A characterization of Einstein manifolds

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Abstract

Let (M, g) be any Riemannian manifold. Our goal is to show that if g and Ricci tensor r_g are no locally constant, if, locally, their product is non-negative (respectively, non-positive), and if its scalar curvature s_g is non-negative (respectively, non-positive), then (M, g) is an Einstein manifolds. This result is a generalization of the characterization for compacts Einstein manifolds given by Hilbert [3].

Key words: Einstein manifolds; stochastic analisys on manifolds.

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1 Introduction

Let (M, g) be a Riemannian manifold. Let us denote the Ricci tensor field by r_g and the scalar curvature by s_g. M is called an Einstein manifold if, for every vector fields X, Y on M, there exists a constant such that

$$r_g(X, Y) = \lambda g(X, Y).$$

We call the metric g of Einstein metric.

When M is a smooth compact manifold, Einstein manifolds can be characterized characterized variationally, via the Hilbert action on the space of all unit volume metrics. More exactly, g is an Einstein metric if and only if it is a cirtical point of normalized Hilbert fuctional. The best general reference here is the [1].

Let $x \in M$ and $(U, x_1, \ldots, x_n)$ be a local coordinate system around x. If g is an Einstein metric whit non-negative constant λ, a natural consequence in U is

$$r_{g,ij} g_{ij} \geq 0, \tag{1}$$

where $r_{g,ij}$ and g_{ij}, for $i, j = 1, \ldots n$, are the coordinates of r_g and g in U. In the case that g is an Einstein metric whit non-positive constant λ, we have in U that

$$r_{g,ij} g_{ij} \leq 0. \tag{2}$$

We will use these natural assumptions to prove our main Theorem. Furthermore, we will need the following assumption under the metrics.

(I) Throughout the paper, any metric g on a smooth manifold M is not locally constant.

Using stochastic analisys on manifolds, we show the following result for any Riemannian manifold.

Theorem : Let M be a smooth manifold and g a metric on M which satisfy (I).

1. If the Ricci tensor field r_g is not locally constant, if it, locally, satisfy (1) and if the curvature scalar s_g is a non-negative constant, then (M, g) is an Einstein manifold.

2. If the Ricci tensor field r_g is not locally constant, if it, locally, satisfy (2) and if the curvature scalar s_g is a non-positive constant, then (M, g) is an Einstein manifold.

2 Stochastic tools

In the following we always consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. We begin for introduce the three most importants process for stochastic analisys in manifolds. See for instance [2] for a complete study about these process. From now on the term smooth means of class C^∞.

Definition 2.1 *Let M be a smooth manifold. A continuos M-valued process X_t is called semimartingale if, for each smooth f on M, the real-valued process $f \circ X_t$ is a semimartingale.*

Let X_t be a semimartingale on M and b be a bilinear form on M. Let $(U, x_1, \ldots, x_n)$ be a local coordinate system on M. In this coordinate b is written as $b_{ij} dx^i \otimes dx^j$, where b_{ij} are smooth function on U. The integral of b along X_t is defined, locally, by

$$\int b(dX, dX) = \int b_{ij} \circ X_t d[X^i, X^j]_t, \tag{3}$$

where $X_t^i = x^1 \circ X_t$.

Using this definition has sense the following definition of martingales in smooth manifolds.

Definition 2.2 *Let M be a smooth manifold with a connection ∇. A semimartingale X_t in M is called a martingale if, for every smooth f,*

$$f \circ X_t - f \circ X_0 - \int \mathrm{Hess} f(dX, dX)$$

is a real local martingale.

In the sequel, we define Brownian motion in a smooth manifold. We observe that Theorem 2.1, 2.2 and 2.3 that follows from definition are general. So they do not depende of our assumption about the metric.

Definition 2.3 *Let (M, g) be a Riemannian manifold. Given $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, a M-valued process B_t is called a g-Brownian motion in (M, g) if B_t is continuos and adapted and, for every smooth f,*

$$f \circ B_t - f \circ B_0 - \frac{1}{2}\int \Delta_g f \circ B_t dt$$

is a local martingale.

Given a point x in (M, g), there always exists a g-Brownian motion B_t is M, starting at x, defined on $[0, \zeta[$ for some complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ and some stopping time $\zeta > 0$.

There exists the following Lévy characterization for g-Bronwnian motion in manifolds (see Theorem 5.18 in [2]).

Theorem 2.1 *A M-valued semimartingale B_t is a Brownian motion if and only if it is a martingale and, for every smooth functions f and h,*

$$[f \circ B_t, h \circ B_t] = \int g(\operatorname{grad} f(B_t), \operatorname{grad} h(B_t))dt.$$

In the sequel, we characterize stochastically the constant scalar curvature.

Theorem 2.2 *Let (M, g) be a Riemannian manifold. The scalar curvature s_g is constant if and only if, for every g-Brownian motion B_t, $s_g(B_t) - s_g(B_0)$ is local martingale.*

Proof: Suppose that the scalar curature s_g is constant. Then $\Delta_g s_g = 0$. From Definition 2.3 we conclude that

$$s_g(B_t) - s_g(B_0)$$

is a real local martingale, where B_t is any g-Brownian motion.

Conversely, suppose that, for every g-Brownian motion B_t, $s_g(B_t) - s_g(B_0)$ is a real local martingale. Thus, applying the expectation $\mathbb{E}$ we see that

$$\mathbb{E}[s_g(B_t) - s_g(B_0)] = \mathbb{E}[s_g(B_0) - s_g(B_0)] = 0.$$

Therefore $s_g(B_t) = s_g(B_0)$, $\mathbb{P}$-almost sure. Since B_t is an arbitrary g-Brownian motion, it follows that s_g is constant. □

The following Lemma, which demonstration is found in [2, Lemma 5.20], is fundamental below.

Lemma 2.3 *If B_t is a g-Brownian motion, then , for every bilinear form b,*

$$\int b(dB, dB) = \int \operatorname{tr} b(B_t)dt$$

Let us denote $\mathcal{B}$ the subspace of symmetric bilinear forms on a smooth manifold M. Let g be a metric on M which satisfy (I). For any local coordinate system $(U, x_1, \ldots, x_n)$ we define

$$\mathcal{B}^{g+} := \{b \in \mathcal{B} : b \text{ is no constant on } U \text{ and } b_{ij}(y)g_{ij}(y) \geq 0, \forall y \in U\}$$

and

$$\mathcal{B}^{g-} := \{b \in \mathcal{B} : b \text{ is no constant on } U \text{ and } b_{ij}(y)g_{ij}(y) \leq 0, \forall y \in U\}.$$

It is clear that $\mathcal{B}^{g+} \neq \emptyset$ and $\mathcal{B}^{g-} \neq \emptyset$. We observe that $\mathcal{B}^{g+}$ and $\mathcal{B}^{g-}$ are defined over all local coordinate system.

Proposition 2.4 *Let (M, g) be a Riemannian manifold such that g satisfy (I).*

1. *Let $b, b' \in \mathcal{B}^{g+}$. If $\int b(dB, dB) = \int b'(dB, dB)$, for every g-Brownian motion B_t, then $b = b'$.*
2. *Let $b, b' \in \mathcal{B}^{g-}$. If $\int b(dB, dB) = \int b'(dB, dB)$, for every g-Brownian motion B_t, then $b = b'$.*

Proof: **1.** Let $b, b' \in \mathcal{B}^{g}$ and $(U, x_1, \ldots, x_n)$ be a local coordinate system on M. In this coordinate, the components b_{ij} and b'_{ij} of b and b', respectively, satisfy

$$b_{ij}g_{ij} \geq 0 \quad b'_{ij}g_{ij} \geq 0.$$

Since b_{ij} and b'_{ij} are smooth, in some coordinate system $(U', x_1, \ldots, x_n)$ we have that $b_{ij} - b'_{ij} \geq 0$ or $b'_{ij} - b_{ij} \geq 0$. So

$$(b_{ij} - b'_{ij})g_{ij} \geq 0 \;\; or \;\; (b'_{ij} - b_{ij})g_{ij} \geq 0 \tag{4}$$

Suppose that $b_{ij} - b'_{ij} \geq 0$ in $(U', x_1, \ldots, x_n)$. Thus, from (3), for any g-Brownian motion B_t with intial value B_0 in U', we see that

$$\sum_{ij=1}^{n} \int (b_{ij} - b'_{ij})(B_t)\, d[B^i, B^j] = 0.$$

From Theorem 2.1 we conclude that

$$0 = \sum_{ij=1}^{n} \int (b_{ij} - b'_{ij})(B_t)\, g(\operatorname{grad} x^i, \operatorname{grad} x^j) dt = \sum_{ij=1}^{n} \int (b_{ij} - b'_{ij})(B_t) g_{ij}(B_t) dt. \tag{5}$$

In the second equality we use the musical isomorphism (see page 30 in [1]) to show that $g(\operatorname{grad} x^i, \operatorname{grad} x^j) = g_{ij}$ in U'.

As, in U', each term of (5) is non-negative we have $\int (b_{ij} - b'_{ij})(B_t) g_{ij}(B) dt = 0$, $i, j = 1, \ldots, n$. Since dt is the Lebegue measure and $(b_{ij} - b'_{ij})(B_t) g_{ij}(B_t)$ is continuos for each $\omega \in \Omega$, $b_{ij}(B_t) = b'_{ij}(B_t)$. Because B_t is any g-Brownian motion with intial value B_0 in U' we conclude that $b_{ij} = b'_{ij}$ in U'. As at each $x \in M$ we can found some neighborhood with condition (4) we conclude that $b = b'$.

2. The proof is similar to proof of item 1. □

Corollary 2.5 *Let (M, g) be a Riemannian manifold such that g satisfy (I). Let $b \in \mathcal{B}^g$. If $\int b(dB, dB) = 0$, for every g-Brownian motion B_t, then $b = 0$.*

The next example shows the necessity of assumptions (I) under the metrics and the non-constancy of bilinear symmetric forms.

Example 2.1 *Let $\mathbb{R}^3$ and (x_1, x_2, x_3) a global coordinate system to $\mathbb{R}^3$. Consider the canonical metric g on $\mathbb{R}^3$. Then, it follows that $g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$. Let b, b' be two diferent symmetric bilinear forms given by*

$$[b_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix} \quad [b'_{ij}] = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

It is clear that $b_{ij} g_{ij} \geq 0$ and $b'_{ij} g_{ij} \geq 0$ for $i, j = 1, 2, 3$. Thus $b, b' \in \mathcal{B}^{g+}$. But, for every Brownian motion B_t in $\mathbb{R}^3$, we have that

$$\int b(dB, dB) = \int b'(dB, dB).$$

In the same way, we can take $\beta = -b$ and $\beta' = -b'$ and to show that $\beta, \beta' \in \mathcal{B}^{g-}$. Again, for every Brownian motion B_t in $\mathbb{R}^3$, we have that

$$\int \beta(dB, dB) = \int \beta'(dB, dB).$$

3 Einstein manifolds

We begin recalling the definition of Ricci tensor field for a metric and Einstein Manifold.

Definition 3.1 *The Ricci curvature tensor r of a Riemannian manifold (M, g) is the 2-tensor*

$$r_g(X, Y) = \operatorname{tr}(Z \to R(X, Z)Y),$$

where tr *denotes the trace of the linear map $Z \to R(X, Z)Y$.*

Definition 3.2 *A Riemannian manifold (M, g) is Einstein if there exists a real constant λ such that*

$$r_g(X, Y) = \lambda g(X, Y).$$

We call the metric g of Einstein metric.

Now we prove the main Theorem.

Theorem 3.1 *Let M be a smooth manifold and g a metric on M which satisfy (I).*

1. *If the Ricci tensor field r_g is in $\mathcal{B}^{g+}$ and the curvature scalar s_g is a non-negative constant, then (M, g) is an Einstein manifold.*

2. *If the Ricci tensor field r_g is in $\mathcal{B}^{g-}$ and the curvature scalar s_g is a non-positive constant, then (M, g) is an Einstein manifold.*

Proof: **1.** Let (M, g) be a Riemannian manifold. Suppose that the scalar curvature is a non-negative constant, that is, there exists a constant $\lambda \geq 0$ such that $s_g(x) = \lambda$ for all $x \in M$. If n is dimension of M, then

$$s_g(x) = \frac{\lambda}{n} n.$$

As $n = \operatorname{tr} g$ we have

$$s_g(x) = \frac{\lambda}{n} \operatorname{tr} g(x).$$

Applying this equality about an arbitrary g-Brownian motion B_t in M we obtain

$$s_g(B_t) = \frac{\lambda}{n} \operatorname{tr} g(B_t).$$

We now integrate in t each trajectory of B_t, that is,

$$\int s_g(B_t) dt = \int \frac{\lambda}{n} \operatorname{tr} g(B_t) dt.$$

From Lemma 2.3 we conclude that

$$\int r(dB, dB) = \int \frac{\lambda}{n} g(dB, dB). \tag{6}$$

Since $\lambda \geq 0$, we obtain $\frac{\lambda}{n} g \in \mathcal{B}^{g+}$. Because B_t is any g-Brownian motion, from Proposition 2.4, item 1., we conclude that

$$r = \frac{\lambda}{n} g. \tag{7}$$

Thus (M, g) is an Einstein manifold, which complete the proof.

2. The proof is similar to proof of item 1. The differences are that to conclude (7) from (6) we use Proposition 2.4, item 2., and the fact that if $\lambda \leq 0$, then $\frac{\lambda}{n} g \in \mathcal{B}^{g-}$.

□

Similarly, we can use Corollary 2.5 and prove the following.

Corollary 3.2 *Let M be a smooth manifold and g a metric on M which satisfy (I). If the Ricci tensor field r_g is in $\mathcal{B}^g$ and the curvature scalar $s_g \equiv 0$, then (M, g) is a Ricci-flat manifold.*

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