Stochastic characterization of harmonic sections and a Liouville theorem

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Abstract

Let $P(M,G)$ be a principal fiber bundle and $E(M,N,G,P)$ be an associate fiber bundle. Our interested is to study harmonic sections of the projection π_E of E into M. Our first purpose is to give a stochastic characterization of harmonic section from M into E and a geometric characterization of harmonic sections with respect to its equivariant lift. The second purpose is to show a version of Liouville theorem for harmonic sections and to prove that section M into E is a harmonic section if and only if it is parallel.

Key words: harmonic sections; fiber bundles; Liouville theorem, stochastic analisys on manifolds.

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1 Introduction

Let $\pi_E : (E,k) \to (M,g)$ be a Riemannian submmersion and σ be a section of π_E, that is, $\pi_E \circ M = Id_M$. We know that $TE = VE \oplus HE$ such that $VE = \ker(\pi_{\mathrm{E}*})$ and HE is the horizontal bundle ortogonal to VE. C. Wood has studied the harmonic sections in many context, see [14], [15], [16], [17], [18]. To recall, a harmonic sections is a minimal section for the vertical energy functional

$$E(\sigma) = \frac{1}{2}\int_M \|\mathbf{v}\sigma_*\|^2 vol(g),$$

where $\mathbf{v}\sigma_*$ is the vertical component of σ_*. Furthermore, in [14], Wood showed that σ is a minimizer of the vertical energy functional if

$$\tau_\sigma^v = \mathrm{tr}\nabla^{\mathrm{v}}\mathbf{v}\sigma_* = 0,$$

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where ∇^v is the vertical part of Levi-Civita connection on E, since π_E has totally geodesics fibers. Wood called σ a harmonic section if $\tau_\sigma^v = 0$.

In this work, we drop the Riemannian submersion condition of π_E and we mantain the fact that $TE = VE \oplus HE$ and that M is a Riemmanian Manifold. Let ∇^E be a symmetric connection on E, where E is not necessarily a Riemannian manifold. About these conditions we can define harmonic sections in the same way that Wood, only observing that ∇^v is vertical componente of ∇^E. There is no compatibility between ∇^E and Levi-Civita connection on M.

Furthermore, we restrict the context of our study. Let $P(M, G)$ be a Riemannian G-principal fiber bundle over a Riemannian manifold M such that the projection π of P into M is Riemmanian submmersion. Suppose that P has a connection form ω. Let $E(M, N, G, P)$ be an associated fiber bundle of P with fiber N. It is well know that ω yields horizontal spaces on E. Our goal is to study the harmonic sections of projection π_E.

Let $F : P \to N$ be a differential map. We call F a horizontally harmonic map if $\tau_F \circ (H \otimes H) = 0$, where H is the horizontal lift from M into P associated to ω.

Let σ be a section of π_E. It is well know that there exists a unique equivariant lift $F_\sigma : P \to N$ associated to σ. Our first purpose is to give an stochastic characterization for the harmonic section σ and the horizontally harmonic map F_σ. From these stochastic characterizations we show that a section σ of π_E is harmonic section if and only if F_σ is a horizontally harmonic map. This result is an extension of Theorem 1 in [14].

For our second purpose we consider $P(M, G)$ endowed with the Kaluza-Klein metric, M and G with the Brownian coupling property and N with the non-confluence property. About these conditions we show a version of Liouville Theorem and a version of result due to T. Ishiara in [5] to harmonic sections. As applications of our Liouville Theorem we can show the following. If we suppose that M is complete Riemmanian manifold with nonnegative Ricci curvature and its tangent bundle TM is endowed with the Sasaky metric, then the harmonic sections σ of π_{TM} are the 0-section. In the same way we can construct an ambient for Hopf fibrations, with Riemannian structure, such that harmonic sections are the 0-section.

2 Preliminaries

In this work we use freely the concepts and notations of P. Protter [12], E. Hsu [4], P. Meyer [9], M. Emery [2] and [3], W. Kendall [8] and S. Kobayashi and N. Nomizu [6]. We refer the reader to [1] for a complete survey about the objects of this section.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypothesis (see for example [2]). Our basic assumptions is that every stochastic process are continuos.

Definition 2.1 *Let M be a differential manifold. Let X be a process stochastic with valued in M. We call X a semimartingale if, for all f smooth on M, $f(X)$ is a real semimartingale.*

Let M be a differential manifold endowed whit symmetric connection ∇^M. Let X be a semimartingale in M and θ be a 1-form on M defined along X. We denote the Itô integral on M along the semimartingale X by $\int_0^t \theta d^{\nabla^M} X_s$. Let $b \in T^{(2,0)}M$ defined along X. We denote the quadratic integral on M along the semimartingale X by $\int_0^t b\ (dX, dX)_s$.

Let M and N be differential manifolds endowed with symmetric connections ∇^M and ∇^N, respectively. Let $F : M \to N$ be a differential map and θ be a section of TN^*. We have the following geometric Itô formula:

$$\int_0^t \theta\ d^{\nabla^N} F(X_s) = \int_0^t F^*\theta\ d^{\nabla^M} X_s + \frac{1}{2}\int_0^t \beta_F^*\theta\ (dX, dX)_s, \tag{1}$$

where β_F is the second fundamental form of F (see [1] or [13] for the definition of β_F).

Definition 2.2 *Let M be a differential manifold endowed with symmetric connection ∇^M. A semimartingale X with values in M is called a ∇^M-martingale if $\int_0^t \theta\ d^M X_s$ is a real local martingale for all $\theta \in \Gamma(TM^*)$.*

Definition 2.3 *Let M be a Riemannian manifold equipped with metric g. Let B be a semimartingale with values in M, we say that B is a g-Brownian motion in M if B is a ∇^g-martingale, where ∇^g is the Levi-Civita connection of g, and for any section b of $T^{(2,0)}M$ we have that*

$$\int_0^t b(dB, dB)_s = \int_0^t \mathrm{tr\ b_{B_s} ds}. \tag{2}$$

From (1) and (2) we deduce the useful formula:

$$\int_0^t \theta d^{\nabla^N} F(B_s) = \int_0^t F^*\theta d^{\nabla^g} B_s + \frac{1}{2}\int_0^t \tau_F^*\theta_{B_s} ds, \tag{3}$$

where τ_F is the tension field of F.

From formula (2) and Doob-Meyer decomposition it follows that F is an harmonic map if and only if it sends g-Brownian motions to ∇^N-martingales.

Definition 2.4 *Let M be a differential manifold endowed with symmetric connection ∇^M. M has the non-confluence of martingales property if for every filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, M-valued martingales X and Y defined over Ω and every finite stopping time T such that*

$$X_T = Y_T \quad a.s. \text{ we have } X = Y \quad over \quad [0, T].$$

Example 2.1 *Let $M = V$ be a n-dimensional vector space with flat connection ∇^n. Let X and Y be V-valued martingales. Suppose that there are a stopping time τ with respect to $(\mathcal{F}_t)_{t\geq 0}$, $K > 0$ such that $\tau \leq K < \infty$ and $X_\tau = Y_\tau$. Then straightforward calculus shows that $X_t = Y_t$ for $t \in [0, \tau]$.*

Definition 2.5 *A Riemmanian manifold M has the Brownian coupling property if for all $x_0, y_0 \in M$ we can construct a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t; t \geq 0)$ and two Brownian motions X and Y, not necessarily independents, but both adapted to filtration such that*

$$X_0 = x_0, Y_0 = y_0$$

and

$$\mathbb{P}(X_t = Y_t \text{ for some } t \geq 0) = 1.$$

The stopping time $T(X, Y) = \inf\{t > 0; X_t = Y_t\}$ is called coupling time.

Example 2.2 *Let M be a complete Riemannian manifold. In [7], W. Kendall has showed that if M is compact or M has nonnegative Ricci curvature then M has the Brownian coupling property.*

Let M be a Riemmanian manifold with metric g. Consider X and Y two g-Brownian motion in M which satisfies the Brownian coupling property and $X_0 = x, Y_0 = y$, where $x, y \in M$. Denote by $T(X, Y)$ their coupling time. The process $\bar{Y}$ is defined by

$$\bar{Y}_t = \begin{cases} Y_t & , \quad t \leq T(X, Y) \\ X_t & , \quad t \geq T(X, Y). \end{cases} \tag{4}$$

It is imediatelly that $\bar{Y}_0 = y_0$.

Proposition 2.1 *Let M be a Riemannian manifold with metric g. Suppose that M has the Brownian coupling property. Let X, Y be two g-Brownian motions in M which satisfies the Brownian coupling property. Then the process $\bar{Y}$ is a g-Brownian motion in M.*

Proof: It is a straightforward proof from definition of Brownian motion. □

3 Harmonic sections

Let $P(M,G)$ be a principal fiber bundle over M and $E(M,N,G,P)$ be an associate fiber bundle to $P(M,G)$. We denote the canonical projection from $P \times N$ into E by μ, namely, $\mu(p,\xi) = p \cdot \xi$. For each $p \in P$, we have the map $\mu_p : N \to E$ defined by $\mu_p(\xi) = \mu(p,\xi)$. Let $\sigma : E \to M$ be a section of projection π_E, that is, $\pi_E \circ \sigma = Id_M$. There exists a unique equivariante lift $F_\sigma : P \to N$ associated to σ which is defined by

$$F_\sigma(p) = \mu_p^{-1} \circ \sigma \circ \pi(p). \tag{5}$$

The equivariance property of F_σ is given by

$$F_\sigma(p \cdot g) = g^{-1} \cdot F_\sigma(p), \quad g \in G.$$

Let us endow P and M with Riemmanian metrics k and g, respectively, such that $\pi : (P,k) \to (M,g)$ is a Riemmanian submmersion. Let ω be a connection form on P. We observe that the connection form ω yields a horizontal structure on E, that is, for each $b \in E$, $T_bE = V_bE \oplus H_bE$, where $V_bE := \mathrm{Ker}(\pi_{Eb*})$ and H_bE is the horizontal subspace done by ω on E (see for example [6], pp.87). We denote by $\mathbf{v} : TE \to VE$ and $\mathbf{h} : TE \to HE$ the vertical and horizontal projection, respectively.

Let ∇^M denote the Levi-Civita connection on M and ∇^E be a symmetric connection on E. We follow B. O'Neill in [11] to define the Fundamental tensor T for vector fields X and Y on E by

$$T_X Y := \mathbf{h}\nabla^E_{\mathbf{v}X}\mathbf{v}Y + \mathbf{v}\nabla^E_{\mathbf{v}X}\mathbf{h}Y.$$

We are interested in connections ∇^E such that $T \equiv 0$. We observe that when π_E is a Riemannian submmersion the condition $T \equiv 0$ is equivalent to π_E has totally geodesic fibers.

We denote by ∇^v the vertical component of connection ∇^E on TE, that is, for X, Y vector fields on E we have

$$\nabla^v_X Y = \mathbf{v}\nabla^E_X(\mathbf{v}Y).$$

Let us denote ∇^x the induced connection of ∇^E over fiber $\pi_E^{-1}(x)$ for all $x \in M$. We endow N with a connection ∇^N such that, for each $p \in P$, μ_p is an affine map over its image, the fiber $\pi_E^{-1}(x)$ with $\pi(p) = x$.

Let σ be a section of π_E. Write $\sigma_* = \mathbf{v}\sigma_* + \mathbf{h}\sigma_*$, where $\mathbf{v}\sigma_*$ and $\mathbf{h}\sigma_*$ are the vertical and the horizontal component of σ_*, respectively. The second fundamental form for $\mathbf{v}\sigma_*$ is defined by

$$\beta^v_\sigma = \bar{\nabla}^v \circ \mathbf{v}\sigma_* - \mathbf{v}\sigma_* \circ \nabla^M,$$

where $\bar{\nabla}^v$ is the induced connection on $\sigma^{-1}E$. The vertical tension field is given by

$$\tau_\sigma^v = \mathrm{tr}\beta_\sigma^{\mathrm{v}}.$$

In the following we extend the definition given by C. M. Wood [15] of harmonic section.

Definition 3.1 *1. A section σ of π_E is called harmonic section if $\tau_\sigma^v = 0$;*
2. A differential map $F : P \to N$ is called horizontally harmonic if $\tau_F \circ (H \otimes H) = 0$, where H is horizontal lift from M into P.

Definition 3.2 *1. Let $\theta \in TE^*$. We call θ a vertical form if $\theta(X) = 0$ for every horizontal vector field on E.*
2. A E-valued semimartingale X is called a vertical martingale if, for every vertical form θ on E, $\int_0^t \theta d^{\nabla^v} X_s$ is a real local martingale.

Let us denote by β_μ^v the second fundamental form with respect to product connection $\nabla^{P\times N}$ and vertical connection ∇^v, that is,

$$\beta_\mu^v((X_1,\zeta_1),(X_2,\zeta_2)) \quad = \quad \bar{\nabla}^v_{(X_1,\zeta_1)}\mu_*(X_2,\zeta_2) - \mu_*\left(\nabla^{P\times N}_{(X_1,\zeta_1)}(X_2,\zeta_2)\right)$$

for X_1, X_2 vector fields on P and ζ_1, ζ_2 vector fields on N.

Lemma 3.1 *Let μ_p be an affine map, for each $p \in P$. For every point (p,ξ) in $P \times N$ we have that*

(i) *if X is a horizontal vector field on E, then $\mu_{p*}^{-1}(X) = 0$;*

(ii) $\bar{\nabla}^v_{(X_1,\zeta_1)}\mu_*(X_2,\zeta_2)_{\mu(p,\xi)} = \nabla^x_{\mu_{p*}(\zeta_1)}\mu_{p*}(\zeta_2)$, *for X_1, X_2 horizontal vectors fields on P and ζ_1, ζ_2 vectors fields on N;*

(iii) $\beta_\mu^v((X,\zeta),(X,\zeta))_{(p,\xi)}$ *is a horizontal vector field, where X is a horizontal vector field on P and ζ is a vector field on N.*

Proof: **(i)** The proof is straightforward.
(ii) Using definitions of $\bar{\nabla}^v$ and T we deduce that

$$\bar{\nabla}^v_{(X_1,\zeta_1)}\mu_*(X_2,\zeta_2) \quad = \quad T_{\mu_{p*}(\zeta_2)}\mu_{\xi*}(X_1) - \mathbf{v}[\mu_{p*}(\zeta_2), \mu_{\xi*}(X_1)] + \mathbf{v}\nabla^E_{\mu_{p*}(\zeta_1)}\mu_{p*}(\zeta_2).$$

From (i) and the fact that μ_{p*} is a diffeomorphism we see that $[\mu_{p*}(\zeta_2), \mu_{\xi*}(X_1)]$ is not vertical. For this reason and the assumption that $T \equiv 0$ we conclude that

$$\bar{\nabla}^v_{(X_1,\zeta_1)}\mu_*(X_2,\zeta_2) = \nabla^x_{\mu_{p*}(\zeta_1)}\mu_{p*}(\zeta_2),$$

where ∇^x is the induced connection in the fiber $\pi_E^{-1}(x)$ with $\pi(p) = x$.
(iii) Let $(p, \xi) \in P \times N$. Let X be a horizontal vector field on P and ζ be a vector field on N. From (ii) we see that

$$\beta_\mu^v((X,\zeta),(X,\zeta))_{(p,\xi)} = \nabla^x_{\mu_{p*}(\zeta)}\mu_{p*}(\zeta) - \mu_*\left(\nabla^{P\times N}_{(X,\zeta)}(X,\zeta)\right),$$

where $\pi(p) = x$. As $\nabla^{P\times N}_{(X,\zeta)}(X,\zeta) = \nabla^P_X X + \nabla^N_\zeta \zeta$ we have

$$\beta_\mu^x((X,\zeta),(X,\zeta))_{(p,\xi)} = \nabla^v_{\mu_{p*}(\zeta)}\mu_{p*}(\zeta) - \mu_{\xi*}\nabla^P_X X - \mu_{p*}\nabla^N_\zeta \zeta.$$

Since μ_p is an affine map, for each $p \in P$, it follows that

$$\beta_\mu^v((X,\zeta),(X,\zeta))_{(p,\xi)} = -\mu_{\xi*}\nabla^P_X X.$$

As π is a Riemannian submmersion we have

$$\beta_\mu^v((X,\zeta),(X,\zeta))_{(p,\xi)} = -\mu_{\xi*}\mathbf{h}(\nabla^P_X X),$$

where $\mathbf{h}(\nabla^P_X X)$ is the horizontal componente of $\nabla^P_X X$, which completes the proof. □

Now, we relate the geometric and stochastic concepts of harmonic section and horizontally harmonic map.

Theorem 3.1 *Let $P(M,G)$ be a Riemannian principal fiber bundle endowed with a connection form ω and M a Riemannian manifold such that the projection π of P into M is a Riemannian submmersion. Let $E(M,N,G,P)$ be an associated fiber to P endowed with a connection ∇^E such that its Fundamental tensor T is null. Moreover, suppose that N has a connection ∇^N such that μ_p is an affine map for each $p \in P$. Then*

- **(i)** *a E-valued semimartingale X is vertical martingale if and only if $\mu_Y^{-1} \circ X$ is a ∇^N- martingale in N, where $Y = \pi_E(X)^h$ is the horizontal lift of $\pi_E(X)$ to P;*
- **(ii)** *a section σ of π_E is harmonic section if and only if, for every g-Brownian motion B in M, $\sigma(B)$ is a vertical martingale;*
- **(iii)** *a equivariant lift F_σ associated to σ, σ a section of π_E, is horizontally harmonic map if and only if, for every horizontal Brownian motion B^h in P, $F_\sigma(B^h)$ is a ∇^N-martingale.*

Proof: **(i)** Let X be a semimartingale in E and θ be a vertical form on E. Let us denote $\xi = \mu_Y^{-1} \circ X$. As $X = \mu(Y, \xi)$ we have, by geometric Itô formula (1),

$$\begin{aligned}
\int_0^t \theta d^{\nabla^v} X_s &= \int_0^t \theta d^{\nabla^v} \mu(Y_s, \xi_s) \\
&= \int_0^t \mu^* \theta d^{\nabla^{P\times N}} (Y_s, \xi_s) + \frac{1}{2} \int \beta_\mu^{v*} \theta(d(Y_s, \xi_s), d(Y_s, \xi_s)) \\
&= \int_0^t \mu_{Y_s}^* \theta d^{\nabla^N} \xi_s + \int_0^t \mu_{\xi_s}^* \theta d^{\nabla^P} Y_s + \frac{1}{2} \int \beta_\mu^{v*} \theta(d(Y_s, \xi_s), d(Y_s, \xi_s)),
\end{aligned}$$

where third equality follows from Proposition 3.15 in [3]. Since $d^{\nabla^P} Y_s$ is horizontal, it follows that $\int_0^t \mu_{\xi_s}^* \theta d^{\nabla^P} Y_s = 0$. Hence

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \mu_{Y_s}^* \theta d^{\nabla^N} \xi_s + \frac{1}{2} \int \beta_\mu^{v*} \theta(d(Y_s, \xi_s), d(Y_s, \xi_s)).$$

Since θ is vertical form, from Lemma 3.1 we see that

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \mu_{Y_s}^* \theta d^{\nabla^N} \xi_s.$$

So we conclude that $\int_0^t \theta d^{\nabla^v} X_s$ is local martingale if and only if $\int_0^t \mu_{Y_s}^* \theta d^{\nabla N} \xi_s$ is too, and proof is complete.

(ii) Let B be a g-Brownian motion in M and θ be a vertical form on E. By formula (3),

$$\int_0^t \theta \ d^{\nabla^v} \sigma(B_s) = \int_0^t \sigma^* \theta \ d^{\nabla^M} B_s + \frac{1}{2} \int_0^t \tau_\sigma^{v*} \theta(B_s) \ ds.$$

We observe that $\int \sigma^* \theta \ d^{\nabla^M} B_s$ is a real local martingale. Since B and θ are arbitraries, Doob-Meyer decomposition assure that $\int_0^t \theta \ d^{\nabla^v} \sigma(B_s)$ is real local martingale if and only if τ_σ^v vanishes. From definitions of vertical martingale and harmonic section we conclude the proof.

(iii) Let B be a g-Brownian motion in M and B^h be a horizontal Brownian motion in P, that is,

$$dB^h = H_B dB, \tag{6}$$

where H is the horizontal lift of M to P. Set $\theta \in \Gamma(TN^*)$. By geometric Itô formula (1),

$$\int_0^t \theta \ d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t F_\sigma^* \theta \ d^{\nabla^P} B_s^h + \int_0^t \beta_{F_\sigma}^* \theta(dB^h, dB^h)_s.$$

From (6) we see that

$$\int_0^t \theta \ d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t H^* F_\sigma^* \theta \ d^{\nabla^M} B_s + \int_0^t \beta_{F_\sigma}^* \theta(H_B dB, H_B dB)_s.$$

As B is Brownian motion we have

$$\int_0^t \theta \ d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t H^* F_\sigma^* \theta \ d^{\nabla^M} B_s + \int_0^t (\tau_{F_\sigma}^H)^* \theta(B_s) ds,$$

where $\tau_{F_\sigma}^H = \tau_{F_\sigma} \circ (H \otimes H)$. Since θ and B are arbitraries, Doob-Meyer decomposition shows that $\int_0^t \theta d^{\nabla^N} F_\sigma(B_s^h)$ is real local martingale if and only if $\tau_{F_\sigma}^H$ vanishes. From definitions of martingale and horizontally harmonic map we conclude the proof. □

Now we give an extension of the characterization of harmonic sections obtained by C.M. Wood, see Theorem 1 in [15].

Theorem 3.2 *Under the hypotheses of Theorem 3.1, a section σ of π_E is harmonic section if and only if F_σ is horizontally harmonic map.*

Proof: Let B be a arbitrary g-Brownian motion in M and B^h be a horinzontal lift of B in P, see equation (6).

Suppose that σ is a harmonic section. Theorem 3.1, item (ii), shows that $\sigma(B)$ is a vertical martingale. But $\mu_{B^h}^{-1} \circ \sigma(B)$ is a ∇^N-martingale, which follows from Theorem 3.1, item (i). Since $F_\sigma(B^h) = \mu_{B^h}^{-1} \circ \sigma \circ \pi(B^h)$, it follows that $F_\sigma(B^h)$ is a ∇^N-martingale. Finally, Theorem 3.1, item (iii), shows that F_σ is horizontally harmonic map.

Conversely, suppose that F_σ is a horizontally harmonic map. Theorem 3.1, item (iii), shows that $F_\sigma(B^h)$ is a ∇^N-martingale. Since $F_\sigma(B^h) = \mu_{B^h}^{-1} \circ \sigma \circ \pi(B^h)$, it follows that $\mu_{B^h}^{-1} \circ \sigma(B)$ is a ∇^N-martingale. From Theorem 3.1, item (i), we see that $\sigma(B)$ is a vertical martingale. We conclude from Theorem 3.1, item (ii), that σ is a harmonic section. □

4 A Liouville theorem for harmonic sections

We begin this section defining the Kaluza-Klein metric on $P(M, G)$. Let $P(M, G)$ be a principal fiber bundle endowed with a connection form ω, M be a Riemannian manifold with a metric g and h be a bi-invariant metric on G. The Kaluza-Klein metric is defined by

$$k = \pi^* g + \omega^* h. \tag{7}$$

From now on $P(M,G)$ is endowed with the Kaluza-Klein metric.

We will denote by d_P and d_G the Riemannian distance of P and G, respectively.

Lemma 4.1 *Let $P(M,G)$ be a principal fiber bundle whit a Kaluza-Klein metric k, where g is the Riemannian metric on M and h is the bi-invariant metric on G associated to k. The following assertions are holds:*

(i) *Let $\tau : [0,1] \to P$ be a differential curve such that $\tau(t) = u \cdot \mu(t)$ with $\tau(0) = u$ and $\mu(t) \in G$, then*

$$\int_0^1 k(\dot{\tau}(t), \dot{\tau}(t))^{\frac{1}{2}} dt = \int_0^1 h(\dot{\mu}(t), \dot{\mu}(t))^{\frac{1}{2}} dt.$$

(ii) *Let $\tau : [0,1] \to P$ be a differential curve. If γ is a curve in M and if μ is a curve in G such that $\tau = \gamma(t)^h \cdot \mu(t)$, then*

$$\int_0^1 k(\dot{\tau}(t), \dot{\tau}(t))^{\frac{1}{2}} dt \leq \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}} dt + \int_0^1 h(\dot{\mu}(t), \dot{\mu}(t))^{\frac{1}{2}} dt$$

(iii) *Let $x \in M$ and $u, v, w \in \pi^{-1}(x)$. If a, b are points in G such that $v = u \cdot a$ and $w = u \cdot b$, then*

$$d_P(v,w) = d_G(a,b).$$

Proof: **(i)** and **(ii)** The proofs are straightforward.
(iii) Let $\tau : [0,1] \to P$ be a differential curve such that $\tau(0) = v$ and $\tau(1) = w$. Consider a curve γ in M such that $\pi(\tau) = \gamma$. There exists a differential curve μ in G such that $\mu(0) = a$, $\mu(1) = b$ and $\tau = \gamma^h \cdot \mu$. We observe that $\gamma(0) = x$ and $\gamma(1) = x$. This gives $\int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}} dt = 0$. Thus from item (i) and item (ii) we conclude that

$$\int_0^1 k(\dot{\tau}(t), \dot{\tau}(t))^{\frac{1}{2}} dt = \int_0^1 h(\dot{\mu}(t), \dot{\mu}(t))^{\frac{1}{2}} dt.$$

Therefore it is only necessary to consider vertical curves. It follows that $d_P(v,w) = d_P(u \cdot a, u \cdot b) = d_G(a,b)$, by definition of Riemmanian distance. $\square$

Theorem 4.1 *Let $P(M,G)$ be a principal fiber bundle equipped with Kaluza-Klein metric and $E(M,N,G,P)$ be an associated fiber to P. Let ∇^E and ∇^N be connetions on E and N, respectively, such that the Fundamental tensor T is null and μ_p is an affine map for each $p \in P$. Moreover, if N has the non-confluence martingales property and if M and G have the Brownian coupling property, then*

(i) *a section σ of π_E is harmonic section if and only if F_σ is constante map;*

(ii) *the left action of G into N has a fix point if there exists a harmonic section σ of π_E;*

(iii) *a section σ of π_E is harmonic section if and only if σ is parallel.*

Proof: **(i)** We first suppose that F_σ is a constante map. Then it is immediately that $\tau^v_\sigma = 0$, so σ is harmonic section.

Conversely, the proof will be divided into two parts. Firstly, we found a suitable stopping time τ. After, we use τ to prove that F_σ is constant over P.

Choose $x, y \in M$ arbitraries. By assumption about M, there exists two g-Brownian motion X and Y in M such that $X_0 = x$ and $Y_0 = y$, which satisfy the Brownian coupling property. Consequently, the coupling time $T(X,Y)$ is finite. Proposition 2.1 now assures that the process

$$\bar{Y}_t = \begin{cases} Y_t & , \quad t \leq T(X,Y) \\ X_t & , \quad t \geq T(X,Y) \end{cases} \tag{8}$$

is a g-Brownian motion in M.

Let $a, b \in G$ be arbitraries points. Since G has the Brownian coupling property, we have two h-Brownian motion μ and ν in G such that $\mu_0 = a$, $\nu_0 = b$. Moreover, there is a finite coupling time $T(\mu,\nu)$. But the process

$$\bar{\nu}_t = \begin{cases} \nu_t & , \quad t \leq T(\mu,\nu) \\ \mu_t & , \quad t \geq T(\mu,\nu) \end{cases} \tag{9}$$

is a h-Brownian motion in G, which follows from Proposition 2.1.

Set $u, v \in P$ such that $\pi(u) = x$ and $\pi(v) = y$. Consider two horizontal Brownian motion X^h and $\bar{Y}^h$ in P such that $X^h_0 = u$ and $\bar{Y}^h_0 = v$. Define $\tau = T(X,Y) \vee T(\mu,\nu)$. We claim that

$$X^h_t \cdot \mu_t = \bar{Y}^h_t \cdot \bar{\nu}_t, \text{ a.s. } \forall \ \ t \geq \tau. \tag{10}$$

In fact, we need consider two cases. First, suppose that $T(X,Y) \leq T(\mu,\nu)$. For all $t \geq T(\mu,\nu)$ we have

$$d_P(X^h_t \cdot \mu_t, \bar{Y}^h_t \cdot \bar{\nu}_t) = d_P(X^h_t \cdot \mu_t, \bar{Y}^h_t \cdot \mu_t) = d_P(R_{\mu_t} X^h_t, R_{\mu_t} \bar{Y}^h_t).$$

Since k is the Kaluza-Klein metric, it follows that

$$d_P(X^h_t \cdot \mu_t, \bar{Y}^h_t \cdot \bar{\nu}_t) = d_M(X_t, \bar{Y}_t).$$

From (8) we conclude that (10) is satisfied for all $t \geq T(\mu, \nu)$.

In the other side, suppose that $T(X,Y) \geq T(\mu,\nu)$. For all $t \geq T(X,Y)$, Lemma 4.1, item (iii), assures that

$$d_P(X_t^h \cdot \mu_t, \bar{Y}_t^h \cdot \bar{\nu}_t) = d_P(X_t^h \cdot \mu_t, X_t^h \cdot \bar{\nu}_t) = d_G(\mu_t, \bar{\nu}_t).$$

From (9) we conclude that (10) is satisfied for all $t \geq T(X,Y)$.

Setting $t \geq \tau$ we obtain $F_\sigma(X_t^h \cdot \mu_t) = F_\sigma(\bar{Y}_t^h \cdot \bar{\nu}_t)$. Since F_σ is equivariant by right action, $\mu_t^{-1} \cdot F_\sigma(X_t^h) = \bar{\nu}_t^{-1} \cdot F_\sigma(\bar{Y}_t^h)$. Because $\mu_t = \bar{\nu}_t$ for $t \geq \tau$, we conclude that $F_\sigma(X_t^h) = F_\sigma(\bar{Y}_t^h)$.

Since σ is a harmonic section, from Theorem 3.2 we see that F_σ is a horizontally harmonic map. Theorem 3.1 now shows that $F_\sigma(X_t^h)$ and $F_\sigma(\bar{Y}_t^h)$ are ∇^N-martingales in N. Since N has non-confluence martingales property,

$$F_\sigma(X_0^h) = F_\sigma(\bar{Y}_0^h).$$

It follows immediately that $F_\sigma(u) = F_\sigma(v)$. Consequently, F_σ is a constant map.

(ii) Let σ be a harmonic section of π_E. From item (i) there exists $\xi \in N$ such that $F_\sigma(p) = \xi$ for all $p \in P$. We claim that ξ is a fix point. In fact, set $a \in G$. From equivariant property of F_σ we deduce that

$$a \cdot \xi = a \cdot F_\sigma(p) = F_\sigma(p \cdot a^{-1}) = \xi.$$

(iii) Let σ be a section of π_E. Suppose that σ is parallel. Then $\sigma_*(X)$ is horizontal for all $X \in TM$ (see for example [6], pp.114). This gives $\mathbf{v}\sigma_*(X) = 0$. Then it is clear, by definition, that σ is harmonic section.

Suppose that σ is a harmonic section. From item (i) it follows that there exists $\xi \in N$ such that $F_\sigma(p) = \xi$ for all $p \in P$. By definition of equivariant lift,

$$\sigma(x) = \sigma \circ \pi(p) = \mu(p, \xi) = \mu_\xi(p), \quad \pi(p) = x,$$

where μ_ξ is an application from P into E. Let $v \in T_xM$ and let $\gamma(t)$ be a curve in M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then

$$\sigma_*(v) = \left.\frac{d}{dt}\right|_0 \sigma \circ \gamma(t) = \left.\frac{d}{dt}\right|_0 \mu_\xi \circ \gamma^h(t) = \mu_{\xi *}(\dot{\gamma}^h(0)),$$

where γ^h is the horizontal lift of γ into P. Since $\dot{\gamma}^h(0)$ is horizontal vector in P, so is $\mu_{\xi *}(\dot{\gamma}^h(0))$ in E (see for example [6], pp.87). Therefore $\sigma_*(v)$ is horizontal vector. So we conclude that σ is parallel. □

Tangent bundle

Let M be a complete Riemannian manifold which is compact or has non-negative Ricci curvature. Let OM be the ortonormal frame bundle endowed whit the Kaluza-Klein metric. Let TM be the tangent bundle equipped with the Sasaky metric g_s. Thus π_E is a Riemannian submersion with totally geodesic fibers and, for each $p \in P$, μ_p is a isometric map (see for example [10]). From these assumptions and Examples 2.1 and 2.2 it follows that the hypotheses of Theorem 4.1 are satisfied.

Proposition 4.2 *Under conditions stated above, if σ is a harmonic section of π_{TM}, then σ is the 0-section.*

Proof: Let σ be a harmonic section of π_{TM}. By Theorem 4.1, item (i), there exists $\xi \in N$ such that $F_\sigma(u) = \xi$ for all $u \in P$. Moreover, by item (ii) ξ is a fix point of left action of $O(n, \mathbb{R})$ into $\mathbb{R}^n$. We observe that $0 \in \mathbb{R}^n$ is the unique fix point to this left action. Thus get $F_\sigma(u) = 0$. Therefore σ is the 0-section. $\square$

Hopf fibration

Let $S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}$ be a Hopf fibration. It is well know that $S^{2n-1}(\mathbb{CP}^{n-1}, S^1)$ is a principal fiber bundle. We recall that $U(1) \cong S^1$. Let ϕ be the aplication of $U(1) \times \mathbb{C}^m$ into $\mathbb{C}^m$ given by

$$(g, (z_1, \ldots, z_m)) \to g \cdot (z_1, \ldots, z_m) = (gz_1, \ldots, gz_m). \tag{11}$$

Clearly, ϕ is a left action of $U(1)$ into $\mathbb{C}^m$. Thus, we can consider $\mathbb{C}^m$ as standard fiber of associate fiber $E(\mathbb{CP}^{n-1}, \mathbb{C}^m, S^1, S^{2n-1})$, where $E = S^{2n-1} \times_{U(1)} \mathbb{C}^m$. We are considering the canonical scalar product $<,>$ on $\mathbb{C}^n$ and the induced Riemannian metric g on $\mathbb{CP}^{n-1}$. Since $U(1)$ is invariant by $<,>$, there exists one and only one Riemannian metric $\hat{g}$ on E such that π_E is a Riemannian submersion from $(E, \hat{g})$ to (M, g) with totally geodesic fibers isometrics to $(N, <,>)$ (see for example [13]). From these assumptions and examples 2.1 and 2.2 wee see that hypotheses of Theorem 4.1 are holds.

Proposition 4.3 *Under conditions stated above, if σ is a harmonic section of π_E, then σ is the 0-section.*

Proof: We first observe that $(0, \ldots, 0)$ is the unique fix point to the left action (11). Since σ is harmonic section, from Theorem 4.1 we see that F_σ

is constant map and $F_\sigma(p) = (0, \dots, 0)$ for all $p \in S^{2n-1}$. Therefore σ is the 0-section. $\square$

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