 = \mathbb{E}(W)$ and $d_3 = \sqrt{V(W)}$.

(iii) **distribution of order statistics:** $F_r(x)$, for $r \in \{1, 2, \dots, n\}$, is the distribution function of the r^{th} order statistic $X_{(r)}$, such that

$$F_r(x) = \sum_{i=r}^n \binom{n}{i} F^i(x) (1 - F(x))^{n-i}$$

since $F_r(x) = \mathbb{P}(X_{(r)} \leq x) = \mathbb{P}(r \text{ or more values of } X_i \leq x)$; which is a sum of binomial probabilities. Also, using a similar reasoning but now in two dimensions, it can be obtained the joint distribution function of $X_{(r)}, X_{(s)}$, $r < s$, as

$$G_{r,s}(x,y) = \sum_{j=s}^n \sum_{i=r}^j \binom{n}{i, j-i} F^i(x) [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j}$$

that is, the former binomial sum now became a trinomial sum. For details, see David, H.A. (1981).

(iv) **moments in terms of distribution functions:** $\mathbb{E}(X)$ in terms of $F(x)$ is given by

$$\mathbb{E}(X) = \int_0^{\infty} (1 - F(x)) dx - \int_{-\infty}^0 F(x) dx$$

which is a more convenient expression for dealing with order statistics than the more usual way in terms of it's density. It can be verified integrating by parts (see Feller, W., 1966, pg 148 or Mood, A.M.; Graybill, F.A. and Boes, D.C., 1974, pg 65). If X is non-negative, as is the case of the relative range W , by the last formula, $\mathbb{E}(W) = \int_0^{\infty} (1 - F(w)) dw$, where $F(w)$ is the distribution function of W , since the second integral is null.

In the case of second order moments, the formula for variances-covariances in terms of distribution functions, is given by

$$Cov(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (G(x, y) - F(x)F(y)) dx dy$$

where $G(x, y)$ is the joint distribution function of X and Y .

This formula is credited originally to Hoeffding, W.(1940) and is also discussed, for instance, in Jones, M.C. and Balakrishnan, N.(2002) and Cuadras, C.M. (2002).

The expressions for $F_r(x)$ and $\mathbb{E}(X)$ will be used (section 2.2) in the construction of the first Tippett formula, and the expression above for $\mathbb{E}(W)$ will be used in section 3.

(v) **moments of spacings:** the first order moment, $\mathbb{E}(S_r) = \mathbb{E}(X_{(r+1)} - X_{(r)})$ can be obtained using the formula for $\mathbb{E}(X)$ given at (iv) and the expression for $F_r(x)$ given at (iii), resulting (after simple algebra), in

$$\mathbb{E}(S_r) = \int_{-\infty}^{\infty} \binom{n}{r} F^r(x) [1 - F(x)]^{n-r} dx$$

This result (originally presented by Pearson, K.,1902) is the first step in the obtaintion of the Tippett formula for $\mathbb{E}(W)$, which is shown in detail at section 2.2.

The second order moment, $\mathbb{E}(S_r S_s) = \mathbb{E}((X_{(r+1)} - X_{(r)})(X_{(s+1)} - X_{(s)}))$ (which is used in the proof of Tippett's formula for $V(W)$) is obtained (after some tedious algebra) using the formula for $Cov(X, Y)$ in (iv) and the expression for $G_{r,s}(x, y)$ in (iii), resulting in a sort of trinomial formula, given by

$$\mathbb{E}(S_r S_s) = \int_x^{\infty} \int_{-\infty}^y \binom{n}{r, s-r} F^r(x) [F(y) - F(x)]^{s-r} [1 - F(y)]^{n-s} dx dy$$

which can be seen as an extension of the binomial formula for $\mathbb{E}(S_r)$, and is presented in Jones, M.C. and Balakrishnan, N.(2002).

2.2 Relative Range Moments

2.2.1 The First Tippett's Integral: d_2 Mean Formula

Let's first calculate $\mathbb{E}(S_r)$ and after that join the spacings,

$$\begin{aligned} \mathbb{E}(S_r) &= \mathbb{E}(X_{(r+1)} - X_{(r)}) = \mathbb{E}(X_{(r+1)}) - \mathbb{E}(X_{(r)}) \\ &= \left[\int_0^{\infty} (1 - F_{r+1}(x)) dx - \int_{-\infty}^0 F_{r+1}(x) dx \right] - \left[\int_0^{\infty} (1 - F_r(x)) dx - \int_{-\infty}^0 F_r(x) dx \right] \\ &= \int_{-\infty}^{\infty} [F_r(x) - F_{r+1}(x)] dx = \int_{-\infty}^{\infty} \binom{n}{r} F^r(x) [1 - F(x)]^{n-r} dx \end{aligned}$$

Now, joining the spacings, and interchanging sum and integral, we have

$$\begin{aligned} \mathbb{E}(R) &= \mathbb{E}\left(\sum_{r=1}^{n-1} S_r\right) = \sum_{r=1}^{n-1} \mathbb{E}(S_r) = \int_{-\infty}^{\infty} \sum_{r=1}^{n-1} \underbrace{\binom{n}{r} F^r(x) [1 - F(x)]^{n-r}}_{p(r)} dx \\ &= \int_{-\infty}^{\infty} \left[\sum_{r=0}^n p(r) - p(0) - p(n) \right] dx = \int_{-\infty}^{\infty} [1 - [1 - F(x)]^n - F^n(x)] dx \end{aligned}$$

Now, since $W = R/\sigma$, considering $Z = X/\sigma$, we have

$$\mathbb{E}(W) = \frac{1}{\sigma} \int_{-\infty}^{\infty} [1 - [1 - F_Z(z)]^n - F_Z^n(z)] \sigma dz = \int_{-\infty}^{\infty} [1 - [1 - F_Z(z)]^n - F_Z^n(z)] dz \quad (\text{I})$$

where $F_Z(z) = \Phi(z)$ in the normal case. This mean depends only on n and is known in the literature (see for instance, Montgomery, D.C., 2008 or SAS/QC® 9.2 User's Guide, 2008) as the d_2 constant or $d_2(n)$.

2.2.2 The Second Tippett's Integral: d_3 Standard Deviation Formula

Let's use the formula for $\mathbb{E}(S_r S_s)$ from 2.1 - (v) after joining the spacings to form the range R ,

$$\mathbb{E}(R^2) = \mathbb{E} \left[\left(\sum_{r=1}^{n-1} S_r \right) \left(\sum_{s=1}^{n-1} S_s \right) \right] = 2 \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} \mathbb{E}(S_r S_s) = 2 \int_x^{\infty} \int_{-\infty}^y \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} a_{rs}$$

where a_{rs} is the trinomial expression (section 2.1-v), such that

$$\sum_{r=1}^{n-1} \sum_{s=1}^{n-1} a_{rs} = \sum_{r=0}^n \sum_{s=0}^n a_{rs} - \sum_{s=0}^n a_{0s} - \sum_{r=0}^n a_{rn} + a_{0n}$$

$$= 1 - [1 - F(x)]^n - F^n(y) + [F(y) - F(x)]^n \quad ; \text{ then}$$

$$\mathbb{E}(W^2) = 2 \int_u^{\infty} \int_{-\infty}^v \left\{ 1 - [1 - F(u)]^n - F^n(v) + [F(v) - F(u)]^n \right\} du dv \quad (\text{II})$$

where $u = x/\sigma$ and $v = y/\sigma$. Then, $d_3 = \sqrt{\mathbb{E}(W^2) - d_2^2}$ and, in the normal case we have $F(u) = \Phi(u)$, $F(v) = \Phi(v)$, the cdf of the standard normal.

2.3 Relative Range Distribution

2.3.1 The Third Tippett's Integral: Distribution Function for W

$$F_W(w) = \mathbb{P}(W \leq w) = \mathbb{P}((X_{(n)} - X_{(1)})/\sigma \leq w) = \mathbb{P}(X_{(n)} \leq X_{(1)} + \sigma w)$$

From the total probability theorem in continuous version, we have,

$$F_W(w) = \mathbb{E}_X [\mathbb{P}(X_{(n)} \leq X_{(1)} + \sigma w \mid X_{(1)} = x_k)], \quad k = 1, 2, \dots, n$$

$$F_W(w) = \mathbb{E}_X [\mathbb{P}(\cup_{k=1}^n (\text{All } (n-1) \text{ remaining } X_j \in [x, x + \sigma w]))]$$

$F_W(w) = \mathbb{E}_X [n \mathbb{P}^{n-1}(x \leq X_j \leq x + \sigma w)]$, since the events are independent in j and excludent in k ; then, $F_W(w) = n \int_{-\infty}^{\infty} [F(x + \sigma w) - F(x)]^{n-1} f(x) dx$

where, in the normal case, $F(x) = \Phi(x)$ and $f(x) = \varphi(x)$ from the standard normal random variable, resulting

$$F_W(w) = n \int_{-\infty}^{\infty} [\Phi(x + \sigma w) - \Phi(x)]^{n-1} \varphi(x) dx \quad (\text{III})$$

2.3.2 The Fourth Tippett's Integral: Density Function for W

$$f_W(w) = \frac{d}{dw} F_W(w) = n \int_{-\infty}^{\infty} \frac{d}{dw} \left\{ [F(x + \sigma w) - F(x)]^{n-1} \right\} f(x) dx$$

$$f_W(w) = n(n-1) \int_{-\infty}^{\infty} [F(x + \sigma w) - F(x)]^{n-2} f(x + \sigma w) f(x) dx$$

where, in the normal case, $F(x) = \Phi(x)$ and $f(x) = \varphi(x)$ from the standard normal random variable, resulting

$$f_W(w) = n(n-1) \int_{-\infty}^{\infty} [\Phi(x+w) - \Phi(x)]^{n-2} \varphi(x+w) \varphi(x) dx \quad (\text{IV})$$

2.4 The Common Use of the 4 Tippett's Integrals

In order to make clear the importance in the quality control context of these 4 integral formulae (eqs. I, II, III, IV) introduced originally by Tippett, L.H.C. (1925) and used or referenced till nowadays, we resume here some of it's main practical applications.

The d_2 relative range mean is used in the (unbiased) estimation of the process standard deviation σ based on the range statistic R , which is given by $\hat{\sigma} = R/d_2(n)$ in the case of just one sample of size n , or $\hat{\sigma} = \bar{R}/d_2(n)$ where $\bar{R} = \frac{1}{m}(R_1 + \dots + R_m)$ in the case of using m samples of size n . Therefore, the d_2 element appears in the Shewhart 3-sigma \bar{X} and R control charts limits formulas.

The d_3 relative range standard deviation is used in the limits expression for the Shewhart 3-sigma R control chart (see for instance, Montgomery, D.C., 2008, Ch. 6).

The expression for $F_W(w)$, the distribution function of the relative range statistic, is used in the construction of exact limits R control charts and its density $f_W(w)$ is usefull if we want to see the density shape involved in the R control chart.

The drawbacks of these formulas and its implementation, as well as simpler alternative formulas, are presented and discussed in the next section.

3 A More Efficient and Simpler Formulation

3.1 Tippett's Formulae Analysis and It's Reformulation

Let's enumerate the 4 main specific drawbacks in using the Tippett's integrals formulae (eqs. I to IV at section 2) as a theoretical background for the range control chart (limits specification). They are:

- (i) four specific integrals formulae are (unnecessarily) too much formulae.
- (ii) the first two moment formulae have long deductions, involving some measure theory results (expressions for moments in terms of $F(x)$ in 2.1 (iv)).
- (iii) some of the integrals are too long, notably formulae II and IV; in particular, formula II requires a jacobian transformation before a numerical integration routine (like Quadpack, Piessens, R. et al, 1983) can be applied.
- (iv) the more important or essential formulae (eq. III) that gives the relative range distribution function (it's inverse or quantile function gives the exact chart limits) has limited

implementation facilities (tables, software) available in practice.

In order to overcome such drawbacks, we introduce here a reformulation strategy for such formulae based on the following elements:

(i) we keep only formulae III (relative range distribution function) in our new proposed formulation, since this is the only essential element, and discard all the other formulae, without any loss of information.

(ii) for the moments of W , and its mean $\mathbb{E}(W)$ in particular, we don't need any specific formula, since a generic and much simpler relation valid for any non-negative continuous random variable X , like $\mathbb{E}(X) = \int_0^\infty (1 - F(x))dx$, known from basic probability and statistics books (Mood, Graybill and Boes, 1974) is sufficient to obtain $\mathbb{E}(W)$ from $F(w)$; its proof is simple and is presented in section 3.2.

(iii) for the practical (computer) implementation of formulae III for $F(w)$, instead of using directly a numerical integration routine like Quadpack (Piessens, R. et al, 1983), we explore a simple relation that exists between the relative range statistic and the Tukey maximum studentized range statistic (see Tukey, J.W.,1953; Copenhaver and Holland, 1988; Ferreira, D. et al, 2007). In fact, the first is a limiting case of the second. Since this last statistic (distribution function and quantile function) is implemented in some (not all but some) statistical software (as for instance the free R software), we can just use such facilities, with the proper adaptations (details in section 3.3).

(iv) if we want to see the shape (density) of our sample relative range W , instead of differencing $F(w)$ analytically (formulae IV), we can do this numerically for instance by Richardson method (Fornberg and Sloan, 1994) through the R software (see the “grad” function of package “numDeriv” through the CRAN webpage or Crawley, M.J., 2007 for instance) or other software such as Matlab for instance.

3.2 Simpler Formulae for Range Moments

Instead of specific formulae for the relative range moments (Tippett's formulae I and II) we consider simpler generic formulae for moments (in terms of distribution function) valid for any non-negative continuous random variable.

The mean formula (see 2.1 (iv)) for W , is given by $\mathbb{E}(W) = \int_0^\infty (1 - F(w))dw$

Proof: Consider $\int_0^\infty (1 - F(w))dw = \int_0^\infty [\int_w^\infty f(s) ds] dw$

Integrating by parts, where u is the element in brackets and $dv = dw$, then, $du = -f(w)dw$ and $v = w$. Therefore, $\int_0^\infty (1 - F(w))dw = [w \int_w^\infty f(s) ds]_0^\infty + \int_0^\infty wf(w) dw$

Since the second integral is $\mathbb{E}(W)$ and the first one is null at 0, the proof is complete, since $\lim_{w \rightarrow \infty} w \int_w^\infty f(s) ds \leq \lim_{w \rightarrow \infty} \int_w^\infty s f(s) ds = 0$, and the first integral (non-negative) is zero.

Although we don't need other moments of W than the mean for building exact limits

R control charts (and we don't recommend the use of 3-sigma R chart), however, for completeness, we present here the expression of the second moment of W , which is given by,

$$\mathbb{E}(W^2) = \int_0^\infty (1 - F(w))dw^2 = 2 \int_0^\infty w(1 - F(w))dw$$

which is obtained by a sort of "chain rule" starting from the mean formula (see for instance Mardia, K.V., 1965 or Mardia, K.V. and Thompson, J.W., 1972).

In this way, we simplify drastically Tippett's formula II by the expression above. Also, obviously, we have that $d_3 = \sqrt{\mathbb{E}(W^2) - d_2^2}$. This expression is used in the present paper (section 4) only for calculating the 3-sigma limits R control chart in order to show the very bad false alarm properties of such "ad hoc" procedure.

3.3 Practical Computer Implementation of $F(w)$ and its Inverse

A simple way to implement the $F(w)$ distribution (instead of using numerical integration routines like Quadpack for instance) is to explore pre-existing available implementations. However, specific implementations for the relative range distribution in the quality control context, as mentioned before, aren't very frequent. On the other way, some implementations for the studentized range distribution (Tukey statistic) are available in some important softwares (as for instance the R free software and the SAS package) since it has a more central place in statistics (multiple comparison tests of means in one-way or two-way ANOVA).

Therefore, a natural way to take advantage of such available facilities, is to explore the close relationship between the two statistics and corresponding distributions. Since, by definition, the relative range is $W = R/\sigma$, with degree of freedom n (numerator), and the studentized range is $W^* = R/S$, with the same degree of freedom in the numerator, and d.f. = m in the denominator, then as $m \rightarrow \infty$, the sample standard deviation S converges in probability to σ and the two statistics became equivalent. In other words, W is a limiting case of W^* when the denominator d.f. is infinity, and $F_{W^*}(w^*)$ coincide with $F_W(w)$ in the limit.

Of course this is an almost obvious result or property but little explored in practice, since the two statistics are used in different areas of statistics, with different contexts of interpretation. A simpler and more well known result like that is the convergence of a central student-ratio statistic to a standard normal variable.

Therefore, if we have the Tukey statistic distribution (for one-way ANOVA) implemented in some software, such that $F_{W^*}(w^*) = F_{W^*}(w^*, n, m)$ for given values of the arguments (w^* is a quantile and n, m are degrees of freedom), then $F_W(w) \equiv F_W(w, n) = F_{W^*}(w^* = w, n, \infty)$ where w is the given quantile and n is the sample size (in the quality control context). In the ANOVA context n would be the number of pairs of means and m would be the sample size or replications.

For instance, using the R free software through its function $ptukey(w^*, n, m)$ with arguments as above, we obtain $F_W(w, n)$ just considering $m = \infty$, for a given quantile w and sample size n . In practice however, we need its inverse, the quantile function, such that

$F_W(w_0) = p$ for some probability or risk p , where w_0 is the quantile we want (control chart limit).

Considering the R software, we use the function $qtukey(p, n, m)$ just making $m = \infty$, where p is a given probability (in practice, $p = \alpha/2$ and $1 - \alpha/2$) and n is the sample size. The α value (false alarm risk) used in practice is 0.0027, common in the USA, or 0.0020, common in Europe. The numerical methods used in such couple of functions of R software are respectively numerical integration (Quadpack) and secant method (see Piessens, R. et al, 1983; Brent, R.P., 1973 and the CRAN webpage).

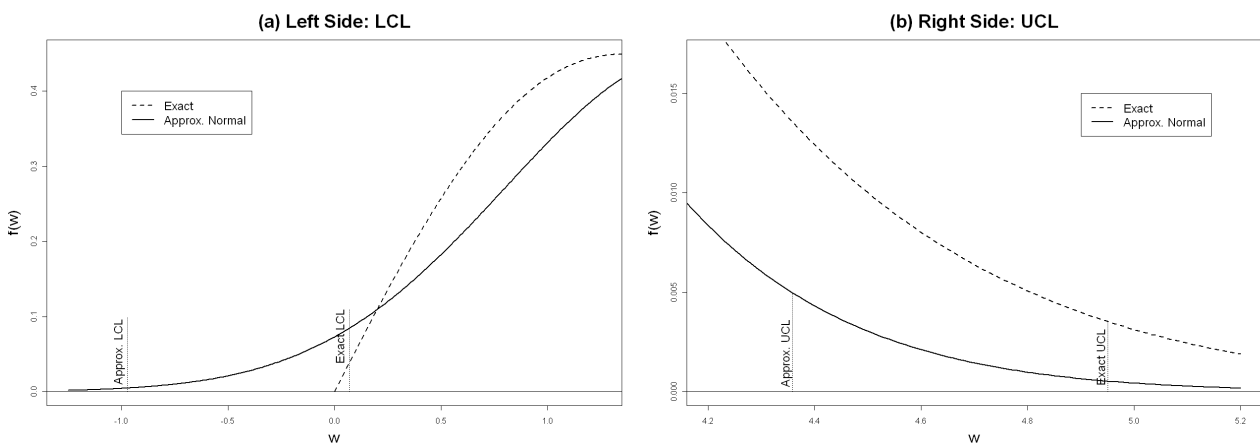
4 False Alarm Study of Range Control Charts

In the last two sections we have revisited the traditional Tippett's integral formulae (used mainly in the construction of Shewhart R charts) and presented a simpler alternative way to work with the range statistic (usefull for the construction of exact limits R charts). Therefore, it's easier now to work with exact limits R charts than with the traditional 3-sigma R charts based on Tippett's moments formulae.

Now, in this section, we apply the procedures proposed in the last section to show the serious drawbacks of using Shewhart (3-sigma normal approx. limits) R charts in terms of false alarm inflation. In fact, although such false alarm inflation is recognized by some authors (for instance see Alwan, L.C., 2000 or Alwan, L.C. and Roberts, H.V., 1995) we show that this drawback is worse than is admitted in the literature.

Before the presentation of the false alarm study itself we show preliminarily (see Figs. 1a and 1b below) the shape (probability density) of the sample relative range statistic involved in both charts (exact and normally approximated) versions.

Figure 1: W Density Function: Exact vs Approx. Normal - n=3



The exact density was built from numerical derivate of the W distribution (through functions “*ptukey*” and “*grad*” of package “*numDeriv*”, from the R software).

From Figure 1a above it's not difficult to see that the gaussian model isn't a very good approximation to the sample range distribution (density). In particular, it shows that small

quantiles (like $\alpha/2 = 0.00135$) which correspond to *LCL* (low control limits) in the *R* control charts, aren't only quite different, but the normal approximation is totally inconsistent with the positive domain of the range statistic, causing obvious drawbacks, among others, the impossibility to detect variability decrease (eventual process improvement). Also, from Figure 1b (which focus the densities upper tails) we can see that big quantiles (as $1 - \alpha/2 = 0.99865$) are under approximated, which is another drawback (increase in control chart false alarm).

In order to show in more detail the comparison of the two versions of the *R* chart (exact limits versus normal approx.) we present initially a numerical illustration of false alarm directly through the control chart itself. After that, it's presented a false alarm comparative study (in terms of alpha risk and Average Run Length - ARL), as a function of the sample size n . One key aspect of our comparative study, is that we compare the two different implementations of the *R* chart considering not only the overall risk of false alarm (crossing the *UCL* or *LCL* under H_0) but also the upper risk separately (crossing the *UCL* under H_0).

In this study and illustration, we consider two different reference values for the α risk (probability of false alarm under the null hypothesis of stable variance $\sigma = \sigma_0$): $\alpha_0 = 0.0027$ and $\alpha_0 = 0.0020$, which are commonly used by the QC community in the USA and Europe respectively.

The Shewhart 3-sigma normal limits for the range is given by

$$\mathbb{E}(R) \pm 3\sigma_R = \sigma d_2 \pm 3\sigma d_3 = (d_2 \pm 3d_3)\sigma \quad , \text{ what means that in practice}$$

$$LCL = \max\{0, (d_2(n) - 3d_3(n))\sigma\} \text{ and } UCL = (d_2(n) + 3d_3(n))\sigma$$

where in numerical applications, σ is estimated from m calibration samples (for instance, $\hat{\sigma} = \bar{R}/d_2$ where $\bar{R} = (R_1 + \dots + R_m)/m$ and m is large, say $m \geq 30$).

Under $H_0 : \sigma = \sigma_0$, the false alarm probability of this 3-sigma *R* chart is given by,

$$\begin{aligned} \alpha \text{ risk} &= 1 - \mathbb{P}(LCL \leq R \leq UCL) = 1 - \mathbb{P}(\sigma_0(d_2 - 3d_3) \leq \sigma_0 W \leq \sigma_0(d_2 + 3d_3)) \\ &= 1 - [F_W(d_2(n) + 3d_3(n)) - F_W(d_2(n) - 3d_3(n))] \end{aligned}$$

if we consider $\alpha_0 = 0.0027$; for $\alpha_0 = 0.0020$ we have 3.09 instead of 3 in the expression above. Also, consequently the upper α risk of false alarm, will be given by,

$$\text{upper } \alpha \text{ risk} = 1 - F_W(d_2(n) + 3d_3(n))$$

The corresponding Average Run Length (average number of samples until the first appearance of false alarm) will be, $ARL(0) = (\alpha \text{ risk})^{-1}$, and $\text{upper } ARL(0) = (\text{upper } \alpha \text{ risk})^{-1}$ since we have a Bernoulli-Geometric process involved.

In order to illustrate the real possibility of false alarm (at a pre-fixed α_0 value) not by chance only but by the non-exactness of the normal approximated Shewhart limits, we present two numerical examples of cases of control charts with two-sided limits (one crossing approx. *UCL* and other crossing approx. *LCL*), both with sample size $n = 5$, at Figure 2(a, b) below, for $\alpha_0 = 0.0027$ and one case of one-sided (upper limit only) control chart (Figure 3a and 3b) for $\alpha_0 = 0.0027$ and 0.0020. In all the charts of Figures 2 and 3, the "vertical dot line" indicate the separation of phase I (m calibration samples), used in the estimation of σ , and phase II, where the new sample range values are confronted with the control limits.

Figure 2: R Control Chart - Two-Sided, $n=5$, $\alpha_0 = 0.0027$

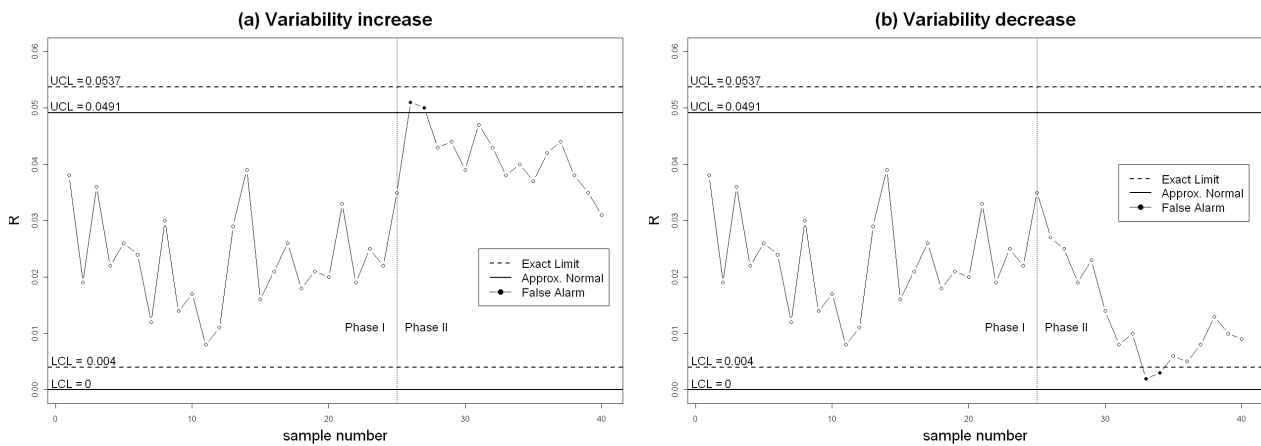


Figure 3: R Control Chart - One-Sided, $n=2$

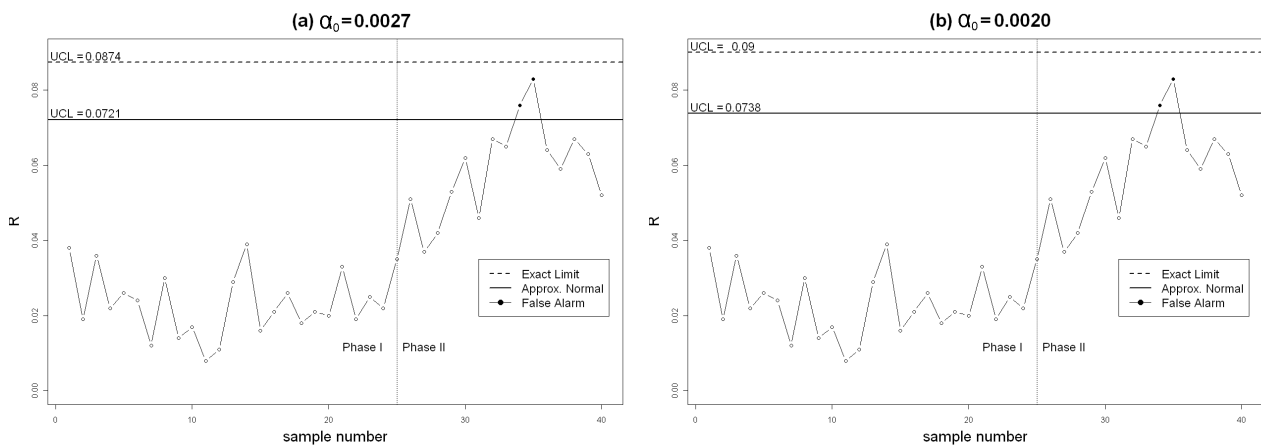


Figure 2a shows a hypothetical false alarm given by the Shewhart control chart where in fact it shouldn't be an alarm if we calculate the limits exactly. Inversely, Figure 2b shows a situation of non-alarm by the Shewhart chart (non-detection of improvement) when in fact it's detected by the exact limits control chart.

In resume, the two charts of Figure 2 show that, the use of misplaced control limits can cause unnecessary false alarm (that could be avoided) or can cause the non-detection of decrease in dispersion (process improvement) which could be detected using proper limits.

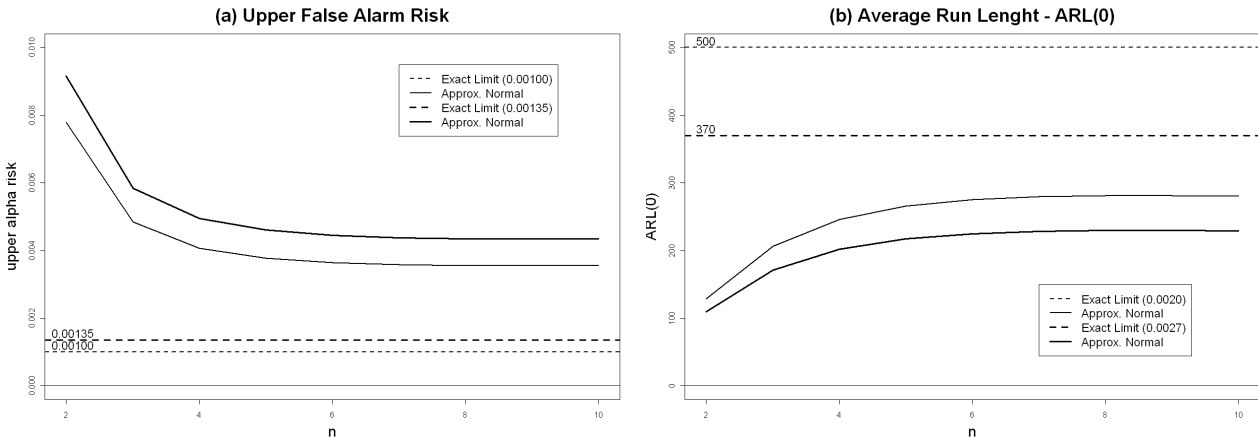
Figures 3a and 3b show also false alarms given by Shewhart charts that could be avoided using correct limits, with the difference that now the charts are "one-sided", with UCL only.

Now, after these previous illustrations of the possible drawbacks of using misplaced control limits in the range chart, lets quantify the damage of these approximations in terms of false alarm risk and average run length (see the formulae previously obtained in this section for such measures of chart performance).

Figure 4a below shows the real false alarm for the Shewhart chart as a function of the sample size n , confronted with the constant false alarm given by the exact limits chart (where $\alpha_0 = 0.0027$ or 0.0020), where the increase in false alarm is evident, mainly for small n . Alternatively, from Figure 4b, we can see the price we pay in terms of decrease of average

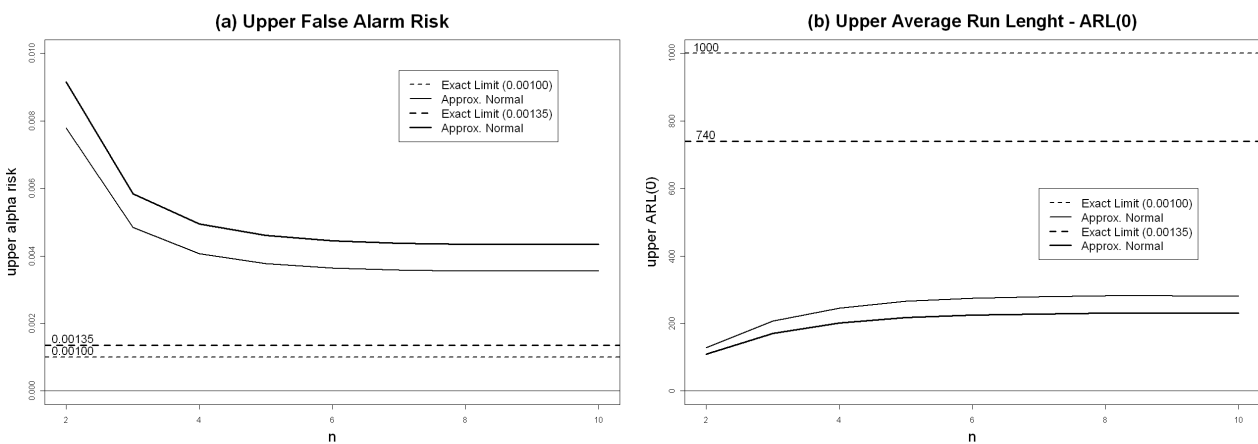
run length (expected number of samples before the occurrence of a false alarm) when we use misplaced control limits.

Figure 4: α risk and ARL : Exact vs Approx. Normal Limits



Now, in order to show the real performance of Shewhart R chart in more detail, we analyse the false alarm risk by crossing the UCL only, which is the more important case in practice and the more fair way to compare the charts, because there is no LCL (is zero) for one of them in small samples. This upper false alarm risk is presented at Figure 5a below ($\alpha_0 = 0.00135$ and 0.0010) and the corresponding $ARL(0)$ at Figure 5b.

Figure 5: Upper α risk and ARL : Exact vs Approx. Normal Limits



From Figure 5a above it's clear that the real increase of false alarm when considering the upper case isolated is bigger than when considering both (upper and low) together at Figure 4a. The same can be seen in terms of ARL , by comparing Figures 4b and 5b, since the last figure shows stronger differences between exact versus approx. approaches.

In order to give the exact numbers of false alarm risk of Shewhart R charts and its percentual increase in relation to the exact limits chart and corresponding ARL values (considering risks together and separated), we present Table 1a and b below, respectively for $\alpha_0 = 0.0027$ and 0.0020 .

Table 1a: α risk and derivatives - Approx. Normal ($\alpha_0 = 0.0027$)

n	total risk	total risk Δ	$ARL(0)$	upper risk	upper risk Δ	upper $ARL(0)$
2	0.00915	238.89%	109	0.00915	577.78%	109
3	0.00584	116.30%	171	0.00584	332.59%	171
4	0.00495	83.33%	202	0.00495	266.67%	202
5	0.00460	70.37%	217	0.00460	240.74%	217
6	0.00445	64.81%	225	0.00445	229.63%	225
7	0.00438	62.22%	228	0.00438	224.44%	228
8	0.00435	61.11%	230	0.00435	222.22%	230
9	0.00435	61.11%	230	0.00434	221.48%	230
10	0.00437	61.85%	229	0.00434	221.48%	230
15	0.00449	66.30%	222	0.00444	228.89%	225
ref.	0.0027	-	370	0.00135	-	740

Table 1b: α risk and derivatives - Approx. Normal ($\alpha_0 = 0.0020$)

n	total risk	total risk Δ	$ARL(0)$	upper risk	upper risk Δ	upper $ARL(0)$
2	0.00780	289.88%	128	0.00780	679.76%	128
3	0.00484	142.10%	206	0.00484	384.20%	206
4	0.00406	103.26%	246	0.00406	306.53%	246
5	0.00377	88.39%	265	0.00377	276.77%	265
6	0.00364	81.81%	275	0.00364	263.63%	275
7	0.00358	78.86%	279	0.00358	257.71%	279
8	0.00355	77.75%	281	0.00355	255.37%	281
9	0.00355	77.70%	281	0.00355	254.95%	281
10	0.00356	78.24%	280	0.00356	255.63%	281
15	0.00367	83.58%	272	0.00365	264.60%	274
ref.	0.0020	-	500	0.0010	-	1000

From table 1a, we see that the α risk of false alarm for the Shewhart chart, instead of keeping constant at $\alpha_0 = 0.0027$ (as in the exact limit control chart), it can have an expressive inflation, ranging from around 60% to near 240%, depending on the sample size n . For instance, for very small samples, say $n = 2$ or 3 (what isn't very uncommon in practice), the actual false alarm risk is more than 2 or 3 times the reference risk α_0 . Equivalently, from this table, the expected number of samples before a false alarm, called Average Run Length or $ARL(0)$ (which value is 370 for the exact limit chart) will be reduced by the same percentages that the risk increases; for instance, for $n = 2$ we have $ARL(0) = 109$ instead of 370, what isn't a nice expectation to have so frequent false alarms.

However, the more important measure of performance in practice and the more fair way

to compare the charts, is through the amount of false alarm risk by crossing the upper control limit. This is also given by table 1a, which shows an even higher increase. The upper risk increase (inflation) ranges from around 220% to near 580%, that is, from 3 to almost 7 times the reference risk of 0.00135 (given by the exact limit chart), depending on the sample size. Equivalently, the reduction of the average number of samples before an upper false alarm, can range from 3 to almost 7 times the reference $ARL(0) = 740$ (exact limit case for any sample size); in this more extreme case, it reduces to 109 when $n = 2$, what is obviously a serious drawback of the Shewhart chart to have so frequent false alarms.

Table 1b ($\alpha_0 = 0.0020$) shows even stronger results than the last table, where the total α risk in the Shewhart chart can reach almost 4 times (290%) more than the reference risk of 0.0020 associated to the exact chart. However, the stronger result of this table, is that the upper α risk of the Shewhart chart suffer an inflation that ranges from around 280% to 680% (that is, 3.5 to almost 8 times) in relation to the reference risk of 0.0010 (exact chart), depending on the sample size.

In summary, if we consider this more adequate measure of chart performance (false alarm upper risk and not the total risk), we can see from these two tables (1a and 1b) that the risk inflation in the Shewhart chart ranges from a minimum for 4 or 5 times the risk of the exact limit chart, to almost 7 or 8 times more, depending on the sample size and the pre-fixed α_0 risk, if 0.0027 or 0.0020. Therefore, the false alarm drawbacks of 3-sigma range charts are much more serious than previously supposed according to the QC literature.

Finally, we present below at table 2, the quantiles of the W distribution for small samples, for two-sided and one-sided limits, with $\alpha_0 = 0.0027$ and 0.0020. With all the facilities provided in this paper, we expect that the QC practitioner can now use range control charts with exact limits and not with misplaced (3-sigma) limits, avoiding these serious drawbacks.

Table 2: **W Quantiles for Small Samples** - $\alpha_0 = 0.0020$ and 0.0027

n	$\alpha/2$		α		$1 - \alpha$		$1 - \alpha/2$	
	0.0010	0.00135	0.0020	0.0027	0.9980	0.9973	0.9990	0.99865
2	0.00177	0.00239	0.00354	0.00478	4.37025	4.24261	4.65351	4.53274
3	0.06024	0.07000	0.08522	0.09903	4.79802	4.67870	5.06345	4.95017
4	0.19945	0.22055	0.25166	0.27838	5.05319	4.93846	5.30880	5.19966
5	0.36739	0.39653	0.43836	0.47338	5.23478	5.12314	5.48375	5.37740
6	0.53474	0.56899	0.61747	0.65751	5.37531	5.26597	5.61933	5.51506
7	0.69135	0.72885	0.78144	0.82451	5.48964	5.38211	5.72975	5.62713
8	0.83483	0.87439	0.92957	0.97450	5.58582	5.47978	5.82273	5.72146
9	0.96551	1.00641	1.06322	1.10929	5.66870	5.56391	5.90291	5.80277
10	1.08458	1.12634	1.18417	1.23093	5.74143	5.63772	5.97331	5.87416

5 Final Discussion and Conclusions

We have presented in this paper three main results that are useful respectively in the theoretical study, computer implementation and practical application of the range control chart for monitoring process variability.

The first point was a drastic simplification in the Tippett's theory (background for range control charts), keeping only the formula for $F(w)$, which is the simplest one, and substituting specific moments formulae for much simpler generic expressions as the case of $\mathbb{E}(W)$. In this way, it's now easy to study exact limits range control charts.

The second point was to show a simple way to implement computationally $F(w)$ exploring its relation with the Tukey studentized range distribution, available in some important statistical software. In this way there is no more problem of shortage of tables or other means of implementation.

The third point was to show that the drawbacks of using 3-sigma (Shewhart) range control charts based on normal approximation are extremely serious in terms of false alarm inflation. In this way we expect that we have convinced that is much better to consider the exact limits charts.

References

- [1] Alwan, L. (2000). *Statistical Process Analysis*. Boston, McGrawHill.
- [2] Alwan, L. C. and Roberts, H. V. (1995). *The Problem of Misplace Control Limits*. Appl. Statist., vol 44, 269-278.
- [3] Brent, R. P. (1973). *Algorithms for Minimization without Derivatives*. N. J. Prentice Hall, chapter 4.
- [4] Brown, R. A. (1974). *Robustness of the Studentized Range Statistic*. Biometrika, vol 61 (1), 171-175.
- [5] Burr, I. W. (1967). *The Effect of Non-normality on Constants for \bar{X} and R Charts*. Industrial Quality Control, 23, 563-568.
- [6] Burr, I. W. (1974). *Applied Statistical Methods*. Academic Press Inc, page 479.
- [7] Copenhaver, M. D. and Holland, B. (1988). *Computation of the Distribution of the Maximum Studentized Range Statistic with Application to Multiple Significance Testing*. J. of Statist. Comput. Simul., vol 30, 1-15.
- [8] Costa, A. F. B., Epprecht, E. K. e Carpinetti, L. C. R. (2004). *Controle Estatístico da Qualidade*. Edit. Atlas, SP - Brasil.
- [9] Crawley, M. J. (2007). *The R Book*. Wiley, London.

- [10] Cuadras, C. M. (2002). *On the Covariance between Functions*. J. of Multiv. Analysis, vol 81 (1), 19-27.
- [11] David, H. A. (1981). *Order Statistics*. 2nd ed., Wiley, NY.
- [12] Devor, R. E.; Chang, T. and Sutherland, J. W. (2007). *Statistical Quality Design and Control*. 2nd ed., Pearson/ Prentice Hall.
- [13] Feller, W. (1966). *An Introduction to Probability Theory and It's Applications*. vol II, 148-149, Wiley, NY.
- [14] Ferreira, D.; Demétrio, C.; Manly, B. and Machado, A. (2007). *Quantiles from the Maximum Studentized Range Distribution*. Biometric Brazilian Journal, vol 25 (1), 117-135.
- [15] Fornberg, B. and Sloan, D. (1994). *Richardson Method for Numerical Derivatives*. Acta Numerica, 203-267.
- [16] Grant, E. L. and Leavenworth, R. S. (1996). *Statistical Quality Control*. 7th ed., McGrawHill, NY.
- [17] Harter, H. L. (1960). *Tables of Range and Studentized Range*. The Annals of Mathematical Statistics, vol 31, 1122-1147.
- [18] Harter, H. L. and Balakrishnan, N. (1997). *Tables for the Use of Range and Studentized Range in Tests of Hypotheses*. CRC Press, Florida, US.
- [19] Hoeffding, W. (1940). *Masstabinvariante Korrelation Theorie*. Schrifte Math. Inst., Univ. Berlin 5, 181-233.
- [20] Jones, M. C. and Balakrishnan, N. (2002). *How are Moments of Spacing Related to Distribution Functions*. J. of Stat. Plan. and Infer., 103, 377-390.
- [21] Kenett, R. and Zachs, S. (1998). *Modern Industrial Statistics: The Design and Control of Quality and Reliability*. Duxbury Press.
- [22] Mardia, K. V. (1965). *Tippet's Formulas and Other Results on Sample Range and Extremes*. Ann. Inst. Stat. Math., vol 17, 85-91.
- [23] Mardia, K. V. and Thompson, J. W. (1972). *Unified Treatment of Moment-formulae*. Sankhya: The Indian J. of Stat., Series A, vol 34, 121-132.
- [24] Matlab 2009. The MathWorks Inc.
- [25] Minitab. USA, Minitab Inc. 2008.
- [26] Montgomery, D.C. (2008). *Introduction to Statistical Quality Control*. 6th ed., Wiley, NY.
- [27] Mood, A. M.; Graybill, F. A. and Boes, D. C. (1974). *Introduction to the Theory of Statistics*. 3rd ed., 64-66, McGrawHill.

- [28] Pearson, K. (1902). *Note on Francis Galton's Problem*. Biometrika 1, 390-399.
- [29] Pearson, E. S. and Hartley, H. O. (1943). *Tables of the Probability Integral of the Studentized Range*. Biometrika, vol 33, 88-89.
- [30] Piessens, R. et al. (1983). *Quadpack: A Subroutine Package for Automatic Integration*. Chapter II: Theoretical Background, Berlin, Springer.
- [31] R Development Core Team (2009). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, URL <http://www.R-project.org> .
- [32] Ryan, T. P. (1989). *Statistical Methods of Quality Improvement*. Wiley, NY.
- [33] SAS/QC® 9.2 (2008). *User's Guide*. Appendix C: Functions, 2267-2294, SAS Institute Inc., Cary, NC.
- [34] Shewhart, W. A. (1926). *Quality Control Charts*. Bell System Technical Journal 5, 593-603.
- [35] Tippett, L. H. C. (1925). *On the Extreme Individuals and the Range of Samples from a Normal Population*. Biometrika, vol 17, 364-387.
- [36] Tukey, J. W. (1953). *The Problem of Multiple Comparisons*. Unpublished memorandum, USA.