LIMIT CYCLES OF RESONANT FOUR-DIMENSIONAL POLYNOMIAL SYSTEMS

JAUME LLIBRE[1], ANA CRISTINA MEREU[2] AND MARCO A. TEIXEIRA[2]

Abstract. We study the bifurcation of limit cycles from four-dimensional centers inside a class of polynomial differential systems. Ours results establish an upper bounded for the number of limit cycles which can be prolonged in function of the degree of the polynomial perturbation considered, up to first-order expansion of the displacement function with respect to small parameter. The main tool for proving such results is the averaging theory.

1. Introduction

The problem of determining the maximum number of limit cycles that a given differential system can have became one of the main topics in the qualitative theory of differential systems.

The second part of the Hilbert's 16th problem is, roughly speaking, to find a uniform upper bound for the number of limit cycles that a planar polynomial differential system with a given degree can have. Related with this problem there exists a special interest in the following question: *How many limit cycles emerge from a perturbation of a planar center?* This problem has been studied by many researchers and many different results have been obtained, see for example the book [2] and the references there in. Our main concern is to bring this problem to higher dimension.

We consider the following problem: *How many limit cycles emerge from the periodic orbits of a center in $\mathbb{R}^4$ when we perturb it inside a given class of polynomial differential systems?* More precisely we consider the four-dimensional linear center

$$\dot{x} = Ax, \tag{1}$$

where

$$A = \begin{pmatrix} 0 & -p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & q & 0 \end{pmatrix},$$

with p and q coprime positive integers. We perturb it as follows

$$\dot{x} = Ax + \varepsilon F(x), \tag{2}$$

where $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ is a small parameter and $F : \mathbb{R}^4 \to \mathbb{R}^4$ is a polynomial vector field $F(x) = (F^1(x), F^2(x), F^3(x), F^4(x))$ of the form $F^k = F_1^k + F_N^k$ where F_i^k,

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$i = 1, N$ are homogeneous polynomial of degree i in the variables $x = (x_1, x_2, x_3, x_4)$ and N is an integer number. Thus system (2) becomes

$$
\begin{array}{lllll}
\dot{x_1} & = & -p\, x_2 & + & \varepsilon(F_1^1(x) + F_N^1(x)), \\
\dot{x_2} & = & p\, x_1 & + & \varepsilon(F_1^2(x) + F_N^2(x)), \\
\dot{x_3} & = & -q\, x_4 & + & \varepsilon(F_1^3(x) + F_N^3(x)), \\
\dot{x_4} & = & q\, x_3 & + & \varepsilon(F_1^4(x) + F_N^4(x)).
\end{array} \tag{3}
$$

We assume that

$$F_m^n = \sum_{i+j+k+l=m} a_{ijkl}^n x_1^i x_2^j x_3^k x_4^l,$$

for $m = 1, N$ and $n = 1, 2, 3, 4$.

Ours main results are the following.

Theorem 1. *Consider p, q coprime integer numbers with $p + q > 2$, $p > 1$ and $N = p + q - 1$. Then for $N \geq 2$ even and $\varepsilon \neq 0$ sufficiently small the following statement holds.*

(a) *If the displacement function of order ε is not identically zero, then the maximum number of limit cycles of the differential system (3) bifurcating from the periodic orbits of the linear differential center (1) is at most $2pq$.*
(b) *There are examples of system (3) having $2pq$ limit cycles bifurcating from the periodic orbits of the linear differential center (1).*

Theorem 2. *Consider p, q coprime integer numbers with $p + q > 2$, $p > 1$ and $N = p + q - 1$. Then for $N \geq 3$ odd and $\varepsilon \neq 0$ sufficiently small, if the displacement function of order ε is not identically zero, then the maximum number of limit cycles of the differential system (3) bifurcating from the periodic orbits of the linear differential center (1) is at most $pq(N + 1)$ if $p \geq 2$. When $p = 1$ this number is $q(N + 2)$.*

The results stated in Theorems 1 and 2 for $p = 1$ are already known, they have been proved in [1]. There is also proved that for $p = 1$ and $\varepsilon \neq 0$ sufficiently small there are differential systems (3) having $q(N + 2)$ limit cycles bifurcating from the periodic orbits of the linear differential center (1) if $N = 3, 5, 7, 9$. Additionally in [1] it is conjectured that the previous results hold for all $N \geq 3$ odd. Here we conjecture that the upper bounds stated in Theorem 2 are always reached for arbitrary p and q.

The proof of these two theorems are based on the averaging method. We will present the averaging method in Section 2. Some preliminary results are given in Section 3. The proofs of Theorems 1 and 2 are given in Sections 4 and 5, respectively.

2. First Order Averaging Method

In this section we present briefly the first order averaging method for computing periodic orbits. For details, see for instance, Theorem 11.5 of [3].

Theorem 3. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon)\ , \qquad x(0) = x_0\ , \tag{4}$$

and

$$\dot{y} = \varepsilon f^0(y)\ , \qquad y(0) = x_0\ , \tag{5}$$

x, y, $x_0 \in D$ an open subset of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f and g are periodic of period T in the variable t, and

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt \ . \tag{6}$$

Suppose

(i) *the vector functions f, $\partial f/\partial x$, $\partial^2 f/\partial x^2$, g and $\partial g/\partial x$ are defined, continuous and bounded by a constant independent on ε in $[0,\infty)\times D$ and $\varepsilon \in (0,\varepsilon_0]$;*

(ii) *T is independent on ε.*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged system (5) such that*

$$\left|\frac{\partial f^0(y)}{\partial y}\right|_{y=p} \neq 0 \ ,$$

then there exists a T–periodic solution $\phi(t,\varepsilon)$ of system (4) which is close to p such that $\phi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.

(b) *If a singular point p of the averaged system (5) is hyperbolic, then for $|\varepsilon| > 0$ sufficiently small the corresponding periodic orbit $\phi(t,\varepsilon)$ of system (4) is hyperbolic and of the same stability type as p.*

3. Preliminary Results

The proofs of Theorems 1 and 2 are based on the first-order averaging method stated in the previous section. In order to apply these results we need a convenient change of variables which writes our system (3) in the standard form for averaging.

Lemma 4. *Changing the variables (x_1,x_2,x_3,x_4) to (θ,r,R,s) by*

$$\begin{array}{llll} x_1 = & r\cos(p\ \theta), & x_2 = & r\sin(p\ \theta), \\ x_3 = & R\cos\left(q(\theta+s)\right), & x_4 = & R\sin\left(q(\theta+s)\right), \end{array}$$

system (3) is transformed into a system of the form

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon H_1(\theta,r,R,s) + \mathcal{O}(\varepsilon^2), \\ \frac{dR}{d\theta} &= \varepsilon H_2(\theta,r,R,s) + \mathcal{O}(\varepsilon^2), \\ \frac{ds}{d\theta} &= \varepsilon H_3(\theta,r,R,s) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{7}$$

where,

$$\begin{aligned} H_1 = &\ (F_1^1 + F_N^1)\cos(p\ \theta) + (F_1^2 + F_N^2)\sin(p\ \theta), \\ H_2 = &\ (F_1^3 + F_N^3)\cos(q(\theta+s)) + (F_1^4 + F_N^4)\sin(q(\theta+s)), \\ H_3 = &\ \frac{1}{qR}\left((F_1^4 + F_N^4)\cos(q(\theta+s)) - (F_1^3 + F_N^3)\sin(q(\theta+s))\right) - \\ &\ \frac{1}{pr}\left((F_1^2 + F_N^2)\cos(p\ \theta) - (F_1^1 + F_N^1)\sin(p\ \theta)\right). \end{aligned}$$

We take ε_f sufficiently small, n arbitrarily large, and $D_n = (1/n,n)\times(1/n,n)\times\mathbb{S}^1$. Then, the vector field of system (7) is well defined and continuous on $\mathbb{S}^1 \times D_n \times$

$(-\varepsilon_f, \varepsilon_f)$ where $\theta, s \in \mathbb{S}^1$, r, $R \in [\frac{1}{n}, n)$ and $\varepsilon \in (-\varepsilon_f, \varepsilon_f)$. Moreover, it is 2π-periodic with respect to θ and analytic with respect to (r, R, s, ε).

Proof. System (3) in the variables (θ, r, R, s) becomes

$$\begin{array}{rcl}\dot{\theta} & = & 1+\varepsilon\dfrac{1}{pr}\left((F_1^2+F_N^2)\cos(p\,\theta)-(F_1^1+F_N^1)\sin(p\,\theta)\right),\\ \dot{r} & = & \varepsilon H_1(\theta,r,R,s),\\ \dot{R} & = & \varepsilon H_2(\theta,r,R,s),\\ \dot{s} & = & \varepsilon H_3(\theta,r,R,s).\end{array} \tag{8}$$

We notice that for $|\varepsilon|$ sufficiently small, $\dot{\theta}(t) > 0$ for each t when $(\theta, r, R, s) \in \mathbb{S}^1 \times D_n$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is easy to see that the right hand side of the new system is well defined and continuos on $\mathbb{R} \times D_n \times (-\varepsilon_f, \varepsilon_f)$, it is 2π-periodic with respect to the independent variable θ and analytic with respect to (r, R, s). Form (7) is obtained after an expansion with respect to the small parameter ε. $\square$

Our next step is to find the corresponding function (6). We will denote it by $h : D_n \to \mathbb{R}^3$, $h = (h_1, h_2, h_3)^T$. For each $i = 1, 2, 3$, the component h_i is defined by formula

$$h_i(r, R, s) = \int_0^{2\pi} H_i(\theta, r, R, s)d\theta,$$

where the functions H_i are given in (7).

In order to calculate the exact expression of h, we use the following lemma where the proof can be seen in [1].

Lemma 5. *Let n be a non-negative integer and α and β be real numbers. The following statements hold.*

(a) $\cos^n \alpha = \displaystyle\sum_{i=0}^{[n/2]} b_i \cos((n-2i)\alpha)$;

(b) $\sin^n \alpha = \begin{cases} \displaystyle\sum_{i=0}^{n/2} b_i \cos((n-2i)\alpha) & \text{if } n \text{ is even;} \\ \displaystyle\sum_{i=0}^{(n-1)/2} b_i \sin((n-2i)\alpha) & \text{if } n \text{ is odd.} \end{cases}$

(c) *The expression $\cos^i \alpha \sin^j \alpha \cos^k \beta \sin^l \beta$, where i, j, k, l are non-negative integers, is equal to*

$$\sum_{m=0}^{[\frac{i+j}{2}]} \sum_{M=0}^{[\frac{k+l}{2}]} c_{mM} \cos\left(((i+j-2m)\alpha) \pm ((k+l-2M)\beta)\right),$$

or

$$\sum_{m=0}^{[\frac{i+j}{2}]} \sum_{M=0}^{[\frac{k+l}{2}]} d_{mM} \sin\left(((i+j-2m)\alpha) \pm ((k+l-2M)\beta)\right),$$

if $j + l$ is even or odd, respectively.

Lemma 6. *The following statements hold.*

(a) If $N = p + q - 1$ is even, then

$$h_1(r, R, s) = a_1 r + r^{q-1} R^p \left(b_1 \sin(pqs) + c_1 \cos(pqs)\right).$$

(b) If $N = p + q - 1$ is odd, then

$$h_1(r, R, s) = a_1 r + r^{q-1} R^p \left(b_1 \sin(pqs) + c_1 \cos(pqs)\right) + \sum_{M=0}^{\frac{N-1}{2}} d_M^1 r^{N-2M} R^{2M}.$$

Here a_1, b_1, c_1 and d_M^1's depend on the coefficients of the perturbation $F(x) = (F^1(x), F^2(x), F^3(x), F^4(x))$.

Proof. We write $H_1 = H_1^1 + H_1^N$ where $H_1^j = F_j^1 \cos(p\theta) + F_j^2 \sin(p\theta)$ and $h_1 = h_1^1 + h_1^N$ with $h_1^j = \frac{1}{2\pi}\int_0^{2\pi} H_1^j(\theta, r, R, s)d\theta$, $j = 1, N$. Thus

$$\begin{aligned}
&h_1^1(r, R, s) = \\
&\sum_{i+j+k+l=1} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^1 r^{i+j} R^{k+l} \cos^{i+1}(p\theta)\sin^j(p\theta)\cos^k(q(\theta+s))\sin^l(q(\theta+s))d\theta + \\
&\sum_{i+j+k+l=1} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^2 r^{i+j} R^{k+l} \cos^{i}(p\theta)\sin^{j+1}(p\theta)\cos^k(q(\theta+s))\sin^l(q(\theta+s))d\theta = \\
&\frac{a_{1000}^1 + a_{0100}^2}{2} r.
\end{aligned}$$

Now we calculate

$$\begin{aligned}
&h_1^N(r, R, s) = \\
&\sum_{i+j+k+l=N} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^1 r^{i+j} R^{k+l} \cos^{i+1}(p\theta)\sin^j(p\theta)\cos^k(q(\theta+s))\sin^l(q(\theta+s))d\theta + \\
&\sum_{i+j+k+l=N} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^2 r^{i+j} R^{k+l} \cos^{i}(p\theta)\sin^{j+1}(p\theta)\cos^k(q(\theta+s))\sin^l(q(\theta+s))d\theta.
\end{aligned}$$

Applying Lemma 5 we have that

$$h_1^N(r, R, s) = \sum_{i+j+k+l=N} r^{i+j} R^{k+l} \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=0}^{\left[\frac{i+j+1}{2}\right]} \sum_{M=0}^{\left[\frac{k+l}{2}\right]} C_{mM}^{ijkl}(\theta)d\theta$$

where $C_{mM}^{ijkl}(\theta)$ is

$$\begin{aligned}
&c_{mM}^{ijkl} \cos\big(((i+j+1-2m)p\theta) \pm ((k+l-2M)q(\theta+s))\big) + \\
&d_{mM}^{ijkl} \sin\big(((i+j+1-2m)p\theta) \pm ((k+l-2M)q(\theta+s))\big),
\end{aligned}$$

with c_{mM}^{ijkl} and d_{mM}^{ijkl} depending on the coefficients of perturbation. All these integrals with respect θ are zero except when

$$p(i+j+1-2m) = q(k+l-2M). \tag{9}$$

Without loss of generality we assume $p < q$. Since p and q are coprime, there exists a positive integer n such that $i + j + 1 - 2m = nq$ and $k + l - 2M = np$. Moreover, it is easy to see that $0 \le i + j + 1 - 2m \le N + 1 = p + q$. Then $nq \le p + q$, that is, $n \le \frac{p+q}{q} < 2$. So there are two possibilities: (1) $k + l - 2M = 0$, or (2) $k + l - 2M = p$.

First we consider the case $N = p + q - 1$ even. If $k + l - 2M = 0$ then $k + l$ is even. So $i + j = N - (k + l)$ is even because N is even. It is a contradiction with (9). If $k + l - 2M = p$ then (9) implies that $k + l + 2m = p$. Thus $m = 0$ and $k + l = p$. So $M = 0$ and $i + j = q - 1$. So in this case we get $h_1^N(r, R, s) = r^{q-1}R^p(b_1 \sin(pqs) + c_1 \cos(pqs))$ with b_1 and c_1 depending on c_{mM}^{ijkl} and d_{mM}^{ijkl} and this shows statement (a).

Now we consider the case $N = p + q - 1$ odd. If $k + l - 2M = 0$ then from (9) we have $k + l + 2m = p + q$. Thus for each M from 0 to $(N - 1)/2$, we obtain the terms $d_M^1 r^{N-2M} R^{2M}$. If $k + l - 2M = p$ then (9) implies that $k + l + 2m = p$. Thus $m = 0$ and $k + l = p$ and consequently $M = 0$ and $i + j = q - 1$. Finally we get $h_1^N(r, R, s) = r^{q-1}R^p(b_1 \sin(pqs) + c_1 \cos(pqs)) + \sum_{M=0}^{\frac{N-1}{2}} r^{N-2M}R^{2M}$ with b_1 and c_1 depending on c_{mM}^{ijkl} e d_{mM}^{ijkl} and this shows statement (b). $\square$

Lemma 7. *The following statements hold.*

(a) If $N = p + q - 1$ is even, then

$$h_2(r, R, s) = a_2 R + r^q R^{p-1} (b_2 \sin(pqs) + c_2 \cos(pqs)).$$

(b) If $N = p + q - 1$ is odd, then

$$h_2(r, R, s) = a_2 R + r^q R^{p-1} (b_2 \sin(pqs) + c_2 \cos(pqs)) + \sum_{M=0}^{\frac{N-1}{2}} d_M^2 r^{N-(2M+1)} R^{2M+1},$$

where a_2, b_2, c_2 and d_M^2's depend on the coefficients of the perturbation.

Proof. As in Lemma 6 we write $H_2 = H_2^1 + H_2^N$ where $H_2^j = F_j^3 \cos(q(\theta + s)) + F_j^4 \sin(q(\theta+s))$ and $h_2 = h_2^1 + h_2^N$ that from Theorem 3 $h_2^j = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} H_2^j(\theta, r, R, s)d\theta$, $j = 1, N$. Thus

$$\begin{aligned} &h_2^1(r, R, s) = \\ &\sum_{i+j+k+l=1} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^3 r^{i+j} R^{k+l} \cos^i(p\theta)\sin^j(p\theta)\cos^{k+1}(q(\theta+s))\sin^l(q(\theta+s))d\theta + \\ &\sum_{i+j+k+l=1} \frac{1}{2\pi}\int_0^{2\pi} a_{ijkl}^4 r^{i+j} R^{k+l} \cos^i(p\theta)\sin^j(p\theta)\cos^{k}(q(\theta+s))\sin^{l+1}(q(\theta+s))d\theta = \\ &\frac{a_{0010}^3 + a_{0001}^4}{2} R. \end{aligned}$$

Now we calculate $h_2^q(r, R, s)$. We find a similar expression to the one obtained in Lemma 6 except that the terms of the integrals are non-necessarily zero, they are given by

$$p(i + j - 2m) = q(k + l + 1 - 2M). \tag{10}$$

First, we consider the case $N = p+q-1$ even. If $k+l+1-2M = 0$ then $k+l$ is odd. So $i+j$ is odd. It is a contradiction with (10). If $k+l+1-2M = p$ then (10) implies that $k + l + 2m = p - 1$. Thus $p - 1 + 2M + 1 + 2m = p$ and $m = M = 0$. So $k + l = p - 1$ and $i + j = q$.

In the case $N = p+q-1$ odd we have the following. If $k+l+1-2M = 0$ then (10) implies $k+l+2m = N$. Thus for each M from 0 to $(N-1)/2$, we obtain the terms $d_M^2 r^{N-(2M+1)} R^{2M+1}$. If $k+l+1-2M = p$ we obtain the same than in the case N even, i.e., $k+l = p-1$ and $i+j = q$.

In short if N even

$$h_2^N(r,R,s) = r^q R^{p-1}(b_2 \sin(pqs) + c_2 \cos(pqs)),$$

and if N odd

$$h_2^N(r,R,s) = r^q R^{p-1}(b_2 \sin(pqs) + c_2 \cos(pqs)) + \sum_{M=0}^{\frac{N-1}{2}} d_M^2 r^{N-(2M+1)} R^{2M+1},$$

with b_2, c_2 and d_M^2's depending on c_{mM}^{ijkl} and d_{mM}^{ijkl}. This concludes the proof of the lemma. $\square$

Lemma 8. *The following statements hold.*

(a) If $N = p+q-1$ is even, then $h_3(r,R,s)$ is

$$a_3 + r^{q-2}R^p \left(b_3 \sin(pqs) + c_3 \cos(pqs)\right) + r^q R^{p-2} \left(d_3 \sin(pqs) + e_3 \cos(pqs)\right).$$

(b) If $N = p+q-1$ is odd, then $h_3(r,R,s)$ is

$$a_3 + r^{q-2}R^p \left(b_3 \sin(pqs) + c_3 \cos(pqs)\right) +$$
$$r^q R^{p-2} \left(d_3 \sin(pqs) + e_3 \cos(pqs)\right) + \sum_{M=0}^{\frac{N-1}{2}} d_M^3 r^{N-(2M+1)} R^{2M},$$

where a_3, b_3, c_3, d_3, e_3 and d_M^3's depend on the coefficients of the perturbation.

Proof. We write $H_3 = H_3^1 + H_3^N$ where

$$H_3^j = \frac{1}{Rq}\left(F_j^4 \cos\left(q(\theta+s)\right) - F_j^3 \sin\left(q(\theta+s)\right)\right) - \frac{1}{rp}\left(F_j^2 \cos\theta - F_j^1 \sin\theta\right),$$

and $h_3 = h_3^1 + h_3^N$ where $h_3^j = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} H_3^j(\theta,r,R,s)d\theta$, $j = 1, N$.

Using the same arguments of Lemmas 6 and 7 we obtain

$$h_3^1(r,R,s) = \frac{a_{0010}^4 - a_{0001}^3}{2q} - \frac{a_{1000}^2 - a_{0100}^1}{2p}.$$

Now we calculate $h_3^N(r,R,s)$. In a similar way to Lemmas 6 and 7 we get two sums of the form

$$\begin{aligned} h_3^N(r,R,s) = & \sum_{i+j+k+l=N} r^{i+j} R^{k+l-1} \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=0}^{\left[\frac{i+j}{2}\right]} \sum_{M=0}^{\left[\frac{k+l+1}{2}\right]} C_{mM}^{ijkl}(\theta) d\theta + \\ & \sum_{i+j+k+l=N} r^{i+j-1} R^{k+l} \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=0}^{\left[\frac{i+j+1}{2}\right]} \sum_{M=0}^{\left[\frac{k+l}{2}\right]} E_{mM}^{ijkl}(\theta) d\theta, \end{aligned}$$

where $C_{mM}^{ijkl}(\theta)$ is

$$c_{mM}^{ijkl} \cos\big(((i+j-2m)p\theta) \pm ((k+l+1-2M)q(\theta+s))\big) +$$
$$d_{mM}^{ijkl} \sin\big(((i+j-2m)p\theta) \pm ((k+l+1-2M)q(\theta+s))\big),$$

and $E_{mM}^{ijkl}(\theta)$ is

$$e_{mM}^{ijkl} \cos\big(((i+j+1-2m)p\theta) \pm ((k+l-2M)q(\theta+s))\big) +$$
$$f_{mM}^{ijkl} \sin\big(((i+j+1-2m)p\theta) \pm ((k+l-2M)q(\theta+s))\big),$$

with c_{mM}^{ijkl}, d_{mM}^{ijkl}, e_{mM}^{ijkl} and f_{mM}^{ijkl} depending on the coefficients of the perturbation.

In the first summand the terms whose integrals are non-necessarily zero are given by

$$p(i+j-2m) = q(k+l+1-2M), \tag{11}$$

and in the second summand by

$$p(i+j+1-2m) = q(k+l-2M). \tag{12}$$

The same arguments used in Lemmas 6 and 7 show that, in the first summand, if N is even then the terms that remain are $r^q R^{p-2}(d_3 \sin(pqs) + e_3 \cos(pqs))$ and if $\sigma = N$ is odd $r^{N-(2M+1)} R^{2M}$ with M from 0 to $(N-1)/2$, and $r^q R^{p-2}(d_3 \sin(pqs) + e_3 \cos(pqs))$.

In the second summand, if $\sigma = N$ is even then the term $r^{q-2} R^p (b_3 \sin(pqs) + c_3 \cos(pqs))$ remains. If $\sigma = N$ is odd we obtain the terms $d_M^3 r^{N-(2M+1)} R^{2M}$ with M from 0 to $(N-1)/2$ and $r^{q-2} R^p (b_3 \sin(pqs) + c_3 \cos(pqs))$. This concludes the proof of the lemma. □

Lemma 9. *Let p, q, α, β, γ and δ be non-negative integers such $\alpha+\beta = q-1$ and $\gamma+\delta = p$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^\alpha(p\theta)\sin^\beta(p\theta)\cos^\gamma(q(\theta+s))\sin^\delta(q(\theta+s))d\theta =$$

$$\begin{cases} \dfrac{(-1)^{\frac{\beta+\delta}{2}}}{2^{p+q-1}}\cos(pqs) & \text{if } \beta,\ \delta \text{ even;} \\ \dfrac{(-1)^{\frac{\beta+\delta-1}{2}}}{2^{p+q-1}}\sin(pqs) & \text{if } \beta \text{ even, } \delta \text{ odd ;} \\ -\dfrac{(-1)^{\frac{\beta+\delta-1}{2}}}{2^{p+q-1}}\sin(pqs) & \text{if } \beta \text{ odd, } \delta \text{ even ;} \\ \dfrac{(-1)^{\frac{\beta+\delta-2}{2}}}{2^{p+q-1}}\cos(pqs) & \text{if } \beta,\ \delta \text{ odd.} \end{cases}$$

Proof. The expression $\cos^\alpha(p\theta)\sin^\beta(p\theta)\cos^\gamma(q(\theta+s))\sin^\delta(q(\theta+s))$ may be written as $\left(\dfrac{e^{ip\theta}+e^{-ip\theta}}{2}\right)^\alpha \left(\dfrac{e^{ip\theta}-e^{-ip\theta}}{2i}\right)^\beta \left(\dfrac{e^{iq(\theta+s)}+e^{-iq(\theta+s)}}{2}\right)^\gamma \left(\dfrac{e^{iq(\theta+s)}-e^{-iq(\theta+s)}}{2i}\right)^\delta$.
In the expansion of this expression we consider only the terms $e^{ip\theta}$, $e^{-ip\theta}$, $e^{iq(\theta+s)}$ and $e^{-iq(\theta+s)}$ such that have the highest degree, i.e., $\alpha+\beta = q-1$ and $\gamma+\delta = p$, because the integral of the other terms on the interval $[0, 2\pi]$ are zero. So we get $\left(\dfrac{e^{ipq\theta}+(-1)^\beta e^{-ipq\theta}}{2^q\ i^\beta}\right)\left(\dfrac{e^{ipq(\theta+s)}+(-1)^\delta e^{-ipq(\theta+s)}}{2^p\ i^\delta}\right)$. Thus

$$\frac{e^{ipq\theta}+(-1)^{\beta}e^{-ipq\theta}}{2^q\ i^{\beta}} = \begin{cases} \dfrac{(-1)^{\frac{\beta}{2}}}{2^{q-1}}\cos(pq\theta) & \text{if } \beta \text{ even},\\ \dfrac{(-1)^{\frac{\beta-1}{2}}}{2^{q-1}}\sin(pq\theta) & \text{if } \beta \text{ odd}.\end{cases}$$

$$\frac{e^{ipq(\theta+s)}+(-1)^{\delta}e^{-ipq(\theta+s)}}{2^p\ i^{\delta}} = \begin{cases} \dfrac{(-1)^{\frac{\delta}{2}}}{2^{p-1}}\cos(pq(\theta+s)) & \text{if } \delta \text{ even},\\ \dfrac{(-1)^{\frac{\delta-1}{2}}}{2^{p-1}}\sin(pq(\theta+s)) & \text{if } \delta \text{ odd}.\end{cases}$$

For β, δ even we get:

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} cos^{\alpha}(p\theta)\sin^{\beta}(p\theta)\cos^{\gamma}(q(\theta+s))\sin^{\delta}(q(\theta+s))d\theta =\\ &\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{(-1)^{\frac{\beta}{2}}}{2^{q-1}}\cos(pq\theta)\right)\left(\frac{(-1)^{\frac{\delta}{2}}}{2^{p-1}}\cos(pq(\theta+s))\right)d\theta =\\ &\frac{(-1)^{\frac{\beta+\delta}{2}}}{2^{p+q-1}}\cos(pqs).\end{aligned}$$

The other cases are similar and we conclude the proof. □

Lemma 10. *The function h_3 of Lemma 8 is such that $b_3 = -c_1/p$, $c_3 = b_1/p$, $d_3 = -c_2/q$ and $e_3 = b_2/q$.*

Proof. Let $a^1_{ijkl}x_1^i x_2^j x_3^k x_4^l$ be a monomial of F^1_N such that $i+j = q-1$ and $k+l = p$. When we compute the expressions of h_1 and h_3, then this monomial appears in h_1 as

$$\frac{1}{2\pi}\int_0^{2\pi} a^1_{ijkl}cos^{i+1}(p\theta)\sin^j(p\theta)\cos^k(q(\theta+s))\sin^l(q(\theta+s))d\theta \tag{13}$$

and in h_3 as

$$\frac{1}{2p\pi}\int_0^{2\pi} a^1_{ijkl}cos^{i}(p\theta)\sin^{j+1}(p\theta)\cos^k(q(\theta+s))\sin^l(q(\theta+s))d\theta. \tag{14}$$

By Lemma 9 we have that (13) and (14) are equal to:

Table 1. The values of the integrals (13) and (14).

	13	14
j, l even	$\dfrac{a^1_{ijkl}(-1)^{j+l}}{2^N}\cos(pqs)$	$-\dfrac{a^1_{ijkl}(-1)^{j+l}}{2^N p}\sin(pqs)$
j par l odd	$\dfrac{a^1_{ijkl}(-1)^{j+l-1}}{2^N}\sin(pqs)$	$\dfrac{a^1_{ijkl}(-1)^{j+l-1}}{2^N p}\cos(pqs)$
j odd, l even	$-\dfrac{a^1_{ijkl}(-1)^{j+l-1}}{2^N}\sin(pqs)$	$\dfrac{a^1_{ijkl}(-1)^{j+l+1}}{2^N p}\cos(pqs)$
j, l, odd	$\dfrac{a^1_{ijkl}(-1)^{j+l-2}}{2^N}\cos(pqs)$	$\dfrac{a^1_{ijkl}(-1)^{j+l}}{2^N p}\sin(pqs)$

For j, l even, the coefficient a^1_{ijkl} of the monomial appears in a sum that determines the coefficient of $r^{q-1}R^p\cos(pqs)$ in h_1, and also appears in a sum that determines the coefficient of $r^{q-2}R^p\sin(pqs)$ in h_3 with the opposite sign and divided by p.

For j even and l odd, the coefficient a^1_{ijkl} of the monomial appears in a sum that determines the coefficient of $r^{q-1}R^p\sin(pqs)$ in h_1, and appears in a sum that determines the coefficient of $r^{q-2}R^p\cos(pqs)$ in h_3 divided by p.

For j, l odd, the coefficient a^1_{ijkl} of the monomial appears in a sum that determines the coefficient of $r^{q-1}R^p\cos(pqs)$ em h_1, and in a sum that determines the coefficient of $r^{q-2}R^p\sin(pqs)$ in h_3 divided by p.

For j odd and l even, the coefficient a^1_{ijkl} of the monomial appears in a sum that determines the coefficient of $r^{q-1}R^p\cos(pqs)$ in h_1, and in a sum that determines the coefficient of $r^{q-2}R^p\sin(pqs)$ in h_3 with the opposite sign and divided by p.

We can do the same for all monomial of F^2_N, F^3_N and F^4_N and finally we prove that $b_3 = -c_1/p$, $c_3 = b_1/p$, $d_3 = -c_2/q$ and $e_3 = b_2/q$. $\square$

In short we have proved the next result.

Proposition 11. *If N is even, then*

$$
\begin{aligned}
h_1(r,R,s) &= a_1 r + r^{q-1}R^p\left(b_1\sin(pqs) + c_1\cos(pqs)\right),\\
h_2(r,R,s) &= a_2 R + r^{q}R^{p-1}\left(b_2\sin(pqs) + c_2\cos(pqs)\right),\\
h_3(r,R,s) &= a_3 + r^{q-2}R^p\left(-\frac{c_1}{p}\sin(pqs) + \frac{b_1}{p}\cos(pqs)\right) +\\
&\qquad r^qR^{p-2}\left(-\frac{c2}{q}\sin(pqs) + \frac{b_2}{q}\cos(pqs)\right).
\end{aligned}
\tag{15}
$$

If N is odd, then

$$
\begin{aligned}
h_1(r,R,s) &= a_1 r + r^{q-1}R^p\left(b_1\sin(pqs) + c_1\cos(pqs)\right) + \sum_{M=0}^{\frac{N-1}{2}} d^1_M r^{N-2M}R^{2M},\\
h_2(r,R,s) &= a_2 R + r^{q}R^{p-1}\left(b_2\sin(pqs) + c_2\cos(pqs)\right) + \sum_{M=0}^{\frac{N-1}{2}} d^2_M r^{N-(2M+1)}R^{2M+1},\\
h_3(r,R,s) &= a_3 + r^{q-2}R^p\left(b_3\sin(pqs) + c_3\cos(pqs)\right)\\
&\quad + r^qR^{p-2}\left(d_3\sin(pqs) + e_3\cos(pqs)\right) + \sum_{M=0}^{\frac{N-1}{2}} d^3_M r^{N-(2M+1)}R^{2M}.
\end{aligned}
\tag{16}
$$

Finally we shall prove Theorems 1 and 2.

4. Proof of Theorem 1

According to Proposition 11, the functions $h_1(r,R,s)$, $h_2(r,R,s)$ and $h_3(r,R,s)$ are given by (15). We consider the change of variables $A = \dfrac{R}{r}$, $B = r^{q-1}R^{p-1}$, $u = \sin(pqs)$, $v = \cos(pqs)$. In these new variables system (15) becomes.

$$\begin{array}{lll}
\tilde{h}_1(A,B,u,v) = & h_1(r,R,s)/r = & a_1 + AB\,(b_1u + c_1v)\,,\\
\tilde{h}_2(A,B,u,v) = & h_2(r,R,s)/r = & a_2A + B\,(b_2u + c_2v)\,,\\
\tilde{h}_3(A,B,u,v) = & Rh_3(r,R,s)/r = & a_3A + B\left(-\dfrac{c_2}{q}u + \dfrac{b_2}{q}v\right) + A^2B\left(-\dfrac{c1}{p}u + \dfrac{b_1}{p}v\right),\\
\tilde{h}_4(A,B,u,v) = & & u^2 + v^2 - 1.
\end{array}$$

We denote $\tilde{h}_i = \tilde{h}_i(A,B,u,v)$ for $i = 1,2,3,4$ and we solve $(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3,\tilde{h}_4) = (0,0,0,0)$.
From $\tilde{h}_2 = 0$ we have

$$B = \frac{-Aa_2}{b_2u + c_2v}.$$

Substituting B in $\tilde{h}_1 = 0$, we obtain

$$A = \sqrt{\frac{a_1(b_2u + c_2v)}{a_2(b_1u + c_1v)}},$$

and so

$$B = \frac{-a_2}{b_2u + c_2v}\sqrt{\frac{a_1(b_2u + c_2v)}{a_2(b_1u + c_1v)}}.$$

Substituting A and B in $\tilde{h}_3 = 0$ we have

$$\frac{B_1u^2 + B_2uv + B_3v^2}{pq(b_2u + c_2v)(b_1u + c_1v)} = 0, \tag{17}$$

where

$$\begin{array}{ll}
B_1 = & pqa_3b_1b_2 + pa_2b_1c_2 + qa_1b_2c_1,\\
B_2 = & pq(a_3b_2c_1 + a_3c_2b_1) + p(a_2c_1c_2 - a_2b_1b_2) + q(a_1c_1c_2 - a_1b_2b_1),\\
B_3 = & pqa_3c_2c_1 - pa_2c_1b_2 - qa_1c_2b_1.
\end{array}$$

The zeros of (17) are $u = v = 0$, or a pair of crossing straight lines passing through the origin. So the maximum number of zeros of (17) and $u^2 + v^2 = 1$ is 4. Observe that for each zero of $(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3,\tilde{h}_4) = (0,0,0,0)$, with $A > 0$ and $B > 0$, we can find pq zeros (r,R,s) of $(h_1,h_2,h_3) = (0,0,0)$. We note that if for a zero (u_0,v_0) of (17) we obtain a solution $B_0 > 0$, then for $(-u_0,-v_0)$ of (17) we have $-B_0 < 0$. This proves that the maximum number of zeros of $(h_1,h_2,h_3) = (0,0,0)$ is $2pq$. So, by Theorem 3, the maximum number of limit cycles obtained via the averaging theory of first order for system (3) is $2pq$. This completes the proof of statement (a).

To find an example of the system (3) possessing $2pq$ limit cycles, we choose the coefficients a^n_{ijkl} of F all zeros except

$$
\begin{aligned}
a^1_{1000} &= -2^{1-p}\,(16+4^p)\,p,\\
a^3_{0010} &= 2^{3-p}\,(1+4^p)\,q,\\
a^4_{0010} &= 30q,\\
a^1_{0q-10p} &= \left(1+4^{2-p}\right)p,\\
a^2_{q-10p0} &= 2^{-p}\,(16+4^p)\,p,\\
a^3_{0q0p-1} &= -4^{2-p}\,(1+4^p)\,q,\\
a^4_{q0p-10} &= -2^{2-p}\,(1+4^p)\,q.
\end{aligned}
$$

Then system (3) becomes

$$
\begin{array}{rcrcl}
\dot{x_1} &=& -p\,x_2 &+& \varepsilon(-2^{1-p}\,(16+4^p)\,p\,x_1 + \left(1+4^{2-p}\right)p\,x_2^{q-1}x_4^p),\\
\dot{x_2} &=& p\,x_1 &+& \varepsilon(2^{-p}\,(16+4^p)\,p\,x_1^{q-1}\,x_3^p),\\
\dot{x_3} &=& -q\,x_4 &+& \varepsilon(2^{3-p}\,(1+4^p)\,q\,x_3 - 4^{2-p}\,(1+4^p)\,q\,x_2^q\,x_4^{p-1}),\\
\dot{x_4} &=& q\,x_3 &+& \varepsilon(30\,q\,x_3 - 2^{2-p}\,(1+4^p)\,qx_1^q\,x_3^{p-1}).
\end{array}
$$

Computing h_1, h_2 and h_3 for this system we obtain

$$
\begin{array}{rl}
h_1(r,R,s) = & -2^{-p}\,(16+4^p)\,pr + r^{q-1}R^p(2^{-p}\,(16+4^p)\,p\cos(pqs) + \left(1+4^{2-p}\right)p\sin(pqs)),\\
h_2(r,R,s) = & 2^{2-p}\,(1+4^p)\,qR + r^qR^{p-1}(-2^{2-p}\,(1+4^p)\,q\cos(pqs) - 4^{2-p}\,(1+4^p)\,q\sin(pqs)),\\
h_3(r,R,s) = & 15 + r^qR^{p-2}(2^{2-p}\,(1+4^p)\sin(pqs) - 4^{2-p}\,(1+4^p)\cos(pqs)) +\\
& r^{q-2}R^p(\left(1+4^{2-p}\right)\cos(pqs) + 2^{-p}\,(16+4^p)\sin(pqs))
\end{array}
$$

The zeros of $(h_1,h_2,h_3)=(0,0,0)$ are

$$
(r,R,s) = \left(1,1,k\frac{2\pi}{pq}\right), \qquad k=0,...,pq-1
$$

and

$$
(r,R,s) = \left(2,2^{p-1},\frac{\pi}{2pq}+k\frac{2\pi}{pq}\right), k=0,...,pq-1
$$

The Jacobian determinant of $h=(h_1,h_2,h_3)$ computed at $\left(1,1,k\frac{2\pi}{pq}\right)$ for $k=0,...,pq-1$ is

$$
32^{3-5p}p^2q^2\,(16+17\;4^p+16^p)^2\,(p+q-2)\neq 0
$$

and computed at $\left(2,2^{p-1},\frac{\pi}{2pq}+k\frac{2\pi}{pq}\right)$ for $k=0,...,pq-1$ is

$$
-32^{1-3p}p^2q^2\,(16+17\;4^p+16^p)^2\,(p+q-2)\neq 0
$$

Applying Theorem 3, the proof of statement (b) is done. Thus we conclude the proof of Theorem 1.

5. Proof of Theorem 2

The functions $h_1(r,R,s)$, $h_2(r,R,s)$ and $h_3(r,R,s)$ are given in Proposition 11. In this following coordinates $A=\dfrac{R}{r},\ B=r^{q-1}R^{p-1},\ u=\sin(pqs),\ v=\cos(pqs)$ we obtain

$$
\begin{aligned}
\tilde{h}_1(A,B,u,v)= \quad & h_1(r,R,s)/r= \quad && a_1+AB\,(b_1u+c_1v)+A^{1-p}B\sum_{M=0}^{\frac{N-1}{2}}d^1_MA^{2M},\\
\tilde{h}_2(A,B,u,v)= \quad & h_2(r,R,s)/r= \quad && a_2A+B\,(b_2u+c_2v)+A^{2-p}B\sum_{M=0}^{\frac{N-1}{2}}d^2_MA^{2M},\\
\tilde{h}_3(A,B,u,v)= \quad & Rh_3(r,R,s)/r= \quad && a_3A+B\left(-\frac{c_2}{q}u+\frac{b_2}{q}v\right)+\\
& && A^2B\left(-\frac{c1}{p}u+\frac{b_1}{p}v\right)+A^{2-p}B\sum_{M=0}^{\frac{N-1}{2}}d^3_MA^{2M},\\
\tilde{h}_4(A,B,u,v)= \quad & && u^2+v^2-1.
\end{aligned}
$$

We solve $(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3)=(0,0,0)$ and we find a solution $B=A^{p-1}B_1(A^2)$, $u=A^{2-p}U(A^2)$, $v=A^{2-p}V(A^2)$, where $B_1(z)$ is the quotient of one polynomial of degree 2 by a polynomial of degree $(N+3)/2$, and $U(z)$ and $V(z)$ are the quotient of one polynomial of degree $(N+1)/2$ by the same polynomial of degree 2.

Substituting u and v in $\tilde{h}_4=0$ we have the following situations.

If $p=1$ then we obtain the quotient of a polynomial of degree $N+2$ in the variable A^2 by a polynomial of degree 4 in A^2. So the maximum number of positive roots A of the numerator of $\tilde{h}_4$ is $N+2$.

If $p=2$ then $u=U(A^2)$ and $v=V(A^2)$. Thus we obtain the quotient of one polynomial of degree $N+1$ in the variable A^2 by a polynomial of degree 4 in A^2. In this case the maximum number of positive roots A of the numerator of $\tilde{h}_4$ is $N+1$.

If $p>2$ then $u=\dfrac{U(A^2)}{A^{p-2}}$ and $v=\dfrac{V(A^2)}{A^{p-2}}$. So we have the quotient of one polynomial of degree $N+1$ in the variable A^2 by a polynomial of degree $p+2$. Since $p+2<N+1$ we obtain that the maximum number of positive roots A of the numerator of $\tilde{h}_4$ is $N+1$.

For each solution A_0 we have at most one $B_0=B(A_0)>0$ and a pair $(u_0,v_0)=(u(A_0),v(A_0))$. For each pair (u_0,v_0) we can find $s_1,...,s_{pq}\in[0,2\pi)$ such that $\sin(pqs_i)=u_0$ and $\cos(pqs_i)=v_0$ for $i=1,...,pq$. So by Theorem 3, the maximum number of limit cycles obtained via the averaging theory of first order for system (3) is $q(N+2)$ if $p=1$ and $pq(N+1)$ if $p\geq 2$. This completes the proof of Theorem 2.

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[2] Departamento de Matemàtica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083-970, Campinas, S.P, Brazil

E-mail address: anameren@ime.unicamp.br,teixeira@ime.unicamp.br