On the variations of the Betti numbers of regular levels of Morse flows

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Abstract

We generalize results in [C-dR] by completely describing how the Betti numbers of the boundary of an orientable manifold vary after attaching a handle, when the homology coefficients are in $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$ with p prime. Next we consider the Ogasa invariant associated with handle decompositions of manifolds. We make use of the above results in order to obtain upper bounds for the Ogasa invariant of product manifolds.

Key words: Betti numbers, handle decomposition, Conley index, Ogasa invariant.

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1 Introduction

It is well known that Morse-Smale systems exhibit gradient-like behaviour. In [Sm], [Mey] it was proved that they possess Lyapunov functions. In fact, Conley [Co] generalized these results by proving the existence of Lyapunov functions for any continuous flow on smooth compact manifolds. Hence the question of understanding the topology of level sets associated with Lyapunov functions is quite natural.

Morse-Smale flows on a smooth n-dimensional manifold M were considered together with a handle decomposition associated with a Lyapunov function in [C-dR]. Thus, after the attachment of a handle corresponding to a singularity (or a round handle corresponding to a periodic orbit) one can consider the effect on the new regular level set. The authors completely describe how the Betti numbers of the level set vary after attaching a (round) handle when the homology coefficients are taken in $\frac{\mathbf{Z}}{2\mathbf{Z}}$. These results were generalized in [Be-Me-dR1] by considering continuous flows associated with Lyapunov functions on n-dimensional manifolds. More specifically, a flow in the isolating block N of an isolated invariant set S with possibly complicated dynamical behavior was considered. The effect on the Betti numbers of the regular level sets corresponding to the incoming N^+ and outgoing N^- boundaries of the flow in N were determined in terms of the homology indices of S.

A new invariant associated with a handle decomposition of a smooth manifold was introduced in [O]. Ogasa suggests that one way of measuring the simplicity of a Morse flow is to compute, for each regular level, the sum of its Betti numbers, and then take the maximum of the obtained values. Given an n-dimensional manifold, its *Ogasa invariant* is then the minimum, over all Morse flows, of these maxima. In other words, a Morse flow realizing the Ogasa invariant of the manifold is one for which the *maximum of the sums of the Betti numbers of each regular level* is *the smallest* possible.

In this paper, we consider handle decompositions of n-dimensional manifolds from a dynamical point of view in order to understand the Ogasa invariant as a *detector* of complicated dynamical behavior as we will motivate subsequently.

Our first result generalizes results in [C-dR] by completely describing the effect that attaching a handle has on the Betti numbers of the boundary, when the homology coefficients are chosen among the most standard ones, that is, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$, with p prime. Since such a description is technical, we state it in a simpler way and refer the reader to Theorem 3.1 for the detailed version.

Theorem 1. *Let N be an n-dimensional manifold with compact orientable boundary $\partial N = N^+ \sqcup N^-$, endowed with a Morse flow entering through the regular level set N^+, exiting through the regular level set N^- and containing a unique singularity of index l inside N. Let the homology coefficients be chosen in $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$, with p prime. Then the Betti numbers of N^+ and N^- are the same except for both β_l and β_{n-1-l} or both β_{l-1} and β_{n-l}, for which the behaviour is classified.*

Roughly speaking, up to few exceptions, attaching a handle of index l can either increase by 1 the l-th Betti number and its dual (i.e. $\beta_l(N^+) = \beta_l(N^-) + 1$ and $\beta_{n-1-l}(N^+) = \beta_{n-1-l}(N^-) + 1$) or can decrease by 1 the $(l-1)$-th Betti number of N^+ and its dual $(n-l)$. The most significant exception is given in the case $n = 2i$ by $l = i$ for which there is also the possibility for all the Betti numbers to keep unchanged (and when it happens we shall speak of *invariant* handles).

On one hand, our generalization implies that

Corollary 2. *All the results and machinery using Conley Index Theory and continuation of Lyapunov graphs developed in [Be-Me-dR1], [Be-Me-dR2], [Be-dR-V] and [Be-dR-Man-V] are still true, independently of the homology coefficients, provided that they be chosen in $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$, with p prime.*

On the other hand, Theorem 1 allows us to use the Ogasa invariant to detect complicated chain recurrent components of a flow in the following sense. Our result tells us that attaching a handle can change the sum of the Betti numbers of the regular levels by 0, 2 or -2. Hence, for instance, if a manifold is known to have Ogasa invariant equal to 32 and we know that the sum of the Betti numbers of each section of a filtration of a gradient-like flow is less than, say, 10, then, it necessarily means that at least one of the isolating neighbourhoods of the filtration cannot be built with less than 22 handles[1], which must reveal a complexity of the chain recurrent component inside it.

The problem of this invariant is that it seems very difficult to be computed, except for some easy examples. In particular it is very difficult to find significant lower bounds. Even for manifolds M which are the connected sum $X \sharp Y$ of two manifolds X and Y, it is very easy to show that the Ogasa invariant of M is less than or equal to the maximum between the Ogasa invariant of X and that of Y (see [O]). However, we want to emphasize that equality might not hold, as $M = \mathbf{CP}^2 \sharp (\mathbf{S}^2 \times \mathbf{S}^2)$ shows[2].

[1] In fact, let us consider any handle decomposition of each of the isolating neighbourhoods of the given gradient-like flow. If we compute the sum of the Betti numbers of the regular levels of the corresponding Morse flow, at least in one of them we must reach at least 32 by definition of the Ogasa invariant. Under our assumptions, abstractly speaking, the most economical way of reaching it would be between two sections for which the sum of the Betti numbers is 10 with 22 handles, 11 of which increase by 2 the sum of the Betti numbers of the boundary in order to reach the value 32, and 11 of which decrease by 2 the sum of the Betti numbers of the boundary in order to go back to 10.

[2] The Ogasa invariant of $\mathbf{CP}^2$ is 2, that of $\mathbf{S}^2 \times \mathbf{S}^2$ is 4, that of M is 2 because M can also be seen as $\mathbf{CP}^2 \sharp (\mathbf{CP}^2 \sharp - \mathbf{CP}^2)$.

In order to investigate possible ways to compute such an invariant, in the second part of this paper we focus on product manifolds $M = P \times Q$.

We first consider a way of building a handle decomposition of the product M from handle decompositions of the factors P and Q. By applying Theorem 1 to this construction we can prove that the following upper bounds hold.

Theorem 3. *Let R denote one of the following rings: $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{R}$. Let P be a p-dimensional closed orientable manifold and let $\mathcal{H}_P$ be a handle decomposition of P of L_P handles. Let Q be a q-dimensional closed orientable manifold and let $\mathcal{H}_Q$ be a handle decomposition of Q realizing the Ogasa invariant of Q denoted by $\nu(Q)$. Let $\beta_k(P;\ R)$ denote the k-th Betti number of P, computed with respect to R, the ring of the homology coefficients. Then:*

1. *if $\mathcal{H}_Q$ contains no invariant handle, we have*

$$\nu(P \times Q;\ R) \leq L_P + (\nu(Q;\ R) - 1) \cdot \sum_{j=0}^{p} \beta_j(P;\ R)$$

2. *if $\mathcal{H}_Q$ contains at least one invariant handle, we have*

$$\nu(P \times Q;\ R) \leq 2\lfloor \frac{L_P}{2} \rfloor + \nu(Q;\ R) \cdot \sum_{j=0}^{p} \beta_j(P;\ R)$$

Note that in the original definition of [O], the homology and, consequently, the Betti numbers, are computed by considering coefficients in $\mathbf{R}$. We have naturally extended the definition by adding the dependence on the coefficient ring R.

Concerning these inequalities, they can be sharp in some cases, e.g. inequality of Item 1 for $\mathbf{S}^1 \times \mathbf{S}^2$, but there are examples for which the opposite inequality doesn't hold (of course, even when interchanging the role of P and Q). An interesting example of the latter case is $L_{3,1} \times \mathbf{S}^2$, where $L_{3,1}$ is the lens space associated with the couple $(3, 1)$ and the possible homology coefficients are $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{2\mathbf{Z}}$. Theorem 3 insures that $\nu(L_{3,1} \times \mathbf{S}^2) \leq 6$, but explicit computation (see Subsection 4.2) shows that $\nu(L_{3,1} \times \mathbf{S}^2) = 4$. The reason why the Ogasa invariant is less than expected is due to the presence of torsion of order 3 in the regular levels. In this sense, this example is new with respect to those in [O].

By looking closer at Theorem 3, other related results and questions naturally arise (see Subsection 4.4 for further details).

Finally, there are two ways of considering a Morse flow to be a "simplest" one: firstly, the classical, by minimizing the number of singularities; secondly, Ogasa's, by minimizing the complexity of the regular levels. It is interesting to compare the two approaches. For instance, for $\mathbf{S}^p \times \mathbf{S}^q$, the Ogasa invariant is four and is achieved by a handle decomposition corresponding to a Morse flow having the minimal number of singularities. In general, it remains an open question for which classes of manifolds, if not all, the Ogasa invariant is achieved by a handle decomposition corresponding to a Morse flow having the minimal number of singularities.

The paper is organized as follows. Section 2 contains background material. Section 3 is devoted to Theorem 1 and its proof. In Section 4 we study product manifolds: in particular we prove Theorem 3 and discuss its consequences.

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2 Notation and general definitions

2.1 Handle decompositions of a manifold

The theory of handle decompositions is very classical. We briefly recall the needed definitions and set our notation. Let $\mathbf{D}^m$ denote the m-dimensional closed ball. We say that the m-dimensional manifold B is obtained by gluing the *index-q handle* h_q to the (maybe empty) m-dimensional manifold A if:

- h_q is homeomorphic to $\mathbf{D}^m$
- B is homeomorphic to the space $A \cup_\Phi h_q$, where

$$\Phi : \mathbf{S}^{q-1} \times \mathbf{D}^{m-q} \subset \partial h_q \longrightarrow \partial A$$

  is a homeomorphism onto its image.

The homeomorphism Φ above is classically called the *attaching map*, its domain is called the *attaching region* of the q-handle h_q, while the complementary set $\mathbf{D}^q \times \mathbf{S}^{m-q-1}$ of ∂h_q is called the *belt region* of h_q.

Let M be a given m-dimensional closed manifold. Roughly speaking, an ordered handle decomposition $\mathcal{H}$ of M is a sequence

$$M_0, M_1, \ldots, M_{L_\mathcal{H}}$$

of $(L_\mathcal{H} + 1)$ m-dimensional manifolds such that:

- $M_0 = \emptyset$, $M_1 = \mathbf{D}^m$ and $M_{L_\mathcal{H}} = M$;
- for all $j = 1, \ldots, L_\mathcal{H}$, M_j is obtained by gluing a handle to M_{j-1}.

More precisely, an *(ordered) handle decomposition* also contains explicitly the information about the gluing, so that in the sequel, an ordered handle decomposition of M will be denoted by

$$\mathcal{H} = \left[(h_0^{(1)}, \varphi_1), \ldots, (h_j^{(l)}, \varphi_l), \ldots, (h_m^{L_\mathcal{H}}, \varphi_{L_\mathcal{H}})\right]$$

where

- $L_\mathcal{H}$ denotes the total number of handles in the decomposition;
- the subscript j in $h_j^{(l)}$ denotes the index of the handle $h_j^{(l)}$;
- the exponent (l) in $h_j^{(l)}$ indicates that the handle $h_j^{(l)}$ is the l-th handle to be glued in the ordered handle decomposition $\mathcal{H}$;
- if M_l is the manifold obtained after gluing the first l handles of the decomposition, then $\varphi_l : \mathbf{S}^{j-1} \times \mathbf{D}^{m-j} \longrightarrow \partial M_{l-1}$ is the attaching map associated with $h_j^{(l)}$, that is, the map describing how the handle $h_j^{(l)}$ is glued to M_{l-1} in order to build M_l; in particular $\varphi_1 : \emptyset \longrightarrow \emptyset$ is always the empty map.

Brackets are there to emphasize that the decomposition is ordered. Sometimes, we shall write, for short, $\mathcal{H} = \left[h_0^{(1)}, \ldots, h_j^{(l)}, \ldots, h_m^{L_\mathcal{H}}\right]$ but it is understood that the gluing comes together with the handle. As an example, the canonical two-handle decomposition of the sphere $\mathbf{S}^n$ will be denoted by $\left[h_0^{(1)}, h_n^{(2)}\right]$ and the underlying $\varphi_2 : \mathbf{S}^{n-1} \longrightarrow \mathbf{S}^{n-1}$ is for instance the identity on $\mathbf{S}^{n-1}$.

2.2 Homology of product manifolds

Let us briefly recall here the *Künneth formula* (see for instance [Mat] or [Br]), which allows us to compute the homology of the product of two manifolds.

Theorem 2.1. *(Künneth formula) Let X and Y be two finitely generated free complexes. Then, for all k we have*

$$H_k(X \times Y) = \left(\bigoplus_{a+b=k} H_a(X) \otimes H_b(Y) \right) \oplus \left(\bigoplus_{a+b=k-1} Tor(H_a(X), H_b(Y)) \right)$$

where $\otimes$ denotes the tensor product*, while $Tor(\cdot, \cdot)$ denotes the* torsion product*.*

In [Mat] one can find the main rules for the computation of these products.

2.3 Homology with coefficients, Betti numbers and Ogasa invariant

In the sequel we shall be interested in considering homology groups with coefficients in an Abelian group G which, for us, will be chosen among $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$.

Theorem 2.2. *(Universal Coefficient Theorem) For any Abelian group G, any simplicial complex X and any integer k, we have*

$$H_k(X;\ G) = (H_k(X;\ \mathbf{Z}) \otimes G) \oplus (Tor(H_{k-1}(X;\ \mathbf{Z}), G))$$

Definition 2.3. *(Betti numbers) Let R denote the ring of the homology coefficients, which can be chosen among $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$. If $R = \mathbf{Z}$, then for $j = 0 \dots (n-1)$ the j-th Betti number of the $(n-1)$-dimensional manifold N, denoted by $\beta_j(N;\ \mathbf{Z})$, is defined as the rank of the Abelian group $H_j(N;\ \mathbf{Z})$. In all the other cases, R is a field, and for $j = 0 \dots (n-1)$ the j-th Betti number of the $(n-1)$-dimensional manifold N, denoted by $\beta_j(N;\ R)$, is defined as the dimension of the vector space $H_j(N;\ R)$.*

Note that, when N is orientable, by the Universal Coefficient Theorem, the Betti numbers relative to $\mathbf{Z}$ are the same as those relative to $\mathbf{Q}$ or $\mathbf{R}$.

Definition 2.4. *(Ogasa invariant, [O]) Let R denote the ring of the homology coefficients, which can be chosen among $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{R}$. For any ordered handle decomposition $\mathcal{H}$ of M, let*

$$\nu_{\mathcal{H}}(M;\ R) = \max_{l=1,\dots,L_{\mathcal{H}}} \left(\sum_{i=0}^{n-1} \beta_i(N_l;\ R) \right)$$

where $\beta_i(N_l;\ R)$ denotes the i-th Betti number of the $(n-1)$-dimensional manifold $N_l = \partial M_l$, which is the boundary of the manifold obtained after attaching the first l handles of $\mathcal{H}$. The Ogasa invariant of the manifold M (relative to R) is the number defined by:

$$\nu(M;\ R) = \min_{\mathcal{H}}\ \nu_{\mathcal{H}}(M;\ R)$$

where the minimum is taken over all of the ordered handle decompositions of M.

Note that in the original paper [O], the author only considers the case where R is $\mathbf{R}$, the field of the real numbers.

3 The effect on the Betti numbers of regular levels after handle attachments

3.1 Main result

In this section we prove the following theorem which describes explicitly how the Betti numbers of the regular levels may change when only a Morse singularity of index l is contained in between them. It is a generalization, concerning the homology coefficients, of the analogous result proved in [C-dR] for coefficients in $\frac{\mathbf{Z}}{2\mathbf{Z}}$.

Theorem 3.1. *Let N be an n-dimensional manifold obtained by attaching a handle of index l to the collar of an $(n-1)$-dimensional closed orientable manifold denoted by N^-. Let N^+ denote the new boundary $\partial N \backslash N^-$. Let R denote the ring of the homology coefficients, which can be chosen among $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{R}$. For all $k = 0, \dots, (n-1)$ let $\beta_k(N^-;\ R)$ (resp. $\beta_k(N^+;\ R)$) denote the k-th Betti number of N^- (resp. N^+), computed with respect to R. Then we have:*

1. *if $l = 0$ then*
$$\begin{cases} \beta_0(N^+;\ R) = \beta_0(N^-;\ R) + 1 & \\ \beta_{n-1}(N^+;\ R) = \beta_{n-1}(N^-;\ R) + 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq 0 \text{ and } k \neq n-1; \end{cases}$$
if $l = n-1$ then
$$\begin{cases} \beta_0(N^+;\ R) = \beta_0(N^-;\ R) - 1 & \\ \beta_{n-1}(N^+;\ R) = \beta_{n-1}(N^-;\ R) - 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq 0 \text{ and } k \neq n-1; \end{cases}$$

2. *if $n = 2i+1$*
   (a) *if $l = i$ then*
$$\textit{either} \quad \begin{cases} \beta_i(N^+;\ R) = \beta_i(N^-;\ R) + 2 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq i \end{cases}$$
$$\textit{or} \quad \begin{cases} \beta_{i-1}(N^+;\ R) = \beta_{i-1}(N^-;\ R) - 1 & \\ \beta_{i+1}(N^+;\ R) = \beta_{i+1}(N^-;\ R) - 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq i-1 \text{ and } k \neq i+1; \end{cases}$$
   (b) *if $l = i+1$ then*
$$\textit{either} \quad \begin{cases} \beta_{i-1}(N^+;\ R) = \beta_{i-1}(N^-;\ R) + 1 & \\ \beta_{i+1}(N^+;\ R) = \beta_{i+1}(N^-;\ R) + 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq i-1 \text{ and } k \neq i+1 \end{cases}$$
$$\textit{or} \quad \begin{cases} \beta_i(N^+;\ R) = \beta_i(N^-;\ R) - 2 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq i; \end{cases}$$

3. *if $n = 2i$ and $l = i$ then*
$$\textit{either} \quad \beta_k(N^+;\ R) = \beta_k(N^-;\ R) \quad \text{for all } k$$
$$\textit{or} \quad \begin{cases} \beta_i(N^+;\ R) = \beta_i(N^-;\ R) + 1 & \\ \beta_{i-1}(N^+;\ R) = \beta_{i-1}(N^-;\ R) + 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq i \text{ and } k \neq i-1 \end{cases}$$
$$\textit{or} \quad \begin{cases} \beta_{i-1}(N^+;\ R) = \beta_{i-1}(N^-;\ R) - 1 & \\ \beta_i(N^+;\ R) = \beta_i(N^-;\ R) - 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq i \text{ and } k \neq i-1; \end{cases}$$

4. for all the other cases, we have

$$\textit{either}\quad \begin{cases} \beta_l(N^+;\ R) = \beta_l(N^-;\ R) + 1 & \\ \beta_{n-1-l}(N^+;\ R) = \beta_{n-1-l}(N^-;\ R) + 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq l \text{ and } k \neq n-1-l \end{cases}$$

$$\textit{or}\quad \begin{cases} \beta_{l-1}(N^+;\ R) = \beta_{l-1}(N^-;\ R) - 1 & \\ \beta_{n-l}(N^+;\ R) = \beta_{n-l}(N^-;\ R) - 1 & \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) & \text{for all } k \neq l-1 \text{ and } k \neq n-l. \end{cases}$$

Corollary 3.2. *The Ogasa invariant of an orientable closed n-dimensional manifold M is always even and if M admits a handle decomposition of $\tilde{L}_M$ non-invariant handles, then, for R equal to $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{R}$, we have that $\nu(M;\ R) \leq \tilde{L}_M$.*

3.2 Preliminary Lemmas

First let us note that, by the Universal Coefficient Theorem, the Betti numbers relative to $\mathbf{R}$ and $\mathbf{Q}$ are, in our setting, the same as those computed with respect to $\mathbf{Z}$. Let us then assume R to be $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime) or $\mathbf{Z}$.

The main ingredient of the proof of Theorem 3.1 will be the study of the following two long exact sequences. The first one, denoted by LES−, concerns the index pair (N, N^-) obtained by attaching a handle of index l to N^-:

$$\ldots \xrightarrow{p^-_{j+1}} H_{j+1}(N, N^-;\ R) \xrightarrow{\partial^-_{j+1}} H_j(N^-;\ R) \xrightarrow{i^-_j} H_j(N;\ R) \xrightarrow{p^-_j} H_j(N, N^-;\ R) \xrightarrow{\partial^-_j} \ldots \quad (1)$$

The second one, denoted by LES+, is obtained by considering the opposite flow, and is therefore related to the index pairs (N, N^+) obtained by attaching a handle of index $(n-l)$ to N^+:

$$\ldots \xrightarrow{p^+_{j+1}} H_{j+1}(N, N^+;\ R) \xrightarrow{\partial^+_{j+1}} H_j(N^+;\ R) \xrightarrow{i^+_j} H_j(N;\ R) \xrightarrow{p^+_j} H_j(N, N^+;\ R) \xrightarrow{\partial^+_j} \ldots \quad (2)$$

Another useful tool will be the following lemma.

Lemma 3.3. *Let R be $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime) or $\mathbf{Z}$. Then,*
for all $k \neq l$ and $k \neq l-1$ we have $\beta_k(N^-;\ R) = \beta_k(N;\ R)$;
for all $k \neq n-1-l$ and $k \neq n-l$ we have $\beta_k(N^+;\ R) = \beta_k(N;\ R)$.

Proof. We know that the only non-zero homology group of the index pair (N, N^-) is $H_l(N, N^-;\ R) = R$. Hence, for all $k \neq l$ and $k \neq l-1$ the long exact sequence LES− splits into $0 \to H_k(N^-;\ R) \to H_k(N;\ R) \to 0$. $H_k(N^-;\ R)$ is therefore isomorphic to $H_k(N;\ R)$, thus implying $\beta_k(N^-;\ R) = \beta_k(N;\ R)$. The analogous statement concerning N^+ and N follows in the same way from the analysis of LES+. $\square$

The following lemma allows us to generalize the result in [C-dR].

Lemma 3.4. *Let R be $\frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime) or $\mathbf{Z}$. If $R = \mathbf{Z}$ let A, B, D and E be arbitrary finitely generated Abelian groups, and let $rank(A)$ denote the rank of A, that is, the dimension of the largest torsion-free subgroup of A. If $R = \frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), let A, B, D and E be arbitrary finite dimensional vector spaces over R, and let $rank(A)$ denote the rank of A, that is, the dimension of A as a vector spaces over R. If the sequence*

$$0 \to A \to B \xrightarrow{p} R \xrightarrow{\partial} D \to E \to 0$$

is exact, then we have:
a) if $\ker \partial = 0$ then $rank(A) = rank(B)$ and $rank(D) = rank(E) + 1$;
b) if $\ker \partial \neq 0$ then $rank(B) = rank(A) + 1$ and $rank(D) = rank(E)$;

Proof. If $\ker \partial = 0$, then the exact sequence splits into the two exact sequences

$$0 \to A \to B \xrightarrow{p} 0 \quad \text{and} \quad 0 \to R \xrightarrow{\partial} D \to E \to 0$$

so that A is isomorphic to B and D is isomorphic to $E \oplus R$, thus implying case a).

If $\ker \partial \neq 0$ and $\ker \partial = R$, then the exact sequence splits into the two exact sequences

$$0 \to A \to B \xrightarrow{p} R \to 0 \quad \text{and} \quad 0 \to D \to E \to 0$$

thus implying B isomorphic to $A \oplus R$ and D isomorphic to E. If $R = \frac{\mathbf{Z}}{p\mathbf{Z}}$ (p prime), this solves case b).

If $R = \mathbf{Z}$, only one more case is left, that is, $\ker \partial \neq 0$ and $\ker \partial = m\mathbf{Z}$, $m \in \mathbf{N}^*$ and $m \neq 1$. Then the sequence splits into the two exact sequences

$$0 \to A \to B \xrightarrow{p} \mathbf{Z} \to \frac{\mathbf{Z}}{m\mathbf{Z}} \to 0 \quad \text{and} \quad 0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to D \to E \to 0$$

where $\cdot m$ denotes the multiplication by m. By the Rank-Nullity theorem, given an exact sequence, the alternating sum of the ranks of the appearing groups is zero, and case b) is done also for $R = \mathbf{Z}$. $\square$

3.3 Proof of Theorem 3.1

Proof of Item 1

Straighforward. If $l = 0$, N^+ is the disjoint union of N^- and $\mathbf{S}^{n-1}$.
The case $l = n$ follows from the previous one by considering the reverse flow, thus interchanging the role of N^+ and N^-.

Proof of Item 2 (a)

Lemma 3.3 implies that for all $k \neq i$, $k \neq i-1$ and $k \neq i+1$, we have $\beta_k(N^-;\ R) = \beta_k(N;\ R) = \beta_k(N^+;\ R)$.

In order to study the remaining indices, we shall consider the two exact sequences extracted from LES$-$ and LES$+$:

$$0 \to H_i(N^-;\ R) \to H_i(N;\ R) \to R \xrightarrow{\partial_i^-} H_{i-1}(N^-;\ R) \to H_{i-1}(N;\ R) \to 0 \qquad (1')$$

$$0 \to H_{i+1}(N^+;\ R) \to H_{i+1}(N;\ R) \to R \xrightarrow{\partial_{i+1}^+} H_i(N^+;\ R) \to H_i(N;\ R) \to 0 \qquad (2')$$

In the sequel we shall study these sequences with the help of Lemma 3.4 and use the equalities below, resulting from Lemma 3.3:

$$\beta_{i+1}(N^-;\ R) = \beta_{i+1}(N;\ R) \quad \text{and } \beta_{i-1}(N^+;\ R) = \beta_{i-1}(N;\ R) \qquad (3)$$

Case 1: $\ker \partial_l^- = 0$ *and* $\ker \partial_{n-l}^+ = 0$. This case cannot occur because it contradicts the Poincaré Duality. We should have $\beta_{i-1}(N^+;\ R) = \beta_{i+1}(N^+;\ R)$, but under our assumptions:
$\beta_{i-1}(N^+;\ R) = \beta_{i-1}(N;\ R)$ by (3);
$\beta_{i-1}(N;\ R) = \beta_{i-1}(N^-;\ R) - 1$ by Lemma 3.4 applied to sequence $(1')$;
$\beta_{i-1}(N^-;\ R) - 1 = \beta_{i+1}(N^-;\ R) - 1$ by the Poincaré Duality;
$\beta_{i+1}(N^-;\ R) - 1 = \beta_{i+1}(N;\ R) - 1$ by (3);
$\beta_{i+1}(N;\ R) - 1 = \beta_{i+1}(N^+;\ R) - 1$ by Lemma 3.4 applied to sequence $(2')$
so that $\beta_{i-1}(N^+;\ R) = \beta_{i+1}(N^+;\ R) - 1$ which is the wanted contradiction.

Case 2: $\ker \partial_l^- = 0$ *and* $\ker \partial_{n-l}^+ \neq 0$. Lemma 3.4 applied to sequences $(1')$ and $(2')$ implies

$$\begin{cases} \beta_i(N;\ R) = \beta_i(N^-;\ R) & \text{and} \quad \beta_{i-1}(N^-;\ R) = \beta_{i-1}(N;\ R) + 1 \\ \beta_{i+1}(N;\ R) = \beta_{i+1}(N^+;\ R) + 1 & \text{and} \quad \beta_i(N^+;\ R) = \beta_i(N;\ R). \end{cases}$$

and, after substituting (3), we get $\begin{cases} \beta_{i-1}(N^+;\ R) = \beta_{i-1}(N^-;\ R) - 1 \\ \beta_{i+1}(N^+;\ R) = \beta_{i+1}(N^-;\ R) - 1 \\ \beta_i(N^+;\ R) = \beta_i(N^-;\ R) \end{cases}$

Case 3: $\ker \partial_l^- \neq 0$ *and* $\ker \partial_{n-l}^+ = 0$. Lemma 3.4 applied to sequences $(1')$ and $(2')$ implies

$$\begin{cases} \beta_i(N;\ R) = \beta_i(N^-;\ R) + 1 \quad \text{and} \quad \beta_{i-1}(N^-;\ R) = \beta_{i-1}(N;\ R) \\ \beta_{i+1}(N;\ R) = \beta_{i+1}(N^+;\ R) \quad \text{and} \quad \beta_i(N^+;\ R) = \beta_i(N;\ R) + 1 \end{cases}$$

and, after substituting (3), we get $\begin{cases} \beta_i(N^+;\ R) = \beta_i(N^-;\ R) + 2 \\ \beta_k(N^+;\ R) = \beta_k(N^-;\ R) \qquad \text{for } k = i-1 \text{ and } k = i+1. \end{cases}$

Case 4: $\ker \partial_l^- \neq 0$ *and* $\ker \partial_{n-l}^+ \neq 0$. The situation here is symmetric to the one of Case 1 and cannot occur. Under our assumptions:
$\beta_{i+1}(N^-;\ R) = \beta_{i+1}(N;\ R)$ by (3);
$\beta_{i+1}(N;\ R) = \beta_{i+1}(N^+;\ R) + 1$ by Lemma 3.4 applied to sequence $(2')$;
$\beta_{i+1}(N^+;\ R) + 1 = \beta_{i-1}(N^+;\ R) + 1$ by the Poincaré Duality;
$\beta_{i-1}(N^+;\ R) + 1 = \beta_{i-1}(N;\ R) + 1$ by (3);
$\beta_{i-1}(N;\ R) + 1 = \beta_{i-1}(N^-;\ R) + 1$ by Lemma 3.4 applied to sequence $(1')$
so that $\beta_{i+1}(N^-;\ R) = \beta_{i-1}(N^-;\ R) + 1$ which contradicts the Poincaré Duality.

Proofs of Item 2 (b), Item 3 and Item 4
These proofs are completely analogous to that of Item 2 (a). For each of the four cases of Item 2 (a), we solve the system in which the unknowns are the Betti numbers of N^+ and N^-, and the equations are given by:

- the thesis of Lemma 3.3
- the thesis of Lemma 3.4 applied to the exact sequences extracted from LES− and LES+:

$$0 \to H_l(N^-;\ R) \to H_l(N;\ R) \to R \xrightarrow{\partial_l^-} H_{l-1}(N^-;\ R) \to H_{l-1}(N;\ R) \to 0$$

$$0 \to H_{n-l}(N^+;\ R) \to H_{n-l}(N;\ R) \to R \xrightarrow{\partial_{n-l}^+} H_{n-l-1}(N^+;\ R) \to H_{n-l-1}(N;\ R) \to 0$$

4 Applications to product manifolds

4.1 Handle decompositions of the product of two manifolds

In what follows we describe a method for constructing a handle decomposition of the product space $P \times Q$, once a handle decomposition of P and a handle decomposition of Q are known.

Proposition 4.1. *Let P be a p-dimensional manifold with handle decomposition $\mathcal{H}_P = \left[(f_0^{(1)}, \varphi_1), \ldots, (f_i^{(l)}, \varphi_l), \ldots, (f_p^{L_P}, \varphi_{L_P})\right]$, and let Q be a q-dimensional manifold with handle decomposition $\mathcal{H}_Q = \left[(g_0^{(1)}, \psi_1), \ldots, (g_j^{(m)}, \psi_m), \ldots, (g_q^{L_Q}, \psi_{L_Q})\right]$. Then $\mathcal{H}_P$ and $\mathcal{H}_Q$ induce a handle decomposition $\mathcal{H}_{P\times Q}$ of the $(p+q)$-dimensional manifold $P \times Q$ such that*

- *the total number of handles of $\mathcal{H}_{P\times Q}$ is $L_P \cdot L_Q$;*
- *all the ($(p+q)$-dimensional) handles h_k of $\mathcal{H}_{P\times Q}$ of index k are of the form*

$$f_i^{(l)} \times g_j^{(m)} \ \text{ with } \ f_i^{(l)} \in \mathcal{H}_P;\ \ g_j^{(m)} \in \mathcal{H}_Q \ \text{ and } \ i + j = k;$$

- *if $n = (m-1)L_P + l$, with $1 \leq l \leq L_P$ and $1 \leq m \leq L_Q$, then the handle $h_k^{(n)}$ in the n-th position in $\mathcal{H}_{P\times Q}$ corresponds to $f_i^{(l)} \times g_j^{(m)}$, where $f_i^{(l)} \in \mathcal{H}_P$ and $g_j^{(m)} \in \mathcal{H}_Q$.*

Proof. Let $h_k^{(n)} \in \mathcal{H}_{P\times Q}$. Then there exist $f_i^{(l)} \in \mathcal{H}_P$ and $g_j^{(m)} \in \mathcal{H}_Q$ such that

$$\underbrace{\mathbf{D}^k \times \mathbf{D}^{p+q-k}}_{h_k^{(n)}} \approx \underbrace{\mathbf{D}^i \times \mathbf{D}^{p-i}}_{f_i^{(l)}} \times \underbrace{\mathbf{D}^j \times \mathbf{D}^{q-j}}_{g_j^{(m)}}$$

As for the attaching region of h_k, since

$$\mathbf{S}^{k-1} = \partial \mathbf{D}^k = \partial(\mathbf{D}^i \times \mathbf{D}^j) = \left(\mathbf{S}^{i-1} \times \mathbf{D}^j\right) \bigcup_{\mathbf{S}^{i-1}\times\mathbf{S}^{j-1}} \left(\mathbf{D}^i \times \mathbf{S}^{j-1}\right)$$

we shall consider the following identification:

$$\underbrace{\mathbf{S}^{k-1} \times \mathbf{D}^{p+q-k}}_{\text{attaching region of } h_k^{(n)}} \approx \left(\left(\mathbf{S}^{i-1} \times \mathbf{D}^j\right) \cup_{\mathbf{S}^{i-1}\times\mathbf{S}^{j-1}} \left(\mathbf{D}^i \times \mathbf{S}^{j-1}\right)\right) \times \mathbf{D}^{p+q-k}$$

$$\approx \left(\underbrace{\mathbf{S}^{i-1} \times \mathbf{D}^{p-i}}_{\text{attaching region of } f_i^{(l)}} \times \underbrace{\mathbf{D}^j \times \mathbf{D}^{q-j}}_{g_j^{(m)}}\right) \bigcup_{\mathbf{S}^{i-1}\times\mathbf{D}^{p-i}\times\mathbf{S}^{j-1}\times\mathbf{D}^{q-j}} \left(\underbrace{\mathbf{D}^i \times \mathbf{D}^{p-i}}_{f_i^{(l)}} \times \underbrace{\mathbf{S}^{j-1} \times \mathbf{D}^{q-j}}_{\text{attaching region of } g_j^{(m)}}\right)$$

Observing that $(P \times Q)_{n-1}$ is given by $(P \times Q_{m-1}) \cup_{P_{l-1}\times Q_{m-1}} (P_{l-1} \times Q_m)$, the gluing of the attaching region of $h_k^{(n)}$ on $\partial((P \times Q)_{n-1})$ will be given by the map

$$\chi : \mathbf{S}^{k-1} \times \mathbf{D}^{p+q-k} \longrightarrow \partial((P \times Q)_{n-1})$$

naturally defined by:

$\chi(x, y) = (\varphi_l(x), \iota_1(y))$ if x belongs to the attaching region of $f_i^{(l)}$, y belongs to $g_j^{(m)}$ minus the attaching region of $g_j^{(m)}$, and where ι denotes the inclusion of $g_j^{(m)}$ in $Q_m \subset Q$;

$\chi(x, y) = (\iota_2(x), \psi_m(y))$ if x belongs to $f_i^{(l)}$ minus the attaching region of $f_i^{(l)}$, y belongs to the attaching region of $g_j^{(m)}$, and where ι denotes the inclusion of $f_i^{(l)}$ in $P_l \subset P$;

$\chi(x, y) = (\varphi_l(x), \psi_m(y))$ if x belongs to the attaching region of $f_i^{(l)}$ and y belongs to the attaching region of $g_j^{(m)}$.

Note that, in particular, a point simultaneously belonging to both the attaching region of $f_i^{(l)}$ and the attaching region of $g_j^{(m)}$ will be attached to a well defined point belonging to $(\partial P_{l-1} \times \partial Q_{m-1}) \subset ((\partial P_{l-1} \times Q_m) \cap (P_l \times \partial Q_{m-1}))$. ☐

Let us emphasize that, in particular, after attaching the first mL_P handles, the obtained manifold is $P \times Q_m$ whose boundary is $P \times \partial Q_m$.

4.2 An example

Example 4.2. Let $L_{a,b}$ denote the lens space associated with the integer co-prime parameters a and b. Let us fix a handle decomposition for $L_{a,b}$, made of four handles $[f_0, f_1, f_2, f_3]$ and associated with its minimal Heegaard splitting. Let us consider the product space $L_{a,b} \times \mathbf{S}^2$. If we fix for $\mathbf{S}^2$ a two-handle decomposition $[g_0, g_2]$, following the procedure given in Proposition 4.1 we get a eight-handle decomposition for $L_{a,b} \times \mathbf{S}^2$. After gluing the first four handles we get:

attached handle $h^{(n)}_{i+j}$:	$h^{(1)}_{0+0}$	$h^{(2)}_{1+0}$
resulting manifold $(P \times Q)_n$:	$\mathbf{D}^3 \times \mathbf{D}^2$	$(\mathbf{S}^1 \times \mathbf{D}^2) \times \mathbf{D}^2$
boundary of the resulting manifold:	$\mathbf{S}^4$	$\mathbf{S}^1 \times \mathbf{S}^3$
non-zero $\mathbf{Z}$-homology of the boundary:	$H_0 = H_4 = \mathbf{Z}$	$H_0 = H_4 = \mathbf{Z}$; $H_1 = H_3 = \mathbf{Z}$
sum of the Betti numbers of the boundary:	2	4

attached handle $h^{(n)}_{i+j}$:	$h^{(3)}_{2+0}$	$h^{(4)}_{3+0}$
resulting manifold $(P \times Q)_n$:	$(L_{a,b} \setminus \mathbf{D}^3) \times \mathbf{D}^2$	$L_{a,b} \times \mathbf{D}^2$
boundary of the resulting manifold:	$(\mathbf{S}^2 \times \mathbf{D}^2) \cup_{\mathbf{S}^2 \times \mathbf{S}^1} ((L_{a,b} \setminus \mathbf{D}^3) \times \mathbf{S}^1)$	$L_{a,b} \times \mathbf{S}^1$
non-zero $\mathbf{Z}$-homology of the boundary:	$H_0 = H_4 = \mathbf{Z}$; $H_1 = H_2 = \frac{\mathbf{Z}}{a\mathbf{Z}}$;	$H_0 = H_4 = \mathbf{Z}$; $H_1 = \mathbf{Z} \oplus \frac{\mathbf{Z}}{a\mathbf{Z}}$; $H_3 = \mathbf{Z}$ $H_2 = \frac{\mathbf{Z}}{a\mathbf{Z}}$
sum of the Betti numbers of the boundary:	2	4

The homology of the third boundary has been computed by using the Mayer-Vietoris exact sequence.

When attaching the remaining four handles, by the symmetry of the decomposition, we find back the same boundaries (the third, the second, the first and the empty set).

Now, choose R among $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$ with p prime not dividing a. Then, the given handle decomposition guarantees that $\nu(L_{a,b} \times \mathbf{S}^2;\ R) \leq 4$. Moreover, we know from [O] that, the fundamental group of the space being non-zero, the opposite inequality also holds. Therefore we have that, for our choices of R,

$$\nu(L_{a,b} \times \mathbf{S}^2;\ R) = 4$$

4.3 Upper bounds for the Ogasa invariant

In this section we want to apply Theorem 3.1 to the construction described in Proposition 4.1 in order to deduce general upper bounds for the Ogasa invariant of the product of two orientable manifolds. We shall prove the formulae of Theorem 3 in the next subsection, while the optimality of these inequalities, as well as the special example $L_{3,1} \times \mathbf{S}^2$, will be discussed thereafter.

4.3.1 Proof of Theorem 3

This subsection is completely devoted to the proof of Theorem 3.

By Proposition 4.1, we have a handle decomposition of $P \times Q$ made of $L_P \cdot L_Q$ handles and denoted by $\mathcal{H}_{P\times Q}$. Moreover, after attaching the first $m \cdot L_P$ handles of $\mathcal{H}_{P\times Q}$, the resulting manifold is $P \times Q_m$ whose boundary is $P \times \partial Q_m$.

By using the Künneth formula and the Universal Coefficient Theorem, we can prove the following formula.

Lemma 4.3. *Let X and Y be closed manifolds of dimension x and y respectively. If R is $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$, then*

$$\sum_{k=0}^{x+y} \beta_k(X \times Y;\ R) = \left(\sum_{i=0}^{x} \beta_i(X;\ R)\right) \cdot \left(\sum_{j=0}^{y} \beta_j(Y;\ R)\right)$$

Therefore, for all $m = 1, \ldots, L_Q$, by Theorem 3.1 we are in one of the following situations.

1. $\sum_{k=0}^{q-1} \beta_k(\partial Q_m;\ R) = \sum_{k=0}^{q-1} \beta_k(\partial Q_{m-1};\ R) + 2.$
   By Lemma 4.3, this means that after attaching the L_P handles of $\mathcal{H}_{P\times Q}$ $\left[h^{((m-1)L_P+1)}, \ldots, h^{(mL_P)}\right]$, we go from $\sum_{k=0}^{p+q-1} \beta_k(P \times \partial Q_{m-1};\ R)$ to

   $$\sum_{k=0}^{p+q-1} \beta_k(P \times \partial Q_m;\ R) = \sum_{k=0}^{p+q-1} \beta_k(P \times \partial Q_{m-1};\ R) + 2\sum_{i=0}^{p} \beta_i(P;\ R)$$

   Increasing the sum of the Betti numbers by $2 \cdot \sum_{i=0}^{p} \beta_i(P;\ R)$ with L_P handles guarantees that in this interval of L_P handles

   $$\max_{l=1,\ldots,L_P} \textstyle\sum_{k=0}^{p+q-1} \beta_k(\partial(P \times Q)_{(m-1)L_P+l};\ R) \leq$$

   $$\leq \textstyle\sum_{k=0}^{p+q-1} \beta_k(\partial(P \times Q)_{mL_P};\ R) + (L_P - \sum_{i=0}^{p} \beta_i(P;\ R))$$

   $$= L_P + \left(\textstyle\sum_{j=0}^{q-1} \beta_j(Q_m;\ R) - 1\right) \cdot \sum_{i=0}^{p} \beta_i(P;\ R)$$

   $$\leq L_P + (\nu(Q;\ R) - 1) \cdot \textstyle\sum_{i=0}^{p} \beta_i(P;\ R)$$

   (see also Figure 1 for the computation of the first inequality).
   Note that the last inequality does not depend on the interval of L_P handles we are dealing with.
2. $\sum_{k=0}^{q-1} \beta_k(\partial Q_m;\ R) = \sum_{k=0}^{q-1} \beta_k(\partial Q_{m-1};\ R) - 2.$
   This case is symmetric to the previous one. Even in the worst situation, the maximum over the considered interval of L_P handles is a value which has already been considered in another interval of L_P handles belonging to Case 1 above. Therefore we can again insure that

   $$\max_{l=1,\ldots,L_P} \textstyle\sum_{k=0}^{p+q-1} \beta_k(\partial(P \times Q)_{(m-1)L_P+l};\ R) \leq$$

   $$\leq L_P + (\nu(Q;\ R) - 1) \cdot \textstyle\sum_{i=0}^{p} \beta_i(P;\ R)$$

3. $\sum_{k=0}^{q-1} \beta_k(\partial Q_m;\ R) = \sum_{k=0}^{q-1} \beta_k(\partial Q_{m-1};\ R).$
   This is the case where the l-th handle $g^{(l)}$ of $\mathcal{H}_Q$ is invariant.

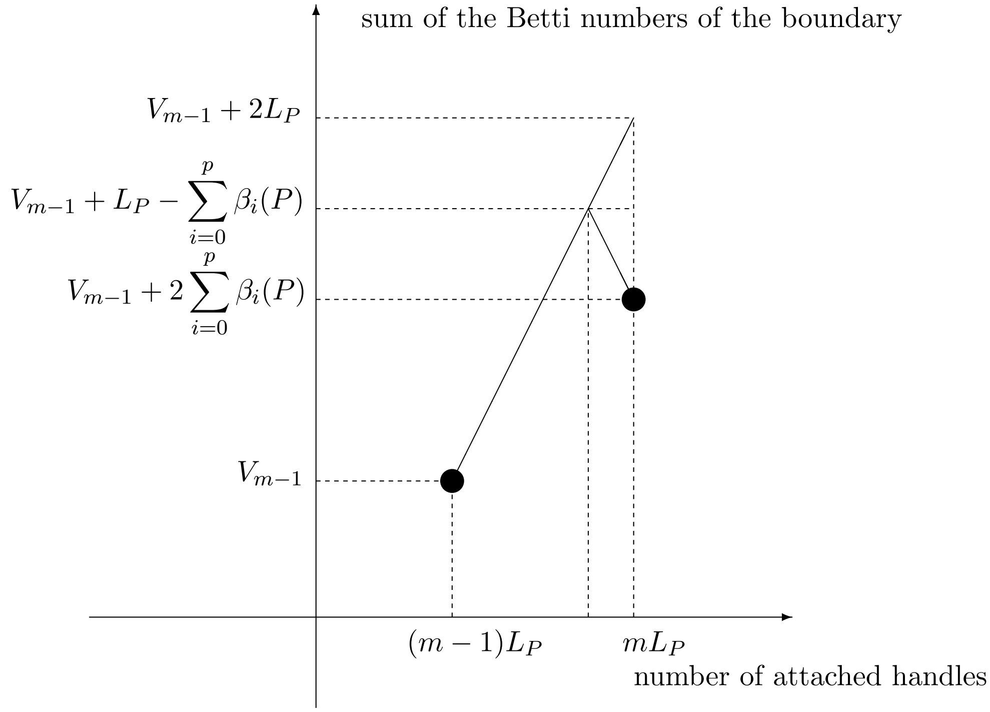

Figure 1: Computation of the inequality of Case 1, where $V_{m-1} = \sum_{k=0}^{p+q-1} \beta_k(P \times \partial Q_{m-1})$.

Leaving the sum of the Betti numbers unchanged after attaching L_P handles guarantees that in this interval of L_P handles

$$\max_{l=1,\dots,L_P} \textstyle\sum_{k=0}^{p+q-1} \beta_k(\partial(P\times Q)_{(m-1)L_P+l};\ R) \leq$$

$$= \left(\textstyle\sum_{j=0}^{q-1} \beta_j(Q_m;\ R)\right) \cdot (\textstyle\sum_{i=0}^{p} \beta_i(P;\ R)) + 2 \cdot \lfloor \frac{L_P}{2} \rfloor$$

$$\leq (\nu(Q;\ R)) \cdot (\textstyle\sum_{i=0}^{p} \beta_i(P;\ R)) + 2 \cdot \lfloor \frac{L_P}{2} \rfloor$$

where the floor function has been used to make the inequality as sharp as possible in the case where $\mathcal{H}_P$ has a unique invariant handle, thus implying L_P odd.

The conclusion of the proof is now straightforward. Item 1 of Theorem 3 follows directly from Cases 1 and 2 above. Item 2 follows directly from Cases 1, 2 and 3 above by considering that $2 \cdot \lfloor \frac{L_P}{2} \rfloor \geq (L_P - \sum_{i=0}^{p} \beta_i(P;\ R))$.

4.4 Comments on Theorem 3

Here are some remarks and consequences of Theorem 3.

- If P is an oriented manifold for which there exists a handle decomposition with $L_P = \sum_{i=0}^{p} \beta_i(P;\ R)$ handles, e.g. a product of spheres, then the inequality of Item 1 of Theorem 3 reduces to

$$\nu(P \times Q;\ R) \leq (\nu(Q;\ R)) \cdot \sum_{j=0}^{p} \beta_j(P;\ R)$$

while that of Item 2 becomes

$$\nu(P \times Q;\ R) \leq (\nu(Q;\ R) + 1) \cdot \sum_{j=0}^{p} \beta_j(P;\ R)$$

because the homology generated in Q by the invariant handle can contribute to the homology to some regular section.

- When considering orientable manifolds, the situation described in Item 2 can occur only when the dimension of Q is a multiple of 4: it is the case for instance of the $4k$-dimensional manifolds $\mathbf{CP}^{2k}$.

  For this reason, let us focus on Item 1.

- If $P \times Q = \mathbf{S}^1 \times \mathbf{S}^2$, equality holds in the inequality of Item 1 if we choose for Q the minimal handle decomposition made of two handles and any homology coefficients.

  Notwithstanding it, let us consider now the product manifold $L_{3,1} \times \mathbf{S}^2$, where $L_{3,1}$ is the lens space associated with the couple $(3, 1)$. Let the homology coefficients be $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\frac{\mathbf{Z}}{p\mathbf{Z}}$ with $p \neq 3$ prime. We are in the case of Item 1 above and the best upper bounds are given by the canonical handle decompositions for $L_{3,1}$ and $\mathbf{S}^2$ described in Example 4.2. According to the choice of the roles of P and Q, by Item 1 we get:

$$\nu(L_{3,1} \times \mathbf{S}^2;\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}}) \leq \underbrace{L_{L_{3,1}}}_{4} + \underbrace{(\nu(\mathbf{S}^2;\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}}) - 1)}_{2} \cdot \underbrace{\sum_{j=0}^{p} \beta_j(L_{3,1};\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}})}_{2} = 6$$

$$\nu(\mathbf{S}^2 \times L_{3,1};\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}}) \leq \underbrace{L_{\mathbf{S}^2}}_{2} + \underbrace{(\nu(L_{3,1};\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}}) - 1)}_{4} \cdot \underbrace{\sum_{j=0}^{p} \beta_j(\mathbf{S}^2;\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}})}_{2} = 8$$

  However, in Example 4.2 we have shown that $\nu(L_{3,1} \times \mathbf{S}^2;\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}}) = 4$, hence in general the opposite inequality doesn't hold.

- The above example shows the delicate interplay between the Betti numbers of a manifold and those of its regular sections. On one hand, the Ogasa invariant of $L_{3,1}$ is somehow greater than expected

$$4 = \nu(L_{3,1};\ R) \gneq \sum_{j=0}^{p} \beta_j(L_{3,1};\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}}) = 2$$

  because of the presence of torsion of order 3 in $\Pi_1(L_{3,1})$. On the other hand, when considering the product with $\mathbf{S}^2$, the Ogasa invariant $\nu(L_{3,1} \times \mathbf{S}^2;\ R \neq \frac{\mathbf{Z}}{3\mathbf{Z}})$ is somehow less than expected, again because of the presence of torsion.

We naturally conclude with the following question. For any m-dimensional orientable manifold M, let L^{min} denote the minimal number of handles needed to have a handle decomposition of M. Let $\mathcal{C}_R$ be the class of such manifolds for which

$$\nu(M;\ R) = \sum_{k=0}^{m} \beta_k(M;\ R) = L^{min}$$

For instance $\mathbf{S}^p \times \mathbf{S}^q$ belongs to $\mathcal{C}_R$, whereas $\mathbf{CP}^2$ does not. Is it true that if P and Q belong to $\mathcal{C}_R$, then

$$\nu(P \times Q;\ R) = \nu(P;\ R) \cdot \nu(Q;\ R)?$$

The inequality $\leq$ is trivial. In particular, answering in the affirmative would prove $\nu(\prod_{j=1}^{\ell} \mathbf{S}^{k_j}) = 2^{\ell}$.

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