Reversibility and Quasi-Homogeneous Normal Forms of Vector Fields

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Abstract

This paper uses tools in Quasi-Homogeneous Normal Form theory to discuss certain aspects of reversible vector fields around an equilibrium point. Our main result provides an algorithm, via Lie Triangle, that detects the non-reversibility of vector fields. As a consequence we answer an intriguing question related to the problems derived from the 16° Hilbert Problem. That is, it is possible to decide whether a planar center is not reversible. Some of the theory developed is also applied to get further results on nilpotent and degenerate polynomial vector fields. We find several families of nilpotent centers which are non-reversible.

1 Introduction and setting of the problem

This paper is focused on the differential systems with time-reversal symmetries. A time-reversal symmetry is one of the fundamental symmetries in natural science and it arises in many branches in physics, see for instance, Lamb and Roberts [15] for a survey on reversible systems and related topics.

In the last decades there has been an increasing interest in the study of systems with time-reversal symmetries. In recent years, a lot of attention has been devoted to understand and use the interplay between dynamics and symmetry properties. Reversible vector fields were first considered by Birkoff, in the beginning of the last century, when he was studying the restricted three body problem. In [13] the theory was formalized by Devaney.

The property of reversibility of a planar vector field is a sufficient condition for a monodromic planar vector field to be a center, and this provides a strong motivation to study the reversibility of vector fields. Moreover, there exists a strong connection between the reversible and the center characteristics of a planar vector field. In fact, it is known that a planar system having a non-degenerate (respectively nilpotent) center at the origin is reversible (respectively orbitally reversible), see Poincaré [18], Berthier and Moussu [9]. In this paper, we also study nilpotent centers which are non-reversible.

Much effort has been dedicated to understand the connection between, centers, analytic integrability and reversibility of a planar vector field, see for instance (Algaba, Gamero and García [2], Berthier and Moussu [9], Berthier, Cerveau and Lins Neto [8], Chavarriga, Giacomin, Giné and Llibre [12], Llibre and Medrado, [16], Strozyna and Zoladek, [20], Zoladek, [22], Teixeira and Yang, [21], and references therein).

On the other hand, much work has been done in the study of planar polynomial vector fields by means of techniques in the quasi-homogeneous normal form theory, see for instance, Gasull and Torregrosa [14], Algaba, Gamero and García [2], Algaba, Freire, Gamero and García [3], [4], [5].

In this paper, our main aim is to establish a discussion involving reversible vector fields and quasi-homogeneous normal forms theory.

We now need to introduce some definitions and terminology.

- An involution is a diffeomorphism $\sigma \in \mathcal{C}^\infty(U_0 \subset \mathbb{R}^n, \mathbb{R}^n)$, such that $\sigma \circ \sigma = Id$, where U_0 is a small neighborhood of $0 \in \mathbb{R}^n$. Denote $\mathrm{Fix}\,(\sigma) = \{\mathbf{x} \in U_0 \,|\sigma(\mathbf{x}) = \mathbf{x}\}$ This set is a local sub-manifold of $\mathbb{R}^n$ and we are assuming throughout the paper that $\dim\left(\mathrm{Fix}\,(\sigma)\right) = n-1$.

- We say that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, or the vector field $\mathbf{F}$ is reversible if there is an involution σ, $\sigma(\mathbf{0}) = \mathbf{0}$, such that $\sigma_*\mathbf{F} = -\mathbf{F}$.

- We say that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, or the vector field $\mathbf{F}$ is orbitally reversible if there exists an involution σ and a function $f \in \mathcal{C}^\infty(U_0 \subset \mathbb{R}^n, \mathbb{R})$, $f(\mathbf{0}) = 1$ such that $\sigma_*\left(f\mathbf{F}\right) = -f\mathbf{F}$.

- We say that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, or the vector field $\mathbf{F}$ is reversible with respect to the coordinate x_i (or just $R_{x_i}-$reversible) , $i = 1, \cdots, n$, if it is reversible with respect to the involution

  $$R_{x_i}(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) = (x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n).$$

  We mean that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is invariant under a coordinate system given by $x_i \to -x_i$, $t \to -t$ for some i.

We deal with $n-$dimensional systems. Let $\mathbf{F}_0 = (X, Y)$, $X \in \mathbb{R}$, $Y \in \mathbb{R}^{n-1}$, a (germ of) C^r reversible vector field with $\mathbf{F}_0(\mathbf{0}) = \mathbf{0}$, $r > 1$, $r = \infty$ or $r = \omega$. We know (Montgomery-Bochner Theorem, (see [17], pp. 206)) that there exists a coordinate system of class C^r around 0 such that the vector field is expressed as $\mathbf{F}_0(x, y) = (f(x^2, y), xg(x^2, y))$, $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$ with f and g being C^r-functions. So a system is not reversible provided that it cannot be expressed, up to $C^r - conjugacy$, to the above form. This is, roughly speaking, the route we have chosen to conduct this paper.

In summary, in what follows, we give a rough all-over description of the main results of the paper.

- **Necessary conditions of reversibility** We derive that it is enough to use reversible generators in order to calculate necessary conditions for the reversibility of a vector field. (Theorem 2.11). This fact provides a strong simplification to deal with the reversibility problem.
- **Algorithm of non-reversibility**. We exhibit an algorithm, via the Lie triangle, that detects the non-reversibility of the system (Theorem 3.19)
- **Applications**. We apply some of the theory developed to get further results on nilpotent polynomial and degenerate vector fields.

The remaining sections are organized as follows. In Section 2 some terminology, basic concepts and preparatory results are presented. In Section 3 an adequate normal form to detect the reversibility of a vector field is discussed. In section 4 we present some applications on nilpotent and degenerate vectors fields, where the center and the reversibility problem are connected.

2 $N-$reversibility

First of all, we establish some terminology and definitions.

Let $\mathcal{P}_k^{\mathbf{t}}$ be the vector space of real quasi-homogeneous polynomial functions of degree $k \in \mathbb{N}$, respect to the type $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$, i.e., $f \in \mathcal{P}_k^{\mathbf{t}}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ if and only if $f(\epsilon^{t_1} x_1, \cdots, \epsilon^{t_n} x_n) = \epsilon^k f(x_1, \cdots, x_n)$ for all $\epsilon, x_1, \cdots, x_n \in \mathbb{R}$ and $\mathcal{Q}_k^{\mathbf{t}}$ be the vector space of the polynomial quasi-homogeneous vector fields of degree $k \in \mathbb{Z}$, respect to type $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$, i.e., $\mathbf{F} = (Q_1, \cdots, Q_n)^T \in \mathcal{Q}_k^{\mathbf{t}}$, $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, if and only if $Q_i \in \mathcal{P}_{k+t_i}^{\mathbf{t}}$, $\forall i = 1, \cdots, n$. For more details see Bruno [10].

We will denote:

- $\mathcal{R}_k^{\mathbf{t}} := \{\mu_k \in \mathcal{P}_k^{\mathbf{t}} \,|\mu_k(-x, y) = -\mu_k(x, y)\}$.
- $\mathcal{S}_k^{\mathbf{t}} := \{\mu_k \in \mathcal{P}_k^{\mathbf{t}} \,|\mu_k(-x, y) = \mu_k(x, y)\}$, where $x = x_1$ and $y = (x_2, \cdots, x_n)^T$.
- $\mathcal{R}_k^{\mathbf{t}} := \left\{(p, q)^T \in \mathcal{Q}_k^{\mathbf{t}} \,\middle|\, p \in \mathcal{S}_{k+t_1}^{\mathbf{t}}, q_i \in \mathcal{R}_{k+t_i}^{\mathbf{t}}, i = 2, \cdots, n\right\}$, the R_x-reversible quasi-homogeneous vector fields of degree k, where $q = (q_2, \cdots, q_m)^T$.
- $\mathcal{S}_k^{\mathbf{t}} := \left\{(p, q)^T \in \mathcal{Q}_k^{\mathbf{t}} : p \in \mathcal{R}_{k+t_1}^{\mathbf{t}}, q_i \in \mathcal{S}_{k+t_i}^{\mathbf{t}}, i = 2, \cdots, n\right\}$ the R_x-symmetric quasi-homogeneous vector fields of degree k.

In this way we may always consider the decomposition $\mathcal{P}_k^{\mathbf{t}} = \mathcal{R}_k^{\mathbf{t}} \bigoplus \mathcal{S}_k^{\mathbf{t}}$ and $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{R}_k^{\mathbf{t}} \bigoplus \mathcal{S}_k^{\mathbf{t}}$.

Remark 2.1 *Denote now* $\widetilde{\mathbf{U}}_j = \mathrm{Proy}_{\mathcal{R}_j^{\mathbf{t}}}(\mathbf{U})$ *and* $\overline{\mathbf{U}}_j = \mathrm{Proy}_{\mathcal{S}_j^{\mathbf{t}}}(\mathbf{U})$.

Next lemma is a direct consequence of the last definitions.

Lemma 2.2 *Let* $S = \mathrm{diag}(-1, \overbrace{1, \cdots, 1}^{n-1})$. *Then*

a) $\tilde{\mu}_k \in \mathcal{R}_k^{\mathbf{t}}$ *if, and only if,* $\tilde{\mu}_k(S\mathbf{x}) = -\tilde{\mu}_k(\mathbf{x})$.

b) $\overline{\mu}_k \in \mathcal{S}_k^{\mathbf{t}}$ *if, and only if,* $\overline{\mu}_k(S\mathbf{x}) = \overline{\mu}_k(\mathbf{x})$.

c) $\widetilde{\mathbf{P}}_k \in \mathcal{R}_k^{\mathbf{t}}$, *if, and only if,* $\widetilde{\mathbf{P}}_k(S\mathbf{x}) = -S\widetilde{\mathbf{P}}_k(\mathbf{x})$.

d) $\overline{\mathbf{P}}_k \in \mathcal{S}_k^{\mathbf{t}}$, *if, and only if,* $\overline{\mathbf{P}}_k(S\mathbf{x}) = S\overline{\mathbf{P}}_k(\mathbf{x})$.

Lemma 2.3 *Let* $\widetilde{\mathbf{F}}_r \in \mathcal{R}_r^{\mathbf{t}}$, $\overline{\mathbf{F}}_s \in \mathcal{S}_s^{\mathbf{t}}$, $\tilde{\mu}_k \in \mathcal{R}_k^{\mathbf{t}}$, $\overline{\mu}_k \in \mathcal{S}_k^{\mathbf{t}}$. *Hence:*

a) $\nabla\tilde{\mu}_k \cdot \widetilde{\mathbf{F}}_r \in \mathcal{S}_{r+k}^{\mathbf{t}}$.

b) $\nabla\tilde{\mu}_k \cdot \overline{\mathbf{F}}_s \in \mathcal{R}_{s+k}^{\mathbf{t}}$.

c) $\nabla\overline{\mu}_k \cdot \widetilde{\mathbf{F}}_r \in \mathcal{R}_{r+k}^{\mathbf{t}}$.

d) $\nabla\overline{\mu}_k \cdot \overline{\mathbf{F}}_s \in \mathcal{S}_{s+k}^{\mathbf{t}}$.

Proof: This proof is an immediate consequence of Lemma 2.2. We prove only the item **a)**.

If $\tilde{\mu}_k \in \mathcal{R}_k^{\mathbf{t}}$, $\mathbf{y} = S\mathbf{x}$ then:

$$\nabla\tilde{\mu}_k(S\mathbf{x}) = \nabla\tilde{\mu}_k(S\mathbf{x})S = \nabla\tilde{\mu}_k(\mathbf{y})S = -\nabla\tilde{\mu}_k(\mathbf{x}).$$

Hence $\nabla\tilde{\mu}_k(\mathbf{y}) = -\nabla\tilde{\mu}_k(\mathbf{x})S^{-1} = -\nabla\tilde{\mu}_k(\mathbf{x})S$. So:

$$\left(\nabla\tilde{\mu}_k \cdot \widetilde{\mathbf{F}}_r\right)(\mathbf{y}) = \nabla\tilde{\mu}_k(\mathbf{y}) \cdot \widetilde{\mathbf{F}}_r(\mathbf{y}) = \nabla\tilde{\mu}_k(\mathbf{x}) \cdot SS\widetilde{\mathbf{F}}_r(\mathbf{x}) = \nabla\tilde{\mu}_k(\mathbf{x}) \cdot \widetilde{\mathbf{F}}_r(\mathbf{x}) = \left(\nabla\tilde{\mu}_k \cdot \widetilde{\mathbf{F}}_r\right)(\mathbf{x})$$

and finally the proof is done. ∎

In [6] is proved the following result

Theorem 2.4 $\mathbf{F}$ *is reversible if, and only if, there exists an involution* $\sigma_0 \in \mathcal{Q}_0^{\mathbf{t}}$ *and a change of variables close to the identity* Φ *such that* $(\Phi \circ \sigma_0)_* \mathbf{F}$ *is* R_x-*reversible.*

Therefore, we can assume up to change of variables of zero degree that $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots$, with $\widetilde{\mathbf{F}}_r \in \mathcal{R}_r^{\mathbf{t}}$ and study when there exists a change of variables $\Phi = \sum_{j\geq 0}\Phi_j$, $\Phi_j \in \mathcal{Q}_j^{\mathbf{t}}$ and $D\Phi(\mathbf{0}) = Id$, such that $\Phi_*\mathbf{F}$ is R_x-reversible. Otherwise, $\mathbf{F}$ is not reversible.

Along this paper, we will need sometimes to truncate quasi-homogeneous expansions. In this way, for the vector field $\mathbf{F}$, expanded in quasi-homogeneous terms of type $\mathbf{t}$, $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots$, we define its quasi-homogeneous $(r+k)$-jet respect to the type $\mathbf{t}$ by

$$\mathcal{J}^{r+k}(\mathbf{F}) = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots + \mathbf{F}_{r+k}.$$

Sometimes, we will need to pick-up the k-degree quasi-homogeneous term of a vector field. As we have already done, we use subscripts on vector fields to denote its projection on the space of quasi-homogeneous polynomials. For instance, $[\mathbf{F}, \mathbf{G}]_k$ will denote the k-degree quasi-homogeneous term of the Lie product.

Throughout this paper we use the relation between change of variables and the generators associate to the change of variables, more concretely.

Let $\mathbf{y} = \Phi(\mathbf{x})$ be a smooth change of variables. It is known (see [1]) that it can be suspended to a flow $\mathbf{u}_\epsilon(\mathbf{x})$ (where $\mathbf{u}_1(\mathbf{x}) = \Phi(\mathbf{x})$) defined by a vector field $\mathbf{U}$ ($\mathbf{U}$ is called generator of Φ). Recall that $\mathbf{u}(\mathbf{x},\epsilon)$ is the solution of

$$\tfrac{\partial \mathbf{u}(\mathbf{x},\epsilon)}{\partial \epsilon} = \mathbf{U}\left(\mathbf{u}(\mathbf{x},\epsilon)\right), \qquad \mathbf{u}(\mathbf{x},0) = \mathbf{x}.$$

Considering the development (in ϵ) at $\epsilon = 0$:

$$\begin{aligned}
\mathbf{u}(\mathbf{x},\epsilon) &= \mathbf{u}(\mathbf{x},0) + \frac{\partial \mathbf{u}(\mathbf{x},0)}{\partial \epsilon}\epsilon + \frac{1}{2!}\frac{\partial^2 \mathbf{u}(\mathbf{x},0)}{\partial \epsilon^2}\epsilon^2 + \cdots \\
&= \mathbf{x} + \mathbf{U}(\mathbf{u}(\mathbf{x},0))\epsilon + \frac{1}{2!}\left.\frac{\partial \mathbf{U}\left(\mathbf{u}(\mathbf{x},\epsilon)\right)}{\partial \epsilon}\right|_{\epsilon=0}\epsilon^2 + \cdots \\
&= \mathbf{x} + \mathbf{U}(\mathbf{x})\epsilon + \frac{1}{2!}D\mathbf{U}(\mathbf{x})\mathbf{U}(\mathbf{x})\epsilon^2 + \cdots
\end{aligned}$$

So,

$$\Phi(\mathbf{x}) = \mathbf{x} + \mathbf{U}(\mathbf{x}) + \frac{1}{2!}D\mathbf{U}(\mathbf{x})\mathbf{U}(\mathbf{x}) + \cdots \tag{2.1}$$

If we denote the $\Phi-$action over $\mathbf{F}$ as $\Phi_*\mathbf{F}$ or $\mathbf{U}_{**}\mathbf{F}$, it is known (see [1]) that,

$$\mathbf{U}_{**}\mathbf{F} = \mathbf{F} + [\mathbf{F},\mathbf{U}] + \cdots + \tfrac{1}{n!}\overbrace{[\cdots[}^{n \text{ veces}}\,\mathbf{F},\mathbf{U}\,],\cdots,\mathbf{U}\,] + \cdots \tag{2.2}$$

Definition 2.5 *Let $N \in \mathbb{N}$, $\mathbf{F} = \widetilde{\mathbf{F}}_r + \sum_{j\geq 1}\mathbf{F}_{r+j}$, $\mathbf{F}_{r+j} \in \mathcal{Q}^{\mathbf{t}}_{r+j}$. We say that $\mathbf{F}$ is N-reversible if there exists a vector field $\mathbf{U} = \sum_{j\geq 1}\mathbf{U}_j$, $\mathbf{U}_j \in \mathcal{Q}^{\mathbf{t}}_j$ such that $\mathcal{J}^{r+N}\left(\mathbf{U}_{**}\mathbf{F}\right)$ is R_x-reversible.*

Remark 2.6 *It is clear that if there exists $N \in \mathbb{N}$ such that $\mathbf{F}$ is N-reversible but not $(N+1)$-reversible, then $\mathbf{F}$ is non-reversible.*

Our goal now is to find sufficient conditions for non-reversibility of the field $\mathbf{F}$. Consider the above decomposition on each quasi-homogeneous term of $\mathbf{F}$. In this way, we will study the system:

$$\dot{\mathbf{x}} = \widetilde{\mathbf{F}}_r + \sum_{j\geq r+1}\left(\widetilde{\mathbf{F}}_j + \overline{\mathbf{F}}_j\right), \tag{2.3}$$

where $\widetilde{\mathbf{F}}_j \in \mathcal{R}^{\mathbf{t}}_j$ and $\overline{\mathbf{F}}_j \in \mathcal{S}^{\mathbf{t}}_j$.

Lemma 2.7 *Let $\widetilde{\mathbf{F}}_r \in \mathcal{R}^{\mathbf{t}}_r$, $\widetilde{\mathbf{R}}_k \in \mathcal{R}^{\mathbf{t}}_k$ and $\overline{\mathbf{R}}_k \in \mathcal{S}^{\mathbf{t}}_k$. Then:*

(a) $\left[\widetilde{\mathbf{R}}_k, \widetilde{\mathbf{F}}_r\right] \in \mathcal{S}^{\mathbf{t}}_{r+k}$.

(b) $\left[\overline{\mathbf{R}}_k, \widetilde{\mathbf{F}}_r\right] \in \mathcal{R}^{\mathbf{t}}_{r+k}$.

Proof: Let $\widetilde{\mathbf{F}}_r = (f, g)^T$, $g = (g_2, \cdots, g_n)^T$ and $\widetilde{\mathbf{R}}_k = (p, q)^T$, $q = (q_2, \cdots, q_n)^T$. As $\widetilde{\mathbf{F}}_r \in \mathcal{R}_r^{\mathbf{t}}$ and $\widetilde{\mathbf{R}}_k \in \mathcal{R}_r^{\mathbf{t}}$ one has $f \in \mathcal{S}_{r+t_1}^{\mathbf{t}}$, $p \in \mathcal{S}_{k+t_1}^{\mathbf{t}}$, $g_i \in \mathcal{R}_{r+t_i}^{\mathbf{t}}$ and $q_i \in \mathcal{R}_{k+t_i}^{\mathbf{t}}$, $i = 2, \cdots, n$. Recall that

$$\left[\widetilde{\mathbf{R}}_k, \widetilde{\mathbf{F}}_r\right] = \left(\nabla p \widetilde{\mathbf{F}}_r - \nabla f \widetilde{\mathbf{R}}_k, \nabla q_2 \widetilde{\mathbf{F}}_r - \nabla g_2 \widetilde{\mathbf{R}}_k, \cdots, \nabla q_n \widetilde{\mathbf{F}}_r - \nabla g_n \widetilde{\mathbf{R}}_k\right)^T,$$

Now from Lemma 2.3 we obtain **(a)**.

Let us discuss **(b)**. If $\overline{\mathbf{R}}_k = (\tilde{p}, \overline{q})^T \in \mathcal{S}_k^{\mathbf{t}}$, $\overline{q} = (\overline{q}_2, \cdots, \overline{q}_n)^T$, then $\tilde{p} \in \mathcal{R}_{k+t_1}^{\mathbf{t}}$ and $\overline{q}_i \in \mathcal{S}_{k+t_i}^{\mathbf{t}}$, for $i = 2, \cdots, n$.

From Lemma 2.3 one has $\nabla \tilde{p} \widetilde{\mathbf{F}}_r \in \mathcal{S}_{r+k+t_1}^{\mathbf{t}}$, $\nabla f \overline{\mathbf{R}}_k \in \mathcal{S}_{r+k+t_1}^{\mathbf{t}}$, $\nabla \overline{q}_i \widetilde{\mathbf{F}}_r \in \mathcal{R}_{r+k+t_i}^{\mathbf{t}}$, $\nabla g_i \widetilde{\mathbf{R}}_k \in \mathcal{R}_{r+k+t_i}^{\mathbf{t}}$ and the claim **(b)** follows. ■

Lemma 2.8 *Consider* Φ *and* Ψ *coordinates changes given by* $\mathbf{U} = \sum_{j\geq 1} \mathbf{U}_j$ *y* $\mathbf{V} = \sum_{j \geq k} \mathbf{V}_j$ *with* $\mathbf{U}_j, \mathbf{V}_j \in \mathcal{Q}_j^{\mathbf{t}}$, *respectively. Let* $\mathbf{W} = \sum_{j\geq 1} \mathbf{W}_j$, $\mathbf{W}_j \in \mathcal{Q}_j^{\mathbf{t}}$ *be a generator of one of the mappings* $\Psi \circ \Phi$ *or* $\Phi \circ \Psi$. *Then*

$$\begin{aligned} &\mathbf{W}_j = \mathbf{U}_j, \quad \forall j = 1, \cdots, k-1, \\ &\mathbf{W}_k = \mathbf{U}_k + \mathbf{V}_k. \end{aligned}$$

Proof: If Φ is generated by $\mathbf{U}$, equation (2.1) allows us to derive $\Phi(\mathbf{x}) = \mathbf{x} + \sum_{j\geq 1} \Phi_j(\mathbf{x})$ with $\Phi_1 = \mathbf{U}_1$ and $\Phi_j = \mathbf{U}_j + f_j(\mathbf{U}_1, \cdots, \mathbf{U}_{j-1})$ where f_j with $f_j(0) = 0$.

Similarly for $\Psi(\mathbf{x}) = \mathbf{x} + \sum_{j\geq 1} \Psi_j(\mathbf{x})$, with $\Psi_1 = \mathbf{V}_1$ y $\Psi_j = \mathbf{V}_j + f_j(\mathbf{V}_1, \cdots, \mathbf{V}_{j-1})$. In this case, as $\mathbf{V}_1 = \cdots = \mathbf{V}_{k-1} = \mathbf{0}$, one obtains $\Psi_1 \equiv \cdots \equiv \Psi_{k-1} \equiv 0$ y $\Psi_k = \mathbf{V}_k$.

We discuss just the mapping $\Psi \circ \Phi(\mathbf{x})$; the proof for $\Phi \circ \Psi(\mathbf{x})$ is completely analogous.

Let $\Theta(\mathbf{x}) := \Psi \circ \Phi(\mathbf{x}) = \mathbf{x} + \sum_{j\geq 1} \Theta_j(\mathbf{x})$. Using same arguments as above one obtains that both, $\Theta_1 = \mathbf{W}_1$ and $\Theta_j = \mathbf{W}_j + f_j(\mathbf{W}_1, \cdots, \mathbf{W}_{j-1})$, depend only on de $\mathbf{W}_1, \cdots, \mathbf{W}_j$.

If $k = 1$, as $\Theta(\mathbf{x}) = \Psi \circ \Phi(\mathbf{x}) = \Psi\left(\mathbf{x} + \sum_{j\geq 1} \Phi_j(\mathbf{x})\right) = \mathbf{x} + \Psi_1(\mathbf{x}) + \Phi_1(\mathbf{x}) + \cdots$ where $\cdots$ are higher degree (> 1) quasi-homogeneous terms one has $\mathbf{W}_1 = \mathbf{U}_1 + \mathbf{V}_1$.

If $k > 1$, then $\Psi_1 \equiv \cdots \equiv \Psi_{k-1} \equiv 0$ y $\Theta(\mathbf{x}) = \Psi \circ \Phi(\mathbf{x}) = \Psi\left(\mathbf{x} + \sum_{j\geq 1} \Phi_j(\mathbf{x})\right) = \mathbf{x} + \Phi_1(\mathbf{x}) + \cdots + \Phi_{k-1}(\mathbf{x}) + \Psi_k(\mathbf{x}) + \Phi_k(\mathbf{x}) + \cdots$ where $\cdots$ are higher degree ($> k$) quasi-homogeneous terms. Thus $\Theta_1 = \Phi_1, \cdots, \Theta_{k-1} = \Phi_{k-1}$ y $\Theta_k = \Phi_k + \Psi_k$.

In this way we get $\mathbf{W}_1 = \mathbf{U}_1$. Moreover for each j, $2 \leq j \leq k-1$ we have:

$$\Theta_j = \mathbf{W}_j + f_j(\mathbf{W}_1, \cdots, \mathbf{W}_{j-1}) = \mathbf{U}_j + f_j(\mathbf{U}_1, \cdots, \mathbf{U}_{j-1}) = \Phi_j$$

We obtain then $\mathbf{W}_1 = \mathbf{U}_1, \cdots, \mathbf{W}_{k-1} = \mathbf{U}_{k-1}$. Finally the proof of the Proposition follows from

$$\begin{aligned} \Theta_k &= \mathbf{W}_k + f_k(\mathbf{W}_1, \cdots, \mathbf{W}_{k-1}) = \mathbf{W}_k + f_k(\mathbf{U}_1, \cdots, \mathbf{U}_{k-1}) \\ \Psi_k + \Phi_k &= \mathbf{V}_k + \mathbf{U}_k + f_k(\mathbf{U}_1, \cdots, \mathbf{U}_{k-1}). \end{aligned}$$

■

Lemma 2.9 *Let $r \in \mathbb{N}$, $\mathbf{F} = \widetilde{\mathbf{F}}_r + \sum_{j\geq 1} \mathbf{F}_{r+j}$ with $\widetilde{\mathbf{F}}_r \in \mathcal{R}_r^{\mathbf{t}}$, $\mathbf{F}_{r+j} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$ and $\widetilde{\mathbf{F}}_r \neq \mathbf{0}$. Consider $\mathbf{U} = \sum_{j\geq 1} \mathbf{U}_j$ with $\mathbf{U}_j \in \mathcal{Q}_j^{\mathbf{t}}$. There exist $\mathbf{V} = \sum_{j\geq 1} \mathbf{V}_j$, $\mathbf{V}_j \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{V}_r \in \widehat{\mathcal{Q}}_r^{\mathbf{t}}$, where $\widehat{\mathcal{Q}}_r^{\mathbf{t}}$ is a complementary space to $\left\langle \widetilde{\mathbf{F}}_r \right\rangle$ in $\mathcal{Q}_r^{\mathbf{t}}$; that is, $\mathcal{Q}_r^{\mathbf{t}} = \widehat{\mathcal{Q}}_r^{\mathbf{t}} \bigoplus \left\langle \widetilde{\mathbf{F}}_r \right\rangle$, such that $\mathbf{U}_{**}\mathbf{F} = \mathbf{V}_{**}\mathbf{F}$*

Proof: Let Φ be the diffeomorphism generated by $\mathbf{U}$. Let $\mathbf{U}_r = \widehat{\mathbf{U}}_r + \lambda \widetilde{\mathbf{F}}_r$ with $\lambda \in \mathbb{R}$ and $\widehat{\mathbf{U}}_r \in \widehat{\mathcal{Q}}_r^{\mathbf{t}}$. If $\lambda = 0$ the assertion follows; it is enough to take $\mathbf{V} = \mathbf{U}$.

If $\lambda \neq 0$, we select Ψ, the diffeomorphism generated by $\lambda\mathbf{F}$. Observe that

$$(\lambda\mathbf{F})_{**}\mathbf{F} \;=\; \mathbf{F} + [\mathbf{F}, \lambda\mathbf{F}] + \cdots + \frac{1}{n!}\left[\cdots[\mathbf{F}, \lambda\mathbf{F}], \cdots, \lambda\mathbf{F}\right] + \cdots = \mathbf{F}$$

If $\mathbf{V}$ is generator of $\Phi \circ \Psi^{-1}$, then:

$$\mathbf{U}_{**}\mathbf{F} \;=\; \Phi_*\mathbf{F} = \Phi \circ \Psi^{-1} \circ \Psi_*\mathbf{F} = \left(\Phi \circ \Psi^{-1}\right)_* (\Psi_*\mathbf{F}) = \mathbf{V}_{**}\left((\lambda\mathbf{F})_{**}\mathbf{F}\right) = \mathbf{V}_{**}\mathbf{F}$$

On the other hand $\Psi(\mathbf{x}) = \mathbf{x} + \lambda\mathbf{F}_r(\mathbf{x}) + \cdots$. So $\Psi^{-1}(\mathbf{x}) = \mathbf{x} - \lambda\mathbf{F}_r(\mathbf{x}) + \cdots$. Applying Lemma 2.8 one obtains that $\mathbf{V}_r = \mathbf{U}_r - \lambda\mathbf{F}_r = \widehat{\mathbf{U}}_r \in \widehat{\mathcal{Q}}_r^{\mathbf{t}}$. ■

From now on, we will denote, abusing of the language, $\mathcal{R}_r^{\mathbf{t}}$ as a complementary subspace to $< \widetilde{\mathbf{F}}_r >$ in $\mathcal{R}_r^{\mathbf{t}}$

Proposition 2.10 *If the system (2.3) is reversible, then there is a diffeomorphism Ψ generated by a R_x-reversible vector field $\mathbf{V}$, i.e. $\mathbf{V} \in \bigoplus_{j\geq 1} \mathcal{R}_j^{\mathbf{t}}$, such that $\left(\overline{\mathbf{V}_{**}\mathbf{F}}\right)_{r+j} = \mathbf{0}$ for all $j \geq 1$.*

Proof: If $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is reversible, there exists a diffeomorphism Φ generated by $\mathbf{U}$ such that $\left(\overline{\mathbf{U}_{**}\mathbf{F}}\right)_{r+j} = \mathbf{0}$ for all $j \geq 1$.

Let $k_{\mathbf{U}} = \min\left\{j \in \mathbb{N} \,\middle|\, \overline{\mathbf{U}}_j \neq \mathbf{0}\right\}$. If $k_{\mathbf{U}} < +\infty$, we show that there exists a diffeomorphism generated by $\mathbf{V}$ such that $\left(\overline{\mathbf{V}_{**}\mathbf{F}}\right)_{r+j} = \mathbf{0}$ for every $j \geq 1$. Moreover $k_{\mathbf{V}} > k_{\mathbf{U}}$.

Let $\Psi_{k_{\mathbf{U}}}$ be the diffeomorphism generated by $-\overline{\mathbf{U}}_{k_{\mathbf{U}}}$. Consider $\mathbf{V}$ one of the possible generators of $\Psi_{k_{\mathbf{U}}} \circ \Phi$. We show that $\left(\overline{(\Psi_{k_{\mathbf{U}}} \circ \Phi)_* \mathbf{F}}\right)_{r+j} = \mathbf{0}$ for every $j \geq 1$. From Lemma 2.7 cases **b)** and **c)** we just need to prove that:

$$0 \;=\; \left(\overline{\left(-\overline{\mathbf{U}}_{k_{\mathbf{U}}}\right)_{**}(\mathbf{U}_{**}\mathbf{F})}\right)_{r+j} = \left(\left(-\overline{\mathbf{U}}_{k_{\mathbf{U}}}\right)_{**}\left(\overline{\mathbf{U}_{**}\mathbf{F}}\right)\right)_{r+j}$$

This assertion is always true due to $\left(\overline{\mathbf{U}_{**}\mathbf{F}}\right)_{r+j} = \mathbf{0}$ for every $j \geq 1$.

We conclude the present proof by applying the Proposition 2.8 and recalling that $\mathcal{J}^{k_{\mathbf{U}}-1}(\mathbf{V}) = \mathcal{J}^{k_{\mathbf{U}}-1}(\mathbf{U})$ and $\mathbf{V}_{k_{\mathbf{U}}} = \mathbf{U}_{k_{\mathbf{U}}} - \overline{\mathbf{U}}_{k_{\mathbf{U}}}$. ■

Theorem 2.11 $\mathbf{F} = \widetilde{\mathbf{F}}_r + \sum_{j\geq 1} \mathbf{F}_{r+j}$ *is N-reversible, $N \in \mathbb{N}$, provided that there exists a vector field $\widetilde{\mathbf{U}} = \sum_{j=1}^{N} \widetilde{\mathbf{U}}_j$, $\widetilde{\mathbf{U}}_j \in \mathcal{R}_j^{\mathbf{t}}$ such that $\mathcal{J}^{r+N}\left(\widetilde{\mathbf{U}}_{**}\mathbf{F}\right)$ is R_x-reversible.*

Proof: It is enough to apply Lemma 2.9 and Proposition 2.10. ■

Remark 2.12 *Therefore to study the reversibility of a vector field it is enough to use change of variables whose generators are reversibles.*

3 $N-$reversible normal forms and reversibility

The goal of this section is to determine a suitable normal form adequate to our setting. Following the terminology of the last section, let $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{S}_k^{\mathbf{t}} \bigoplus \mathcal{R}_k^{\mathbf{t}}$ for every k. So we write

$$\mathbf{F} \;=\; \widetilde{\mathbf{F}}_r + \sum_{j=1}^{\infty} \left(\widetilde{\mathbf{F}}_{r+j} + \overline{\mathbf{F}}_{r+j} \right), \; \widetilde{\mathbf{F}}_{r+j} \in \mathcal{R}_{r+j}^{\mathbf{t}}, \; \overline{\mathbf{F}}_{r+j} \in \mathcal{S}_{r+j}^{\mathbf{t}}, \tag{3.4}$$

where $\widetilde{\mathbf{F}}_r \neq \mathbf{0}$, $\widetilde{\mathbf{F}}_r \in \mathcal{R}_r^{\mathbf{t}}$, i.e. $\mathbf{F}$ is $0-$reversible.

We take $\mathbf{F}^{(0)} := \mathbf{F} = \widetilde{\mathbf{F}}_r^{(0)} + \left(\widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{(0)} \right) + \cdots$. To simplify the terms $\overline{\mathbf{F}}_{r+1}^{(0)}$, we apply the variable change Φ_1 generated by $\widetilde{\mathbf{U}}_1 \in \mathcal{R}_1^{\mathbf{t}}$. Observe that from Theorem 2.11 we may use generators belonging to $\mathcal{R}_k^{\mathbf{t}}$ instead $\mathcal{Q}_k^{\mathbf{t}}$, and we get

$$(\Phi_1)_* \mathbf{F} \;=\; \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{(0)} + \left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_1 \right] + \cdots$$

Such results suggest to define the following linear operator:

$$\begin{array}{rcl} \overline{\mathbf{L}}^{(1)} : \mathcal{R}_1^{\mathbf{t}} & \longrightarrow & \mathcal{S}_{r+1}^{\mathbf{t}} \\ \widetilde{\mathbf{U}}_1 & \rightarrow & \left[\widetilde{\mathbf{U}}_1, \widetilde{\mathbf{F}}_r^{(0)} \right]. \end{array} \tag{3.5}$$

From Lemma 2.7 **(a)** we know that $\overline{\mathbf{L}}^{(1)}$, is well defined, and it depends on $\widetilde{\mathbf{F}}_r^{(0)}$ and we may write $\overline{\mathbf{L}}^{(1)} = \overline{\mathbf{L}}^{(1)} \left[\widetilde{\mathbf{F}}_r^{(0)} \right]$.

Consider $\overline{\mathbf{F}}_{r+1}^{(0)} = \overline{\mathbf{F}}_{r+1}^{r} + \overline{\mathbf{F}}_{r+1}^{c}$ where $\overline{\mathbf{F}}_{r+1}^{r} \in \mathrm{Im}\left(\overline{\mathbf{L}}^{(1)} \right)$ and $\overline{\mathbf{F}}_{r+1}^{c} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(1)} \right)$, with $\mathcal{S}_{r+1}^{\mathbf{t}} = \mathrm{Im}\left(\overline{\mathbf{L}}^{(1)} \right) \bigoplus \mathrm{Cor}\left(\overline{\mathbf{L}}^{(1)} \right)$. We may select $\widetilde{\mathbf{U}}_1 \in \mathcal{R}_1^{\mathbf{t}}$ such that $\overline{\mathbf{F}}_{r+1}^{r} = \overline{\mathbf{L}}^{(1)} \left(\widetilde{\mathbf{U}}_1 \right)$. So $\mathbf{F}^{(1)} := (\Phi_1)_* \mathbf{F}^{(0)}$ is expressed as:

$$\mathbf{F}^{(1)} \;=\; \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{c} + \cdots$$

If $\overline{\mathbf{F}}_{r+1}^{c} \neq \mathbf{0}$ we will show that the original vector field cannot be 1-reversible. Otherwise we may write

$$\mathbf{F}^{(1)} \;=\; \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \widetilde{\mathbf{F}}_{r+2}^{(1)} + \overline{\mathbf{F}}_{r+2}^{(1)} + \cdots \tag{3.6}$$

To simplify $\overline{\mathbf{F}}_{r+2}^{(1)}$ we apply Φ_2 having as generator the vector field $\widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2$ with $\widetilde{\mathbf{U}}_j \in \mathcal{R}_j^{\mathbf{t}}$ for $j = 1, 2$. We choose $\widetilde{\mathbf{U}}_1 \in \mathrm{Ker}\left(\overline{\mathbf{L}}^{(1)} \right)$ and we get:

$$\begin{aligned} (\Phi_2)_* \mathbf{F}^{(1)} \;=\; & \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_1 \right] + \widetilde{\mathbf{F}}_{r+2}^{(1)} + \overline{\mathbf{F}}_{r+2}^{(1)} + \left[\widetilde{\mathbf{F}}_{r+1}^{(0)}, \widetilde{\mathbf{U}}_1 \right] + \left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_2 \right] \\ & + \frac{1}{2!} \left[\left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_1 \right], \widetilde{\mathbf{U}}_1 \right] + \cdots \\ =\; & \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \widetilde{\mathbf{F}}_{r+2}^{(1)} + \overline{\mathbf{F}}_{r+2}^{(1)} + \left[\widetilde{\mathbf{F}}_{r+1}^{(0)}, \widetilde{\mathbf{U}}_1 \right] + \left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_2 \right] + \cdots \end{aligned}$$

In this way a sequence of linear operators can be defined $\overline{\mathbf{L}}^{(m)}$, $m \in \mathbb{N}$ as follows.

Definition 3.13 *Let $m \in \mathbb{N}$ and*

$$\mathbf{F}^{(m)} = \widetilde{\mathbf{F}}_r^{(m)} + \cdots + \widetilde{\mathbf{F}}_{r+m-1}^{(m)} + \sum_{j \geq m} \left(\widetilde{\mathbf{F}}_{r+j}^{(m)} + \overline{\mathbf{F}}_{r+j}^{(m)} \right),$$

where $\widetilde{\mathbf{F}}_{r+j}^{(m)} \in \mathcal{R}_{r+j}^{\mathbf{t}}$ for each $j \geq 0$ and $\overline{\mathbf{F}}_{r+j}^{(m)} \in \mathcal{S}_{r+j}^{\mathbf{t}}$ for each $j \geq m$. We define the operator $\overline{\mathbf{L}}^{(m)}$ by induction on $m \in \mathbb{N}$. $\overline{\mathbf{L}}^{(1)}$ is defined in (3.5), and the operators $\overline{\mathbf{L}}^{(m)}$ that depend on $\widetilde{\mathbf{F}}_r^{(m)}, \cdots, \widetilde{\mathbf{F}}_{r+m-1}^{(m)}$, i.e. $\overline{\mathbf{L}}^{(m)} = \overline{\mathbf{L}}^{(m)}[\widetilde{\mathbf{F}}_r^{(m)}, \cdots, \widetilde{\mathbf{F}}_{r+m-1}^{(m)}]$ for every $m \geq 1$, are defined by

$$\begin{array}{rcl} \overline{\mathbf{L}}^{(m)} : \mathrm{Ker}\left(\overline{\mathbf{L}}^{(m-1)}\right) \times \mathcal{R}_m^{\mathbf{t}} & \longrightarrow & \mathcal{S}_{r+m}^{\mathbf{t}} \\ \left(\left(\widetilde{\mathbf{R}}_1, \cdots, \widetilde{\mathbf{R}}_{m-1} \right), \widetilde{\mathbf{R}}_m \right) & \rightarrow & \sum_{j=0}^{m-1} \left[\widetilde{\mathbf{R}}_{m-j}, \widetilde{\mathbf{F}}_{r+j} \right]. \end{array}$$

Lemma 2.7 **(a)** allows us to conclude that $\overline{\mathbf{L}}^{(m)}$ is well defined.

Definition 3.14 *Let $\mathbf{F}$, $\mathbf{G}$ be two conjugate vector fields such that $\widetilde{\mathbf{U}}_{**}\mathbf{F} = \mathbf{G}$, $\widetilde{\mathbf{U}} \in \bigoplus_{j \geq 1} \mathcal{R}_j^{\mathbf{t}}$. We say that $\mathbf{G}$ is a $N-$reversible normal form of $\mathbf{F}$ if $\mathbf{G}$ can be expressed as*

$$\mathbf{G} = \widetilde{\mathbf{G}}_r + \cdots + \widetilde{\mathbf{G}}_{r+N-1} + \left(\widetilde{\mathbf{G}}_{r+N} + \overline{\mathbf{G}}_{r+N} \right) + \cdots,$$

with $\widetilde{\mathbf{G}}_r = \widetilde{\mathbf{F}}_r$, $\overline{\mathbf{G}}_{r+N} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(N)}[\widetilde{\mathbf{G}}_r, \cdots, \widetilde{\mathbf{G}}_{r+N-1}] \right)$ and $\widetilde{\mathbf{G}}_{r+j} \in \mathcal{R}_{r+j}^{\mathbf{t}}$, $0 \leq j \leq N$.

Lemma 3.15 *Let $k \in \mathbb{N}$, $k > 1$, $\mathbf{F} = \sum_{j=0}^{k-1} \widetilde{\mathbf{F}}_{r+j} + \left(\widetilde{\mathbf{F}}_{r+k} + \overline{\mathbf{F}}_{r+k} \right) + \cdots$, $\widetilde{\mathbf{F}}_{r+j} \in \mathcal{R}_{r+j}^{\mathbf{t}}$ and $\overline{\mathbf{F}}_{r+k} \in \mathcal{S}_{r+k}^{\mathbf{t}}$. Assume that $\widetilde{\mathbf{U}} = \sum_{j=1}^{\infty} \widetilde{\mathbf{U}}_j$, $\widetilde{\mathbf{U}}_j \in \mathcal{R}_j^{\mathbf{t}}$ such that $\left(\widetilde{\mathbf{U}}_1, \cdots, \widetilde{\mathbf{U}}_{k-1} \right) \in \mathrm{Ker}\left(\overline{\mathbf{L}}^{(k-1)} \right)$. Then:*

$$\begin{aligned} \left(\widetilde{\widetilde{\mathbf{U}}_{**}\mathbf{F}} \right)_{r+j} &= \widetilde{\mathbf{F}}_{r+j}, \quad \textit{for } j = 0, \cdots, k, \\ \left(\overline{\widetilde{\mathbf{U}}_{**}\mathbf{F}} \right)_{r+j} &= \mathbf{0}, \quad \textit{for } j = 0, \cdots, k-1, \\ \left(\overline{\widetilde{\mathbf{U}}_{**}\mathbf{F}} \right)_{r+k} &= \overline{\mathbf{F}}_{r+k} + \sum_{l=1}^{k} \left[\widetilde{\mathbf{F}}_{r+k-l}, \widetilde{\mathbf{U}}_l \right] \end{aligned}$$

Proof: For $j = 0, \cdots, k$, one has

$$\begin{aligned} \left(\widetilde{\mathbf{U}}_{**}\mathbf{F} \right)_{r+j} = {} & \mathbf{F}_{r+j} + \left[\mathbf{F}, \widetilde{\mathbf{U}} \right]_{r+j} + \frac{1}{2!} \left[\left[\mathbf{F}, \widetilde{\mathbf{U}} \right], \widetilde{\mathbf{U}} \right]_{r+j} + \frac{1}{3!} \left[\left[\left[\mathbf{F}, \widetilde{\mathbf{U}} \right], \widetilde{\mathbf{U}} \right], \widetilde{\mathbf{U}} \right]_{r+j} + \cdots \\ & + \frac{1}{n!} \left[\cdots \left[\mathbf{F}, \widetilde{\mathbf{U}} \right], \cdots, \widetilde{\mathbf{U}} \right]_{r+j} \end{aligned}$$

Since $\left(\widetilde{\mathbf{U}}_1, \cdots, \widetilde{\mathbf{U}}_{k-1}\right) \in \operatorname{Ker}\left(\overline{\mathbf{L}}^{(k-1)}\right)$, one has:

$$\begin{aligned}
\mathbf{0} &= \left[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{U}}_1\right] = \left[\mathbf{F}, \widetilde{\mathbf{U}}\right]_{r+1}, \\
\mathbf{0} &= \sum_{l=1}^{2} \left[\widetilde{\mathbf{F}}_{r+1-l}, \widetilde{\mathbf{U}}_l\right] = \left[\mathbf{F}, \widetilde{\mathbf{U}}\right]_{r+2}, \\
&\vdots \\
\mathbf{0} &= \sum_{l=1}^{k-1} \left[\widetilde{\mathbf{F}}_{r+k-1-l}, \widetilde{\mathbf{U}}_l\right] = \left[\mathbf{F}, \widetilde{\mathbf{U}}\right]_{r+k-1}.
\end{aligned}$$

and so the proof is complete. ∎

The following result proves that the $(r+N)-$jet of any two $N-$reversible normal form is unique, modulus reversible generators.

Theorem 3.16 *Let*

$$\begin{aligned}
\mathbf{F} &= \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{F}}_{r+1} + \cdots + \widetilde{\mathbf{F}}_{r+N-1} + \left(\widetilde{\mathbf{F}}_{r+N} + \overline{\mathbf{F}}_{r+N}\right) + \cdots, \\
\mathbf{G} &= \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{G}}_{r+1} + \cdots + \widetilde{\mathbf{G}}_{r+N-1} + \left(\widetilde{\mathbf{G}}_{r+N} + \overline{\mathbf{G}}_{r+N}\right) + \cdots
\end{aligned}$$

such that

$$\overline{\mathbf{F}}_{r+N} \in \operatorname{Cor}\left(\overline{\mathbf{L}}^{(N)}[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{F}}_{r+1}, \cdots, \widetilde{\mathbf{F}}_{r+N-1}]\right),$$
$$\overline{\mathbf{G}}_{r+N} \in \operatorname{Cor}\left(\overline{\mathbf{L}}^{(N)}[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{G}}_{r+1}, \cdots, \widetilde{\mathbf{G}}_{r+N-1}]\right).$$

Assume that there exists a formal series $\widetilde{\mathbf{U}} = \sum_{j\geq 1} \widetilde{\mathbf{U}}_j$ *with* $\widetilde{\mathbf{U}}_j \in \mathcal{R}_j^{\mathbf{t}}$ *for all* $j \geq 1$ *such that* $\left(\widetilde{\mathbf{U}}_{**}\mathbf{F}\right)_{r+j} = \mathbf{G}_{r+j}$ *for all* $j = 1, \cdots, N$. *Then*

$$\widetilde{\mathbf{G}}_{r+j} = \widetilde{\mathbf{F}}_{r+j} \text{ for all } j = 1, \cdots, N \text{ and } \overline{\mathbf{G}}_{r+N} = \overline{\mathbf{F}}_{r+N}$$

Proof: As $\mathbf{G}_{r+j} = \left(\widetilde{\mathbf{U}}_{**}\mathbf{F}\right)_{r+j}$, for $j = 1, \cdots, N$ one obtains:

$$\begin{aligned}
\mathbf{G}_{r+j} &= \mathbf{F}_{r+j} + \left[\mathbf{F}, \widetilde{\mathbf{U}}\right]_{r+j} + \frac{1}{2!}\left[\left[\mathbf{F}, \widetilde{\mathbf{U}}\right], \widetilde{\mathbf{U}}\right]_{r+j} + \cdots \\
&\quad + \frac{1}{n!}\left[\cdots\left[\mathbf{F}, \widetilde{\mathbf{U}}\right], \cdots, \widetilde{\mathbf{U}}\right]_{r+j} + \cdots
\end{aligned} \tag{3.7}$$

Hence:

$$\overline{\mathbf{G}}_{r+j} = \overline{\mathbf{F}}_{r+j} + \overline{\left[\mathbf{F}, \widetilde{\mathbf{U}}\right]}_{r+j} + \frac{1}{2!}\overline{\left[\left[\mathbf{F}, \widetilde{\mathbf{U}}\right], \widetilde{\mathbf{U}}\right]}_{r+j} + \cdots + \frac{1}{n!}\overline{\left[\cdots\left[\mathbf{F}, \widetilde{\mathbf{U}}\right], \cdots, \widetilde{\mathbf{U}}\right]}_{r+j} + \cdots$$

$$\widetilde{\mathbf{G}}_{r+j} = \widetilde{\mathbf{F}}_{r+j} + \widetilde{\left[\mathbf{F}, \widetilde{\mathbf{U}}\right]}_{r+j} + \frac{1}{2!}\left[\widetilde{\left[\mathbf{F}, \widetilde{\mathbf{U}}\right]}, \widetilde{\mathbf{U}}\right]_{r+j} + \cdots + \frac{1}{n!}\left[\cdots\widetilde{\left[\mathbf{F}, \widetilde{\mathbf{U}}\right]}, \cdots, \widetilde{\mathbf{U}}\right]_{r+j} + \cdots$$

Observe that these infinite sums are well defined. As $\overline{\mathbf{G}}_r = \cdots = \overline{\mathbf{G}}_{r+N-1} = \mathbf{0}$, for $1 \leq j \leq N-1$ one has:

$$\mathbf{0} = \overline{\left[\mathbf{F}, \widetilde{\mathbf{U}}\right]}_{r+j} + \frac{1}{2!}\overline{\left[\left[\mathbf{F}, \widetilde{\mathbf{U}}\right], \widetilde{\mathbf{U}}\right]}_{r+j} + \cdots + \frac{1}{n!}\overline{\left[\cdots\left[\mathbf{F}, \widetilde{\mathbf{U}}\right], \cdots, \widetilde{\mathbf{U}}\right]}_{r+j} + \cdots$$

For $j = 1$, taking into account the quasi-homogeneous degree in the non-reversible part, the equation (3.7) is written as:

$$\mathbf{0} \;=\; \left[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{U}}_1\right]_{r+1} = -\overline{\mathbf{L}}^{(1)}\left(\widetilde{\mathbf{U}}_1\right),$$

Hence $\widetilde{\mathbf{U}}_1 \in \mathrm{Ker}\left(\overline{\mathbf{L}}^{(1)}\right)$.

Consider now the reversible terms for $j = 1$. From Lemma 2.7 case **(a)**, we get:

$$\widetilde{\mathbf{G}}_{r+1} \;=\; \widetilde{\mathbf{F}}_{r+1} + \widetilde{\left[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{U}}_1\right]}_{r+1} = \widetilde{\mathbf{F}}_{r+1}.$$

For $j = 2$, considering in (3.7) the reversible terms, one has:

$$\begin{aligned}\mathbf{0} \;&=\; \left[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{U}}_2\right] + \left[\widetilde{\mathbf{F}}_{r+1}, \widetilde{\mathbf{U}}_1\right] + \tfrac{1}{2!}\left[\left[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{U}}_1\right], \widetilde{\mathbf{U}}_1\right] \\ &=\; \left[\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{U}}_2\right] + \left[\widetilde{\mathbf{F}}_{r+1}, \widetilde{\mathbf{U}}_1\right] = -\overline{\mathbf{L}}^{(2)}\left(\widetilde{\mathbf{U}}_1, \widetilde{\mathbf{U}}_2\right)\end{aligned}$$

So $\left(\widetilde{\mathbf{U}}_1, \widetilde{\mathbf{U}}_2\right) \in \mathrm{Ker}\left(\overline{\mathbf{L}}^{(2)}\right)$.

Considering now the reversible terms in (3.7) for $j = 2$, from Lemma 2.7, case **(a)**, one gets $\widetilde{\mathbf{F}}_{r+2} = \widetilde{\mathbf{G}}_{r+2}$.

Arguing in the same way as above we have $\widetilde{\mathbf{F}}_{r+N-1} = \widetilde{\mathbf{G}}_{r+N-1}$ and $\left(\widetilde{\mathbf{U}}_1, \cdots, \widetilde{\mathbf{U}}_{N-1}\right) \in \mathrm{Ker}\left(\overline{\mathbf{L}}^{(N-1)}\right)$.

From Lemma 3.15, the non-reversible terms of the equation (3.7), for $j = N$, are:

$$\overline{\mathbf{G}}_{r+N} \;=\; \overline{\mathbf{F}}_{r+N} + \sum_{j=0}^{N-1}\left[\widetilde{\mathbf{F}}_{r+j}, \widetilde{\mathbf{U}}_{N-j}\right] = \overline{\mathbf{F}}_{r+N} - \overline{\mathbf{L}}^{(N)}\left(\widetilde{\mathbf{U}}_1, \cdots, \widetilde{\mathbf{U}}_N\right)$$

Thus $\overline{\mathbf{G}}_{r+N} - \overline{\mathbf{F}}_{r+N} \in \mathrm{Im}\left(\overline{\mathbf{L}}^{(N)}\right) \cap \mathrm{Cor}\left(\overline{\mathbf{L}}^{(N)}\right) = \{\mathbf{0}\}$ and so $\overline{\mathbf{G}}_{r+N} = \overline{\mathbf{F}}_{r+N}$.

Finally the reversible terms of (3.7), for $j = N$, are:

$$\widetilde{\mathbf{G}}_{r+N} \;=\; \widetilde{\mathbf{F}}_{r+N} + \sum_{j=0}^{N}\widetilde{\left[\widetilde{\mathbf{F}}_{r+j}, \widetilde{\mathbf{U}}_{N-j}\right]} = \widetilde{\mathbf{F}}_{r+N}.$$

This finishes the proof.

■

Lemma 3.17 *Let $k, N \in \mathbb{N}$, $k \leq N$. If*

$$\mathbf{F} = \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{F}}_{r+1} + \cdots + \widetilde{\mathbf{F}}_{r+k-1} + \left(\widetilde{\mathbf{F}}_{r+k} + \overline{\mathbf{F}}_{r+k}\right) + \cdots,$$

is N-reversible then

$$\overline{\mathbf{F}}_{r+k} \in \operatorname{Im}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]\right)$$

Proof: Consider $\overline{\mathbf{F}}_{r+k} = \overline{\mathbf{F}}^{r}_{r+k} + \overline{\mathbf{F}}^{c}_{r+k}$ with $\overline{\mathbf{F}}^{r}_{r+k} \in \operatorname{Im}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]\right)$ and $\overline{\mathbf{F}}^{c}_{r+k} \in \operatorname{Cor}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]\right)$.

So there is a $k-$*upla* $\left(\widetilde{\mathbf{R}}_1, \cdots, \widetilde{\mathbf{R}}_k\right)$ in the domain of the operator $\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]$ satisfying

$$\overline{\mathbf{F}}^{r}_{r+k} = \overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]\left(\widetilde{\mathbf{R}}_1, \cdots, \widetilde{\mathbf{R}}_k\right).$$

Let Φ be the diffeomorphism generated by $\sum_{j=1}^{k} \widetilde{\mathbf{R}}_j$. From Lemma 3.15 one gets

$$\begin{aligned}
\Phi_*\mathbf{F} &= \widetilde{\mathbf{F}}_r + \cdots + \widetilde{\mathbf{F}}_{r+k-1} + \widetilde{\mathbf{F}}_{r+k} + \overline{\mathbf{F}}^{r}_{r+k} - \overline{\mathbf{L}}^{(k)}_1[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]\left(\widetilde{\mathbf{R}}_1, \cdots, \widetilde{\mathbf{R}}_k\right) \\
&\quad + \overline{\mathbf{F}}^{c}_{r+k} + \cdots \\
&= \widetilde{\mathbf{F}}_r + \cdots + \widetilde{\mathbf{F}}_{r+k-1} + \widetilde{\mathbf{F}}_{r+k} + \overline{\mathbf{F}}^{c}_{r+k} + \cdots
\end{aligned}$$

As $\Phi_*\mathbf{F}$ is N-reversible and $N \geq k$ we conclude that $\Phi_*\mathbf{F}$ is also k-reversible. Hence, there is $\widetilde{\mathbf{U}} = \sum_{j=1}^{k} \widetilde{\mathbf{U}}_j$, $\widetilde{\mathbf{U}}_j \in \mathcal{R}^{\mathbf{t}}_j$ such that $\overline{\left(\widetilde{\mathbf{U}}_{**}\left(\Phi_*\mathbf{F}\right)\right)}_{r+j} = \mathbf{0}$ for $j = 0, \cdots, k$. Denoting $\mathbf{G} = \widetilde{\mathbf{U}}_{**}\left(\Phi_*\mathbf{F}\right)$ then

$$\mathbf{G} = \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{G}}_{r+1} + \cdots + \widetilde{\mathbf{G}}_{r+k} + \overline{\mathbf{G}}_{r+k} + \cdots$$

with $\overline{\mathbf{G}}_{r+k} = \mathbf{0}$.

Theorem 3.16 implies that $\widetilde{\mathbf{F}}_{r+j} = \widetilde{\mathbf{G}}_{r+j}$, for $j = 1, \cdots, k$ and $\overline{\mathbf{F}}^{c}_{r+k} = \mathbf{0}$ and so we finish the proof. ■

The following proposition defines a procedure which provides us the necessary conditions of reversibility up to a defined order.

Proposition 3.18 *let $N \in \mathbb{N}$, $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots$ a vector field. There exists $\widetilde{\mathbf{U}} = \sum_{j=1}^{N} \widetilde{\mathbf{U}}_j$ with*

$$\widetilde{\mathbf{U}}_{**}\mathbf{F} = \widetilde{\mathbf{F}}_r + \sum_{j\geq 1}\left(\widetilde{\mathbf{F}}_{r+j} + \overline{\mathbf{F}}_{r+j}\right)$$

such that if $\overline{\mathbf{F}}_{r+j} = \mathbf{0}$ $j = 0, \cdots, k-1$, then $\overline{\mathbf{F}}_{r+k} \in \operatorname{Cor}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+k-1}]\right)$, for all $k = 1, \cdots, N$.

Proof: Let $\mathbf{F}^{(0)} := \mathbf{F} = \widetilde{\mathbf{F}}_r^{(0)} + \sum_{j\geq 1}\left(\widetilde{\mathbf{F}}_{r+j}^{(0)} + \overline{\mathbf{F}}_{r+j}^{(0)}\right)$, we are going to prove that there exists a sequence of change of variables near the identity $\Phi^{(k)}$, $k = 1, \cdots, N$ such that $\mathbf{F}^{(k)} := \Phi_*^{(k)}\mathbf{F}^{(k-1)}$, verifies the following properties:

(a) $\Phi^{(k)}$ is generated by $\widetilde{\mathbf{U}}^{(k)} = \widetilde{\mathbf{U}}_1^{(k)} + \cdots + \widetilde{\mathbf{U}}_k^{(k)}$, where $\left(\widetilde{\mathbf{U}}_1^{(k)}, \cdots, \widetilde{\mathbf{U}}_k^{(k)}\right)$, belong to the domain of the operator $\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]$.

(b) If $\mathbf{F}^{(k)} := \mathbf{F}_r^{(k)} + \sum_{j\geq 1}\left(\widetilde{\mathbf{F}}_{r+j}^{(k)} + \overline{\mathbf{F}}_{r+j}^{(k)}\right)$ then it is verified:

$$\overline{\mathbf{F}}_{r+j}^{(k)} = \mathbf{0}, j = 0, \cdots, k-1 \quad \text{if, and only if,} \quad \overline{\mathbf{F}}_{r+j}^{(k-1)} = \mathbf{0}, j = 0, \cdots, k-1,$$

and in this case it has $\widetilde{\mathbf{F}}_{r+j}^{(k)} = \widetilde{\mathbf{F}}_{r+j}^{(k-1)}, j = 0, \cdots, k$.

(c) If $\overline{\mathbf{F}}_{r+j}^{(k)} = \mathbf{0}$ $j = 0, \cdots, k-1$ then $\overline{\mathbf{F}}_{r+k}^{(k)} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(0)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]\right)$.

Let us prove this statement by induction. For $k = 1$, we consider the decomposition

$$\overline{\mathbf{F}}_{r+1}^{(0)} = \overline{\mathbf{F}}_{r+1}^{r} + \overline{\mathbf{F}}_{r+1}^{c} \quad \text{with} \quad \begin{cases} \overline{\mathbf{F}}_{r+1}^{r} \in \mathrm{Im}\left(\overline{\mathbf{L}}^{(1)}[\widetilde{\mathbf{F}}_r^{(0)}]\right) \\ \widetilde{\mathbf{F}}_{r+1}^{c} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(1)}[\widetilde{\mathbf{F}}_r^{(0)}]\right) \end{cases}$$

So, there exists $\widetilde{\mathbf{U}}_1^{(1)}$ such that $\overline{\mathbf{F}}_{r+1}^{r} = \overline{\mathbf{L}}^{(1)}[\widetilde{\mathbf{F}}_r^{(0)}]\left(\widetilde{\mathbf{U}}_1^{(1)}\right)$.

Taking $\Phi^{(1)}$ generated by $\widetilde{\mathbf{U}}_1^{(1)}$ (it is verified **(a)** and $\mathbf{F}^{(1)} = \Phi_*^{(1)}\mathbf{F}^{(0)}$. It has that $\widetilde{\mathbf{F}}_{r+1}^{(1)} = \widetilde{\mathbf{F}}_{r+1}^{(0)}$, therefore is obtained **(b)** and **(c)**).

Let us assume that **(a)**, **(b)** and **(c)** are verified for $k-1 < N$ and let us prove these statements for k.

We consider the decomposition:

$$\overline{\mathbf{F}}_{r+k}^{(k-1)} = \overline{\mathbf{F}}_{r+k}^{r} + \overline{\mathbf{F}}_{r+k}^{c} \quad \text{with} \quad \begin{cases} \overline{\mathbf{F}}_{r+k}^{r} \in \mathrm{Im}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]\right) \\ \overline{\mathbf{F}}_{r+k}^{c} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]\right) \end{cases}.$$

So, there is $\left(\widetilde{\mathbf{U}}_1^{(k)}, \cdots, \widetilde{\mathbf{U}}_k^{(k)}\right)$ in the domain of $\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]$ such that

$$\overline{\mathbf{F}}_{r+k}^{r} = \overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]\left(\widetilde{\mathbf{U}}_1^{(k)}, \cdots, \widetilde{\mathbf{U}}_k^{(k)}\right)$$

Considering $\Phi^{(k)}$ generated by $\widetilde{\mathbf{U}}^{(k)} = \widetilde{\mathbf{U}}_1^{(k)} + \cdots + \widetilde{\mathbf{U}}_k^{(k)}$ (therefore **(a)** is verified) and $\mathbf{F}^{(k)} = \Phi_*^{(k)}\mathbf{F}^{(k-1)}$. Let us prove **(b)**.

- If $\overline{\mathbf{F}}_{r+j}^{(k-1)} = \mathbf{0}$ for $j = 0, \cdots, k-1$ then by Lemma 3.15 it has $\overline{\mathbf{F}}_{r+j}^{(k)} = \mathbf{0}$ for $j = 0, \cdots, k-1$ and $\widetilde{\mathbf{F}}_{r+j}^{(k)} = \widetilde{\mathbf{F}}_{r+j}^{(k-1)}$ for $j = 0, \cdots, k$. This is the sufficient condition.

- We prove the necessary condition by induction on j, $0 \le j \le k-1$. For $j=0$ it is trivial. Assuming the condition $j-1$. Let us prove for j. In fact, if we express the vector fields

$$\begin{aligned}
\mathbf{F}^{(k-1)} &= \widetilde{\mathbf{F}}_r^{(k-1)} + \cdots + \widetilde{\mathbf{F}}_{r+j-1}^{(k-1)} + \left(\overline{\mathbf{F}}_{r+j}^{(k-1)} + \widetilde{\mathbf{F}}_{r+j}^{(k-1)}\right) + \left(\overline{\mathbf{F}}_{r+j+1}^{(k-1)} + \widetilde{\mathbf{F}}_{r+j+1}^{(k-1)}\right) + \cdots, \\
\mathbf{F}^{(k)} &= \widetilde{\mathbf{F}}_r^{(k-1)} + \cdots + \widetilde{\mathbf{F}}_{r+j-1}^{(k-1)} + \widetilde{\mathbf{F}}_{r+j}^{(k-1)} + \left(\overline{\mathbf{F}}_{r+j+1}^{(k)} + \widetilde{\mathbf{F}}_{r+j+1}^{(k)}\right) + \cdots.
\end{aligned}$$

We have to prove that $\overline{\mathbf{F}}_{r+j}^{(k-1)} = \mathbf{0}$ and $\widetilde{\mathbf{F}}_{r+j+1}^{(k)} = \widetilde{\mathbf{F}}_{r+j+1}^{(k-1)}$. We know that $\left(\widetilde{\mathbf{U}}_1^{(k)}, \cdots, \widetilde{\mathbf{U}}_j^{(k)}\right) \in \mathrm{Ker}\left(\overline{\mathbf{L}}^{(j)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+j-1}^{(k-1)}]\right)$ and by Lemma 3.15, it has

$$\mathbf{0} = \overline{\mathbf{F}}_{r+j}^{(k)} = \left(\overline{\widetilde{\mathbf{U}}_{**}^{(k)}\mathbf{F}^{(k-1)}}\right)_{r+j} = \overline{\mathbf{F}}_{r+j}^{(k-1)} + \sum_{l=1}^{j}\left[\widetilde{\mathbf{F}}_{r+j-l}^{(k-1)}, \widetilde{\mathbf{U}}_l^{(k)}\right] = \overline{\mathbf{F}}_{r+j}^{(k-1)},$$

hence $\overline{\mathbf{F}}_{r+j}^{(k-1)} = \mathbf{0}$.

In this case, applying Lemma 3.15 again, we obtain

$$\widetilde{\mathbf{F}}_{r+j+1}^{(k)} = \left(\widetilde{\widetilde{\mathbf{U}}_{**}^{(k)}\mathbf{F}^{(k-1)}}\right)_{r+j+1} = \widetilde{\mathbf{F}}_{r+j+1}^{(k-1)}$$

Finally, we prove the property **(c)**. By applying Lemma 3.15, we have

$$\begin{aligned}
\overline{\mathbf{F}}_{r+k}^{(k)} &= \left(\overline{\widetilde{\mathbf{U}}_{**}^{(k)}\mathbf{F}^{(k-1)}}\right)_{r+k} = \overline{\mathbf{F}}_{r+k}^{(k-1)} + \sum_{l=1}^{k}\left[\widetilde{\mathbf{F}}_{r+k-l}^{(k-1}, \widetilde{\mathbf{U}}_l^{(k)}\right] \\
&= \overline{\mathbf{F}}_{r+k}^{r} + \overline{\mathbf{F}}_{r+k}^{c} - \overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]\left(\widetilde{\mathbf{U}}_1^{(k)}, \cdots, \widetilde{\mathbf{U}}_k^{(k)}\right) = \overline{\mathbf{F}}_{r+k}^{c}
\end{aligned}$$

So, $\overline{\mathbf{F}}_{r+k}^{(k)} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(k)}[\widetilde{\mathbf{F}}_r^{(k-1)}, \cdots, \widetilde{\mathbf{F}}_{r+k-1}^{(k-1)}]\right)$, to obtain the result it is enough to apply induction hypothesis and property **(b)**.

To finish the proof it is enough to consider $\widetilde{\mathbf{U}}$ a generator of the change of variables $\Phi^{(N)} \circ \Phi^{(N-1)} \circ \cdots \circ \Phi^{(1)}$. ■

The following result characterizes the $k-$reversibility of a vector field and ensures the existence of a $(k+1)-$reversible normal for this vector field.

Theorem 3.19 *Let $N \in \mathbb{N}$, $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots$ a vector field and $\widetilde{\mathbf{U}}$ the generator defined in Proposition 3.18 (i.e., $\widetilde{\mathbf{U}}_{**}\mathbf{F} = \widetilde{\mathbf{F}}_r + \sum_{j\ge1}\left(\widetilde{\mathbf{F}}_{r+j} + \overline{\mathbf{F}}_{r+j}\right)$, verifying the properties of the Proposition 3.18). Then:*

$\mathbf{F}$ is $k-$reversible, $k \le N$ if, and only if, $\overline{\mathbf{F}}_{r+j} = \mathbf{0}$, $j = 1, \cdots, k$.

Moreover, in this case $\widetilde{\mathbf{U}}_{**}\mathbf{F}$ is a $(k+1)-$reversible normal form of $\mathbf{F}$.

Proof: The sufficient condition is trivial. Let us prove the necessary condition. We assume $\mathbf{F}$ is $k-$reversible and $\overline{\mathbf{F}}_{r+j} \ne \mathbf{0}$ for some $j = 1, \cdots, k$. We define $j_0 :=$

$\min\left\{j = 1, \cdots, k : \overline{\mathbf{F}}_{r+j} \neq \mathbf{0}\right\}$. By hypothesis $\overline{\mathbf{F}}_{r+j_0} \in \mathrm{Cor}\left(\overline{\mathbf{L}}^{(j_0)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+j_0-1}]\right)$ and therefore $\mathbf{U}_{**}\mathbf{F}$ is a j_0-reversible normal form. By using Lemma 3.17 $\overline{\mathbf{F}}_{r+j_0} \in \mathrm{Im}\left(\overline{\mathbf{L}}^{(j_0)}[\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{F}}_{r+j_0-1}]\right)$, then $\overline{\mathbf{F}}_{r+j_0} = \mathbf{0}$, and this is a contradiction. ■

Taking into account the Hilbert Basis Theorem and the above Theorem, we obtain the following result.

Corollary 3.20 *Let $\mathcal{F} = \left\{\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots | \mathbf{F}\text{ polinomial vector field}\right\}$ a family of polynomial vector fields. Then there exists $M \in \mathbb{N}$ such that:*
$\mathbf{F} \in \mathcal{F}$ is reversible if, and only if, $\mathbf{F}$ is $M-$reversible.

Remark 3.21 *The minimum natural number $M \in \mathbb{N}$ which exists by Corollary 3.20 is called reversibility order of $\mathcal{F}$.*

4 Applications

There are inside the theory of planar vector fields three important objects: analytic integrability, the center-focus problem and reversibility. Moreover, they are themselves intrinsically and closely related.

In fact, it is well known that the vector field $(-y, x)^T + \cdots$ has center at the origin if and only if it is reversible, (Poincaré theorem)(see [18]). In this sense, the reversibility of some families of vector fields are known. For instance, the reversibility of $(-y, x)^T + (P_m(x, y), Q_m(x, y))^T$ for $m = 2$ and $m = 3$, is known (see Bautin [7], Sibirskii [19], Zoladek [23], [24], and references therein). However, this doesn't happen for the monodromic nilpotent case. In this case due to Berthier and Moussu theorem (see [9]), the vector field $(-y, x^{2n+1})^T + \cdots$ has a center at the origin if and only if it is orbitally reversible. Therefore the monodromic and reversible planar systems of type nilpotent are centers. However, the reverse claim is not true. We discuss some counterexamples in this section.

In the first part of this section we discuss the reversibility of the following nilpotent systems:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \sigma x^{4q+1} \end{pmatrix} + \begin{pmatrix} a_1xy + a_2x^{2q+2} \\ b_1y^2 + b_2x^{2q+1}y + b_3x^{4q+2} \end{pmatrix}, \tag{4.8}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \sigma x^{4q-1} \end{pmatrix} + \begin{pmatrix} a_1xy + a_2x^{2q+1} \\ b_1y^2 + b_2x^{2q}y + b_3x^{4q} \end{pmatrix}, \tag{4.9}$$

with $\sigma = \pm 1$, $q \in \mathbb{N}$.

The analytic integrability and the center problem for these systems in the case $\sigma = -1$ (monodromic case) has been studied in [11] and [14], respectively.

In the second part of this section we study the reversibility of a family of systems with null linear part:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + \begin{pmatrix} a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 \\ b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 \end{pmatrix}. \tag{4.10}$$

Remark 4.22 *In Algaba et al [3], a recursive procedure to compute quasi-homogeneous normal forms under equivalence, that uses the Lie triangle, is presented. The computation of the $N-$reversible normal form can be accomplished by adapting this procedure to the case of reversible transformations, by using Proposition 3.18*

We have the following result.

Theorem 4.23 *The system (4.8) is reversible if and only if one of the following conditions is satisfied:*

a) $b_1 = -a_1$, $a_2 = 0$, $b_3 = -\sigma a_1$.

b) $a_1 = b_1 = b_3 = 0$, *(reversible to the change* $x \to -x$, $t \to -t$*).*

c) $a_2 = b_2 = 0$, *(reversible to the change* $y \to -y$, $t \to -t$*).*

d) $a_1 + 2b_1 = 0$, $b_2 = -2(q+1)a_2$.

Proof: Let $\mathbf{F}$ be the vector field given by (4.8). Observe that $\mathbf{F}$ is a sum of two quasi-homogeneous vector field of type $\mathbf{t} = (1, 2q+1)$ and degree $2q$ and $2q+1$, respectively. The first quasi-homogeneous term, $\widetilde{\mathbf{F}}_{2q} := (y, \sigma x^{4q+1})^T$ is already in a desired simplified form (it is R_x- and R_y- reversible) and these are the unique zero degree involutions which carry the first term of the vector field to R_x-reversible.

We start seeking the R_x-reversibility. To obtain reversible conditions we take the generator $\widetilde{\mathbf{U}} = \sum_{j\geq 1} \widetilde{\mathbf{U}}_j$, $\widetilde{\mathbf{U}}_j \in \mathcal{R}_j^{\mathbf{t}}$, where: $\widetilde{\mathbf{U}}_1 = (\alpha_1 x^2, \alpha_2 xy)^T$, $\widetilde{\mathbf{U}}_2 = (0, \alpha_3 x^{2q+3})^T$, $\widetilde{\mathbf{U}}_3 = (\alpha_4 x^4, \alpha_5 x^3 y)^T$, $\widetilde{\mathbf{U}}_4 = (\chi_{\{q=1\}}\alpha_6 x^2 y, \alpha_7 x^{2q+5} + \chi_{\{q=1\}}\alpha_8 xy^2)^T$ and $\chi_{\{q=n\}}$ is 1 when $q = n$ and 0 otherwise. Choosing adequately α_i, $i = 1, \cdots, 8$ as specified in the Proposition 3.18, we obtain:

$$\begin{aligned}
\widetilde{\mathbf{U}}_{**}\mathbf{F} \;&=\; \mathbf{F}^{(1)}(\mathbf{x}) = \widetilde{\mathbf{F}}_{2q} + \left\{ \widetilde{\mathbf{F}}_{2q+1} + \lambda^{(1)} \begin{pmatrix} 0 \\ x^{4q+2} \end{pmatrix} \right\} + \left\{ \widetilde{\mathbf{F}}_{2q+2} + \frac{1}{2}\lambda^{(2)} \begin{pmatrix} 0 \\ x^{2q+2}y \end{pmatrix} \right\} \\
&\quad + \left\{ \widetilde{\mathbf{F}}_{2q+3} + \frac{1}{24(2q+1)}\lambda^{(3)} \begin{pmatrix} 0 \\ x^{4q+4} \end{pmatrix} \right\} + \left\{ \widetilde{\mathbf{F}}_{2q+4} - \frac{1}{12}\lambda^{(4)} \begin{pmatrix} 0 \\ x^{2q+4}y \end{pmatrix} \right\} \\
&\quad + \left\{ \widetilde{\mathbf{F}}_{2q+5} + \frac{1}{480}\lambda^{(5)} \begin{pmatrix} 0 \\ x^{4q+6} \end{pmatrix} \right\} + \cdots .
\end{aligned}$$

So, by applying Theorem 3.19, $\mathbf{F}$ is $5-$reversible if, and only if, $\lambda^{(j)} = 0$ for $1 \leq j \leq 5$.

The equations $\lambda_1^{(1)} = \lambda_1^{(2)} = 0$, are equivalent to:

$$b_3 = -\tfrac{\sigma}{2}\left[(4q+1)a_1 + (4q-1)b_1\right], \tag{4.11}$$

$$(2q+1)(b_1+a_1)b_2 + 2\left[(2q^2+3q-1)b_1 + q(2q+3)a_1\right]a_2 = 0, \tag{4.12}$$

We have the following possibilities:

(1) $a_2 = b_1 + a_1 = 0$. In this case (4.8) by means of the coordinates change $u = x$, $v = y + a_1 xy$ the system is expressed by:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \sigma u^{4q+1} \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 u^{2q+1} v \end{pmatrix} + \begin{pmatrix} 0 \\ -\sigma a_1^2 u^{4q+3} \end{pmatrix}, \sigma = \pm 1, q \in \mathbb{N},$$

that is R_u-reversible. This is the situation described in **(a)**.

(2) $a_2 = b_2 = 0$. In this case the system is R_y-reversible. This is the situation described in **(c)**.

(3) $a_1 = b_1 = 0$, $a_2 \neq 0$. In this case the system is R_x-reversible. This is the situation described in **(b)**.

(4) $a_2(a_1+b_1) \neq 0$. Assuming the condition (4.11) and $b_2 = -\frac{2\left[(2q^2+3q-1)b_1+q(2q+3)a_1\right]a_2}{(2q+1)(b_1+a_1)}$, i.e., the condition (4.12), we get

$$\begin{aligned}\lambda^{(3)} &= -(2q+1)\sigma(b_1+a_1)\left[2(4q+1)(2q+1)(4q+3)a_1^2\right.\\ &\left.+(128q^3+96q^2-4q-9)b_1a_1+(64q^3-8q+3)b_1^2\right]\\ &+24q\left[(2q+3)(4q+1)a_1+(8q^2+14q+1)b_1\right]a_2^2,\end{aligned}$$

- If $(2q+3)(4q+1)a_1 + (8q^2+14q+1)b_1 = 0$ then

$$\lambda^{(3)} = \frac{8(4q+5)(40q^3+74q^2+31q+3)(2q+1)\sigma}{(4q+1)^2(2q+3)^3}b_1^3 \neq 0$$

So, $\mathbf{F}$ is non-reversible.

- If $(2q+3)(4q+1)a_1 + (8q^2+14q+1)b_1 \neq 0$, $\lambda^{(3)} = 0$ is equivalent:

$$a_2^2 = \frac{(2q+1)\sigma(b_1+a_1)\left[2(4q+1)(2q+1)(4q+3)a_1^2+(128q^3+96q^2-4q-9)b_1a_1+(64q^3-8q+3)b_1^2\right]}{24q[(2q+3)(4q+1)a_1+(8q^2+14q+1)b_1]} \quad (4.13)$$

Assuming (4.11), (4.12) and (4.13), one obtains

$$\lambda^{(4)} = a_2(2b_1+a_1)\overbrace{\left[(4q^2+10q+5)b_1^2+2q(2q-1)a_1^2+(8q^2+8q-5)a_1b_1\right]}^{E}.$$

As the discriminant of E is $-16q^2-40q+25<0$, for $q\in\mathbb{N}$, $\lambda^{(4)}=0$ only if $2b_1+a_1=0$. In this case the condition (4.13) is $a_2^2 = \frac{b_1^2\sigma(16q^2+28q+9)}{24q}$, which is not possible for $\sigma=-1$.

If $\sigma = 1$, $a_1+2b_1 = 0$ y $a_2^2 = \frac{b_1^2(16q^2+28q+9)}{24q}$, one obtains

$$\lambda^{(5)} = -\frac{b_1^5(4q+7)(256q^5-128q^4-256q^3+428q^2-105q-135)}{3q} \neq 0.$$

So, $\mathbf{F}$ is non-reversible.

Now we deal with the R_y-reversibility. By means of $x\leftrightarrow y$, $y\leftrightarrow x$, the system (4.8) takes the expression:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\sigma y^{4q+1}\\ x\end{pmatrix} + \begin{pmatrix}b_1x^2+b_2xy^{2q+1}+b_3y^{4q+2}\\ a_1xy+a_2y^{2q+2}\end{pmatrix}, \sigma=\pm1. \quad (4.14)$$

Let $\mathbf{F}$ be the vector field given by (4.14). Observe that $\mathbf{F}$ is a sum of two quasi-homogeneous vector field of type $\mathbf{t}=(2q+1,1)$ and degree $2q$ and $2q+1$, respectively. The first quasi-homogeneous term, $\widetilde{\mathbf{F}}_{2q} := (\sigma y^{4q+1}, x)^T$ is already in a desired simplified form (it is R_x-reversible). Choosing adequately $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2$, as specified in the Proposition 3.18, we obtain:

$$\begin{aligned}\widetilde{\mathbf{U}}_{**}\mathbf{F}(\mathbf{x}) &= \begin{pmatrix} \sigma y^{4q+1} \\ x \end{pmatrix} + \left\{ \widetilde{\mathbf{F}}_{2q+1} + \lambda^{(1)} \begin{pmatrix} xy^{2q+1} \\ 0 \end{pmatrix} \right\} \\ &+ \left\{ \widetilde{\mathbf{F}}_{2q+2} - \lambda^{(2)} \begin{pmatrix} xy^{2q+2} \\ 0 \end{pmatrix} \right\} + \cdots .\end{aligned}$$

So, by applying Theorem 3.19, $\mathbf{F}$ is $2-$reversible if, and only if, $\lambda^{(1)} = \lambda^{(2)} = 0$, or equivalently

$$b_2 = -2(q+1)a_2, \tag{4.15}$$
$$a_2(2b_1 + a_1) = 0. \tag{4.16}$$

Assuming (4.15), from (4.16) we derive the following possibilities:

(1) $a_2 = 0$. In this case (4.8) is R_y-reversible and so it is reversible. This situation is described in **(c)**.

(2) $a_1 = -2b_1$. In this case the change of variables $u = x - 2b_1xy + a_2y^{2q+2}$, $v = y$ transforms the systems (4.8) in

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{b_1u^2 - \sigma v^{4q+1} + [4\sigma b_1 - b_3]v^{4q+2} + 2\left[2b_3b_1 - 2\sigma b_1^2 - (q+1)a_2^2\right]v^{4q+3} + \left[3b_1a_2^2 - 4b_3b_1^2 + 4qb_1a_2^2\right]v^{4q+4}}{2b_1v - 1} \\ u \end{pmatrix},$$

$\sigma = \pm 1, q \in \mathbb{N}$, that is R_u-reversible. This situation is described in **(d)**.

This completes the proof. ∎

Remark 4.24 *Notice that the reversibility order of the families (4.8) and (4.14) are* 5 *and* 2*, respectively.*

Theorem 4.25 *The system (4.9) is reversible if and only if one of the following conditions is satisfied:*

a) $a_1 = b_1 = 0$*,* $b_2 = -(2q+1)a_2$*.*

b) $a_2 = b_2 = 0$*, (reversible to the change* $y \to -y$*,* $t \to -t$*).*

Proof: Let $\mathbf{F}$ be the vector field given by (4.9). $\mathbf{F}$ is a sum of two quasi-homogeneous vector field of type $\mathbf{t} = (1, 2q)$ and degree $2q-1$ and $2q$, respectively. The first quasi-homogeneous term, $\widetilde{\mathbf{F}}_{2q-1} := (y, \sigma x^{4q-1})^T$ is already in a desired simplified form (it is R_x- and R_y- reversible) and these are the unique zero degree involutions which carry the first term of the vector field to R_x-reversible.

We start seeking the R_x−reversibility. Choosing adequately $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2 + \widetilde{\mathbf{U}}_3 + \widetilde{\mathbf{U}}_4 + \widetilde{\mathbf{U}}_5$, as specified in the Proposition 3.18, we obtain:

$$\begin{aligned}
\widetilde{\mathbf{U}}_{**}\mathbf{F} &= \begin{pmatrix} y \\ \sigma x^{4q-1} \end{pmatrix} + \left\{ \widetilde{\mathbf{F}}_{2q} + \lambda_1^{(1)} \begin{pmatrix} 0 \\ x^{4q} \end{pmatrix} + \lambda_2^{(1)} \begin{pmatrix} 0 \\ x^{2q}y \end{pmatrix} \right\} + \widetilde{\mathbf{F}}_{2q+1} \\
&+ \left\{ \widetilde{\mathbf{F}}_{2q+2} - \tfrac{1}{2(2q+3)} \lambda_1^{(3)} \begin{pmatrix} x^{2q+3} \\ 0 \end{pmatrix} - \tfrac{1}{24} \lambda_2^{(3)} \begin{pmatrix} 0 \\ x^{4q+2} \end{pmatrix} \right\} + \widetilde{\mathbf{F}}_{2q+3} \\
&+ \left\{ \widetilde{\mathbf{F}}_{2q+4} - \tfrac{1}{1440(2q+3)} \lambda_1^{(5)} \begin{pmatrix} 0 \\ x^{4q+4} \end{pmatrix} - \tfrac{1}{24(2q+3)} \lambda_2^{(5)} \begin{pmatrix} 0 \\ x^{2q+4}y \end{pmatrix} \right\} + \cdots .
\end{aligned}$$

So, by Theorem 3.19, $\mathbf{F}$ is reversible provided that $\lambda_i^{(j)} = 0$ for $1 \leq j \leq 5$, $i = 1, 2$.

The relations $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$ are equivalent to:

$$b_3 = -\tfrac{\sigma}{2} \left[(4q-3)b_1 + (4q-1)a_1 \right], \qquad b_2 = -(2q+1)a_2. \tag{4.17}$$

Assuming (4.17) one obtains

$$\begin{aligned}
\lambda_1^{(3)} &= a_2(2b_1 + a_1) \left[(2q+1)b_1 + (2q-1)a_1 \right], \\
\lambda_2^{(3)} &= -12 \left[(8q^2 + 2q + 1)b_1 + (2q+1)(4q-1)a_1 \right] a_2^2 + \sigma(b_1 + a_1) \times \\
& \quad \left[4q(16q^2-1)a_1^2 + (64q^3 - 96q^2 + 40q - 1)b_1^2 + (128q^3 - 96q^2 - 4q + 1)a_1 b_1 \right].
\end{aligned}$$

From these equations we get the following possibilities:

(1) $a_2 = 0$. In this case the system (4.9) is R_y−reversible. This situation is described in **(b)**.

(2) $a_1 = b_1 = 0$. In this case the system (4.9) is reversible since the change $u = x$, $v = y + a_2 x^{2q+1}$ carries it to a R_u-reversible form. This situation is described in **(a)**

(3) $2b_1 + a_1 = 0$, $a_1 b_1 a_2 \neq 0$, in this case $\lambda_1^{(3)} = 0$ and

$$\lambda_2^{(3)} = -b_1(4q+3)(-b_1^2\sigma(16q^2+12q-1) + 12(2q-1)a_2^2)$$

If $\sigma = -1$, the equation $\lambda_2^{(3)} = 0$ has no real roots. When $\sigma = 1$, the equation $\lambda_2^{(3)} = 0$ it is equivalent to

$$a_2^2 = \tfrac{b_1^2(16q^2+12q-1)}{12(2q-1)}$$

in this case,

$$\lambda_1^{(5)} = -\tfrac{b_1^5(4q+5)(512q^5 - 1536q^4 + 1280q^3 + 600q^2 - 1162q + 81)}{36(2q-1)} \neq 0,$$

so, $\mathbf{F}$ is non-reversible.

(4) $(2q+1)b_1 + (2q-1)a_1 = 0$, $a_1 b_1 a_2 \neq 0$. in this case $\lambda_1^{(3)} = 0$ and

$$\lambda_2^{(3)} = \tfrac{8a_1 q(a_1^2\sigma(200q^2-94q+9)-24q(2q+1)^2 a_2^2)}{(1+2q)^3}$$

If $\sigma = -1$, the equation $\lambda_2^{(3)} = 0$ has no real roots. When $\sigma = 1$, the equation $\lambda_2^{(3)} = 0$ it is equivalent to

$$a_2^2 = \tfrac{a_1^2(200q^2-94q+9)}{24q(2q+1)^2}$$

in this case,

$$\lambda_2^{(5)} = -\tfrac{8a_1^4 a_2 q(2q-1)(2q-3)(2q+3)}{(2q+1)^3} \neq 0,$$

so, $\mathbf{F}$ is non-reversible.

We now deal with the R_y-reversibility. Applying to (4.9) the change $x \leftrightarrow y$, $y \leftrightarrow x$, one has:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \sigma y^{4q-1} \\ x \end{pmatrix} + \begin{pmatrix} b_1 x^2 + b_2 x y^{2q} + b_3 y^{4q} \\ a_1 x y + a_2 y^{2q+1} \end{pmatrix}, \sigma = \pm 1. \tag{4.18}$$

If $\mathbf{F}$ denotes the vector field given in (4.18), then it is a sum of two quasi-homogeneous vector field of type $\mathbf{t} = (2q, 1)$ and degree $2q-1$ and $2q$ respectively. The first quasi-homogeneous term, $\widetilde{\mathbf{F}}_{2q-1} := (\sigma y^{4q-1}, x)^T$ is already in a desired simplified form (it is R_x-reversible). Choosing adequately $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2$, as specified in the Proposition 3.18, we obtain:

$$\begin{aligned} \widetilde{\mathbf{U}}_{**}\mathbf{F} &= \begin{pmatrix} \sigma y^{4q-1} \\ x \end{pmatrix} + \left\{ \widetilde{\mathbf{F}}_{2q} + \lambda^{(1)} \begin{pmatrix} x y^{2q} \\ 0 \end{pmatrix} \right\} \\ &+ \left\{ \widetilde{\mathbf{F}}_{2q+1} - 2\lambda_1^{(2)} \begin{pmatrix} x y^{2q+1} \\ 0 \end{pmatrix} - 2\lambda_2^{(2)} \begin{pmatrix} 0 \\ y^{2q+2} \end{pmatrix} \right\} + \cdots. \end{aligned}$$

So, by Theorem 3.19, $\mathbf{F}$ is reversible provided that $\lambda^{(1)} = \lambda_1^{(2)} = \lambda_2^{(2)} = 0$ or equivalently:

$$\begin{aligned} b_2 + 2(q+1)a_2 &= 0, \\ a_2\left[b_1 - (4q+3)a_1\right] &= 0, \\ a_1 a_2 &= 0. \end{aligned}$$

From these equations we get the following possibilities:

(1) $a_2 = 0$. In this case the system (4.9) is R_y-reversible. This situation is described in **(b)**.

(2) $a_1 = b_1 = 0$. In this case the system (4.9) is reversible since the change $u = x + a_2 y^{2q+1}$, $v = y$ carries it to a R_u-reversible form. This situation is described in **(a)**

$\blacksquare$

Remark 4.26 *Notice that the families (4.9) and (4.18) have reversibility order equal to* 5 *and* 2*, respectively.*

Remark 4.27 *It is known that the nilpotent centers are orbitally reversible (see [9]), but can not be reversible. We give some examples of nilpotent centers which are non-reversible.*

The results **(a)**, **(b)**, **(c)**, **(d)**, *of the Theorem 4.23, correspond to the families* **(ii)**, **(vi)**, **(i)** *and* **(v)**, *of Proposition 3.4 of [14], respectively. Thus the cases* **(iii)**, **(iv)**, **(vii)**, **(viii)**, **(ix)**, **(x)**, **(xi)** *and* **(xii)** *are non-reversible centers.*

In a similar way the cases **(b)** *and* **(a)**, *of Theorem 4.25, correspond to families* **(i)** *and a sub-family in* **(ii)**, *of Theorem 3.3 of [14], respectively. So the remaining cases are non-reversible centers.*

Next we study the reversibility of a system with null linear part.

Theorem 4.28 *The system (4.10) is reversible if and only if one of the following conditions is satisfied:*

a) $b_0 + a_3 = b_1 + a_2 = b_2 + a_1 = b_3 + a_0 = 0.$

b) $9b_3 - 9a_0 + 2a_1 = b_2 + a_1 = 3a_2 + 9a_0 - 2a_1 = b_1 + 3a_0 = 3a_3 - 6a_0 + 5a_1 = 9b_0 - 18a_0 - 11a_1 = 0$

Proof: Observe that the first quasi-homogeneous term is reversible respect to $y = x$. To transform the first quasi-homogeneous term in one R_x-reversible, we apply the change of variables: $u = \frac{\gamma}{2}(x+y)$, $v = \frac{\gamma}{2}(-x+y)$, $t = \gamma\tau$. The transformed system is:

$$\left(\begin{array}{c} u' \\ v' \end{array}\right) = \left(\begin{array}{c} u^2+v^2 \\ -2uv \end{array}\right) + \left(\begin{array}{c} A_0u^3 + A_1u^2v + A_2uv^2 + A_3v^3 \\ B_0u^3 + B_1u^2v + B_2uv^2 + B_3v^3 \end{array}\right), \quad (4.19)$$

where $' = \frac{d}{d\tau}$ and $A_0 = \frac{a_0+a_1+a_2+a_3+b_0+b_1+b_2+b_3}{2\gamma}$, $A_1 = -\frac{3a_0+a_1-a_2-3a_3+3b_0+b_1-b_2-3b_3}{2\gamma}$, $A_2 = \frac{3a_0-a_1-a_2+3a_3+3b_0-b_1-b_2+3b_3}{2\gamma}$, $A_3 = \frac{-a_0+a_1-a_2+a_3-b_0+b_1-b_2+b_3}{2\gamma}$, $B_0 = -\frac{a_0+a_1+a_2+a_3-b_0-b_1-b_2-b_3}{2\gamma}$, $B_1 = \frac{3a_0+a_1-a_2-3a_3-3b_0-b_1+b_2+3b_3}{2\gamma}$, $B_2 = \frac{-3a_0+a_1+a_2-3a_3+3b_0-b_1-b_2+3b_3}{2\gamma}$, $B_3 = \frac{a_0-a_1+a_2-a_3-b_0+b_1-b_2+b_3}{2\gamma}$.

Let $\mathbf{F}$ be the vector field given by (4.19). Then it is a sum of two quasi-homogeneous vector field of type $\mathbf{t} = (1,1)$ and degree 1 and 2, respectively. The first quasi-homogeneous term, $\widetilde{\mathbf{F}}_1 := (u^2+v^2, -2uv)^T$ is already in a desired simplified form (it is R_x-reversible). In fact, it is a Hamiltonian vector field $\widetilde{\mathbf{F}}_1 = \mathbf{X}_h$, with $h(u,v) = -\frac{1}{3}v^3 - u^2v$ and $\mathbf{X}_h(x,y) := \left(-\frac{\partial h}{\partial v}, \frac{\partial h}{\partial u}\right)^T$.

We can use the parameter γ, to obtain $B_3 = 0$ or $B_3 = 1$. Choosing adequately $\widetilde{\mathbf{U}} = \sum_{i=1}^{7} \widetilde{\mathbf{U}}_i$, as specified in the Proposition 3.18, we obtain:

$$\widetilde{\mathbf{U}}_{**}\mathbf{F} = \left(\begin{array}{c} u^2+v^2 \\ -2uv \end{array}\right) + \left\{\widetilde{\mathbf{F}}_2 + 2\lambda_1^{(1)}\left(\begin{array}{c} u^3 \\ 0 \end{array}\right) + \tfrac{2}{3}\lambda_2^{(1)}\left(\begin{array}{c} 0 \\ v^3 \end{array}\right)\right\}$$

$$+\left\{\widetilde{\mathbf{F}}_3-\tfrac{5}{2}\lambda_1^{(2)}\begin{pmatrix}0\\u^4\end{pmatrix}-\tfrac{5}{2}\lambda_2^{(2)}\begin{pmatrix}0\\u^2v^2\end{pmatrix}\right\}+\left\{\widetilde{\mathbf{F}}_4-\tfrac{1}{3}\lambda_1^{(3)}\begin{pmatrix}0\\u^4v\end{pmatrix}\right\}$$
$$+\left\{\widetilde{\mathbf{F}}_5-\tfrac{1}{3}\lambda_1^{(4)}\begin{pmatrix}0\\u^6\end{pmatrix}-\tfrac{1}{3}\lambda_2^{(4)}\begin{pmatrix}0\\u^4v^2\end{pmatrix}\right\}+\left\{\widetilde{\mathbf{F}}_6+\lambda_1^{(5)}\begin{pmatrix}0\\u^6v\end{pmatrix}\right\}$$
$$+\left\{\widetilde{\mathbf{F}}_7+\lambda_1^{(6)}\begin{pmatrix}0\\u^6v^2\end{pmatrix}\right\}+\left\{\widetilde{\mathbf{F}}_8+\lambda_1^{(7)}\begin{pmatrix}0\\u^8v\end{pmatrix}+\lambda_2^{(7)}\begin{pmatrix}0\\u^6v^3\end{pmatrix}\right\}+\cdots.$$

So, by Theorem 3.19, $\mathbf{F}$ is reversible provided that $\lambda_i^{(j)}=0$ for $1\le j\le 7$, $i=1,2$.

The equations $\lambda_1^{(1)}=\lambda_2^{(1)}=\lambda_1^{(2)}=\lambda_2^{(2)}=0$, are equivalent to:

$$\begin{aligned}A_0&=0,\\B_1&=3B_3+A_2,\\-(4B_0+5A_1-15A_3)B_3-(3B_0+5A_3)A_2&=0, &(4.20)\\(3A_3-4B_2-A_1)B_3-(B_2-A_3)A_2&=0. &(4.21)\end{aligned}$$

From these equations we get the following possibilities:

(1) $B_3=A_2=0$. In this case (4.19) is R_u-reversible. This case is included in **(a)**.

(2) $A_2\neq 0$, $B_3=B_2-A_3=3B_0+5A_3=0$. Assuming these conditions, $\lambda_1^{(3)}=0$ is equivalent to:

$$A_2^2 = \tfrac{5}{3}A_3(13A_3+6A_1). \tag{4.22}$$

Assuming (4.22), one obtains

$$\begin{aligned}\lambda_1^{(4)}&=\tfrac{5}{24}A_3A_2(1980A_3A_1+2917A_3^2-630A_1^2)\\\lambda_2^{(4)}&=-\tfrac{15}{4}A_3A_2(A_1+5A_3)(5A_1+19A_3),\end{aligned}$$

Since $A_2\neq 0$, there exists not solution.

(3) $B_3=1$, $A_2=0$, $B_0=\frac{5}{4}(3A_3-A_1)$, $B_2=\frac{1}{4}(3A_3-A_1)$.

Assuming these conditions, one obtain

$$\lambda_1^{(3)} = \tfrac{45}{4}(A_1-3A_3)(A_1+3A_3).$$

- If $A_1=3A_3$, $\mathbf{F}^{(1)}$ then:

$$\begin{aligned}\lambda_1^{(4)}&=-567A_3\\\lambda_2^{(4)}&=0.\end{aligned}$$

If $A_3\neq 0$, $\mathbf{F}$ is not reversible. Otherwise $\lambda_1^{(5)}=\lambda_1^{(6)}=0$ and

$$\lambda_1^{(7)}=\lambda_2^{(7)}=\tfrac{25664}{25}\neq 0.$$

Therefore (4.19) is not reversible.

- If $A_1 = -3A_3$, then:

$$\begin{aligned}\lambda_1^{(4)} &= -\tfrac{81}{16}A_3(845A_3^2+144)\\ \lambda_2^{(4)} &= \tfrac{9}{4}A_3(105A_3^2-128).\end{aligned}$$

If $A_3 \neq 0$, $\mathbf{F}$ is not reversible. Otherwise $\lambda_1^{(5)} = \lambda_1^{(6)} = 0$ and

$$\lambda_1^{(7)} = 3\lambda_2^{(7)} = \tfrac{25664}{25} \neq 0.$$

Therefore (4.19) is not reversible.

(4) $B_3 = 1$, $A_2 \neq 0$. From the equations (4.20), (4.21), we obtain:

$$\begin{aligned}A_1 &= (A_3-B_2)A_2 - 4B_2 + 3A_3 && (4.23)\\ A_3 &= -\tfrac{3}{10}B_0 + \tfrac{1}{2}B_2 + \tfrac{2}{5}\tfrac{5B_2-B_0}{A_2} && (4.24)\end{aligned}$$

Assuming (4.23), (4.24) one obtains

$$\begin{aligned}\lambda_1^{(3)} &= -\tfrac{3}{4}\tfrac{A_2+3}{A_2}\left[8A_2^2(A_2+6)+15(A_2+4)(A_2-6)B_2^2\right.\\ &\quad\left.-(6A_2^2-116A_2-192)B_2B_0-3(A_2+2)(3A_2+4)B_0^2\right],\\ \lambda_1^{(4)} &= \tfrac{3}{400}\tfrac{1}{A_2^2}\left\{B_2\left[-875(A_2-6)(A_2+3)(A_2-3)(A_2+4)^2B_2^2\right.\right.\\ &\quad+50(A_2+4)(A_2+3)(21A_2^3-143A_2^2-164A_2+1806)B_0B_2\\ &\quad\left.+4200A_2(A_2+4)(A_2+3)(A_2-3)\right]\\ &\quad+B_0\left[3(3A_2+4)(84A_2^4+468A_2^3+1601A_2^2+2852A_2+2604)B_0^2\right.\\ &\quad+(2205A_2^5+9450A_2^4-13780A_2^3-143970A_2^2-384960A_2-342720)B_0B_2\\ &\quad\left.\left.-40A_2(A_2+3)(56A_2^3+119A_2^2-81A_2-252)\right]\right\},\\ \lambda_2^{(4)} &= \tfrac{A_2+3}{4A_2^2}\left[75(A_2+4)(7A_2^2-38A_2-56)B_2^3\right.\\ &\quad+(225A_2^4-375A_2^3-1350A_2^2+18240A_2+37920)B_0B_2^2\\ &\quad+3(3A_2+4)(30A_2^3+19A_2^2-470A_2-1096)B_0^2B_2\\ &\quad+3(3A_2^2+13A_2+26)(3A_2+4)^2B_0^3+8A_2^2(3A_2+4)(13A_2-14)B_0\\ &\quad\left.-40A_2^2(11A_2^2+38A_2+72)B_2\right]\end{aligned}$$

From $\lambda_1^{(3)} = 0$ we get the following possibilities:

- If $A_2 = -3$ one obtains

$$\begin{aligned}\lambda_1^{(4)} &= -\tfrac{525}{16}B_0^2(B_0-3B_2),\\ \lambda_2^{(4)} &= 0.\end{aligned}$$

From these equations we get the following possibilities:

  – If $B_0 = 0$ then (4.19) is:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u^2+v^2 \\ -2uv \end{pmatrix} + \begin{pmatrix} -B_2u^2v - 3uv^2 - \frac{B_2}{6}v^3 \\ B_2uv^2 + v^3 \end{pmatrix}$$

the change of variables, $x_1 = u + \frac{v^2}{-2+B_2 v}$, $x_2 = v$, transforms this system in

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ -2x_1x_2 \end{pmatrix} + \begin{pmatrix} -B_2x_1^2x_2 - \frac{x_2^3(4B_2+30x_2-4B_2^2x_2-12B_2x_2^2+B_2^3x_2^2)}{6(-2+B_2x_2)^2} \\ B_2x_1x_2^2 \end{pmatrix}$$

which is R_{x_1}−reversible. This situation is described in **b)**

- If $B_0 = 3B_2 \neq 0$ then $\lambda_1^{(4)} = \lambda_2^{(4)} = \lambda_1^{(5)} = \lambda_1^{(6)} = 0$ but $\lambda_1^{(7)} \neq 0$. Hence $\mathbf{F}$ is not reversible.

- If $A_2 + 3 \neq 0$, using Gröbner basis for the equations $\lambda_1^{(3)} = \lambda_1^{(4)} = \lambda_2^{(4)} = \lambda_1^{(5)} = 0$, we get $A_2 = 0$ as unique solution, and this is a contradiction.

This completes the proof. ∎

Remark 4.29 *Notice that the family (4.19) has reversibility order equal to* 7.

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