Classification of reversible-equivariant vector fields in 4D with applications to normal forms and first integrals.

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Abstract

This paper uses tools in group theory and symbolic computing to give a classification of the representations of finite groups with order lower than 9 that can be derived from the study of local reversible-equivariant vector fields in $\mathbb{R}^4$. Based on such approach we exhibit, for each element in this class of dynamical systems, a simplified Belitiskii normal form and establish conditions for the existence of first integrals.

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Contents

1 Introduction and statement of the problem

We begin by introducing the matrices

$$A(\alpha,\beta) = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{pmatrix}, \ \alpha,\beta \in \mathbb{R}, \ \alpha\beta \neq 0. \tag{1}$$

Let $\mathfrak{X}_0(\mathbb{R}^4)$ denote the set of all germs of C^∞ vector fields in $\mathbb{R}^4$ singular at origin and define

$$\mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4) = \{X \in \mathfrak{X}_0(\mathbb{R}^4);\ DX(0) = A(\alpha,\beta)\}.$$

Let $X \in \mathfrak{X}_0(\mathbb{R}^4)$ be a *reversible vector field*, that is, there exists a C^∞ involutive diffeomorphism $\varphi : \mathbb{R}^4, 0 \to \mathbb{R}^4, 0$ such that

$$D\varphi(x)X(x) = -X(\varphi(x))$$

and $\dim Fix(\varphi) = 2$, as a local submanifold of $\mathbb{R}^4$. By Montgomery-Bochner Theorem (MB), we can take the involution φ linear:

Theorem 1 (Montgomery-Bochner, [Bo],[BMo],[Ca]). *Let G be a compact group of C^k diffeomorphisms defined on a $C^{k\geq 1}$ manifold $\mathcal{M}$. Suppose that all diffeomorphisms in G have a common fixed point, say x_0. Then, there exists a C^k coordinate system h around x_0 such that all diffeomorphisms in G are linear with respect to h.*

If γ is an orbit for the reversible vector field X, then it is not hard to show that $\varphi(\gamma(-t))$ is also an orbit. So, the phase portrait of a reversible vector field is symmetric with respect to the symmetry axis $Fix(\varphi)$, like in the Figure 1.

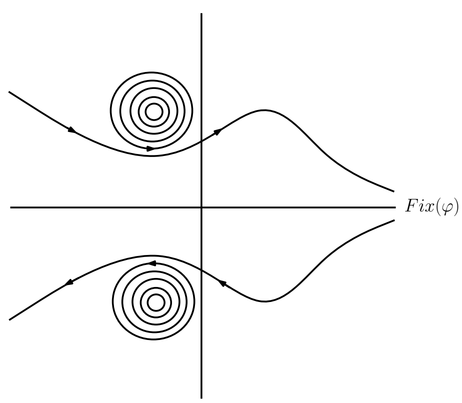

Figure 1: Schematic representation of a reversible vector field.

Reversible vector fields were first considered by Birkhoff [Bi], in the beginning of last century, when he was studying the restricted three body problem. Some decades ago, the theory has been formalized by Devaney [De]. A list of classical results in the Hamiltonian world has been adapted to the reversible universe, for example, the main theorems in KAM Theory ([P], [S]), the Lyapunov Center Theorem ([CL1]) and a lot of methods dealing with heteroclinic and homoclinic orbits. We refer to [LR] for a survey in reversible systems and related topics.

Let X be a vector field in $\mathbb{R}^{2n}$ and $\varphi, \psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be involutive diffeomorphisms. We say that X is (φ, ψ)-*reversible-equivariant* (or just (φ, ψ)-*reversible*) if X is φ-reversible and ψ-reversible. The phase portrait of a reversible-equivariant vector field has two "axis" of symmetry, like in the Figure 2. If G denotes the group $G = \langle \varphi, \psi \rangle$, we say that the vector field X is G-reversible-equivariant (or G-reversible). In this paper we restrict ourselves to the case $n = 2$ ($\mathbb{R}^4$).

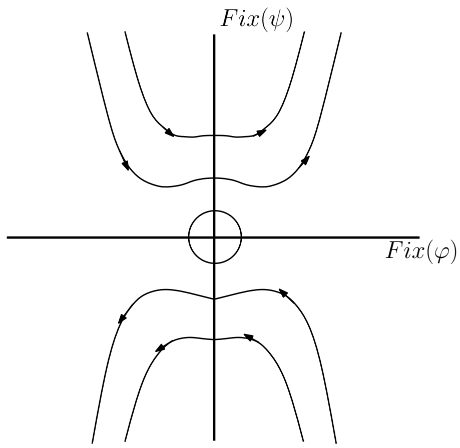

Figure 2: Example of a reversible-equivariant vector field.

Recall [T], where is proved that two involutions in $\mathbb{R}^2, 0$ are simultaneously C^0 linearizable, provided that its composition is a hyperbolic diffeomorphism at 0.

In this paper we classify all possible linearizations of the group G, in the sense of Montgomery-Bochner's Theorem, when G is a finite group and $|G| \leq 9$. We deduce from our classification that the reversible-equivariant structure can simplify some normal forms of vector fields $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$. Moreover, conditions for a reversible-equivariant polynomial vector field to admit a polynomial first integral are presented.

Problems related to stability of pairs of involutions and applications to reversible systems and discontinuous vector fields have been treated in [MMT] and references therein. Also in [O] the $\mathbb{D}_n$-reversibility for real homeomorphisms is studied. We remark that one of our main results (Theorem A) is a extension of some results in [BM]. More specifically, in [BM] just the representations of $\mathbb{D}_n$ for n even are exhibited whereas our result makes explicit the representations for any n.

All of the calculations have been performed using basic algebraic computing methods for solving nonlinear polynomial systems, with algorithms implemented in the software Maple 12 (SAGE interface).

2 Main Results

Let G be a compact group generated by two distinct involutions $\varphi, \psi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\dim Fix(\varphi) = \dim Fix(\psi) = 2$.

Lemma A: If $G = \langle \varphi, \psi \rangle$ is finite, then $(\varphi\psi)^n = Id$ for some $n \geq 2$. So $G \cong \mathbb{D}_n$, where $\mathbb{D}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We remark that among the groups $\mathbb{D}_n$, only $\mathbb{D}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian.

Definition 2. *Given a finitely generated group $G = \langle g_1, \ldots, g_l \rangle$ with a representation $\rho : G \to M_{4\times 4}(\mathbb{R})$ and a vector field $X \in \mathfrak{X}_0(\mathbb{R}^4)$, we say that the representation ρ is compatible with the definition of G-reversibility if $\rho(g_j)X(x) = -X(\rho(g_j))$, for all $j = 1, \ldots, l$.*

Theorem A: For $n = 2, 3, 4$ we present all of the 4-dimensional representations of $G = \mathbb{D}_n$ that are compatible with the definition of G-reversibility for $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$. Moreover, corresponding to each one of such representations, we exibith a non-trivial G-reversible vector field.

We remark that in [CL2] is proved that for every $n, k \in \mathbb{N}$ there exist at least one representation of $\mathbb{D}_n$ in dimension $2k$, with involutive generators S, R given by

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, R = \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) & 0 & 0 \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 & 0 \\ 0 & 0 & \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ 0 & 0 & \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix}.$$

Theorem B: For each group G in Theorem A, the Belitskii normal form for the G-reversible vector fields is exhibited.

As an application of the above theorems we present the following:

Theorem C: If X is a quadratic polynomial G-reversible vector field then conditions for its integrability (by quadratic first integrals) are presented.

We will prove Theorem B just in the case $G = \mathbb{D}_4$ and Theorem C in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{D}_2$. The proofs of remaining cases are similar.

3 Reversible normal forms

In this section we recall some basic concepts and techniques of normalization of vector fields, that will be needed in Section 6.

Let $X \in \mathfrak{X}_0(\mathbb{R}^n)$ and $A = DX(0)$. Assume that A is in Jordan normal form with eigenvalues $\lambda_1, \ldots, \lambda_n$. We say that the monomial

$$x_1^{k_1} \cdots x_n^{k_n} \frac{\partial}{\partial x_j}, \ k_i \in \mathbb{N}$$

is resonant of order $r = k_1 + \ldots + k_n$ if $r \geq 2$ and

$$\lambda_j = k_1\lambda_1 + \ldots + k_n\lambda_n.$$

Next we state 3 classical results in the theory of normal forms.

Theorem 3 (Poincaré-Dulac I, [CLW]). *Any vector field*

$$X(x) = Ax + f_2(x) + \ldots,$$

with f_k a vector-valued homogeneous polynomial of degree k, is formally conjugated to a vector field

$$X^N(x) = Ax + w_2(x) + \ldots + w_k(x) + \ldots,$$

with w_k consisting only of resonant monomials of order k. Moreover, the coefficients that appear in w_k only depend on the coefficients of f_k.

For $B \in \mathbb{R}^{n\times n}$, define

$$L_B(h) := Dh(x)Bx - Bh(x),$$

the homological operator. Then the Poincaré-Dulac Theorem can be stated as

Theorem 4 (Poincaré-Dulac II, [Br]). *Any vector field*

$$X(x) = Ax + f(x),$$

with $f(x) = o(|x|^2)$, is formally conjugated to a vector field

$$X^N(x) = Ax + h(x),$$

with $h(x) = o(|x|^2)$ verifying

$$L_S(h) \equiv 0,$$

where S is the semisimple part of A. The vector field X^N will be called the Poincaré-Dulac Normal Form (PDNF) of X.

Consider $A = S + N$ the unique decomposition of the matrix A in its semisimple and nilpotent parts. Recall that the Poincaré-Dulac Theorem does not deal with N. In the case where $N \neq 0$, we have the Belitskii Theorem:

Theorem 5 (Belitskii,[Be]). *Any vector field*

$$X(x) = Ax + f(x),$$

with $f(x) = o(|x|^2)$, is formally conjugate to a vector field

$$X^B(x) = Ax + h(x),$$

with $h(x) = o(|x|^2)$ verifying

$$L_{A^T}(h) \equiv 0,$$

where A^T is the transpose of A. The vector field X^B is called the Belitskii Normal Form (BNF) of X.

Note that $L_{A^T}(h) \equiv 0$ if and only if $L_S(h) \equiv 0$ and $L_{N^T}(h) \equiv 0$. When $N = 0$, the Belitskii and Poincaré-Dulac Theorems are exactly the same. Throughout this paper we always refer to the Belitskii normal form, even if $N = 0$.

4 Setting the problem

Let us fix $\alpha, \beta \in \mathbb{R}$ and $A = A(\alpha, \beta)$ given in (1). Let $\varphi, \psi : \mathbb{R}^4, 0 \to \mathbb{R}^4$ be involutions. Now consider X a vector field in $\mathbb{R}^4$ with $X(0) = 0$ and $DX(0) = A$. Suppose that X is (φ, ψ)-reversible, that is,

$$D\varphi(x)X(x) = -X(\varphi(x))$$

and

$$D\psi(x)X(x) = -X(\psi(x)).$$

Writing $X(x) = Ax + X_2(x) + \ldots$, where $X_k(x)$ is a vector field whose coordinates are homogeneous polynomials of degree k, we deduce that

$$D\varphi(x)Ax = -A(\varphi(x))$$

and

$$D\psi(x)Ax = -A(\psi(x)).$$

Suppose that the group $\langle \varphi, \psi \rangle$ is finite.

So, if a vector field is reversible-equivariant, the same occurs with its linear approximation. Now consider a normal form X^N of X. It is straightforward to prove that one can find a reversible-equivariant normal form for a reversible-equivariant vector field.

So, if A is reversible-equivariant, then we can always write the vector field X^N as

$$X^N(x) = Ax + X_2^N(x) + \ldots + X_k^N(x) + \ldots,$$

where X_j^N can be taken reversible-equivariant with respect to the same involutions as A.

This reduces the problem of finding linear involutions corresponding to φ and ψ to the problem of finding linear involutions that make reversible-equivariant the linear approximation of X.

From Montgomery-Bochner's Theorem we may choose a coordinate system such that $\varphi = R_0$ is linear.

So our problem is: fixed a linear involution R_0, how many (and what are) linear involutions S_0 do exist such that $\langle \varphi, \psi \rangle \cong \langle R_0, S_0 \rangle$?

5 Some Group Theory

In this section we prove Lemma A and Theorem A. Recall that in Theorem A the list of groups to be considered is: (a) $\mathbb{Z}_2 \times \mathbb{Z}_2$, (b) $\mathbb{D}_3$ and (c) $\mathbb{D}_4$.

5.1 Proof of Lemma A

Let $G_0 = \langle a, b \rangle$ be a finite group generated by a and b ($a \neq b$) with $a^2 = 1$ and $b^2 = 1$. Suppose that G is a finite group. By Lagrange's Theorem (see [L]), $|G|$ is even, so $|G| = 2m$ for some $m \geq 1$. Note that G contains at least the distinct elements $1, a, b, ab$.

If $ab = ba$, then $(ab)^2 = 1$, $|G| = 4$ and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{D}_2$ ($m = 2$). If not, then $\{1, a, b, ab, ba, aba\} \subseteq G$.

If $aba = bab$ then $(ab)^3 = 1$ and $G \cong \mathbb{D}_3$ ($m = 3$). If not, then $abab \in G$.

If $abab = baba$ then $(ab)^4 = 1$ and $G = \mathbb{D}_4$ ($m = 4$). As $|G| < \infty$, this sequence will stop and G is a Dihedral Group.

5.2 Proof of Theorem A: Case $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{D}_2$

Fix the matrix (written in the basis given by CMB Theorem)

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Note that $R_0^2 = Id$ and $R_0 A = -AR_0$. We need to determine all possible involutive matrices $S_0 \in \mathbb{R}^{4\times 4}$ such that

$$S_0 A = -AS_0$$

and

$$\langle R_0, S_0 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Note that the relation $\langle R_0, S_0 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is equivalent to $R_0 S_0 = S_0 R_0$ and $S_0^2 = Id$. Put

$$S_0 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}. \tag{3}$$

The relations $S_0 A = -AS_0$, $S_0^2 = Id$ and $R_0 S_0 = S_0 R_0$ are represented by the following systems of polynomial equations:

$$\left\{ \begin{array}{lcl} a_1^2 - 1 + c_1 a_3 & = & 0 \\ a_1 c_1 + c_1 c_3 & = & 0 \\ b_2 d_2 + d_2 d_4 & = & 0 \\ a_3 a_1 + c_3 a_3 & = & 0 \\ b_4 b_2 + d_4 b_4 & = & 0 \\ b_2^2 + d_2 b_4 - 1 & = & 0 \\ c_1 a_3 + c_3^2 - 1 & = & 0 \\ d_2 b_4 + d_4^2 - 1 & = & 0 \end{array} \right. \qquad \left\{ \begin{array}{lcl} -a_1\alpha - b_2\alpha & = & 0 \\ -c_1\beta - \alpha d_2 & = & 0 \\ b_2\alpha + a_1\alpha & = & 0 \\ d_2\beta + \alpha c_1 & = & 0 \\ -a_3\alpha - \beta b_4 & = & 0 \\ -c_3\beta - d_4\beta & = & 0 \\ b_4\alpha + \beta a_3 & = & 0 \\ d_4\beta + c_3\beta & = & 0 \end{array} \right. \tag{4}$$

Lemma 6. *System* (4) *has 4 solutions:*

$$S_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Proof. This can be done in Maple 12 using the `Reduce` function from the Groebner package and the usual Maple's `solve` function. We remark that the solution S_4 is degenerate, i.e., $S_4 = R_0$. Moreover, we remark that the above representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not equivalent. □

Now we state the main result for $\mathbb{Z}_2 \times \mathbb{Z}_2$-reversible vector fields.

Theorem 7. *Let $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$ be a germ of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-reversible vector field. Then there exists $j \in \{1, 2, 3\}$ such that X is (R_0, S_j)-reversible.*

Now let us give a characterization of the vector fields which are (R_0, S_j)-reversible. Let us fix

$$X(x) = A(\alpha, \beta)x + (f_1(x), f_2(x), f_3(x), f_4(x))^T, \tag{5}$$

with $x \equiv (x_1, x_2, y_1, y_2)$. The proof of next results will be omitted.

Corollary 8. *The vector field (5) is (R_0, S_1)-reversible if and only if the functions f_j satisfies*

$$\begin{cases} f_1(x) &= -f_1(x_1, -x_2, y_1, -y_2) &= f_1(-x_1, x_2, -y_1, y_2) \\ f_2(x) &= f_2(x_1, -x_2, y_1, -y_2) &= -f_2(-x_1, x_2, -y_1, y_2) \\ f_3(x) &= -f_3(x_1, -x_2, y_1, -y_2) &= f_3(-x_1, x_2, -y_1, y_2) \\ f_4(x) &= f_4(x_1, -x_2, y_1, -y_2) &= -f_4(-x_1, x_2, -y_1, y_2). \end{cases}$$

In particular, $f_{1,3}(x_1, 0, y_1, 0) \equiv 0$ and $f_{2,4}(0, x_2, 0, y_2) \equiv 0$.

Corollary 9. *The vector field (5) is (R_0, S_2)-reversible if and only if the functions f_j satisfy*

$$\begin{cases} f_1(x) &= -f_1(x_1, -x_2, y_1, -y_2) &= f_1(-x_1, x_2, y_1, -y_2) \\ f_2(x) &= f_2(x_1, -x_2, y_1, -y_2) &= -f_2(-x_1, x_2, y_1, -y_2) \\ f_3(x) &= -f_3(x_1, -x_2, y_1, -y_2) &= -f_3(-x_1, x_2, y_1, -y_2) \\ f_4(x) &= f_4(x_1, -x_2, y_1, -y_2) &= f_4(-x_1, x_2, y_1, -y_2). \end{cases} \tag{6}$$

In particular, $f_{1,3}(x_1, 0, y_1, 0) \equiv 0$ and $f_{2,3}(0, x_2, y_1, 0) \equiv 0$.

Corollary 10. *The vector field (5) is (R_0, S_3)-reversible if and only if the functions f_j satisfy*

$$\begin{cases} f_1(x) &= -f_1(x_1, -x_2, y_1, -y_2) &= -f_1(x_1, -x_2, -y_1, y_2) \\ f_2(x) &= f_2(x_1, -x_2, y_1, -y_2) &= f_2(x_1, -x_2, -y_1, y_2) \\ f_3(x) &= -f_3(x_1, -x_2, y_1, -y_2) &= f_3(x_1, -x_2, -y_1, y_2) \\ f_4(x) &= f_4(x_1, -x_2, y_1, -y_2) &= -f_4(x_1, -x_2, -y_1, y_2). \end{cases} \tag{7}$$

In particular, $f_{1,3}(x_1, 0, y_1, 0) \equiv 0$ and $f_{1,4}(x_1, 0, 0, y_2) \equiv 0$.

5.3 Proof of Theorem A: Case $G_0 = \mathbb{D}_3$

As above we fix the matrix

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

Now we need to determine all possible involutive matrices $S_0 \in \mathbb{R}^{4\times 4}$ such that

$$S_0 A = -A S_0$$

and

$$\langle R_0, S_0\rangle \cong \mathbb{D}_3.$$

Considering again

$$S_0 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}, \tag{9}$$

the equations $S_0A+AS_0=0$, $S_0^2-Id=0$ and $(R_0S_0)^3-Id=0$ are equivalent to another huge system of equations (like system (4)).

Lemma 11. *The system generated by the above conditions has* 3 *non degenerate solutions:*

$$S_1 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},\ S_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Proof. Again, the proof can be done in Maple 12 using the `Reduce` function from the Groebner package and the usual Maple's `solve` function. □

At this point, we can state the following:

Theorem 12. *Let $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$ be a $\mathbb{D}_3$-reversible vector field. Then there exists $j \in \{1,2,3\}$ such that X is (R_0, S_j)-reversible.*

Next section deals with the characterization of the $\mathbb{D}_4$-reversible vector fields. The analysis of the $\mathbb{D}_3$-reversible case will be omitted since it is very similar to the $\mathbb{D}_4$-reversible case and this last is more interesting (there are more representations).

5.4 Proof of Theorem A: Case $G_0 = \mathbb{D}_4$

Fix the matrix

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

Our aim is to determine all of the possible involutive matrices $S_0 \in \mathbb{R}^{4\times 4}$ such that

$$S_0A = -AS_0$$

and

$$\langle R_0, S_0\rangle \cong \mathbb{D}_4.$$

Considering again

$$S_0 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}, \tag{11}$$

the equations $S_0A + AS_0 = 0$, $S_0^2 - Id = 0$ and $(R_0S_0)^4 - Id = 0$ are represented by a system bigger than (4) (see the Appendix) having 12 non degenerate solutions, arranged in the following way:

$$\Xi_1 = \left\{ \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\}, \ \Xi_2 = \left\{ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\}$$

$$\Xi_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\}, \ \Xi_4 = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\}$$

$$\Xi_5 = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \ \Xi_6 = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

Recall that the above arrangement has obeyed the rule:

Lemma 13. $S_i, S_j \in \Xi_k \Leftrightarrow \langle R_0, S_i \rangle = \langle R_0, S_j \rangle$.

For each $i \in \{1, \ldots, 6\}$, denote by S_i some element of Ξ_i. The proof of the next result follows immediately from the above lemmas.

Theorem 14. *Let $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$ be a germ of a $\mathbb{D}_4$-reversible vector field. Then there exists $j \in \{1, \ldots, 6\}$ such that X is (R_0, S_j)-reversible.*

Now we present some results in the sense of Corollary 8 applied to $\mathbb{D}_4$-reversible vector fields. We will just work with some linearized groups; the other cases are similar.

Let us fix again

$$X(x) = A(\alpha, \beta)x + (g_1(x), g_2(x), g_3(x), g_4(x))^T, \tag{12}$$

with $x \equiv (x_1, x_2, y_1, y_2)$. Keeping the same notation of Section 5.2, we have now $\langle R_0, S_j \rangle \cong \mathbb{D}_4$.

Corollary 15. *The vector field* (12) *is (R_0, S_1)-reversible if and only if the functions g_j satisfy*

$$\begin{cases} g_1(x) &= -g_1(x_1, -x_2, y_1, -y_2) &= -g_2(x_2, x_1, y_1, -y_2) \\ g_2(x) &= g_2(x_1, -x_2, y_1, -y_2) &= -g_1(x_2, x_1, y_1, -y_2) \\ g_3(x) &= -g_3(x_1, -x_2, y_1, -y_2) &= -g_3(x_2, x_1, y_1, -y_2) \\ g_4(x) &= g_4(x_1, -x_2, y_1, -y_2) &= g_4(x_2, x_1, y_1, -y_2) \end{cases} \tag{13}$$

In particular $g_{1,3}(x_1, 0, y_1, 0) \equiv 0$ and $g_{2,4}(0, x_2, 0, y_2) \equiv 0$.

6 Applications

6.1 Normal forms (Proof of Theorem B)

Let $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$ be a $\mathbb{D}_4$-reversible vector field and X^N its reversible-equivariant BNF. To compute the expression of X^N, we have to consider the following possibilities of the parameter $\lambda = \alpha\beta^{-1}$:
(i) $\lambda \notin \mathbb{Q}$,
(ii) $\lambda = 1$,
(iii) $\lambda = pq^{-1}$, with p, q integers with $(p, q) = 1$.

In the case (i), one can show that the normal forms for the reversible and reversible-equivariant cases are essencially the same. This means that any reversible field with such linear approximation is automatically reversible-equivariant. In view of this, case (i) is not interesting, and its analysis will be omitted. We just observe that case (ii) will not be discussed here because of its deep degeneracy. The range of its homological operator

$$L_{A(\alpha,\alpha)} : H^k \to H^k$$

is a very low dimensional subspace of H^k.

Our goal is to focus on the case (iii). Put $\alpha = p$ and $\beta = q$, with $p, q \in \mathbb{Z}$ and $(p, q) = 1$. How to compute a normal form which applies for all $\mathbb{D}_4$-reversible vector fields, without choosing specific involutions?

According to the results in the last section, it suffices to show that X^N satisfies

$$R_0(X^N(x)) = -X^N(R_0(x))$$

and

$$S_j(X^N(x)) = -X^N(S_j(x)),\ j = 1, \ldots, 6,$$

with S_j given on Lemma 13, as the fixed choice for the representative of Ξ_i.

First of all, we consider complex coordinates $(z_1, z_2) \in \mathbb{C}^2$ instead of $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$:

$$\left\{ \begin{array}{rcl} z_1 &=& x_1 + ix_2 \\ z_2 &=& y_1 + iy_2 \end{array} \right. \tag{14}$$

We will write $\Re(z)$ for the real part of the complex number z and $\Im(z)$ for its imaginary part.

Define

$$\left\{ \begin{array}{rcll} \Delta_1 &=& z_1\overline{z_1} & (= x_1^2 + x_2^2) \\ \Delta_2 &=& z_2\overline{z_2} & (= y_1^2 + y_2^2) \\ \Delta_3 &=& z_1^q\overline{z}_2^p & \\ \Delta_4 &=& \overline{\Delta}_3 & \end{array} \right.$$

Note that each Δ_j corresponds to a relation represented by

$$\Gamma_1^1\lambda_1 + \Gamma_2^1\lambda_2 + \Gamma_1^2\overline{\lambda}_1 + \Gamma_2^2\overline{\lambda}_2 = 0,\ \text{where } \Gamma_i^j \in \mathbb{N}.$$

It is not hard to see that the complex Belitskii normal form for X in this case is expressed by

$$\left\{ \begin{array}{rcl} \dot{z}_1 &=& piz_1 + z_1 f_1(\Delta_1, \Delta_2, \Delta_3, \Delta_4) + \overline{z}_1^{q-1} z_2^p f_2(\Delta_1, \Delta_2, \Delta_3, \Delta_4) \\ \dot{z}_2 &=& qiz_2 + z_2 g_1(\Delta_1, \Delta_2, \Delta_3, \Delta_4) + z_1^q \overline{z}_2^{p-1} g_2(\Delta_1, \Delta_2, \Delta_3, \Delta_4), \end{array} \right. \tag{15}$$

with f_j, g_j without linear and constant terms.

Now we consider the effects of $\mathbb{D}_4$-reversibility on the system (15). Writing our involutions in complex coordinates, we derive immediately that

Lemma 16. *Let*

$$\begin{array}{rclrcl}
\varphi_0(z_1,z_2) & = & -(\overline{z_1},\overline{z_2}) & & & \\
\varphi_1(z_1,z_2) & = & (\overline{iz_1},\overline{z_2}) & \varphi_2(z_1,z_2) & = & -(\overline{z_1},\overline{iz_2}) \\
\varphi_3(z_1,z_2) & = & (\overline{z_1},-\overline{iz_2}) & \varphi_4(z_1,z_2) & = & -(\overline{iz_1},\overline{iz_2}) \\
\varphi_5(z_1,z_2) & = & -(\overline{iz_1},\overline{z_2}) & \varphi_6(z_1,z_2) & = & (-\overline{iz_1},\overline{iz_2})
\end{array}$$

Then each group $\langle\varphi_0,\varphi_j\rangle$ corresponds to $\langle R_0, S_j\rangle$.

To compute a $\mathbb{D}_4$-reversible normal form for a vector field, one has first to define which of the groups in Lemma 16 can be used to do the calculations. Now we establish a normal form of a $\mathbb{D}_4$-reversible and $p:q$-resonant vector field X, depending only on p,q and not on the involutions generating $\mathbb{D}_4$:

Theorem 17. *Let p,q be odd numbers with $pq>1$ and $X\in\mathfrak{X}_0^{(p,q)}(\mathbb{R}^4)$ be a $\mathbb{D}_4$-reversible vector field. Then X is formally conjugated to the following system:*

$$\left\{\begin{array}{rcl}
\dot{x_1} & = & -px_2 - x_2\sum_{i+j=1}^{\infty} a_{ij}\Delta_1^i\Delta_2^j \\
\dot{x_2} & = & px_1 + x_1\sum_{i+j=1}^{\infty} a_{ij}\Delta_1^i\Delta_2^j \\
\dot{y_1} & = & -qy_2 - y_2\sum_{i+j=1}^{\infty} b_{ij}\Delta_1^i\Delta_2^j \\
\dot{y_2} & = & qy_1 + y_1\sum_{i+j=1}^{\infty} b_{ij}\Delta_1^i\Delta_2^j,
\end{array}\right. \tag{16}$$

with $a_{ij}, b_{ij}\in\mathbb{R}$ depending on $j^kX(0)$, for $k=i+j$.

Remark 18. *The hypothesis on p,q given in Theorem 17 can be relaxed. In fact, if p,q satisfies the following conditions*

$$\left\{\begin{array}{l}
q\equiv_4 1 \text{ or } q\equiv_4 3 \text{ or } (q\equiv_4 0 \text{ and } p+q=2k+1) \text{ or } (q\equiv_4 2 \text{ and } p+q=2k) \\
p\equiv_4 1 \text{ or } p\equiv_4 2 \text{ or } p\equiv_4 3 \\
p\equiv_4 1 \text{ or } p\equiv_4 3 \text{ or } (p\equiv_4 0 \text{ and } q=2k+1) \text{ or } (p\equiv_4 2 \text{ and } q=2k)
\end{array}\right.$$

then the conclusions of Theorem 17 are also valid.

Remark 19. *The normal form* (16) *coincides (in the nonlinear terms) with the normal form of a reversible vector field $X\in\mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$ with $\alpha\beta^{-1}\notin\mathbb{Q}$. Remember that this fact allowed us to discard the case $\alpha\beta^{-1}\notin\mathbb{Q}$ in page 11.*

Remark 20. *The claim that a_{ij}, b_{ij} depends only on the coefficients of the terms of degree $i+j$ in the Taylor series of X will be not proved here, but it is just a straightforward calculation. In [Br], Bruno makes explicit the dependence of the normal form coefficients with the original Taylor's series coefficients.*

The proof of Theorem 17 (even with the hypothesis of Remark 18) is based on a sequence of lemmas. The idea is just to show that with some hypothesis on p and q, all the coefficients of Δ_3 and Δ_4 in the reversible-equivariant analogous of (15) need to be zero.

First let us focus on the monomials that are never killed by the reversible-equivariant structure.

Lemma 21. *Let $v = az_j\Delta_1^m\Delta_2^n\frac{\partial}{\partial z_j}$, $a \in \mathbb{C}$. So, for any $j \in \{0,\ldots,6\}$, the φ_j-reversibility implies $\overline{a} = -a$ (or $\Re(a) = 0$). In particular, these terms are always present (generically) in the normal form.*

Proof. From

$$\varphi_0\left(az_1\Delta_1^m\Delta_2^n\frac{\partial}{\partial z_1}\right) = -\overline{a}\ \overline{z_1}\Delta_1^m\Delta_2^n\frac{\partial}{\partial z_1}$$

and

$$az_1\Delta_1^m\Delta_2^n\frac{\partial}{\partial z_1}\bigg|_{(-\overline{z_1},-\overline{z_2})} = -a\overline{z_1}\Delta_1^m\Delta_2^n\frac{\partial}{\partial z_1}$$

follows that $-\overline{a} = a$. □

Now let us see what happens with the monomials of type $(\overline{z_1})^{q-1}z_2^p\frac{\partial}{\partial z_1}$. Only for this kind of monomials, we will present a complete proof. Moreover, we will give the statement and the demonstration in the direction of Remark 18.

Lemma 22. *Let $v = a\overline{z_1}^{q-1}z_2^p\frac{\partial}{\partial z_1}$, $a \in \mathbb{C}$. So, we establish the following tables:*

reversibility	*hypothesis on p, q*	*conditions on a*
φ_0	$p+q$ *even*	$\Re(a) = 0$
	$p+q$ *odd*	$\Im(a) = 0$
φ_1	$q \equiv_4 0$	$\Re(a) = 0$
	$q \equiv_4 1$	$\Re(a) = -\Im(a)$
	$q \equiv_4 2$	$\Im(a) = 0$
	$q \equiv_4 3$	$\Re(a) = \Im(a)$
φ_2	$p \equiv_4 0,\ q$ *even*	$\Re(a) = 0$
	$p \equiv_4 0,\ q$ *odd*	$\Im(a) = 0$
	$q \equiv_4 1,\ q$ *even*	$\Re(a) = \Im(a)$
	$q \equiv_4 1,\ q$ *odd*	$\Re(a) = -\Im(a)$
	$q \equiv_4 2,\ q$ *even*	$\Im(a) = 0$
	$q \equiv_4 2,\ q$ *odd*	$\Re(a) = 0$
	$q \equiv_4 3,\ q$ *even*	$\Re(a) = -\Im(a)$
	$q \equiv_4 3,\ q$ *odd*	$\Re(a) = \Im(a)$
φ_3	$p \equiv_4 0$	$\Re(a) = 0$
	$p \equiv_4 1$	$\Re(a) = \Im(a)$
	$p \equiv_4 2$	$\Im(a) = 0$
	$p \equiv_4 3$	$\Re(a) = -\Im(a)$

reversibility	*hypothesis on* p, q	*conditions on* a
φ_4	$p+q \equiv_4 0,\ q$ *even*	$\Re(a)=0$
	$p+q \equiv_4 0,\ q$ *odd*	$\Im(a)=0$
	$p+q \equiv_4 1,\ q$ *even*	$\Re(a)=\Im(a)$
	$p+q \equiv_4 1,\ q$ *odd*	$\Re(a)=-\Im(a)$
	$p+q \equiv_4 2,\ q$ *even*	$\Im(a)=0$
	$p+q \equiv_4 2,\ q$ *odd*	$\Re(a)=0$
	$p+q \equiv_4 3,\ q$ *even*	$\Re(a)=-\Im(a)$
	$p+q \equiv_4 3,\ q$ *odd*	$\Im(a)=\Im(a)$
φ_5	$q \equiv_4 0,\ p+q$ *even*	$\Re(a)=0$
	$q \equiv_4 0,\ p+q$ *odd*	$\Im(a)=0$
	$q \equiv_4 1,\ p+q$ *even*	$\Re(a)=\Im(a)$
	$q \equiv_4 1,\ p+q$ *odd*	$\Re(a)=-\Im(a)$
	$q \equiv_4 2,\ p+q$ *even*	$\Im(a)=0$
	$q \equiv_4 2,\ p+q$ *odd*	$\Re(a)=0$
	$q \equiv_4 3,\ p+q$ *even*	$\Re(a)=-\Im(a)$
	$q \equiv_4 3,\ p+q$ *odd*	$\Re(a)=\Im(a)$
φ_6	$p+q \equiv_4 0$	$\Re(a)=0$
	$p+q \equiv_4 1$	$\Re(a)=-\Im(a)$
	$p+q \equiv_4 2$	$\Im(a)=0$
	$p+q \equiv_4 3$	$\Re(a)=\Im(a)$

Proof. Let us give the proof for φ_2-reversibility. The proof of any other case is similar. Note that

$$\begin{cases} \varphi_2(v(z_1,z_2)) & = & -\overline{a} z_1^{q-1}\overline{z_2}^p \frac{\partial}{\partial z_1} \\ v(\varphi_2(z_1,z_2)) & = & a(-1)^{q-1} i^p z_1^{q-1}\overline{z_2}^p \end{cases}$$

Then, from $\varphi_2(v(z)) = -v(\varphi_2(z))$ we have $\overline{a} = (-1)^{q-1} i^p a$. Now we apply the hypotheses on p, q and the proof follows in a straightforward way. □

Next corollary is the first of a sequence of results establishing that some monomial does not appear in the normal form:

Corollary 23. *Let $X \in \mathfrak{X}_0^{(p,q)}(\mathbb{R}^4)$ be a (φ_0, φ_j)-reversible vector field. Then if*

- $q \equiv 1$ *mod* 4 *or*
- $q \equiv 3$ *mod* 4 *or*
- $q \equiv 0$ *mod* 4 *and* $p+q$ *odd or*
- $q \equiv 2$ *mod* 4 *and* $p+q$ *even,*

then the normal form of X does not contain monomials of the form

$$a_1 \overline{z_1}^{nq-1} z_2^n p \frac{\partial}{\partial z_1}, \quad a_2 z_1^{mq} \overline{z}_2^{mp-1} \frac{\partial}{\partial z_2}, \ a_1, a_2 \in \mathbb{C}. \tag{17}$$

Proof. Observe that the φ_0, φ_j-reversibility implies that the coefficients in (17) satisfy

$$\Re(a_j) = \Im(a_j) = 0.$$

$\square$

Remark 24. *Note that if p, q are odd with $pq > 1$, then they satisfy the hypothesis of Corollary 23.*

The following results can be proved in a similar way as we have done in Lemma 22 and Corollary 23.

Proposition 25. *Let $X \in \mathfrak{X}_0^{(p,q)}(\mathbb{R}^4)$ be a (φ_0, φ_j)-reversible vector field. If one of the following conditions is satisfied:*

(i) $q \equiv 1$ *mod* 4,

(ii) $q \equiv 3$ *mod* 4,

(iii) $q \equiv 0$ *mod* 4 *and* $p + q$

(iv) $q \equiv 2$ *mod* 4 *and* $p + q$ *even,*

then the normal form of X, given in (15)*, does not have monomials of type*

$$z_1(z_1^q \overline{z}_2^p)^m \frac{\partial}{\partial z_1}, \quad z_2(z_1^q \overline{z}_2^p)^m \frac{\partial}{\partial z_2}, \quad m \geq 1.$$

Proposition 26. *Let $X \in \mathfrak{X}_0^{(p,q)}(\mathbb{R}^4)$ be a (φ_0, φ_j)-reversible vector field. If one of the following conditions is satisfied*

(i) $q \equiv 1$ *mod* 4,

(ii) $q \equiv 3$ *mod* 4,

(iii) $q \equiv 0$ *mod* 4 *and* $p + q$ *odd,*

(iv) $q \equiv 2$ *mod* 4 *and* $p + q$ *even,*

then the normal form of X, given in (15)*, does not have monomials of type*

$$z_1(\overline{z_1^q \overline{z}_2^p})^m \frac{\partial}{\partial z_1}, \quad z_2(\overline{z_1^q \overline{z}_2^p})^m \frac{\partial}{\partial z_2}, \quad m \geq 1.$$

Remark 27. *In fact, the conditions imposed on λ in the last results are needed just to assure the (φ_0, φ_j)-reversibility of the vector field X with $j = 1$. For $2 \leq j \leq 6$, the normal form only contains monomials of type $z_j \Delta_1^m \Delta_2^n \dfrac{\partial}{\partial z_j}$.*

Now, to prove Theorem 17, we have just to combine all lemmas, corollaries and propositions given above.

Proof. (*of Theorem 17*) Note that the conditions imposed on λ in Theorem 17 fit the hypothesis of Corollary 23 and Propositions 25 and 26. So, if p, q are odd numbers with $pq > 1$, then the normal form just have monomials of type

$$z_j \Delta_1^m \Delta_2^n \frac{\partial}{\partial z_j}, \ j = 1, 2, \ m, n > 1.$$

$\square$

6.2 Integrability (Proof of Theorem C)

Let X be a reversible-equivariant polynomial vector field with $\deg(X) \leq 2$. Then

$$X(x) = X^1(x)\frac{\partial}{\partial x_1} + X^2(x)\frac{\partial}{\partial x_2} + X^3(x)\frac{\partial}{\partial y_1} + X^4(x)\frac{\partial}{\partial y_2}$$

with $x = (x_1, x_2, y_1, y_2)$ and X^j be homogeneous polynomial of degree at most 2. Let us suppose that the eigenvalues of $DX(0)$ are all non zero.

Let

$$H(x_1, x_2, y_1, y_2) = \sum_{i+j+k+l=1}^{d} h_{ijkl} x_1^i x_2^j y_1^k y_2^l$$

be a polynomial first integral of X with degree d. Then we have the relation $XH = 0$, that is,

$$X^1\frac{\partial H}{\partial x_1} + X^2\frac{\partial H}{\partial x_2} + X^3\frac{\partial H}{\partial y_1} + X^4\frac{\partial H}{\partial y_2} = 0 \tag{18}$$

for H non zero.

Collecting the monomials in (18), their coefficients depend on the coefficients of X^1, X^2, X^3, X^4 and H. So they gives rise to a algebraic system. We should solve this system in terms of the coefficients of H.

If this system has a solution, then every reversible-equivariant polynomial vector field in $\mathbb{R}^4$ would have a first integral. Although it is not always true, we give conditions on the coefficients of the polinomials X^j that allow us to find first integrals. For that, we need to solve a (non linear) system in the coefficients of (18), that depends upon the coefficients of X^j's and H.

6.2.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$-reversibility

Proposition 28. *Let $X \in \mathfrak{X}_0^{(\alpha,\beta)}(\mathbb{R}^4)$ be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-reversible vector field with $\lambda = \alpha\beta^{-1} \notin \mathbb{Q}$. Suppose that X is a polynomial field with $\deg(X) \leq 3$. Then*

(i) if X is (R_0, S_1)-reversible case, then it may be represented by

$$\left\{\begin{array}{rcl}
\dot{x_1} & = & -\alpha x_2 + a_{0003}{y_2}^3 + a_{0021}{y_1}^2 y_2 + a_{0102}x_2{y_2}^2 + a_{0120}x_2{y_1}^2 + a_{0201}{x_2}^2 y_2 + a_{0300}{x_2}^3 \\
 & + & a_{1011}x_1 y_1 y_2 + a_{1110}x_1 x_2 y_1 + a_{2001}{x_1}^2 y_2 + a_{2100}{x_1}^2 x_2 \\
\dot{x_2} & = & \alpha x_1 + b_{0012}y_1{y_2}^2 + b_{0030}{y_1}^3 + b_{0111}x_2 y_1 y_2 + b_{0210}{x_2}^2 y_1 + b_{1002}x_1{y_2}^2 + b_{1020}x_1{y_1}^2 \\
 & + & b_{1101}x_1 x_2 y_2 + b_{1200}x_1{x_2}^2 + b_{2010}{x_1}^2 y_1 + b_{3000}{x_1}^3 \\
\dot{y_1} & = & -\beta y_2 + c_{0003}{y_2}^3 + c_{0021}{y_1}^2 y_2 + c_{0102}x_2{y_2}^2 + c_{0120}x_2{y_1}^2 + c_{0201}{x_2}^2 y_2 + c_{0300}{x_2}^3 \\
 & + & c_{1011}x_1 y_1 y_2 + c_{1110}x_1 x_2 y_1 + c_{2001}{x_1}^2 y_2 + c_{2100}{x_1}^2 x_2 \\
\dot{y_2} & = & \beta y_1 + d_{0012}y_1{y_2}^2 + d_{0030}{y_1}^3 + d_{0111}x_2 y_1 y_2 + d_{0210}{x_2}^2 y_1 + d_{1002}x_1{y_2}^2 + d_{1020}x_1{y_1}^2 \\
 & + & d_{1101}x_1 x_2 y_2 + d_{1200}x_1{x_2}^2 + d_{2010}{x_1}^2 y_1 + d_{3000}{x_1}^3
\end{array}\right. .$$

Moreover, it has a first integral if and only if one of the following conditions holds:
(a) $c_{0120} = c_{0300} = c_{1110} = c_{2100} = d_{1002} = d_{1101} = d_{1200} = d_{3000} = 0$, $d_{0012} = -c_{0003}$, $d_{0030} = -c_{0021}$, $d_{0111} = -c_{0102}$, $d_{0210} = -c_{0201}$, $d_{1020} = -c_{1011}$, $d_{2010} = -c_{2001}$ or

(b) $a_{0003} = a_{0021} = a_{1011} = a_{2001} = b_{0012} = b_{0030} = b_{0111} = b_{0210} = 0$, $b_{1002} = -a_{0102}$, $b_{1020} = -a_{0120}$, $b_{1101} = -a_{0201}$, $b_{1200} = a_{0300}$, $b_{2010} = -a_{1110}$, $b_{3000} = -a_{2100}$ *or*
(c) $b_{1200} = -a_{0300}$, $b_{3000} = -a_{2001}$, $d_{0012} = -c_{0003}$, $d_{0030} = -c_{0021}$, $c_{0120} = -\mu b_{0030}$, $c_{0300} = -\mu b_{0210}$, $c_{1110} = -\mu(a_{0120}+b_{1020})$, $c_{2100} = -\mu(a_{1110}+b_{2010})$, $d_{0111} = -c_{0102}-\mu b_{0012}$, $d_{0210} = -c_{0201} - \mu b_{0111}$, $d_{1002} = -\mu a_{0003}$, $d_{1020} = -a_{0021} - \mu c_{1011}$, $d_{1101} = \mu(a_{0102} + b_{1002})$, $d_{1200} = -\mu(a_{0201} + b_{1101})$, $d_{2010} = -\mu a_{1011} - c_{2001}$, $d_{3000} = -\mu a_{2001}$, *for* $\mu \in \mathbb{R}_*$.

Moreover, the first integrals are given by

$$\begin{cases} (a)\ H_1(x_1, x_2, y_1, y_2) = \nu(y_1^2 + y_2^2), \\ (b)\ H_2(x_1, x_2, y_1, y_2) = \rho(x_1^2 + x_2^2), \\ (c)\ H_2(x_1, x_2, y_1, y_2) = \mu(y_1^2 + y_2^2) + x_1^2 + x_2^2, \end{cases}$$

where ν, μ, ρ *are non zero real parameters.*

(ii) if X *is* (R_0, S_2)*-reversible case,* X *writes as*

$$\begin{cases} \dot{x_1} &=& -\alpha x_2 + a_{0110}x_2y_1 + a_{1001}x_1y_2 + a_{0102}x_2{y_2}^2 + a_{0120}x_2{y_1}^2 + a_{0300}{x_2}^3 \\ &+& a_{1011}x_1y_1y_2 + a_{2100}{x_1}^2x_2 \\ \dot{x_2} &=& \alpha x_1 + b_{0101}x_2y_2 + b_{1010}x_1y_1 + b_{0111}x_2y_1y_2 + b_{1002}x_1{y_2}^2 + b_{1020}x_1{y_1}^2 \\ &+& b_{1200}x_1{x_2}^2 + b_{3000}{x_1}^3 \\ \dot{y_1} &=& -\beta y_2 + c_{0011}y_1y_2 + c_{1100}x_1x_2 + c_{0003}{y_2}^3 + c_{0021}{y_1}^2y_2 + c_{0201}{x_2}^2y_2 \\ &+& c_{1110}x_1x_2y_1 + c_{2001}{x_1}^2y_2 \\ \dot{y_2} &=& \beta y_1 + d_{0002}{y_2}^2 + d_{0020}{y_1}^2 + d_{0200}{x_2}^2 + d_{2000}{x_1}^2 + d_{0012}y_1{y_2}^2 + d_{0030}{y_1}^3 \\ &+& d_{0210}{x_2}^2y_1 + d_{1101}x_1x_2y_2 + d_{2010}{x_1}^2y_1 \end{cases}$$

and has a first integral if and only if one of the following holds:
(a) $a_{1001} = 0, a_{1011} = 0$, $b_{0101} = 0$, $b_{0111} = 0$, $b_{1002} = -a_{0102}$, $b_{1010} = -a_{0110}$, $b_{1020} = -a_{0120}$, $b_{1200} = -a_{0300}$, $b_{3000} = -a_{2100}$ *or*
(b) $d_{0002} = 0$, $d_{0012} = -c_{0003}$, $d_{0020} = -c_{0011}$, $d_{0030} = -c_{0021}$, $d_{0200} = 0$, $d_{0210} = -c_{0201}$, $d_{1101} = 0$, $d_{2000} = 0$, $d_{2010} = -c_{2001}$ *or*
(c) $a_{1001} = -\frac{h_{0002}d_{2000}}{h_{0200}}$, $a_{1011} = -\dfrac{h_{0002}\,(d_{2010} + c_{2001})}{h_{0200}}$, $b_{0101} = -\dfrac{h_{0002}d_{0200}}{h_{0200}}$, $b_{0111} = -\dfrac{h_{0002}\,(d_{0210} + c_{0201})}{h_{0200}}$, $b_{1002} = -\dfrac{h_{0200}a_{0102} + h_{0002}d_{1101}}{h_{0200}}$, $b_{1010} = -\dfrac{h_{0200}a_{0110} + h_{0002}c_{1100}}{h_{0200}}$, $b_{1020} = -\dfrac{h_{0200}a_{0120} + h_{0002}c_{1110}}{h_{0200}}$, $b_{1200} = -a_{0300}$, $b_{3000} = -a_{2100}$, $d_{0002} = 0$, $d_{0012} = -c_{0003}$, $d_{0020} = -c_{0011}$, $d_{0030} = -c_{0021}$, *where* h_{0200}, h_{0002} *are real parameters.*

(iii) if X *is* (R_0, S_3)*-reversible case,* X *writes as*

$$\begin{cases} \dot{x_1} &=& -\alpha x_2 + a_{0011}y_1y_2 + a_{1100}x_1x_2 + a_{0102}x_2{y_2}^2 + a_{0120}x_2{y_1}^2 \\ &+& a_{0300}{x_2}^3 + a_{1011}x_1y_1y_2 + a_{2100}{x_1}^2x_2 \\ \dot{x_2} &=& \alpha x_1 + b_{0002}{y_2}^2 + b_{0020}{y_1}^2 + b_{0200}{x_2}^2 + b_{2000}{x_1}^2 + b_{0111}x_2y_1y_2 \\ &+& b_{1002}x_1{y_2}^2 + b_{1020}x_1{y_1}^2 + b_{1200}x_1{x_2}^2 + b_{3000}{x_1}^3 \\ \dot{y_1} &=& -\beta y_2 + c_{0110}x_2y_1 + c_{1001}x_1y_2 + c_{0003}{y_2}^3 + c_{0021}{y_1}^2y_2 \\ &+& c_{0201}{x_2}^2y_2 + c_{1110}x_1x_2y_1 + c_{2001}{x_1}^2y_2 \\ \dot{y_2} &=& \beta y_1 + d_{0101}x_2y_2 + d_{1010}x_1y_1 + d_{0012}y_1{y_2}^2 + d_{0030}{y_1}^3 \\ &+& d_{0210}{x_2}^2y_1 + d_{1101}x_1x_2y_2 + d_{2010}{x_1}^2y_1 \end{cases}$$

and it has a first integral if and only if one of the following conditions holds:
(a) $c_{0110} = 0$, $c_{1110} = 0$, $d_{0012} = -c_{0003}$, $d_{0030} = -c_{0021}$, $d_{0101} = 0$, $d_{0210} = -c_{0201}$, $d_{1010} = -c_{1001}$, $d_{1101} = 0$, $d_{2010} = -c_{2001}$ *or*
(b) $a_{0011} = 0$, $a_{1011} = 0$, $b_{0002} = 0$, $b_{0020} = 0$, $b_{0111} = 0$, $b_{0200} = 0$, $b_{1002} = -a_{0102}$, $b_{1020} = -a_{0120}$, $b_{1200} = -a_{0300}$, $b_{2000} = -a_{1100}$, $b_{3000} = -a_{2100}$ *or*
(c) $b_{0200} = 0$, $b_{1200} = -a_{0300}$, $b_{2000} = -a_{1100}$, $b_{3000} = -a_{2100}$, $c_{0110} = -\dfrac{h_{0200}b_{0020}}{h_{0020}}$, $c_{1110} = -\dfrac{h_{0200}\left(b_{1020}+a_{0120}\right)}{h_{0020}}$, $d_{0012} = -c_{0003}$, $d_{0030} = -c_{0021}$, $d_{0101} = -\dfrac{h_{0200}b_{0002}}{h_{0020}}$, $d_{0210} = -\dfrac{h_{0020}c_{0201}+h_{0200}b_{0111}}{h_{0020}}$, $d_{1010} = -\dfrac{h_{0200}a_{0011}+h_{0020}c_{1001}}{h_{0020}}$, $d_{1101} = -\dfrac{h_{0200}\left(a_{0102}+b_{1002}\right)}{h_{0020}}$, $d_{2010} = -\dfrac{h_{0200}a_{1011}+h_{0020}c_{2001}}{h_{0020}}$, *where* h_{0200}, h_{0020} *are real parameters.*

Remark 29. *The first integrals in cases (ii) and (iii) are similar to these in (i).*

Proof. We discuss the "details" for (i), since the other cases are quite similar. Using the relations given in Corollaries 8,9 and 10, we deduce the expression for the 2-jet of the vector fields. The polynomial system in the coefficients of (18) involves 45 equations and 50 variables. We add two equations (inequalities) to avoid the degenerate cases and solve everything in Maple 12. Then we get the result (see Appendix). □

7 Conclusions

We have classified the involutions that make a vector field $X \in \mathfrak{X}_0^{(p,q)}(\mathbb{R}^4)$ (φ,ψ)-reversible when the order of the group $\langle\varphi,\psi\rangle$ is smaller than 9.

As a consequence of the results obtained in Theorem A, we find a normal form for $\mathbb{D}_4$-reversible vector fields in $\mathbb{R}^4$, according to their resonances. In this part we have used some results from Normal Form Theory. We also have given some conditions to a $\mathbb{Z}_2\times\mathbb{Z}_2$-polynomial reversible-equivariant vector field admit a first integral. In this part, we have used some Groebner basis methods to solve the algebraic systems in the space of the jets.

We remark that the same approach can be made to the discrete version of the problem, or when the singularity is not elliptic (see for example [JT] and [AQR]).

One can easily generalize the results presented here mainly in two directions: for vector fields on higher dimensional spaces and for groups with higher order. In both cases the hard missions are to face the normal form calculations and to solve some very complicate system of algebraic equations.

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Appendix

Here we will show the polynomial system omitted in Section 5.4, corresponding to the equations $S_0A + AS_0 = 0$, $S_0^2 - Id = 0$ and $S_0R_0 - (R_0S_0)^3 = 0$, as in the page 10.

The system is:

$d_1b_4 + c_1b_3 + b_1b_2 + a_1b_1 = 0, \quad d_1c_4 + c_1c_3 + b_1c_2 + a_1c_1 = 0, \quad d_1d_4 + c_1d_3 + b_1d_2 + a_1d_1 = 0, \quad d_2a_4 + c_2a_3 + a_2b_2 + a_1a_2 = 0$

$d_2c_4 + c_2c_3 + b_2c_2 + c_1a_2 = 0, \quad d_2d_4 + c_2d_3 + b_2d_2 + d_1a_2 = 0, \quad d_3a_4 + a_3c_3 + a_2b_3 + a_1a_3 = 0, \quad d_3b_4 + b_3c_3 + b_2b_3 + b_1a_3 = 0$

$d_3d_4 + c_3d_3 + d_2b_3 + d_1a_3 = 0, \quad a_4d_4 + a_3c_4 + a_2b_4 + a_1a_4 = 0, \quad b_4d_4 + b_3c_4 + b_2b_4 + b_1a_4 = 0, \quad c_4d_4 + c_3c_4 + c_2b_4 + c_1a_4 = 0$

$-\alpha a_2 + \alpha b_1 = 0, \quad -\alpha b_2 - \alpha a_1 = 0, \quad -\alpha c_2 + \beta d_1 = 0, \quad -\alpha d_2 - \beta c_1 = 0, \quad \alpha b_2 + \alpha a_1 = 0, \quad \beta d_2 + \alpha c_1 = 0, \quad -\beta c_2 + \alpha d_1 = 0$

$-\beta a_4 + \alpha b_3 = 0, \quad -\beta b_4 - \alpha a_3 = 0, \quad -\beta c_4 + \beta d_3 = 0, \quad -\beta d_4 - \beta c_3 = 0, \quad \alpha b_4 + \beta a_3 = 0, \quad -\alpha a_4 + \beta b_3 = 0, \quad \beta d_4 + \beta c_3 = 0$

$d_1a_4 + c_1a_3 + b_1a_2 + {a_1}^2 = 1, \quad d_2b_4 + c_2b_3 + {b_2}^2 + b_1a_2 = 1, \quad d_3c_4 + {c_3}^2 + c_2b_3 + c_1a_3 = 1, \quad {d_4}^2 + d_3c_4 + d_2b_4 + d_1a_4 = 1$

$a_1 - d_1a_4d_4 + d_1a_3c_4 - d_1a_2b_4 + c_1d_3a_4 - c_1a_3c_3 + c_1a_2b_3 - b_1d_2a_4 + b_1c_2a_3 - b_1a_2b_2 + 2a_1d_1a_4 - 2a_1c_1a_3 + 2a_1b_1a_2 - {a_1}^3 = 0$

$-b_1 - d_1b_4d_4 + d_1b_3c_4 - d_1b_2b_4 + c_1d_3b_4 - c_1b_3c_3 + c_1b_2b_3 - b_1d_2b_4 + b_1c_2b_3 - b_1{b_2}^2 + b_1d_1a_4 - b_1c_1a_3 + {b_1}^2a_2 + a_1d_1b_4 - a_1c_1b_3 +$
$a_1b_1b_2 - {a_1}^2b_1 = 0$

$c_1 - d_1c_4d_4 + d_1c_3c_4 - d_1c_2b_4 + c_1d_3c_4 - c_1{c_3}^2 + c_1c_2b_3 + c_1d_1a_4 - {c_1}^2a_3 - b_1d_2c_4 + b_1c_2c_3 - b_1b_2c_2 + b_1c_1a_2 + a_1d_1c_4 - a_1c_1c_3 + a_1b_1c_2 - {a_1}^2c_1 =$
0

$-d_1 - d_1{d_4}^2 + d_1d_3c_4 - d_1d_2b_4 + {d_1}^2a_4 + c_1d_3d_4 - c_1c_3d_3 + c_1d_2b_3 - c_1d_1a_3 - b_1d_2d_4 + b_1c_2d_3 - b_1b_2d_2 + b_1d_1a_2 + a_1d_1d_4 - a_1c_1d_3 +$
$a_1b_1d_2 - {a_1}^2d_1 = 0$

$a_2 + d_2a_4d_4 - d_2a_3c_4 - c_2d_3a_4 + c_2a_3c_3 + b_2d_2a_4 - b_2c_2a_3 + a_2d_2b_4 - a_2c_2b_3 + a_2{b_2}^2 - d_1a_2a_4 + c_1a_2a_3 - b_1{a_2}^2 - a_1d_2a_4 + a_1c_2a_3 -$
$a_1a_2b_2 + {a_1}^2a_2 = 0$

$-b_2 + d_2b_4d_4 - d_2b_3c_4 - c_2d_3b_4 + c_2b_3c_3 + 2b_2d_2b_4 - 2b_2c_2b_3 + {b_2}^3 - d_1a_2b_4 + c_1a_2b_3 - b_1d_2a_4 + b_1c_2a_3 - 2b_1a_2b_2 + a_1b_1a_2 = 0$

$c_2 + d_2c_4d_4 - d_2c_3c_4 - c_2d_3c_4 + c_2{c_3}^2 + c_2d_2b_4 - {c_2}^2b_3 + b_2d_2c_4 - b_2c_2c_3 + {b_2}^2c_2 - d_1a_2c_4 - c_1d_2a_4 + c_1c_2a_3 + c_1a_2c_3 - c_1a_2b_2 - b_1a_2c_2 + a_1c_1a_2 =$
0

$-d_2 + d_2{d_4}^2 - d_2d_3c_4 + {d_2}^2b_4 - c_2d_3d_4 + c_2c_3d_3 - c_2d_2b_3 + b_2d_2d_4 - b_2c_2d_3 + {b_2}^2d_2 - d_1d_2a_4 + d_1c_2a_3 - d_1a_2d_4 - d_1a_2b_2 + c_1a_2d_3 -$
$b_1a_2d_2 + a_1d_1a_2 = 0$

$a_3 - d_3a_4d_4 + c_3d_3a_4 + a_3d_3c_4 - a_3{c_3}^2 - d_2b_3a_4 + c_2a_3b_3 - a_2d_3b_4 + a_2b_3c_3 - a_2b_2b_3 + d_1a_3a_4 - c_1{a_3}^2 + b_1a_2a_3 + a_1d_3a_4 - a_1a_3c_3 +$
$a_1a_2b_3 - {a_1}^2a_3 = 0$

$-b_3 - d_3b_4d_4 + c_3d_3b_4 + b_3d_3c_4 - b_3{c_3}^2 - d_2b_3b_4 + c_2{b_3}^2 - b_2d_3b_4 + b_2b_3c_3 - {b_2}^2b_3 + d_1a_3b_4 - c_1a_3b_3 + b_1d_3a_4 - b_1a_3c_3 + b_1b_2a_3 +$
$b_1a_2b_3 - a_1b_1a_3 = 0$

$c_3 - d_3c_4d_4 + 2c_3d_3c_4 - {c_3}^3 - d_2b_3c_4 - c_2d_3b_4 + 2c_2b_3c_3 - b_2c_2b_3 + d_1a_3c_4 + c_1d_3a_4 - 2c_1a_3c_3 + c_1a_2b_3 + b_1c_2a_3 - a_1c_1a_3 = 0$

$-d_3 - d_3{d_4}^2 + {d_3}^2c_4 + c_3d_3d_4 - {c_3}^2d_3 - d_2d_3b_4 - d_2b_3d_4 + d_2b_3c_3 + c_2b_3d_3 - b_2d_2b_3 + d_1d_3a_4 + d_1a_3d_4 - d_1a_3c_3 + d_1a_2b_3 - c_1a_3d_3 +$
$b_1d_2a_3 - a_1d_1a_3 = 0$

$$a_4 + a_4 {d_4}^2 - d_3 a_4 c_4 - a_3 c_4 d_4 + a_3 c_3 c_4 + d_2 a_4 b_4 - c_2 a_3 b_4 + a_2 b_4 d_4 - a_2 b_3 c_4 + a_2 b_2 b_4 - d_1 {a_4}^2 + c_1 a_3 a_4 - b_1 a_2 a_4 - a_1 a_4 d_4 + a_1 a_3 c_4 - a_1 a_2 b_4 + {a_1}^2 a_4 = 0$$

$$- b_4 + b_4 {d_4}^2 - d_3 b_4 c_4 - b_3 c_4 d_4 + b_3 c_3 c_4 + d_2 {b_4}^2 - c_2 b_3 b_4 + b_2 b_4 d_4 - b_2 b_3 c_4 + {b_2}^2 b_4 - d_1 a_4 b_4 + c_1 b_3 a_4 - b_1 a_4 d_4 + b_1 a_3 c_4 - b_1 b_2 a_4 - b_1 a_2 b_4 + a_1 b_1 a_4 = 0$$

$$c_4 + c_4 {d_4}^2 - d_3 {c_4}^2 - c_3 c_4 d_4 + {c_3}^2 c_4 + d_2 b_4 c_4 + c_2 b_4 d_4 - c_2 c_3 b_4 - c_2 b_3 c_4 + b_2 c_2 b_4 - d_1 a_4 c_4 - c_1 a_4 d_4 + c_1 c_3 a_4 + c_1 a_3 c_4 - c_1 a_2 b_4 - b_1 c_2 a_4 + a_1 c_1 a_4 = 0$$

$$- d_4 + {d_4}^3 - 2 d_3 c_4 d_4 + c_3 d_3 c_4 + 2 d_2 b_4 d_4 - d_2 b_3 c_4 - c_2 d_3 b_4 + b_2 d_2 b_4 - 2 d_1 a_4 d_4 + d_1 a_3 c_4 - d_1 a_2 b_4 + c_1 d_3 a_4 - b_1 d_2 a_4 + a_1 d_1 a_4 = 0$$

$$a_1 - d_1 a_4 d_4 + d_1 a_3 c_4 - d_1 a_2 b_4 + c_1 d_3 a_4 - c_1 a_3 c_3 + c_1 a_2 b_3 - b_1 d_2 a_4 + b_1 c_2 a_3 - b_1 a_2 b_2 + 2 a_1 d_1 a_4 - 2 a_1 c_1 a_3 + 2 a_1 b_1 a_2 - {a_1}^3 = 0$$

$$- b_1 - d_1 b_4 d_4 + d_1 b_3 c_4 - d_1 b_2 b_4 + c_1 d_3 b_4 - c_1 b_3 c_3 + c_1 b_2 b_3 - b_1 d_2 b_4 + b_1 c_2 b_3 - b_1 {b_2}^2 + b_1 d_1 a_4 - b_1 c_1 a_3 + {b_1}^2 a_2 + a_1 d_1 b_4 - a_1 c_1 b_3 + a_1 b_1 b_2 - {a_1}^2 b_1 = 0$$

The solutions of the system above were given in the Lemma 11.