A Pluzhnikov's Theorem, Brownian motions and Martingales in Lie Group with skew-symmetric connections

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Abstract

Let G be a Lie Group with a left invariant connection such that its connection function is skew-symmetric. Our main goal is to show a version of Pluzhnikov's Theorem for this kind of connection. To this end, we use the stochastic logarithm. More exactly, the stochastic logarithm gives characterizations for Brownian motions and Martingales in G, and these characterzations are used to prove Pluzhnikov's Theorem.

Key words: harmonic maps; Lie groups; stochastic analisys on manifolds.

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1 Introduction

Let G be a Lie group, $\mathfrak{g}$ its lie algebra and ω_G the Maurer-Cartan form on G. K. Nomizu, in [7], has proved that there is an one-to-one association between left invariant connection in G and bilinear applications α from $\mathfrak{g} \otimes \mathfrak{g}$ into $\mathfrak{g}$, which is called connection function. In this work, we are only interested in skew-symmetric connection functions.

Our main goal is to prove a version of the following Theorem, in the Riemannian case, that was proved by A.I. Pluzhnikov in [9].

Theorem *Let M be a Riemannian manifold, G a Lie group with a left invariant connection ∇^G such that its connection function α is skew-symmetric, ω_G the Maurer-Cartan form on G and $F : M \to G$ a smooth map. Then F is harmonic if and only if*

$$d^* F^* \omega_G = 0,$$

where d^* is the co-differential operator on M.

The proof of this Theorem is based in a stochastic tool: the stochastic logarithm. It was introduced, in [5], by M. Hawkim-Dowek and D. Lépingle. Being X_t a semimartingale with valued in G, the stochastic logarithm, denoted by $\log X_t$, is a semimartingale in the Lie algebra $\mathfrak{g}$.

The key of proof of Theorem above is the characterization of martingales and Brownian motions in terms of stochastic logarithm. In fact, if we take a left invariant connection on G with skew-symmetric connection function α or a bi-invariant metric k, we have the following:

Theorem: **(i)** *A G-valued semimartingale X_t is a ∇^G-martingale if and only if $\log X_t$ is a $\nabla^{\mathfrak{g}}$-martingale, where $\nabla^{\mathfrak{g}}$ is the connection on $\mathfrak{g}$ given by α.*
(ii) *A semimartingale B_t is a k-Brownian motion if and only in $\log B_t$ is a $<,>$-Brownian motion, where $<,>$ is the scalar product in $\mathfrak{g}$ associated to k.*

This paper is organized as follow: in section 2 we give a brief exposition of stochastic calculus on manifold. In section 3 our main results are stated and proved.

2 Preliminaries

We begin by recalling some fundamental facts on stochastic calculus on manifolds, we shall use freely concepts and notations of M. Emery [6] and P. Protter [10]. In [3], we find a complete survey of the stochastic properties in this section.

Let $(\Omega, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space with usual hypothesis (see for instance [6]). In this work we mean smooth as C^∞.

Definition 2.1 *Let M be a differential manifold and X_t a continuos stochastic process with values in M. We call X_t a semimartingale if, for all f smooth function, $f(X_t)$ is a real semimartingale.*

Let M be a smooth manifold with a connection ∇^M, X_t a semimartingale with values in M, θ a section of TM^* and b a section of $T^{(2,0)}M$. The Stratonovich integral of θ along X_t is denoted by $\int_0^t \theta \delta X_s$, the Itô integral of θ along X by $\int_0^t \theta d^{\nabla^M} X_s$. Let $(U, x_1, \ldots, x_n)$ be a local coordiante system on M. Then in U we can write $b = b_{ij} dx^i \otimes dx^j$, where b_{ij} are smooth functions on U. We define the quadratic integral of b along of X_t, locally, by

$$\int_0^t b\ (dX, dX)_s = \int_0^t b_{ij}(X_s) d[X_s^i, X_s^j],$$

where $X^i = x_i \circ X$, for $i = 1, \dots n$.

Definition 2.2 *Let M be a smooth manifold with a connection ∇^M. A semimartingale X with values in M is called a ∇^M-martingale if $\int \theta\ d^{\nabla^M} X$ is a real local martingale for all $\theta \in \Gamma(TM^*)$.*

Definition 2.3 *Let M be a Riemannian manifold with a metric g. Let B be a semimartingale with values in M. We say that B is a g-Brownian motion in M if B is a ∇^g-martingale, being ∇^g the Levi-Civita connection of g, and for any section b of $T^{(2,0)}M$ we have*

$$\int_0^t b(dB, dB)_s = \int_0^t \operatorname{tr} b(B_s) ds. \tag{1}$$

Following, we state the stochastic tools that are necesary to establish our main results. Firstly, we observed that

$$\int_0^t b\ (dX, dX)_s = \int_0^t b^s\ (dX, dX)_s,$$

where b^s is the symmetric part of b.

Let M be a smooth manifold with a connection ∇^M and θ a section of TM^*. We have the Stratonovich-Itô formula of conversion

$$\int_0^t \theta \delta X_s = \int_0^t \theta d^{\nabla^M} X_s + \frac{1}{2} \int_0^t \nabla^M \theta\ (dX, dX)_s. \tag{2}$$

When (M, g) is a Riemannian manifold and B_t is a g-Brownian motion in M we deduce from (1) and (2) the Manabe's formula:

$$\int_0^t \theta \delta B_s = \int_0^t \theta d^{\nabla^M} B_s + \frac{1}{2} \int_0^t d^* \theta(B_s) ds, \tag{3}$$

where d^* is the co-differential on M.

Let M and N be manifolds, θ be a section of TN^*, b be a section of $T^{(2,0)}N$ and $F : M \to N$ be a smooth map. For a semimartingale X_t in M, we have the following Itô formulas for Stratonovich and quadratic integrals:

$$\int_0^t \theta\ \delta F(X) = \int_0^t F^* \theta\ \delta X \tag{4}$$

and

$$\int_0^t b\,(dF(X), dF(X)) = \int_0^t F^*b\,(dX, dX). \tag{5}$$

Let M and N be smooth manifolds endowed with connections ∇^M and ∇^N, respectively. Let $F : M \to N$ be a smooth map and $F^{-1}(TN)$ the induced bundle. We denote by $\nabla^{N'}$ the unique connection on $F^{-1}(TN)$ induced by ∇^N (see for example Proposition I.3.1 in [8]). The bilinear mapping $\beta_F : TM \times TM \to TN$ defined by

$$\beta_F(X,Y) = \nabla_X^{N'} F_*(Y) - F_*(\nabla_X^M Y) \tag{6}$$

is called the second fundamental form of F (see for example definition I.4.1.1 in [8]). F is said affine map if β_F is null.

When (M, g) is a Riemannian manifold, we define the tension field τ_F of F by $\tau_F = \operatorname{tr}\beta_F$. We call F a harmonic map if $\tau_F \equiv 0$. We observe that N is not necessarily a Riemannian manifold to define harmonic map. But this definition is an extension of one gives by energy functional.

Let M and N be smooth manifold with connections ∇^M and ∇^N. The Itô geometric formula is given by:

$$\int_0^t \theta d^{\nabla^N} F(X_s) = \int_0^t F^*\theta d^{\nabla^M} X_s + \frac{1}{2}\int_0^t \beta_F^*\theta(dX, dX)_s. \tag{7}$$

If (M, g) is Riemannian manifold and if B_t is a g-Brownian motion in M, then, from Itô geometric formula and (1) we deduce that

$$\int_0^t \theta d^{\nabla^N} F(B_s) = \int_0^t F^*\theta d^{\nabla^M} B_s + \frac{1}{2}\int_0^t \tau_F^*\theta(B_s)ds. \tag{8}$$

From Itô geometric formula and Doob-Meyer decomposition from real semimartingales we deduce the following stochastic characterizations for affine a harmonic maps:

(i) F is an affine map if and only if it sends ∇^M-martingales to ∇^N-martingales.

(ii) If (M, g) is a Riemmanian manifold, then $F : M \to N$ is a harmonic map if and only if it sends g-Brownian motions to ∇^N-martingales.

3 Pluzhnikov's theorem, Brownian motions and martingales

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let us denote by ω_G the Maurer-Cartan form on G. Let X_t be a semimartingale in G. The stochastic logarithm of the semimartingale X_t (with $X_0 = e$) is the semimartingale, in the Lie algebra $\mathfrak{g}$, given by

$$\log X_t = \int_0^t \omega_G \delta X_s.$$

For the convenience of the reader we repeat the following two results from [4], thus making our exposition self-contained.

Lemma 3.1 *Let G and H be two Lie groups. If $\varphi : G \to H$ is a homomorphism then*

$$\varphi^* \omega_H = \varphi_* \omega_G,$$

where ω_G and ω_H be Maurer-Cartan form on G and H, respectively.

Proof: Once $\varphi(L_{g^{-1}}(h)) = L_{\varphi(g)^{-1}}(\varphi(h))$, chain rule implies that

$$L_{\varphi(g)^{-1}*}(\varphi_*(v)) = \varphi_*(L_{g^{-1}*}(v)).$$

Proposition 3.1 *Let G and H be two Lie groups and $\varphi : G \to H$ be a homomorphism of Lie groups. If X_t is a G-valued semimartingale then*

$$\log \varphi(X_t) = \varphi_* \log X_t.$$

Proof: Let ω_G and ω_H be the Maurer-Cartan form on G and H, respectively. From (4) we see that $\log \varphi(X_t) = \int_0^t \varphi^* \omega_H \delta X_s$. Applying Lemma 3.1 we obtain $\log \varphi(X_t) = \int_0^t \varphi_* \omega_G \delta X_s$. Thus, $\log \varphi(X_t) = \varphi_* \log X_t$. □

In [7], K. Nomizu proved the existence of correspondence between left invariant connections ∇^G on G and bilinear applications $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, which is given by $\nabla^G_X Y = \alpha(X, Y)$ for all $X, Y \in \mathfrak{g}$. The bilinear application α is called the connection function associated to ∇^G.

Proposition 3.2 *For every bilinear application $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ there exists only one connection $\nabla^{\mathfrak{g}}$ associated to α.*

Proof: Let X, Y be a vector fields in $\mathfrak{g}$. We define

$$\nabla^{\mathfrak{g}}_X Y = \alpha(X, Y),$$

and

$$\nabla^{\mathfrak{g}}_{fX} Y = f\alpha(X, Y) \text{ and } \nabla^{\mathfrak{g}}_X fY = X(f)Y + f\alpha(X, Y),$$

for f smooth function on $\mathfrak{g}$. It is clear that $\nabla^{\mathfrak{g}}$ is a connection. Conversely, let $\nabla^{\mathfrak{g}}$ be a connection on $\mathfrak{g}$. Then it is sufficient to define $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ as

$$\alpha(X, Y) = \nabla^{\mathfrak{g}}_X Y.$$

It is obvious that α is bilinear. □

From now on we only work with skew-symmetric bilinear application $\alpha : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, and we call the associated connections ∇^G and $\nabla^{\mathfrak{g}}$ to α the skew-symmetric connections.

Lemma 3.2 *Let ∇^G be a left invariant connection on G and $\nabla^{\mathfrak{g}}$ a connection on $\mathfrak{g}$ such that its connection function α is skew-symmetric.*

1. *If θ is a left-invariant 1-form on G, then the symmetric part of $\nabla^G\theta$ is null.*
2. *If θ is a 1-form in $\mathfrak{g}^*$, then the symmetric part of $\nabla^{\mathfrak{g}}\theta$ is null.*

Proof: *1.* We first observe that $\nabla\theta$ is a tensor, so it is sufficiente to proof for $X, Y \in \mathfrak{g}$. Let us denote $S\nabla\theta$ the symmetric part of $\nabla\theta$. By definition of dual connection,

$$\begin{aligned} S\nabla\theta(X, Y)(g) &= \frac{1}{2}(X\theta(Y) + Y\theta(X) - \theta(\nabla_X Y + \nabla_Y X)(g)) \\ &= -\frac{1}{2}\theta(\alpha(X, Y) + \alpha(Y, X))(g) \end{aligned}$$

Since α is skew-symmetric, $S\nabla\theta(X, Y) = 0$
2. The proof is similar to item 1. □

We now prove a characterization of martingales with values in G through association with martingales with values in $\mathfrak{g}$.

Theorem 3.3 *Let G be a Lie group with a left invariant connection ∇^G and $\nabla^{\mathfrak{g}}$ a connection on Lie algebra $\mathfrak{g}$ such that its connection function α is skew-symmetric. Let M_t be a G-valued semimartingale. Then M_t is a ∇^G-martingale if and only if $\log M_t$ is a $\nabla^{\mathfrak{g}}$-martingale.*

Proof: We first suppose that M_t is a ∇^G-martingale. By definition of stochastic logarithm,

$$\log M_t = \int_0^t \omega_G \delta M_s.$$

Applying the formula of conversion (2) we obtain

$$\log M_t = \int_0^t \omega_G d^{\nabla^G} M_s + \frac{1}{2}\int_0^t \nabla^G \omega_G (dM, dM)_s.$$

Lemma 3.2 now assures that $\nabla^G \omega_G (dM, dM)_t = 0$, because the Maurer-Cartan is a left-invariant form. Thus

$$\log M_t = \int_0^t \omega_G d^{\nabla^G} M_s.$$

We observe that $\log M_t$ is a local martingale. For $\theta \in \mathfrak{g}^*$ we have that

$$\int_0^t \theta \delta \log M_t = \int_0^t \theta \omega_G d^{\nabla^G} M_s.$$

From formula of conversion (2) we see that

$$\int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log M_t + \frac{1}{2}\int \nabla^{\mathfrak{g}} \theta (d \log M_s, d \log M_s) = \int_0^t \theta \omega_G d^{\nabla^G} M_s.$$

Lemma 3.2 leads to $\int \nabla^{\mathfrak{g}} \theta (d \log M_s, d \log M_s) = 0$. Thus

$$\int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log M_s = \int_0^t \theta \omega_G d^{\nabla^G} M_s.$$

Since $\int_0^t \theta \omega_G d^{\nabla^G} M_s$ is a real local martingale, we conclude that $\log M_t$ is a $\nabla^{\mathfrak{g}}$-martingale.

Conversely, let θ be a left invariant 1-form in G. Using the formula of conversion (2) and Lemma 3.2 leads to

$$\int_0^t \theta d^{\nabla^G} M_s = \int_0^t \theta \delta M_s.$$

Writing $\theta_g = \theta_e \circ \omega_G$ we obtain

$$\int_0^t \theta d^{\nabla^G} M_s = \int_0^t \theta_e \omega_G (\delta M_s).$$

By definition of logarithm,

$$\int_0^t \theta d^{\nabla^G} M_s = \int_0^t \theta_e \delta \log M_s.$$

Applying the formula of conversion (2) we see that

$$\int_0^t \theta d^{\nabla^G} M_s = \int_0^t \theta_e d^{\nabla^{\mathfrak{g}}} \log M_s + \frac{1}{2}\int_0^t \nabla^{\mathfrak{g}}\theta_e(d\log M_s, d\log M_s),$$

being $\nabla^{\mathfrak{g}}$ the connection on $\mathfrak{g}$ yielded by connection function α. From Lemma 3.2 it follows that $\nabla^{\mathfrak{g}}\theta_e = 0$. Thus

$$\int_0^t \theta d^{\nabla^G} M_s = \int_0^t \theta_e d^{\nabla^{\mathfrak{g}}} \log M_s.$$

Since $\log M_t$ is a $\nabla^{\mathfrak{g}}$-martingale, we conclude that M_t is a ∇^G-martingale. $\square$

The next corollary is a direct consequence of theorem above, but it is not possible to show its converse with the tools that we are using here.

Corollary 3.4 *Let G be a Lie group with a left invariant connection ∇^G, which has a skew-symmetric connection function α. If M_t is a ∇^G-martingale, then $\log M_t$ is a local martingale in $\mathfrak{g}$.*

Example 3.1 *Let α be the connection function null. Then, from Theorem 3.3 we conclude that M_t is a ∇^G-martingale if and only if $\log M_t$ is local martingale in $\mathfrak{g}$. It was first proved by M. Arnaudon in [2].*

We know that there exists an one-to-one association between bi-invariant metrics on Lie group G and Ad_G-invariant scalar products $<,>$ on Lie algebra $\mathfrak{g}$. We will use this to give the following characterization for Brownian motion in G.

Theorem 3.5 *Let G be a Lie group whit a bi-invariant metric k. Let B_t be a semimartingale in G. Then B_t is a Brownian motion in G if and only in $\log B_t$ is a $<,>$-Brownian motion in $\mathfrak{g}$.*

Proof: We first observe that the Levi-Civita connection associated to metric k is given by

$$\nabla^k_X Y = \frac{1}{2}[X,Y]$$

for all $X, Y \in \mathfrak{g}$ (see for example [1]).

Suppose that B_t is a k-Brownian motion. From definition and Theorem 3.3 we know that $\log B_t$ is a $\nabla^{\mathfrak{g}}$-martingale in $\mathfrak{g}$, where $\nabla^{\mathfrak{g}}$ is connection generate by $\frac{1}{2}[\cdot,\cdot]$. It remains to prove that $\int_0^t b(d\log B, d\log B)_s = \int_0^t \mathrm{tr}\,(\log B_s)ds$, where b is a bilinear form in $\mathfrak{g}$. In fact, let $(x_1,\ldots,x_n)$ be a global coordinates system of $\mathfrak{g}$. Thus, we can write $b = b_{ij} dx^i \otimes dx^j$, where b_{ij} are smooth functions on $\mathfrak{g}$. By definition,

$$\begin{aligned}
\int_0^t b(d\log B, d\log B)_s &= \int_0^t b_{ij}(\log B_s)[\log B_s^i, \log B_s^j] \\
&= \int_0^t b_{ij}(\log B_s) d\int_0^s [\log B_r^i, \log B_r^j] \\
&= \int_0^t b_{ij}(\log B_s) d\int_0^t dx^i \otimes dx^j (d\log B_r, d\log B_r) \\
&= \int_0^t b_{ij}(\log B_s) d\int_0^t dx^i \otimes dx^j (L_{B_r^{-1}*} dB_r, L_{B_r^{-1}*} dB_r) \\
&= \int_0^t b_{ij}(\log B_s) d\int_0^t dx^i \circ L_{B_r^{-1}*} \otimes dx^j \circ L_{B_r^{-1}*} (dB_r, dB_r),
\end{aligned}$$

where we used the Theorem 3.8 of [6] in the second and third equality. Being B_t a Brownian motion,

$$\int_0^t b(d\log B, d\log B)_s = \int_0^t b_{ij}(\log B_s) d\int_0^s \operatorname{tr}(dx^i \circ L_{B_r^{-1}*} \otimes dx^j \circ L_{B_r^{-1}*})(B_r) dr.$$

As k is a bi-invariant metric we have

$$\begin{aligned}
\int_0^t b(d\log B, d\log B)_s &= \int_0^t b_{ij}(\log B_s) d\int_0^s k^{ij}(\log B_r) dr \\
&= \int_0^t b_{ij}(\log B_s) k^{ij}(\log B_s) ds = \int_0^t \operatorname{tr} b(\log B_s) ds,
\end{aligned}$$

where k^{ij} are the coeficients of inverse matrix $(<,>_{ij})$. Therefore $\log B_t$ is a $<,>$-Brownian motion in $\mathfrak{g}$.

Conversely, suppose that $\log B_t$ is a $<,>$-Brownian motion in $\mathfrak{g}$. It remains to prove (1). For each $b \in T^{(0,2)}(G)$,

$$\int_0^t b(dB, dB)_s = \int_0^t b(L_{B_s*} L_{B_s^{-1}*} dB_s, L_{B_s*} L_{B_s^{-1}*} dB_s).$$

By definition of logarithm,

$$\int_0^t b(dB, dB)_s = \int_0^t L_{B_s}^* b(d\log B, d\log B)_s.$$

As $\log B_t$ is a $<,>$-Brownian motion we have

$$\int_0^t b(dB, dB)_s = \int_0^t tr(L_{B_s}^* b)(\log B_s) ds.$$

Being k bi-invariant metric, we have

$$\int_0^t b(dB, dB)_s = \int_0^t \operatorname{tr} b(B_s) ds.$$

Thus B_t is a k-Brownian motion in G. □

As consequence of Theorem above, every k-Brownian motion in G yields a standart Brownian motion in $\mathfrak{g}$, but, as Corollary 3.4, we can not show the converse with these arguments.

Corollary 3.6 *Let G be a Lie group whit a bi-invariant metric k. If B_t is a k-Brownian motion in G, then $\log B_t$ is a Brownian motion in $\mathfrak{g}$.*

Proof: It follows from Corollary 3.4 that if B_t is a ∇^k-martingale, where ∇^k is the Levi-Civita connection associated to metric k, then $\log B_t$ is a local martingale in $\mathfrak{g}$. By Levi's characterization of n-dimensional Brownian motion remains to prove that $[\log B_t^i, \log B_t^j]_t = \delta_j^i t$ (see [10] for more details). In fact, we make

$$[\log B_t^i, \log B_t^j] = \int_0^t d[\log B_s^i, \log B_s^j]$$

and we apply the first part of the demonstration of Theorem 3.5 to conclude the proof. □

As an application of Theorems 3.3 and 3.5 we prove the useful results. Someone will be able to show the next Proposition whit geometric arguments.

Proposition 3.7 *Let G be a Lie group and H a Lie group with a left invariant connection ∇^H, which has a skew-simmetric connection function α and $\varphi : H \to G$ an homorphism of Lie groups. We have the following assertions:*

- **(i)** *If G has a left invariant connection ∇^G such that its connection function is skew-symmetric and if φ_{e*} commutes with α, then every homomorphism $\varphi : H \to G$ is an affine map.*
- **(ii)** *If G has a bi-invariant metric k and if φ_{e*} commutes with α, then every homomorphism $\varphi : H \to G$ is a harmonic map.*

Proof: **(i)** Let M_t be a ∇^H-martingale in H. It is sufficient to show that $\varphi(M_t)$ is a ∇^G- martingale. In fact, Theorem (3.5) shows that $\log M_t$ is a $\nabla^{\mathfrak{h}}$-martingale in the Lie algebra $\mathfrak{h}$. By Proposition 3.1,

$$\log \varphi(M_t) = \varphi_{e*}(\log M_t).$$

Since φ_{e*} commute with α, from Itô geometric formula we deduce that $\log \varphi(M_t)$ is a $\nabla^{\mathfrak{g}}$-martingale in the Lie algbra $\mathfrak{g}$. Theorem 3.3 shows that $\varphi(M_t)$ is a ∇^G-martingale.

(ii) Let B_t be a k-Brownian motion in G. From stochastic characterization for harmonic maps is suficient to show that $\varphi(B_t)$ is a ∇^G- martingale. In fact, Theorem (3.3) shows that $\log B_t$ is a $\nabla^{\mathfrak{h}}$-martingale. By Proposition 3.1,

$$\log \varphi(B_t) = \varphi_{e*}(\log B_t).$$

Because φ_{e*} commutes with α, the Itô formula assures that $\log \varphi(B_t)$ is a $\nabla^{\mathfrak{g}}$-martingale. Theorem 3.3 shows that $\varphi(B_t)$ is a ∇^G-martingale.

Example 3.2 *Let G, H be two Lie groups. If we equippe G whit a connection $\nabla^G_X Y = c_1[X, Y]$ for some $c_1 \in [0,1]$ and, $X, Y \in \mathfrak{g}$, and if we endow H with a connection $\nabla^H_{\tilde{X}} \tilde{Y} = c_2[\tilde{X}, \tilde{Y}]$ for some $c_2 \in [0,1]$, $\tilde{X}, \tilde{Y} \in \mathfrak{h}$, then every homomorphism of Lie groups $\varphi : H \to G$ is an affine map. When H has a bi-invariant metric, every homomorphism of Lie groups $\varphi : H \to G$ is a harmonic map.*

The next Lemma is necessary in the proof the Pluzhnikov's Theorem. We observe that it is true, because we work in the Lie algebra context.

Lemma 3.3 *Let (M, g) be a Riemannian manifold, G a Lie group, ω_G the Maurrer-Cartan form on G and $F : M \to G$ a smooth map. Then*

$$d^* F^* \omega_G^* \theta = \theta d^* F^* \omega_G,$$

for every θ 1-form on $\mathfrak{g}$, where d^* is the co-differential operator on M.

Proof: From definition of co-differential d^*, for any orthonormal frame field $\{e_1, \ldots, e_n\}$ on M, we have

$$d^* F^* \omega_G^* \theta = -\sum_{i=1}^{n} (\nabla^g_{e_i} F^* \omega_G^* \theta)(e_i),$$

where ∇^g is the Levi-Civita connection associated to metric g. By definition of dual connection,

$$\begin{aligned} d^* F^* \omega_G^* \theta &= -\sum_{i=1}^{n} (\nabla^g_{e_i} (F^* \omega_G^* \theta(e_i)) - F^* \omega_G^* \theta(\nabla^g_{e_i} e_i)) \\ &= -\sum_{i=1}^{n} (e_i \theta(F^* \omega_G(e_i)) - \theta(F^* \omega_G \nabla^g_{e_i} e_i)) \end{aligned}$$

Since $\theta : \mathfrak{g} \to \mathbb{R}$ is a linear application, we obtain

$$\begin{aligned} d^* F^* \omega_G^* \theta &= \theta(-\sum_{i=1}^{n} (\nabla^g_{e_i}(F^* \omega_G(e_i)) - F^* \omega_G^* \nabla^g_{e_i} e_i)) \\ &= \theta(d^* F^* \omega_G), \end{aligned}$$

where we used the definition of co-differential in the last equality. □

Finally, we will prove a version of Pluzhnikov's Theorem (see [9]) to skew-symmetric connections.

Theorem 3.8 *Let M be a Riemannian manifold, G a Lie group with a left invariant connection ∇^G such that its connection function α is skew-symmetric, ω_G the Maurer-Cartan form on G and $F : M \to G$ a smooth map. Then F is harmonic if and only if*

$$d^* F^* \omega_G = 0,$$

where d^* is the co-differential operator on M.

Proof: Suppose that F is a harmonic map. From stochastic characterization for harmonic maps we have, for every g-Brownian motion B_t in M, that $F(B_t)$ is a ∇^G-martingale in G. From Theorem 3.3 we see that $\log F(B_t)$ is a $\nabla^{\mathfrak{g}}$-martingale, where $\nabla^{\mathfrak{g}}$ is the connection given by α in $\mathfrak{g}$. Let θ be a 1-form on $\mathfrak{g}$. From formula of conversion (2) we deduce that

$$\begin{aligned} \int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log F(B_s) &= \int_0^t \theta \delta \log F(B_s) - \frac{1}{2} \int_0^t \nabla^{\mathfrak{g}} \theta(d \log F(B_s), d \log F(B_s)) \\ &= \int_0^t \omega_G^* F^* \theta \delta B_s - \frac{1}{2} \int_0^t \nabla^{\mathfrak{g}} \theta(d \log F(B_s), d \log F(B_s)), \end{aligned}$$

where we used the definition of stochastic logarithm and property (4) in the second equality. Because $\nabla^{\mathfrak{g}}$ is given by α and α is skew-symmetric Lemma 3.2 assures that

$$\int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log F(B_s) = \int_0^t \omega_G^* F^* \theta \delta B_s.$$

Manabe's formula (3) now yields

$$\int_0^t \theta d^{\nabla \mathfrak{g}} \log F(B_s) = \int_0^t F^* \omega_G^* \theta d^{\nabla^g} B_s + \frac{1}{2} \int_0^t d^* \omega_G^* F^* \theta(B_s) ds.$$

Since $\log F(B_t)$ is a $\nabla^{\mathfrak{g}}$-martingale, from Doob-Meyer decomposition (see for instance [10]) we deduce that

$$\int_0^t d^* F^* \omega_G^* \theta(B_s) dt = 0.$$

Since B_s is an arbitrary g-Brownian motion, it follows that $d^* F^* \omega_G^* \theta = 0$, where d^* is the co-differential operator on M. From Lemma 3.3 we see that $\theta(d^* F^* \omega_G) = 0$. Being θ an arbitrary 1-form on $\mathfrak{g}$, we conclude that

$$d^* F^* \omega_G = 0.$$

Conversely, suppose that $d^* F^* \omega_G^* = 0$. We want to show, for every g-Brownian motion B_s in M, that $F(B_s)$ is a ∇^G-martingale in G. To this end, we will show that $\log F(B_s)$ is a $\nabla^{\mathfrak{g}}$-martingale in $\mathfrak{g}$ and we will conclude from Theorem 3.3 our assertion. In fact, for $\theta \in \mathfrak{g}^*$ we can repeat to arguments above and to obtain

$$\int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log F(B_s) \quad = \quad \int_0^t F^* \omega^* \theta d^{\nabla^g} B_s + \frac{1}{2} \int_0^t d^* \omega^* F^* \theta(B_s) ds.$$

From Lemma 3.3 and the hypothesis we get

$$\int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log F(B_s) = \int_0^t F^* \omega^* \theta d^{\nabla^g} B_s.$$

Because B_t is a g-Brownian motion, by definition, $\int_0^t \theta d^{\nabla^{\mathfrak{g}}} \log F(B_s)$ is a local martingale. Furthermore, $\log F(B_s)$ is a $\nabla^{\mathfrak{g}}$-martingale in $\mathfrak{g}$, and the proof follows. □

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