INVARIANT TORI FULFILLED OF PERIODIC ORBITS FOR 4-DIMENSIONAL $\mathcal{C}^2$ DIFFERENTIAL SYSTEMS IN PRESENCE OF RESONANCE

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Abstract. We provide an algorithm for studying invariant tori fulfilled of periodic orbits of a perturbed system which emerge from the set of periodic orbits of an unperturbed linear system in $p : q$ resonance. We illustrate the algorithm with an application.

1. Introduction and statement of the results

One of the main problems in general perturbation theory is to detect how persistent are some given properties. In other words we want to translate some dynamical properties from the unperturbed system to the perturbed one. Frequently the unperturbed system is linear and the objects to be continued to the perturbed system are equilibria, periodic orbits or invariant tori. Bifurcations appear when the non persistence occurs.

Our goal in this note is to provide an algorithm for studying the invariant tori fulfilled of periodic orbits of the perturbed system which emerge from the set of periodic orbits of the unperturbed system.

We consider the four-dimensional linear center

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1}$$

where

$$A = \begin{pmatrix} 0 & -p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & q & 0 \end{pmatrix},$$

where $\mathbf{x} = (x, y, z, w) \in \mathbb{R}^4$, and p and q are coprime positive integers. Clearly all orbits of system (1) are periodic with the exception of its unique singular point located at the origin of coordinates. We say that the periodic orbits of this center are in *resonance* $p : q$.

We perturb system (1) as follows

$$\dot{\mathbf{x}} = A\mathbf{x} + \varepsilon F(\mathbf{x}), \tag{2}$$

where $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ is a small parameter and $F : U \to \mathbb{R}^4$ is a $\mathcal{C}^2$ map defined on an open subset U of $\mathbb{R}^4$ containing the origin.

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The algorithm, for studying the invariant tori fulfilled of periodic orbits of the perturbed system (2) which emerge from the set of periodic orbits of the unperturbed system (1), is based in a classical result for studying the periodic orbits of a differential system using the averaging theory. As an application of this algorithm we shall prove the following result. As usual we denote the circle by $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{R})$.

Theorem 1. *Consider the differential system*

$$
\begin{aligned}
\dot{x} &= -y - \varepsilon(z+1)\left(y + 2w + \frac{4x}{x^2+y^2}\right),\\
\dot{y} &= x + \varepsilon y\left(w + 1 - \frac{2}{x^2+y^2}\right),\\
\dot{z} &= -2w,\\
\dot{w} &= 2z + \varepsilon(y+1)w\left(\frac{2}{w^2+z^2} - \frac{1}{2}\right),
\end{aligned} \tag{3}
$$

defined in

$$\{\mathbf{x} = (x,y,z,w) \in \mathbb{R}^4 : \mathbf{x} \neq (x,y,0,0) \textit{ and } \mathbf{x} \neq (0,0,z,w)\},$$

and where $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ is a small parameter. The unperturbed system (3) with $\varepsilon = 0$ is a linear center in $\mathbb{R}^4$ whose periodic orbits are in resonance $1:2$. For $\varepsilon \neq 0$ sufficiently small the perturbed system (3) has a 2-dimensional invariant torus fulfilled of periodic orbits which tend to the torus

$$\{(x,y,z,w) \in \mathbb{R}^4 : x^2+y^2 = 6 + 4\cos\varphi,\ z^2+w^2 = 4 \textit{ with } \varphi \in \mathbb{S}^1\}, \tag{4}$$

when $\varepsilon \to 0$.

In section 2 we present the result of the averaging theory that we need. The algorithm for studying the invariant tori fulfilled of periodic orbits of the perturbed system (2) is described in section 3. Finally in section 4 we prove Theorem 1.

2. Basic result on averaging theory

The key tool for proving the algorithm is the averaging theory. For a general introduction to the averaging theory and related topics see the books [1, 3, 4, 5]. But the result that we shall use is presented in what follows.

We consider the differential system

$$\dot{\mathbf{x}}(t) = \varepsilon F(t, \mathbf{x}(t)) + \varepsilon^2 R(t, \mathbf{x}(t), \varepsilon), \tag{5}$$

with $\mathbf{x} \in U \subset \mathbb{R}^n$, U a bounded domain and $t \geq 0$. Moreover, we assume that $F(t,\mathbf{x})$ and $R(t,\mathbf{x},\varepsilon)$ are T–periodic in t.

The averaged system associated to system (5) is defined by

$$\dot{\mathbf{y}}(t) = \varepsilon f(\mathbf{y}(t)), \tag{6}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F(s,\mathbf{y})ds. \tag{7}$$

The next theorem says us under which conditions the singular points of the averaged system (6) provide T–periodic orbits of system (5). For a proof see Theorem 2.6.1 of [4], Theorems 11.5 and 11.6 of [5], and Theorem 4.1.1 of [2].

Theorem 2. *We consider system* (5) *and assume that the vector functions* F, R, $D_{\mathbf{x}}F_1$, $D^2_{\mathbf{x}}F_1$ *and* $D_{\mathbf{x}}R$ *are continuous and bounded by a constant* M *(independent of* ε*) in* $[0,\infty)\times U$ *with* $-\varepsilon_0<\varepsilon<\varepsilon_0$. *Moreover, we suppose that* F *and* R *are* T*–periodic in* t, *with* T *independent of* ε.

(a) *If* $a\in U$ *is a singular point of the averaged system* (6) *such that* $\det(D_{\mathbf{x}}f(a))\neq 0$ *then, for* $|\varepsilon|>0$ *sufficiently small there exists a unique* T*–periodic solution* $\mathbf{x}_\varepsilon(t)$ *of system* (5) *such that* $\mathbf{x}_\varepsilon(0)\to a$ *as* $\varepsilon\to 0$.

(b) *If the singular point* a *of the averaged system* (6) *is hyperbolic then, for* $|\varepsilon|>0$ *sufficiently small, the corresponding periodic solution* $\mathbf{x}_\varepsilon(t)$ *of system* (5) *is hyperbolic and of the same stability type as* a.

3. The algorithm

In this section we describe the algorithm for studying the invariant tori fulfilled of periodic orbits of the perturbed system (2) which emerge from the set of periodic orbits of the unperturbed system (1).

Doing a rescalling of the independent variable by q, we can assume that the linear part of the differential system (2) is given by the matrix

$$\begin{pmatrix} 0 & -n & 0 & 0\\ n & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix},$$

where $n=-p/q$.

Let $F=(F_1,F_2,F_3,F_4)$. Then changing the variables (x,y,z,w) to (r,θ,R,φ) by

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=R\cos\left(\theta+\frac{(1-n)}{n}\varphi\right),\quad w=R\sin\left(\theta+\frac{(1-n)}{n}\varphi\right),$$

system (2) is transformed into the system

$$\begin{aligned}\dot r &= \varepsilon G_1(r,\theta,R,\varphi),\\ \dot\theta &= n+\varepsilon G_2(r,\theta,R,\varphi),\\ \dot R &= \varepsilon G_3(r,\theta,R,\varphi),\\ \dot\varphi &= n+\varepsilon G_4(r,\theta,R,\varphi),\end{aligned}\tag{8}$$

with

$$\begin{aligned}G_1 =&\ \varepsilon\left[\cos\theta\,\overline{F}_1(r,\theta,R,\varphi)+\sin\theta\,\overline{F}_2(r,\theta,R,\varphi)\right],\\ G_2 =&\ n+\varepsilon\frac{1}{r}\left[\cos\theta\,\overline{F}_2(r,\theta,R,\varphi)-\sin\theta\,\overline{F}_1(r,\theta,R,\varphi)\right],\\ G_3 =&\ \varepsilon\left[\cos\left(\theta+\frac{(1-n)}{n}\varphi\right)\overline{F}_3(r,\theta,R,\varphi)+\sin\left(\theta+\frac{(1-n)}{n}\varphi\right)\overline{F}_4(r,\theta,R,\varphi)\right],\\ G_4 =&\ n+\varepsilon\frac{n}{1-n}\left[\frac{1}{R}\left(\cos\left(\theta+\frac{(1-n)}{n}\varphi\right)\overline{F}_4(r,\theta,R,\varphi)-\right.\right.\\ &\left.\sin\left(\theta+\frac{(1-n)}{n}\varphi\right)\overline{F}_3(r,\theta,R,\varphi)\right)+\\ &\left.\frac{1}{r}\left(\sin\theta\,\overline{F}_1(r,\theta,R,\varphi)-\cos\theta\,\overline{F}_2(r,\theta,R,\varphi)\right)\right],\end{aligned}$$

where

$$\overline{F}_k(r,\theta,R,\varphi)=F_k\left(r\cos\theta, r\sin\theta, R\cos\left(\theta+\frac{(1-n)}{n}\varphi\right), R\sin\left(\theta+\frac{(1-n)}{n}\varphi\right)\right),$$

for $k=1,2,3,4$.

We change the independent variable t in system (8) by taking θ as the new independent variable. Thus system (8) becomes

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon G_1(r,\theta,R,\varphi)+\mathcal{O}(\varepsilon^2),\\ \frac{dR}{d\theta} &= \varepsilon G_3(r,\theta,R,\varphi)+\mathcal{O}(\varepsilon^2),\\ \frac{d\varphi}{d\theta} &= 1+\varepsilon\big(G_4-G_2\big)(r,\theta,R,\varphi)+\mathcal{O}(\varepsilon^2),\end{aligned}\tag{9}$$

From the last equation of system (9) we have that any solution $(r(\theta),R(\theta),\varphi(\theta))$ of system (9) is of the form $\varphi(\theta)=\theta+\varphi_0+\mathcal{O}(\varepsilon)$. Substituting this expression of $\varphi(\theta)$ into system (9), it reduces to

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon G_1(r,\theta,R,\theta+\varphi_0)+\mathcal{O}(\varepsilon^2),\\ \frac{dR}{d\theta} &= \varepsilon G_3(r,\theta,R,\theta+\varphi_0)+\mathcal{O}(\varepsilon^2).\end{aligned}\tag{10}$$

Now we shall study the periodic orbits of system (10) applying to it Theorem 2 of the averaging theory. So we compute the averaged system of system (10) and we get

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon g_1(r,R,\varphi_0),\\ \frac{dR}{d\theta} &= \varepsilon g_3(r,R,\varphi_0),\end{aligned}\tag{11}$$

where

$$g_k(r,R,\varphi_0)=\frac{1}{2\pi}\int_0^{2\pi} G_k(r,\theta,R,\theta+\varphi_0),\quad k=1,3.$$

Assume that for every $\varphi_0\in\mathbb{S}^1$ the averaged system (11) has a singular point $(r(\varphi_0),R(\varphi_0))$ such that

$$\det\left(\frac{\partial(g_1,g_2)}{\partial(r,R)}\bigg|_{r=r(\varphi_0),R=R(\varphi_0))}\right)\neq 0.\tag{12}$$

Then, applying Theorem 2 to the φ_0-parametric differential system (10), we get that system (10) for $\varepsilon\neq 0$ sufficiently small and for each $\varphi_0\in\mathbb{S}^1$ has a unique periodic orbit

$$\Big(r\big(\theta;(r(\varphi_0),R(\varphi_0))\big),R\big(\theta;(r(\varphi_0),R(\varphi_0))\big)\Big),$$

such that

$$\Big(r\big(0;(r(\varphi_0),R(\varphi_0))\big),R\big(0;(r(\varphi_0),R(\varphi_0))\big)\Big)\to(r(\varphi_0),R(\varphi_0))\quad\text{when } \varepsilon\to 0.$$

Going back to differential system (9), we obtain that this system for $\varepsilon\neq 0$ sufficiently small has a continuous family of periodic orbits depending on the parameter

$\varphi_0 \in \mathbb{S}^1$, i.e. we get that system (9) has an invariant torus fulfilled of periodic orbits. Consequently the differential systems (8) and (2) (which are the same system (9) in other variables) have an invariant torus fulfilled of periodic orbits.

This completes the algorithm for detecting invariant torus fulfilled of periodic orbits of system

4. Proof of Theorem 1

In this section we apply the algorithm described in section 3 to the differential system (3). For this system $n = 1/2$ and system (8) becomes

$$\begin{aligned}
\dot{r} = &\quad -\varepsilon\frac{1}{4r}\big(6 - r^2 + (r^2+2)\cos(2\theta) + 2R\cos(\varphi-\theta) + 4R\cos(\varphi+\theta) + \\
&\quad 2R\cos(\varphi+3\theta) + r(2R\sin\varphi + r\sin(2\theta) + 2R(r\cos(\varphi+2\theta)\sin\theta + \\
&\quad R\cos\theta\sin(2(\varphi+\theta)) + \sin(\varphi+2\theta)))\big),\\
\dot{\theta} = &\quad \frac{1}{2} + \varepsilon\frac{1}{2r^2}\sin\theta\big(2R\cos\varphi + (r^2+2)\cos\theta + 2R\cos(\varphi+2\theta) + \\
&\quad r(2R(R\cos(\varphi+\theta)+1)\sin(\varphi+\theta) + r(\sin\theta + R\sin(\varphi+2\theta)))\big),\\
\dot{R} = &\quad -\varepsilon\frac{1}{4R}\left(R^2-4\right)(r\sin\theta+1)\sin^2(\varphi+\theta),\\
\dot{\varphi} = &\quad \frac{1}{2} + \varepsilon\bigg(-\frac{1}{8R^2}(R^2-4)(r\sin\theta+1)\sin(2(\varphi+\theta)) - \\
&\quad \frac{1}{2r^2}\sin\theta\big(2R\cos\varphi + (r^2+2)\cos\theta + 2R\cos(\varphi+2\theta) + \\
&\quad r(2R(R\cos(\varphi+\theta)+1)\sin(\varphi+\theta) + r(\sin\theta + R\sin(\varphi+2\theta)))\big)\bigg).
\end{aligned}$$

Now we write system (9):

(13)

$$\begin{aligned}
\frac{dr}{d\theta} = &\quad -\varepsilon\frac{1}{4r}\big(6 - r^2 + (r^2+2)\cos(2\theta) + 2R\cos(\varphi_0) + 4R\cos(2\theta+\varphi_0) + \\
&\quad 2R\cos(4\theta+\varphi_0) + r(2R\sin(\theta+\varphi_0) + r\sin(2\theta) + 2R(r\cos(3\theta+\varphi_0)\sin\theta + \\
&\quad R\cos\theta\sin(2(2\theta+\varphi_0)) + \sin(3\theta+\varphi_0)))\big) + \mathcal{O}(\varepsilon^2),\\
\frac{dR}{d\theta} = &\quad -\varepsilon\frac{1}{4R}\left(R^2-4\right)(1+r\sin\theta)\sin^2(2\theta+\varphi_0) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Computing the averaged system of system (13), as it is indicated in (11), we get

$$\begin{aligned}
\frac{dr}{d\theta} = &\quad \varepsilon\frac{1}{2r}(r^2 - 6 - 2R\cos\varphi_0),\\
\frac{dR}{d\theta} = &\quad -\varepsilon\frac{R^2-4}{4R}.
\end{aligned} \tag{14}$$

The unique singular point of system (14) with r and R positive is

$$r = \sqrt{6+4\cos\varphi_0}, \qquad R = 2.$$

For this singular point the determinant (12) is always $-1/2$ independently of φ_0.

In short from the last part of the algorithm it follows the statement of Theorem 1.

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