# On the linear transport subject to random velocity fields

F. A. Dorini [a] M. C. C. Cunha [b]

[a] *Department of Mathematics, Federal University of Technology - Paraná, Brazil*

[b] *Department of Applied Mathematics, State University of Campinas, Brazil.*

**Abstract**

This paper deals with the random linear transport equation for which the velocity and the initial condition are random functions. Expressions for the density and joint density functions of the transport equation solution are given. We also verify that in the Gaussian time-dependent velocity case the probability density function (PDF) of the solution satisfies a convection-diffusion equation with a time-dependent diffusion coefficient. Examples are included.

*Key words:* transport equation, random velocity, Gaussian processes.

## 1 Introduction

In this paper, we deal with the random one-dimensional transport problem

$$Q_t(x,t) + V(t)\, Q_x(x,t) = 0 \quad (t > 0,\ x \in \mathbb{R}), \qquad Q(x,0) = Q_0(x), \tag{1}$$

where $V(t)$ is the random velocity and $Q_0(\mathbf{x})$ is the random initial condition. We suppose the natural hypothesis of independence between the velocity and the initial condition. Transport equations arise in the modeling of a wide variety of phenomena that involve advective transport of substances or wave motions [1,2]. In the typical situation of uncertainties in the transport velocity or/and in the initial condition, the random transport equation provides a better description of the process.

Several authors have studied problems related to (1). Most of approaches include methods by which one seeks the statistical moments of the solution (e.g. see [1–6], and the references there in). The main effort is usually concentrated on the derivation of appropriate differential equations for average quantities using, in general, small perturbations with some kind of closure. Another approach is to solve numerically appropriate equations for representative sets of realizations of random fields and to average computed functions. This approach is the so-called Monte Carlo method (e.g. see [6]) which has the advantage of applying to a very broad range of both linear and nonlinear problems. However, the large volume of calculation, the numerical errors in solving the deterministic equations, and the difficulty for generalizing the results limit the significance of this approach.

*Email addresses:* `fabio.dorini@gmail.com` (F. A. Dorini), `cunha@ime.unicamp.br` (M. C. C. Cunha).

The purpose of this paper is to present some new results about the random solution of (1). In Section 2 we couple the total probability theorem with the characteristic method to find the PDF of the solution of (1). In Section 3 we present results and examples in which the velocity in (1) is a Gaussian process. We straightforward verify that the PDF of the solution satisfies a convection-diffusion equation with a time-dependent diffusion coefficient. Although differential equations for moments and density functions have been presented in the literature for some particular cases, we believe that our methodology gives a new insight on the subject and is more direct. Finally, in Section 4 we obtain the two-point joint density function of the solution of (1).

## 2 The probability density function of $Q(x,t)$

For each realization $V(t,\omega)$, of the random velocity, and $Q_0(x,\omega)$, of the initial condition, the solution at a fixed $(x,t)$, $Q(x,t,\omega)$, is constant along the characteristics, *i.e*,

$$Q(x,t,\omega) = Q_0(x_0,\omega), \quad \text{where} \quad x_0 = x - \int_0^t V(\tau,\omega)d\tau.$$

Thus, the solution of (1) can be expressed as $Q(x,t) = Q_0(x - A(t))$, where we denote $A(t)$ as

$$A(t) = \int_0^t V(\tau)d\tau. \tag{2}$$

The concept of conditional probability will play a role in calculating the cumulative function of $Q(x,t)$, $F_Q(q;x,t)$, for a fixed $(x,t)$. In fact, by the *Law of Total Probability* [7] we can write

$$F_Q(q;x,t) = \mathcal{P}(Q(x,t) \le q) = \mathrm{E}_{X_0}[\mathcal{P}(Q(x,t) \le q \mid X_0)], \tag{3}$$

where $X_0$ is given by $X_0(x,t) = x - A(t)$, $\mathrm{E}_{X_0}$ denotes the expected value relative to random variable $X_0$, $\mathcal{P}$ denotes the probability measure, and $\mathcal{P}(U \mid V)$ denotes the conditional probability of $U$ given $V$. By the characteristic method we observe that $Q(x,t) \le q$ given that $X_0 = x_0$ is equivalent to $Q_0(x_0) \le q$. Thus, from (3),

$$F_Q(q;x,t) = \int_{-\infty}^{+\infty} \mathcal{P}(Q_0(x_0) \le q)\, f_{X_0}(x_0)\, dx_0 = \int_{-\infty}^{+\infty} F_{Q_0}(q;x_0)\, f_{X_0}(x_0)\, dx_0, \tag{4}$$

where $F_{Q_0}(q;x_0)$ is the cumulative function of $Q_0(x_0)$. Taking the derivative with respect to $q$, we obtain

$$f_Q(q;x,t) = \int_{-\infty}^{+\infty} f_{Q_0}(q;x_0)\, f_{X_0}(x_0)\, dx_0. \tag{5}$$

Recalling that $A(t)$ is a random variable for $t$ fixed, we have

$$F_{X_0}(x_0) = \mathcal{P}(x - A(t) \le x_0) = \mathcal{P}(A(t) \ge (x - x_0)) = 1 - F_{A(t)}(x - x_0), \tag{6}$$

and by the differentiation with respect to $x_0$ we arrive at $f_{X_0}(x_0) = f_{A(t)}(x - x_0)$. Then, substituting $f_{X_0}(x_0)$ in (5), we obtain

$$f_Q(q;x,t) = \int_{-\infty}^{+\infty} f_{A(t)}(x - x_0)\, f_{Q_0}(q;x_0)\, dx_0. \tag{7}$$

The arguments so far summarized prove the following result:

**Proposition 1** *The PDF of the solution of (1) at a fixed $(x,t)$, $f_Q(q;x,t)$, is given by (7).*

**Corollary 2** *The m-th moment, $\mu^m(x,t)$, $m \geq 1$, of the solution of (1) is given by*

$$\mu^m(x,t) = \int_{-\infty}^{+\infty} q^m \, f_Q(q;x,t) \, dq = \int_{-\infty}^{+\infty} f_{A(t)}(x-x_0) \, \mu_0^m(x_0) \, dx_0, \tag{8}$$

*where $\mu_0^m(x)$ is the m-th moment of $Q_0(x)$.*

**Remark 3** *In the case where $V(t) = V$ we obtain $A(t) = Vt$, $f_{A(t)}(x) = (1/t) f_V(x/t)$, and*

$$f_Q(q;x,t) = \int_{-\infty}^{+\infty} f_V(v) \, f_{Q_0}(q; x - vt) \, dv = \mathrm{E}_V[f_{Q_0}(q; x - Vt)]. \tag{9}$$

**Remark 4** *In the case where $Q(x,0) = g(x)$, a deterministic function, we can express (7) as*

$$f_Q(q;x,t) = \int_{-\infty}^{+\infty} f_{A(t)}(x-x_0) \, \delta(g(x_0) - q) \, dx_0, \tag{10}$$

*where $\delta$ is the Dirac (delta) distribution. Furthermore, if $g(x)$ is a smooth function, if the equation $g(x) - q = 0$ has $n$ isolated zeros, $x_{j,q}$, $j = 1, 2, ..., n$, and if $g'(x)$ does not vanish at each of the zeros, we have (e.g. see [8])*

$$\delta(q - g(x)) = \sum_{j=1}^{n} \frac{\delta(x - x_{j,q})}{|g'(x_{j,q})|}, \quad \textit{and} \tag{11}$$

$$f_Q(q;x,t) = \int_{-\infty}^{+\infty} f_{A(t)}(x-x_0) \sum_{j=1}^{n} \frac{\delta(x_0 - x_{j,q})}{|g'(x_{j,q})|} \, dx_0 = \sum_{j=1}^{n} \frac{1}{|g'(x_{j,q})|} \, f_{A(t)}(x - x_{j,q}). \tag{12}$$

Expressions (7), (8), and (12) point out that practical calculations demand the knowledge of the PDF of $A(t)$. With this in mind, let us present now some results to be used in the next section. From (2) we obtain (e.g. see [7])

$$\mu(t) = \mathrm{E}[A(t)] = \int_0^t \mathrm{E}[V(\tau)] d\tau \quad \text{and} \quad \sigma^2(t) = \mathrm{Var}[A(t)] = \int_0^t \int_0^t \mathrm{Cov}(s,\tau) ds d\tau, \tag{13}$$

where $\mathrm{E}[V(t)]$ is the mean and $\mathrm{Cov}(t,\tau)$ is the covariance function of $V(t)$. Also, if $\mathrm{E}[V(t)]$ and $\mathrm{Cov}(t,\tau)$ are continuous functions it follows that

$$\frac{d[\mu(t)]}{dt} = \mathrm{E}[V(t)] \quad \text{and} \quad \frac{d[\sigma^2(t)]}{dt} = 2 \int_0^t \mathrm{Cov}(t,\tau) d\tau. \tag{14}$$

## 3 The Gaussian velocity case

If we assume $V(t)$ Gaussian then $A(t)$ in (2) is also a Gaussian random variable for each $t$ [7]; its mean and variance are given by (13). In this case we have the following result:

**Proposition 5** *The PDF of $Q(x,t)$, $f_Q(q;x,t)$, satisfies the convection-diffusion equation*

$$(f_Q)_t + \mathrm{E}[V(t)](f_Q)_x = \left( \int_0^t Cov(t,\tau) d\tau \right) (f_Q)_{xx}. \tag{15}$$

**PROOF.** Differentiating $f_Q(q;x,t)$ in (7) conveniently, we can see that if

$$\frac{\partial}{\partial t} f_{A(t)}(x) + \mathrm{E}[V(t)] \frac{\partial}{\partial x} f_{A(t)}(x) = \left( \int_0^t \mathrm{Cov}(t,\tau) d\tau \right) \frac{\partial^2}{\partial x^2} f_{A(t)}(x), \tag{16}$$

then the result in (15) follows. In fact, since $A(t) \sim N(\mu(t), \sigma(t))$ its density function is

$$f_{A(t)}(x) = \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left[-\frac{(x-\mu(t))^2}{2\sigma^2(t)}\right], \tag{17}$$

where $\mu(t)$ and $\sigma(t)$ are given in (13). The differentiation of $f_{A(t)}(x)$ yields

$$\frac{\partial}{\partial x} f_{A(t)}(x) = -f_{A(t)}(x)\left[\frac{x-\mu(t)}{\sigma^2(t)}\right], \quad \frac{\partial^2}{\partial x^2} f_{A(t)}(x) = \frac{f_{A(t)}(x)}{\sigma^2(t)}\left[-1+\frac{(x-\mu(t))^2}{\sigma^2(t)}\right], \text{ and}$$
$$\frac{\partial}{\partial t} f_{A(t)}(x) = f_{A(t)}(x)\left[-\frac{\sigma'(t)}{\sigma(t)} + \frac{(x-\mu(t))}{\sigma^2(t)}\mu'(t) + \frac{(x-\mu(t))^2}{\sigma^3(t)}\sigma'(t)\right].$$

Therefore, using (14),

$$\begin{aligned}\frac{\partial}{\partial t} f_{A(t)}(x) + \mathrm{E}[V(t)]\,\frac{\partial}{\partial x} f_{A(t)}(x) &= \frac{\partial}{\partial t} f_{A(t)}(x) + \mu'(t)\,\frac{\partial}{\partial x} f_{A(t)}(x) = \\ &= f_{A(t)}(x)\left[-\frac{\sigma'(t)}{\sigma(t)} + \frac{(x-\mu(t))^2}{\sigma^3(t)}\sigma'(t)\right] = \frac{\sigma'(t)}{\sigma(t)} f_{A(t)}(x)\left[-1+\frac{(x-\mu(t))^2}{\sigma^2(t)}\right] = \\ &= \sigma(t)\sigma'(t)\frac{\partial^2}{\partial x^2} f_{A(t)}(x) = \frac{1}{2}\frac{d(\sigma^2(t))}{dt}\frac{\partial^2}{\partial x^2} f_{A(t)}(x) = \left(\int_0^t \mathrm{Cov}(t,\tau)d\tau\right)\frac{\partial^2}{\partial x^2} f_{A(t)}(x),\end{aligned}$$

and the result follows.

□

**Corollary 6** *The m-th moment, $\mu^m(x,t)$, $m \geq 1$, of the solution of (1) also satisfies (15), i.e.,*

$$(\mu^m)_t + \mathrm{E}[V(t)](\mu^m)_x = \left(\int_0^t Cov(t,\tau)d\tau\right)(\mu^m)_{xx}. \tag{18}$$

**Example 7** We consider problem (1) with inicial condition defined by $g(x) = 1$, if $x > 0$, and $g(x) = 0$, if $x < 0$. This initial condition has been used by several authors to study mixing zones of substance concentrations. From (10) we have

$$f_Q(q;x,t) = \delta(q-1)\int_{-\infty}^{0} f_{A(t)}(x-x_0)\,dx_0 + \delta(q)\int_0^{+\infty} f_{A(t)}(x-x_0)\,dx_0, \tag{19}$$

*i.e.*, $Q(x,t)$ is the Bernoulli random variable with

$$\begin{aligned}\mathcal{P}(Q(x,t)=1) &= \int_{-\infty}^{0} f_{A(t)}(x-x_0)\,dx_0 = \int_x^{+\infty} f_{A(t)}(\theta)\,d\theta = \\ &= \frac{1}{\sqrt{2\pi}\sigma(t)}\int_x^{+\infty}\exp\left[-\frac{(\theta-\mu(t))^2}{2\sigma^2(t)}\right]d\theta = \frac{1}{\sqrt{\pi}}\int_{\frac{x-\mu(t)}{\sqrt{2}\sigma(t)}}^{+\infty} \mathrm{e}^{-\theta^2}\,d\theta = \frac{1}{2}\mathrm{erfc}\left(\frac{x-\mu(t)}{\sqrt{2}\sigma(t)}\right),\end{aligned} \tag{20}$$

where $\mathrm{erfc}(x)$ is the complementary error function. Moreover, from (8) or (19)–(20), and observing that $\mu^m(x,0) = g(x)$ for all $m$, we arrive at

$$\mu^m(x,t) = \frac{1}{2}\mathrm{erfc}\left(\frac{x-\mu(t)}{\sqrt{2}\sigma(t)}\right). \tag{21}$$

**Example 8** In this example, the Gaussian velocity is defined by its mean, $\mathrm{E}[V(t)]$, and the exponentially decaying covariance function, $\mathrm{Cov}_V(t,\tau) = \sigma_V^2 \exp(-|t-\tau|/\lambda)$. The covariance function is parameterized by the variance, $\mathrm{Var}[V(t)] = \sigma_V^2$, and by the correlation length, $\lambda > 0$, which governs the decay rate of the time correlation. Let $g(x) = \alpha\, e^{-\beta x^2}$ $(\alpha, \beta > 0)$, be the initial condition. Therefore, we may find $f_Q(q; x, t)$ by using (12). For $0 < q < \alpha$ the equation $g(x) = q$ has two roots, $x_{1,q} = [(1/\beta)\ln(\alpha/q)]^{1/2}$ and $x_{2,q} = -[(1/\beta)\ln(\alpha/q)]^{1/2}$. For the function $g(x)$, we find $|g'(x_{j,q})| = 2\beta q\,[(1/\beta)\ln(\alpha/q)]^{1/2}$ $(j = 1, 2)$. Thus, from (12) and (17) we arrive at the exact PDF:

$$f_Q(q; x, t) = \frac{1}{\sqrt{8\pi}\beta\sigma(t)\rho(q)q}\left\{\exp\left[-\frac{(x-\rho(q)-\mu(t))^2}{2\sigma(t)^2}\right] + \exp\left[-\frac{(x+\rho(q)-\mu(t))^2}{2\sigma(t)^2}\right]\right\}, \quad (22)$$

where $\rho(q) = [(1/\beta)\ln(\alpha/q)]^{1/2}$. Evidently, $f_Q(q; x, t) = 0$ for $q \notin (0, \alpha)$. Also, from (13), we have

$$\sigma(t) = \sigma_V\left\{\int_0^t\int_0^t \exp\left[-\frac{|s-\tau|}{\lambda}\right]ds d\tau\right\}^{\frac{1}{2}} = \sqrt{2}\lambda\sigma_V\left[\exp\left(-\frac{t}{\lambda}\right) + \frac{t}{\lambda} - 1\right]^{\frac{1}{2}}. \quad (23)$$

**Example 9** Let $V(t)$ be the Gaussian white noise with zero mean and power spectrum $\eta$. In this case, $A(t)$ is also Gaussian with zero mean, $\eta t$ variance and $f_{A(t)}(\theta) = (2\pi\eta t)^{-1/2}\exp[-\theta^2/(2\eta t)]$. By (14) we arrive at $\int_0^t \mathrm{Cov}(t, \tau)\, d\tau = \eta/2$, and (15) becomes $(f_Q)_t = (\eta/2)\,(f_Q)_{xx}$. This diffusive equation with $f_Q(q; x, 0) = f_{Q_0}(q; x)$ can be solved by the Green's function approach:

$$f_Q(q; x, t) = \frac{1}{\sqrt{2\pi\eta t}}\int_{-\infty}^{+\infty}\exp\left[-\frac{(x-x_0)^2}{2\eta t}\right] f_{Q_0}(q; x_0)\, dx_0. \quad (24)$$

This solution agrees with (7) with $f_{A(t)}$ being the Green function.

## 4 The joint probability density function

Let $Q_1 = Q(x_1, t)$ and $Q_2 = Q(x_2, t)$ be the random solutions of (1) at $(x_1, t)$ and $(x_2, t)$, $t > 0$, respectively. As known, second-order properties of a random process can give significant information about the process such as the correlation of $Q_1$ and $Q_2$, that demands the joint density function of these random variables. The joint cumulative function is given by

$$F_Q(q_1, q_2; x_1, x_2, t) = \mathcal{P}(Q_1 \le q_1,\ Q_2 \le q_2) = \mathrm{E}_{X_0,Y_0}[\mathcal{P}(Q_1 \le q_1,\ Q_2 \le q_2 \mid X_0,\ Y_0)], \quad (25)$$

where, again, we have used the *Law of Total Probability* [7]. Here, $X_0$ and $Y_0$ are the random functions $X_0(x_1, t) = x_1 - A(t)$ and $Y_0(x_2, t) = x_2 - A(t)$. As before, by the characteristic method, $Q_1 \le q_1$ and $Q_2 \le q_2$ given that $X_0 = x_0$ and $Y_0 = y_0$ is equivalent to $Q_0(x_0) \le q_1$ and $Q_0(y_0) \le q_2$, where $Q_0$ is the initial condition in (1). Therefore, from (25)

$$F_Q(q_1, q_2; x_1, x_2, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{Q_0}(q_1, q_2; x_0, y_0)\, f_{X_0,Y_0}(x_0, y_0)\, dx_0 dy_0, \quad (26)$$

where $f_{X_0,Y_0}(x_0, y_0)$ is the joint PDF of $X_0$ and $Y_0$, and $F_{Q_0}(q_1, q_2; x_0, y_0)$ is the joint cumulative function of $Q_0(x_0)$ and $Q_0(y_0)$. Taking the second-order mixed derivative above, we arrive at

$$f_Q(q_1, q_2; x_1, x_2, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Q_0}(q_1, q_2; x_0, y_0)\, f_{X_0,Y_0}(x_0, y_0)\, dx_0 dy_0. \quad (27)$$

To determine the joint PDF, $f_{X_0,Y_0}$, we start with

$$\begin{aligned} F_{X_0,Y_0}(x_0, y_0) &= \mathcal{P}(X_0 \le x_0,\ Y_0 \le y_0) = \mathcal{P}(x_1 - A(t) \le x_0,\ x_2 - A(t) \le y_0) = \\ &= \mathcal{P}(A(t) \ge x_1 - x_0,\ A(t) \ge x_2 - y_0) = \mathcal{P}(A(t) \ge \varphi) = 1 - F_{A(t)}(\varphi), \end{aligned} \quad (28)$$

where $\varphi = \max\{x_1 - x_0,\ x_2 - y_0\} = \max\{u, v\}$, with $u = x_1 - x_0$ and $v = x_2 - y_0$. Taking the derivative of $\varphi$, in the sense of distributions, we have that $\partial\varphi/\partial u = H(u-v)$ and $\partial\varphi/\partial v = H(v-u)$, where $H$ is the Heaviside function. Moreover, $H'(\alpha) = \delta(\alpha)$, the Dirac distribution. Consequently,

$$\frac{\partial\varphi}{\partial x_0} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x_0} = -H(u-v), \qquad \frac{\partial\varphi}{\partial y_0} = \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y_0} = -H(v-u) \quad \text{and} \quad \frac{\partial^2\varphi}{\partial y_0 \partial x_0} = -\delta(u-v).$$

Finally, taking the mixed derivative in (28) we find

$$f_{X_0,Y_0}(x_0, y_0) = -\frac{\partial^2 F_A(t)(\varphi)}{\partial y_0 \partial x_0} = f_{A(t)}(\varphi)\delta(u-v) = f_{A(t)}(u)\delta(u-v), \tag{29}$$

since $h(\alpha)\delta(\alpha) = h(0)\delta(\alpha)$. Substituting this expression in (27) we obtain

$$\begin{aligned} f_Q(q_1, q_2; x_1, x_2, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Q_0}(q_1, q_2; x_1 - u, x_2 - v)\, f_{A(t)}(u)\, \delta(u-v)\, du\, dv = \\ &= \int_{-\infty}^{\infty} f_{Q_0}(q_1, q_2; x_1 - a, x_2 - a)\, f_{A(t)}(a)\, da. \end{aligned} \tag{30}$$

**Proposition 10** *Let $Q_1 = Q(x_1, t)$ and $Q_2 = Q(x_2, t)$ be the random solutions of (1) at $(x_1, t)$ and $(x_2, t)$, $t > 0$, respectively.The joint PDF of these random variables is given by (30). Furthermore, the covariance of the solution of (1) at $(x_1, t)$ and $(x_2, t)$, $t > 0$, is given by*

$$Cov[Q(x_1, t),\ Q(x_2, t)] = \int_{-\infty}^{\infty} Cov[Q_0(x_1 - a), Q_0(x_2 - a)]\, f_{A(t)}(a)\, da. \tag{31}$$


## Acknowledgements

We thank the Brazilian Council for Development of Science and Technology (CNPq) for support through grant $472992/2008-2$.


## References


[1] G. Dagan, Flow and Transport in Porous Formations, Springer, Heidelberg, 1989.

[2] L. I. Piterbarg, A. G. Ostrovskii, Advection and Diffusion in Random Media, Kluwer Academic Publishers, Dordrecht, 1997.

[3] F. A. Dorini, F. Furtado, M. C. C. Cunha, On the evaluation of moments for solute transport by random velocity fields, Applied Numerical Mathematics 79 (12) (2009) 2994–2998.

[4] M. Shvidler, K. Karasaki, Exact averaging of stochastic equations for transport in random velocity field, Transport in Porous Media 50 (2003) 223–241.

[5] M. Shvidler, K. Karasaki, Probability density functions for solute transport in random fields, Transport in Porous Media 50 (2003) 243–266.

[6] D. Zhang, Stochastic Methods for Flow in Porous Media - Coping with Uncertainties, Academic Press, San Diego, 2002.

[7] A. Papoulis, Probability, Random Variables, and Stochastic Processes, 2nd Edition, McGraw-Hill, Inc., New York, 1984.

[8] E. Zauderer, Partial Differential Equations of Applied Mathematics, Pure and Applied Mathematics, John Wiley & Sons, New York, 1983.