On Equivalent Expressions for the Faraday's Law of Induction

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Abstract

In this paper we give a rigorous proof of the equivalence of some different forms of Faraday's law of induction clarifying some misconceptions on the subject and emphasizing that many derivations of this law appearing in textbooks and papers are only valid under very special circunstances and not satisfactory under a mathematical point of view.

1 Introduction

Let $\mathbf{\Gamma}_t$ a smooth closed curve in $\mathbb{R}^3$ with parametrization $\mathbf{x}(t,\ell)$ which is here supposed to represent a filamentary closed circuit which is moving in a convex and simply-connected (open) region $U \subset \mathbb{R}^3$ where at time t as measured in an inertial frame[1], there are an electric and a magnetic fields $\mathbf{E} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, $(t, \mathbf{x}) \mapsto \mathbf{E}(t, \mathbf{x}) \in \mathbb{R}^3$ and $\mathbf{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, $(t, \mathbf{x}) \mapsto \mathbf{B}(t, \mathbf{x}) \in \mathbb{R}^3$. We suppose that when in motion the closed circuit may be eventually *deforming. Let* $\mathbf{\Gamma}$ be a smooth closed curve in $\mathbb{R}^3$ with parametrization $\mathbf{x}(\ell)$ representing the filamentary circuit at $t = 0$. Then, the smooth curve $\mathbf{\Gamma}_t$ is given by $\mathbf{\Gamma}_t = \sigma_t(\mathbf{\Gamma})$ where σ_t (see details below) is the *flow* of a velocity vector field $\mathbf{v} : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}^3$, which describes the motion (and deformation) of the closed circuit. It is an empirical fact known as Faraday's Law of Induction that on the closed loop $\mathbf{\Gamma}_t$ acts an induced *electromotive force* $\mathcal{E}$ such that

$$\mathcal{E} = -\frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{B} \cdot \mathbf{n}\, da, \tag{1}$$

where $\mathcal{S}_t$ is a smooth surface on $\mathbb{R}^3$ such that $\mathbf{\Gamma}_t$ is its boundary and $\mathbf{n}$ is the normal vector field on $\mathcal{S}_t$. We write $\mathbf{\Gamma}_t = \partial \mathcal{S}_t$ with $\mathbf{\Gamma} = \partial \mathcal{S}$. Now, on each

[1] For a mathematical defintion of an inertial reference frame in Minkowski spacetime see, e.g., [14, 15].

element of $\mathbf{\Gamma}_t$ the force acting on a unit charge which is moving with velocity $\mathbf{v}(t, \mathbf{x}(t, s))$ is given by the Lorentz force law. Thus[2] the *emf* $\mathcal{E}$ is by definition:

$$\mathcal{E} = \int_{\mathbf{\Gamma}_t} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}, \tag{2}$$

where $d\mathbf{l} := \frac{\partial \mathbf{x}(t,s)}{\partial s} d\ell$ and Faraday's law reads:

$$\int_{\mathbf{\Gamma}_t} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\mathcal{S}_t} \mathbf{B} \cdot \mathbf{n} \; da. \tag{3}$$

We want to prove that Eq.(3) is equivalent to

$$\int_{\mathbf{\Gamma}_t} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{S}_t} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \; da, \tag{4}$$

from where it trivially follows the differential form of Faraday's law, i.e.,

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{5}$$

Those statements will be proved in Section 3, but first we shall need to recall a few mathematical results concerning differentiable vector fields, in Section 2.

2 Some Identities Involving the Integration of Differentiable Vector Fields

Let $U \subset \mathbb{R}^3$ be a convex and simply-connected (open) region, $\mathbf{X} : \mathbb{R} \times U \to \mathbb{R}^3, (t, \mathbf{x}) \mapsto \mathbf{X}(t, \mathbf{x})$ be a generic differentiable vector field and let $\mathbf{v} : \mathbb{R} \times U \to \mathbb{R}^3$ be a differentiable *velocity vector field* of a *fluid flow*. An *integral line*[3] of $\mathbf{v}$ passing through a given $\mathbf{x} \in \mathbb{R}^3$ is a smooth curve $\sigma_{\mathbf{x}} : \mathbb{R} \to \mathbb{R}^3$, $t \to \sigma_{\mathbf{x}}(t) = \sigma(t, \mathbf{x})$ which at $t = 0$ is at $\mathbf{x}$ (i.e., $\sigma_{\mathbf{x}}(0) = \mathbf{x}$) and such that its tangent vector at $\sigma(t, \mathbf{x})$ is

$$\frac{\partial}{\partial t} \sigma(t, \mathbf{x}) = \mathbf{v}(t, \sigma(t, \mathbf{x})). \tag{6}$$

Let moreover $\sigma_t : U \to \mathbb{R}^3, \sigma_t(\mathbf{x}) = \sigma(t, \mathbf{x})$. We call σ_t the *fluid flow map*. Let $J = (0, 1) \in \mathbb{R}$ and let $\mathbf{\Gamma}$ be a closed loop parametrized by $\mathbf{\Gamma} : J \to \mathbb{R}^3$, $\ell \mapsto \mathbf{\Gamma}(\ell) := \mathbf{x}(\ell)$ and denote by $\Gamma_t = \sigma_t(\Gamma)$ the loop *transported* by the flow. Then

$$\sigma(t, \mathbf{x}(\ell)) := \mathbf{x}(t, \ell) \tag{7}$$

is clearly a parametrization of $\mathbf{\Gamma}_t$. We have the proposition:

Proposition

[2] In this paper we use a system of units such that the numerical value of the speed of light is $c = 1$.

[3] Also called a stream line.

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l}=\int_{\Gamma_t}\frac{D}{Dt}\mathbf{X}\cdot d\mathbf{l}+\int_{\Gamma_t}\mathbf{X}\cdot[(d\mathbf{l}\cdot\nabla)\mathbf{v}], \tag{8a}$$
$$=\int_{\Gamma_t}\frac{D}{Dt}\mathbf{X}\cdot d\mathbf{l}+\int_{\Gamma_t}[\mathbf{X}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{l}+\int_{\Gamma_t}[(\mathbf{X}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l} \tag{8b}$$
$$=\int_{\Gamma_t}\frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l}-\int_{\Gamma_t}[\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot d\mathbf{l}, \tag{8c}$$

where
$$\frac{d}{dt}\mathbf{X}=\frac{D}{Dt}\mathbf{X}:=\frac{\partial}{\partial t}\mathbf{X}+(\mathbf{v}\cdot\nabla)\mathbf{X} \tag{9}$$
is the so-called *material derivative*[4] and
$$d\mathbf{l}=\frac{\partial}{\partial \ell}\sigma(t,\mathbf{x}(\ell))d\ell=\frac{\partial\mathbf{x}(\mathbf{t},\ell)}{\partial\ell}d\ell \tag{10}$$
is the tangent line element[5] of Γ_t at $\sigma(t,\mathbf{x}(\ell))$.
Proof. We can write

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l}=\frac{d}{dt}\int_0^1\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\frac{\partial}{\partial\ell}\sigma(t,\mathbf{x}(\ell))d\ell$$
$$=\int_0^1\frac{d}{dt}\left[\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\right]\cdot\frac{\partial}{\partial\ell}\sigma(t,\mathbf{x}(\ell))d\ell$$
$$+\int_0^1\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial\ell}\sigma(t,\mathbf{x}(\ell))d\ell. \tag{11}$$

Now, taking into account that for each $\mathbf{x}(\ell)$, $\frac{\partial}{\partial t}\sigma(t,\mathbf{x})=\mathbf{v}(t,\sigma(t,\mathbf{x}(\ell)))$ we have
$$\frac{D}{Dt}\left[\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\right]=\frac{\partial}{\partial t}\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))+(\mathbf{v}\cdot\nabla)\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell))), \tag{12}$$
hence, the first term in the right side of Eq.(11) can be written as
$$\int_0^1\frac{d}{dt}\left[\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\right]\cdot\frac{\partial}{\partial\ell}\sigma(t,\mathbf{x}(\ell))d\ell=\int_{\Gamma_t}[\frac{\partial}{\partial t}\mathbf{X}+(\mathbf{v}\cdot\nabla)\mathbf{X}]\cdot d\mathbf{l}=\int_{\Gamma_t}\frac{D}{Dt}\mathbf{X}\cdot d\mathbf{l}. \tag{13}$$

[4] Mind that the *material derivative* is a derivative taken along a path σ_t with tangent vector $\mathbf{v}|_{\sigma_x}$. It is frequently used in fluid mechanics, where it describes the total time rate of change of a given quantity as viewed by a fluid particle moving on σ_x.
[5] Take notice that $d\mathbf{l}$ is not an explicit function of the cartesian coordinates (x,y,z).

Also writing $\sigma(t,\mathbf{x}(\ell)) = (x^1(t,\ell), x^2(t,\ell), x^3(t,\ell))$ we see that the last term in Eq.(11) can be written as:

$$\int_0^1 \mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial \ell}\sigma(t,\mathbf{x}(\ell))d\ell = \int_0^1 \mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\left[\frac{\partial}{\partial \ell}\mathbf{v}(t,\sigma(t,\mathbf{x}(\ell)))d\ell\right]$$
$$= \int_{\Gamma_t} \mathbf{X}\cdot[(d\mathbf{l}\cdot\nabla)\mathbf{v}]. \tag{14}$$

We now recall that for arbitrary differentiable vector fields $\mathbf{a},\mathbf{b} : U \to \mathbb{R}^3$ it holds

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}). \tag{15}$$

Setting $\mathbf{a} = d\mathbf{l}$ and $\mathbf{b} = \mathbf{v}$ and noting that $(\mathbf{v}\cdot\nabla)d\mathbf{l} = \mathbf{v}\times(\nabla\times d\mathbf{l}) = \mathbf{0}$, it implies that

$$(d\mathbf{l}\cdot\nabla)\mathbf{v} = \nabla(d\mathbf{l}\cdot\mathbf{v}) - d\mathbf{l}\times(\nabla\times\mathbf{v}). \tag{16}$$

We need also to recall the well known identity

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \tag{17}$$

which implies setting $\mathbf{a} = \mathbf{X}$, $\mathbf{b} = d\mathbf{l}$ and $\mathbf{c} = (\nabla\times\mathbf{v})$, that

$$-\mathbf{X}\cdot[d\mathbf{l}\times(\nabla\times\mathbf{v})] = -d\mathbf{l}\cdot[(\nabla\times\mathbf{v})\times\mathbf{X}], \tag{18}$$

and also the *not* so well known identity[6]

$$\mathbf{X}\cdot[\nabla(d\mathbf{l}\cdot\mathbf{v})] = [(\mathbf{X}\cdot\nabla)\mathbf{v}]\cdot d\mathbf{l}, \tag{19}$$

to write that

$$\int_{\Gamma_t} \mathbf{X}\cdot[(d\mathbf{l}\cdot\nabla)\mathbf{v}] = -\int_{\Gamma_t} \mathbf{X}\cdot[d\mathbf{l}\times(\nabla\times\mathbf{v})] + \int_{\Gamma_t} [(\mathbf{X}\cdot\nabla)\mathbf{v}]\cdot d\mathbf{l}$$
$$= \int_{\Gamma_t} [\mathbf{X}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{l} + \int_{\Gamma_t} [(\mathbf{X}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l}. \tag{20}$$

Finally, using Eq.(13) and Eq.(20) completes the proof of Eq.(8a) and Eq.(8b). Also, from Eq.(8b) it follows if we recall Eq.(15) that

$$\frac{d}{dt}\int_{\Gamma_t} \mathbf{X}\cdot d\mathbf{l}$$
$$= \int_{\Gamma_t} \frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l} + \int_{\Gamma} [(\mathbf{v}\cdot\nabla)\mathbf{X})]\cdot d\mathbf{l} + \int_{\Gamma_t} [\mathbf{X}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{l} + \int_{\Gamma} [(\mathbf{X}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l}$$
$$= \int_{\Gamma_t} \frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l} - \int_{\Gamma_t} [\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot d\mathbf{l}.$$

[6] See the Appendix for a proof of this identity

from where the proof of Eq.(8c) follows immediately. ■

Remark Before proceeding, we recall that if $\mathbf{X} = \mathbf{v}$ we have

$$\frac{d}{dt}\int_{\Gamma_t} \mathbf{v}\cdot d\mathbf{l} = \int_{\Gamma_t} \frac{D}{dt}\mathbf{v}\cdot d\mathbf{l}, \tag{21}$$

a result that is known in fluid mechanics as *Kelvin's circulation theorem (* see, e.g., *[2, 16]).*

Now,

$$\frac{d}{dt}\int_{\Gamma_t} \mathbf{X}\cdot d\mathbf{l} = \frac{d}{dt}\int_{\mathcal{S}_t} (\nabla\times\mathbf{X})\cdot\mathbf{n}\ da, \tag{22}$$

where, if $\mathcal{S}$ is a smooth surface such that $\partial\mathcal{S} = \Gamma$, then $\mathcal{S}_t = \sigma_t(\mathcal{S})$. Also $\mathbf{n}$ is the normal vector field to S_t. Then using Eq.(8c) we can write:

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{S}_t} (\nabla\times\mathbf{X})\cdot\mathbf{n}\ da &= \int_{\Gamma_t} \frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l} - \int_{\Gamma_t} [\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot d\mathbf{l} \\ &= \int_{\mathcal{S}_t} \frac{\partial}{\partial t}(\nabla\times\mathbf{X})\cdot\mathbf{n}\ da - \int_{\mathcal{S}_t} \nabla\times[\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot\mathbf{n}\ da\end{aligned} \tag{23}$$

Also, denoting $\mathbf{Y} := \nabla\times\mathbf{X}$ we can write

$$\frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{Y}\cdot\mathbf{n}\ da = \int_{\mathcal{S}_t} \left[\frac{\partial}{\partial t}\mathbf{Y} - \nabla\times(\mathbf{v}\times\mathbf{Y})\right]\cdot\mathbf{n}\ da \tag{24}$$

Despite Eq.(24), for a general differentiable vector field $\mathbf{Z} : \mathbb{R}\times U \to \mathbb{R}^3$ such that $\nabla\cdot\mathbf{Z} \neq 0$ we have

$$\frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{Z}\cdot\mathbf{n}\ da = \int_{\mathcal{S}_t} \left[\frac{\partial}{\partial t}\mathbf{Z} + \mathbf{v}(\nabla\cdot\mathbf{Z}) - \nabla\times(\mathbf{v}\times\mathbf{Z})\right]\cdot\mathbf{n}\ da, \tag{25}$$

the so-called *Helmholtz identity* [4] . Note that the identity is also mentioned in [5] . A proof of Helmholtz identity can be obtained using arguments similar to the ones used in the proof of Eq.(8a). Some textbooks quoting Helmholtz identity are [1, 6, 10, 17, 18]. However, we emphasize that the proof of Faraday's law of Induction presented in all the textbooks just quoted are always for very particular situations and definitively not satisfactory form a mathematical point of view.

We now want to use the above results to prove Eq.(3) and Eq.(4).

3 Proofs of Eq.(3) and Eq.(4)

We start remembering that in Maxwell theory we have that the $\mathbf{E}$ and $\mathbf{B}$ fields are derived from potentials, i.e.,

$$\begin{aligned}\mathbf{E} &= -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \\ \mathbf{B} &= \nabla\times\mathbf{A},\end{aligned} \tag{26}$$

where $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is the scalar potential and $\mathbf{A} : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ is the (magnetic) vector potential. If Eq.(26) is taken into account we can immediately derive Eq.(3). All we need is to use the results just derived in Section 2 taking $\mathbf{X} = \mathbf{A}$. Indeed, the first line of Eq.(23) then becomes

$$\frac{d}{dt}\int_{\mathcal{S}_t} (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \int_{\mathbf{\Gamma}_t} \frac{\partial}{\partial t}\mathbf{A} \cdot d\mathbf{l} - \int_{\mathbf{\Gamma}_t} [\mathbf{v} \times (\nabla \times \mathbf{A})] \cdot d\mathbf{l},$$

or

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{B} \cdot \mathbf{n}\, da &= \int_{\mathbf{\Gamma}_t} \frac{\partial}{\partial t}\mathbf{A} \cdot d\mathbf{l} - \int_{\mathbf{\Gamma}_t} (\mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l} \\ &= \int_{\mathbf{\Gamma}_t} \left(\frac{\partial}{\partial t}\mathbf{A} + \nabla\phi - \mathbf{v} \times \mathbf{B}\right) \cdot d\mathbf{l} \\ &= -\int_{\mathbf{\Gamma}_t} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \end{aligned} \tag{27}$$

To obtain Eq.(4) we recall that from the second line of Eq.(23) we can write (using Stokes theorem)

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{B} \cdot \mathbf{n}\, da &= \int_{\mathcal{S}_t} \frac{\partial}{\partial t}\mathbf{B} \cdot \mathbf{n}\, da - \int_{\mathcal{S}_t} \nabla \times [\mathbf{v} \times \mathbf{B}] \cdot \mathbf{n}\, da \\ &= \int_{\mathcal{S}_t} \frac{\partial}{\partial t}\mathbf{B} \cdot \mathbf{n}\, da - \int_{\mathbf{\Gamma}_t} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \end{aligned} \tag{28}$$

Comparing the second member of Eq.(27) and Eq.(28) we get Eq.(4), i.e.,

$$\int_{\mathbf{\Gamma}_t} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{S}_t} \frac{\partial}{\partial t}\mathbf{B} \cdot \mathbf{n}\, da, \tag{29}$$

from where the differential form of Faraday's law follows.

Remark 1 *We end this section by recalling that in the physical world the real circuits are not filamentary and worse, are not described by smooth closed curves. However, if the closed curve representing a 'filamentary circuit' is made of finite number of sections that are smooth, we can yet apply the above formulas with the integrals meaning Lebesgue integrals.*

4 Conclusions

Recently a paper [12] titled 'Faraday's Law via the Magnetic Vector Potential', has been commented in [7] and replied in [13]. Thus, the author of [12], claims to have presented an "alternative" derivation for Faraday's law for a filamentary circuit which is moving with an arbitrary velocity and which is changing its shape, using directly the vector potential $\mathbf{A}$ instead of the magnetic field $\mathbf{B}$ and the electric field $\mathbf{E}$ (which is the one presented in almost all textbooks).

Now, [7] correctly identified that the derivation in [12] is wrong, and that author agreed with that in [13]. Here we want to recall that a presentation of Faraday's law in terms of the magnetic vector potential $\mathbf{A}$ already appeared in Maxwell treatise [8], using big formulas involving the components of the vector fields involved. We recall also that a formulation of Faraday's law in terms of $\mathbf{A}$ using modern vector calculus has been given by Gamo more than 30 years ago [3]. In Gamo's paper (not quoted in [7, 12, 13]) Eqs.(8c) appears for the special case in which $\mathbf{X} = \mathbf{A}$ (the vector potential) and $\mathbf{B} = \nabla \times \mathbf{A}$ (the magnetic field), i.e.,

$$\frac{d}{dt}\int_{\mathbf{\Gamma}_t} \mathbf{A}\cdot d\mathbf{l} = \int_{\mathbf{\Gamma}_t} \frac{\partial}{\partial t}\mathbf{A}\cdot d\mathbf{l} - \int_{\mathbf{\Gamma}_t} [\mathbf{v}\times(\nabla\times\mathbf{A})]\cdot d\mathbf{l}. \tag{30}$$

Thus, Eq.(30) also appears in [12] (it is there Eq.(9)). However, in footnote 3 of [12] it is said that Eq.(30) is equivalent to "$\frac{d}{dt}\int_{\Gamma_t} \mathbf{A}\cdot d\mathbf{l} = \int_{\Gamma_t} \frac{D}{Dt}\mathbf{A}\cdot d\mathbf{l}$ ", where the term $\int_{\Gamma}[(\mathbf{A}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l}$ is missing. This is the error that has been observed by authors [7], which also presented a proof of Eq.(8b), which however is not very satisfactory from a mathematical point of view, that being one of the reasons why we decided to write this note presenting a correct derivation of Faraday's Law in terms of $\mathbf{A}$ and its relation with Helmholtz formula. Another reason is that there are still people (e.g., [11]) that do not understand that Eq.(3) and Eq.(4) are equivalent and think that Eq.(3) implies the form of Maxwell equations as given by Hertz, something that we know since a long time that is wrong [9].

We also want to observe that Jackson's proof of Faraday's law using 'Galilean invariance' is valid only for a filamentary circuit moving without deformation with a constant velocity. The proof we presented is general and valid in Special Relativity, since it is based on trustful mathematical identities and in the Lorentz force law applied in the laboratory frame with the motion and deformation of the filamentary circuit mathematically well described.

A Proof of the Identity in Eq.(19)

We know from Eq.(16) that

$$\nabla(d\mathbf{l}\cdot\mathbf{v}) = (d\mathbf{l}\cdot\nabla)\mathbf{v} + d\mathbf{l}\times(\nabla\times\mathbf{v}) \tag{31}$$

Let $\{e^1, e^2, e^3\}$ be an orthonormal base of $\mathbb{R}^3$. We can write, using *Einstein Notation,*

$$(\nabla\times\mathbf{v}) = e^i\partial_i\times\mathbf{v} = e^i\times\partial_i\mathbf{v}, \tag{32}$$

where $\nabla = (\partial_1, \partial_2, \partial_3) = e^1\frac{\partial}{\partial x_1} + e^2\frac{\partial}{\partial x_2} + e^3\frac{\partial}{\partial x3} = e^i\partial_i$, with $\partial_i = \frac{\partial}{\partial x_i}$. It follows then

$$d\mathbf{l}\times(\nabla\times\mathbf{v}) = d\mathbf{l}\times(e^i\times\partial_i\mathbf{v}). \tag{33}$$

Using the known identity $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ in Eq.(33), we obtain

$$d\mathbf{l}\times(e^i\times\partial_i\mathbf{v}) = (d\mathbf{l}\cdot\partial_i\mathbf{v})e^i - (d\mathbf{l}\cdot e^i)\partial_i\mathbf{v}. \tag{34}$$

In the other hands, considering $d\mathbf{l} = (dl_1, dl_2, dl_3) = dl_i e^i$, we have

$$(d\mathbf{l}\cdot\nabla)\mathbf{v} = (dl_i \partial_i)\mathbf{v} = (d\mathbf{l} \cdot e^i)\partial_i \mathbf{v}. \tag{35}$$

Hence, substituting Eq.(34) and Eq.(35) in Eq.(31), we can rewrite it as

$$\begin{aligned}\nabla(d\mathbf{l} \cdot \mathbf{v}) &= (d\mathbf{l} \cdot e^i)\partial_i \mathbf{v} + (d\mathbf{l} \cdot \partial_i \mathbf{v})e^i - (d\mathbf{l}\cdot e^i)\partial_i \mathbf{v} \\ &= (d\mathbf{l} \cdot \partial_i \mathbf{v})e^i.\end{aligned} \tag{36}$$

From this last result, its easy to see that

$$\begin{aligned}\mathbf{X}\cdot [\nabla (d\mathbf{l} \cdot \mathbf{v})] = \mathbf{X}\cdot \left[(d\mathbf{l} \cdot \partial_i \mathbf{v})e^i\right] &= X_i(d\mathbf{l} \cdot \partial_i)\mathbf{v} = d\mathbf{l} \cdot (X_i \partial_i)\mathbf{v} \\ &= d\mathbf{l}\cdot [(\mathbf{X}\cdot\nabla)\mathbf{v}] = [(\mathbf{X}\cdot\nabla)\mathbf{v}] \cdot d\mathbf{l},\end{aligned}$$

where $\mathbf{X} = (X_1, X_2, X_3) = X_i e^i$.

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