Reversal symmetries for planar vector fields

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Abstract

In this paper, we formulate a comprehensive study of relevant properties of reversible vector fields. As a consequence, we prove that the reversibility of the first non-zero quasi-homogeneous term, respect to some types, of a vector field is a necessary condition for the reversibility of the vector field. We also provide a straightforward characterization of the reversibility for quasi-homogeneous vector fields. Finally, as an application of our previous results, we analyze some special polynomial and nilpotent systems, including examples which are centers and non-reversible.

1 Introduction and setting of the problem

In the last decades there has been a surging interest in the study of systems with time-reversal symmetries. Symmetry properties arise naturally and frequently in dynamical systems. In recent years, a lot of attention has been devoted to understand and use the interplay between dynamics and symmetry properties. Reversible vector fields were first considered by Birkoff, in the beginning of last century, when he was studying the restricted three body problem. Some decades ago, the theory has been formalized by Devaney, [8]. We refer to Lamb and Roberts, [9] for a survey in reversible systems and related topics. Many authors have dedicated to understand the connection between, centers, analytic integrability and reversibility, see for instance (Algaba, Gamero and García [1], Berthier and Moussu [4], Berthier, Cerveau and Lins Neto [3], Chavarriga, Giacomin, Giné and Llibre [7], Strozyna and Zoladek, [13], Zoladek, [15], Teixeira and Yang, [14], and references therein).

In this paper, we are concerned to establish a discussion involving reversible vector fields and quasi-homogeneous normal forms theory.

We deal with two dimensional systems. Let $\mathbf{F} = (X, Y)$ be a (germ of) C^r reversible vector field with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, $r > 1$, $r = \infty$ or $r = \omega$. We know by Montgomery-Bochner Theorem (see [11]) that there exists a coordinate system of class C^r such that the vector field is expressed as $\mathbf{F}(x, y) = (yf(x, y^2), g(x, y^2))$ with f *and* g being C^r-functions. So a system is not reversible provided that it cannot be expressed, up to $C^r - conjugacy$, in the above form. This is, roughly speaking, the route we have chosen to conduct this paper.

We now need to introduce some definitions and terminology.

- An involution is a diffeomorphism $\sigma \in \mathcal{C}^\infty(U_0 \subset \mathbb{R}^n, \mathbb{R}^n)$, such that $\sigma \circ \sigma = Id$, where U_0 is a small neighborhood of $\mathbf{0} \in \mathbb{R}^n$. Denote $\mathrm{Fix}\,(\sigma) = \{\mathbf{x} \in U_0\,|\sigma(\mathbf{x}) = \mathbf{x}\}$ This set is a local sub-manifold of $\mathbb{R}^n$ and we are assuming throughout the paper that $\dim\left(\mathrm{Fix}\,(\sigma)\right) = n-1$.

- We say that $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, or $\mathbf{F}$ is reversible or σ-reversible, if there is an involution σ, $\sigma(\mathbf{0}) = \mathbf{0}$, such that $\sigma_*\mathbf{F} = -\mathbf{F}$.

- We say that $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, or $\mathbf{F}$ is linear reversible if $\mathbf{F}$ is σ-reversible with σ a linear involution.

- We say that $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, or $\mathbf{F}$ is axis-reversible when it is R_{x_i}-reversible for some $i = 1, \cdots, n$, where R_{x_i} is the following involution

$$R_{x_i}(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) = (x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n).$$

A planar vector field $\mathbf{F}$, respect to the type $\mathbf{t} = (t_1, t_2)$, can be expanded as:

$$\mathbf{F} = \mathbf{F}_{r(\mathbf{t})} + \mathbf{F}_{r(\mathbf{t})+1} + \cdots, \quad \mathbf{F}_{r(\mathbf{t})} \not\equiv \mathbf{0}, \quad \text{and} \quad r(\mathbf{t}) \in \mathbb{Z},$$

where $\mathbf{F}_{r(\mathbf{t})}$ is called the first quasi-homogeneous term of $\mathbf{F}$ respect to the type $\mathbf{t}$, and we denote by $\mathbf{F}_k$, $k \in \mathbb{Z}$, a polynomial quasi-homogeneous vector field in the plane of degree k respect to the type $\mathbf{t}$. When the type $\mathbf{t}$ is fixed we use $\mathbf{F} = \mathbf{F}_r + \cdots$.

It is known the role of the first quasi-homogeneous term of a vector field in the following cases: center problem and analytic integrability problem of a vector field. In fact the monodromic character of the first quasi-homogeneous term respect to some type is a sufficient condition for the monodromic character of the original vector field (see [10]). On the other hand, if a vector field is analytically integrable then its first quasi-homogeneous term respect to any type is analytically integrable (see [2]). In this paper we investigate the role of the first quasi-homogeneous term inside the reversible universe.

Summarizing, in what follows we give a rough overall description of the main results of the paper.

- **Reversibility with respect to the first term**. Let $\mathbf{F}$ be reversible and expressed in quasi-homogeneous terms as $\mathbf{F} = \mathbf{F}_r + H.O.T.$ where a suitable type $\mathbf{t}$ is chosen. Then $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is also reversible. (Proposition 3.13).

- **Relation between reversibility and axis-reversibility of the first term**. Let $\mathbf{F}$ be reversible and $\mathbf{t} = (t_1, t_2)$ be a suitable type in the range of values. Then there exists a change of variables, Φ, which depends on t such that $\Phi_*\mathbf{F}_{r(\mathbf{t})}$ is axis-reversible. (Theorem 3.14).

- **Conditions for quasi-homogeneous reversibility**. Some necessary and sufficient conditions for quasi-homogeneous reversibility are exhibited (Propositions 4.19 and 4.21).

The remaining sections are organized as follows. In Section 2 some terminology, basic concepts and auxiliary results are presented. In Section 3, a discussion on Newton Diagram is given as well as some preparatory results on reversibility. In Section 4, results on reversibility of quasi-homogeneous planar vector fields are discussed, and we apply previous results to get some useful information on polynomial models and nilpotent systems.

2 Background and preparatory results

First of all, we establish some terminology and definitions.

Let $\mathcal{P}_k^{\mathbf{t}}$ be the vector space of real quasi-homogeneous polynomial functions of degree $k \in \mathbb{N}$, respect to the type $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$, i.e., $f \in \mathcal{P}_k^{\mathbf{t}}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ if and only if $f(\epsilon^{t_1} x_1, \cdots, \epsilon^{t_n} x_n) = \epsilon^k f(x_1, \cdots, x_n)$ for all $\epsilon, x_1, \cdots, x_n \in \mathbb{R}$ and $\mathcal{Q}_k^{\mathbf{t}}$ be the vector space of the polynomial quasi-homogeneous vector fields of degree $k \in \mathbb{Z}$, respect to type $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$, i.e., $\mathbf{F} = (Q_1, \cdots, Q_n)^T \in \mathcal{Q}_k^{\mathbf{t}}$, $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, if and only if $Q_i \in \mathcal{P}_{k+t_i}^{\mathbf{t}}$, $\forall i = 1, \cdots, n$.

It is obvious the following result:

Lemma 2.1 *Let* $E = \operatorname{diag}(\epsilon^{t_1}, \cdots, \epsilon^{t_n})$. *Then:*

(a) $f \in \mathcal{P}_k^{\mathbf{t}}$ *if and only if* $f(E\mathbf{x}) = \epsilon^k f(\mathbf{x})$, *for all* $x \in \mathbb{R}^n$.

(b) $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$ *if and only if* $\mathbf{F}(E\mathbf{x}) = \epsilon^k E\mathbf{F}(\mathbf{x})$, *for all* $x \in \mathbb{R}^n$.

Lemma 2.2 *Let* $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$, $E = \operatorname{diag}(\epsilon^{t_1}, \cdots, \epsilon^{t_n})$. *Then,* $D\mathbf{F}(E\mathbf{x}) = \epsilon^k ED\mathbf{F}(\mathbf{x})E^{-1}$.

Proof: Let $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$. So, $\mathbf{F}(E\mathbf{x}) = \epsilon^k E\mathbf{F}(\mathbf{x})$. If we derive this expression with respect to $\mathbf{x}$ we get $ED\mathbf{F}(E\mathbf{x}) = \epsilon^k ED\mathbf{F}(\mathbf{x})$. So the proof is achieved. ∎

Lemma 2.3 *Consider the following decompositions:* $\mathbf{F} = \sum_{j=r}^{\infty} \mathbf{F}_j$, $\sigma = \sum_{j=i_0}^{\infty} \sigma_j$; $\mathbf{F}_j, \sigma_j \in \mathcal{Q}_j^{\mathbf{t}}$, *with* $r, i_0 \in \mathbb{Z}$ *y* $j_0 \in \mathbb{N}$. *Then by means of the re-scaling,* $\mathbf{x} = E\mathbf{y}$, $E = \operatorname{diag}(\epsilon^{t_1}, \cdots, \epsilon^{t_n})$, *the equation* $D\sigma(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) = -\mathbf{F}(\sigma(\mathbf{x}))$, *is transformed in:*

$$\epsilon^{i_0} \sum_{i=0}^{\infty} \epsilon^i \sum_{j=0}^{i} D\sigma_{i+i_0}(\mathbf{y}) \cdot \mathbf{F}_{r+j-i}(\mathbf{y}) = -\sum_{j=0}^{\infty} \epsilon^j \mathbf{F}_{r+j} \left(\epsilon^{i_0} \sum_{i=0}^{\infty} \epsilon^i \sigma_{i+i_0}(y) \right) \tag{2.1}$$

Proof: Applying the re-scaling $\mathbf{x} = E\mathbf{y}$ one has:

$$D\sigma(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) \;=\; D\sigma(E\mathbf{y}) \cdot \mathbf{F}(E\mathbf{y}) = \left[\sum_{i=0}^{\infty} D\sigma_{i+i_0}(E\mathbf{y}) \right] \left[\sum_{j=0}^{\infty} \mathbf{F}_{r+j}(E\mathbf{y}) \right]$$

From Lemma 2.1 **b)** and Lemma 2.2 one gets:

$$\begin{aligned} D\sigma(\mathbf{x})\cdot\mathbf{F}(\mathbf{x}) &= E\left[\sum_{i=0}^{\infty}\epsilon^{i+i_0}D\sigma_{i+i_0}(\mathbf{y})\right]E^{-1}E\left[\sum_{j=0}^{\infty}\epsilon^{r+j}\mathbf{F}_{r+j}(\mathbf{y})\right] \\ &= \epsilon^{r+i_0}E\left[\sum_{i=0}^{\infty}\epsilon^{i}D\sigma_{i+i_0}(\mathbf{y})\right]\left[\sum_{j=0}^{\infty}\epsilon^{j}\mathbf{F}_{r+j}(\mathbf{y})\right] \\ &= \epsilon^{r+i_0}E\sum_{i=0}^{\infty}\epsilon^{i}\sum_{j=0}^{i}D\sigma_{i+i_0}(\mathbf{y})\cdot\mathbf{F}_{r+j-i}(\mathbf{y}). \end{aligned}$$

Using again Lemma 2.1 **b)**, one obtains:

$$\begin{aligned} \mathbf{F}(\sigma(\mathbf{x})) &= \mathbf{F}(\sigma(E\mathbf{y})) = \sum_{j=0}^{\infty}\mathbf{F}_{r+j}\left(\sum_{i=0}^{\infty}\sigma_{i+i_0}(E\mathbf{y})\right) = \sum_{j=0}^{\infty}\mathbf{F}_{r+j}\left(E\epsilon^{i_0}\sum_{i=0}^{\infty}\epsilon^{i}\sigma_{i+i_0}(\mathbf{y})\right) \\ &= E\sum_{j=0}^{\infty}\epsilon^{r+j}\mathbf{F}_{r+j}\left(\epsilon^{i_0}\sum_{i=0}^{\infty}\epsilon^{i}\sigma_{i+i_0}(\mathbf{y})\right) = E\epsilon^{r}\sum_{j=0}^{\infty}\epsilon^{j}\mathbf{F}_{r+j}\left(\epsilon^{i_0}\sum_{i=0}^{\infty}\epsilon^{i}\sigma_{i+i_0}(\mathbf{y})\right). \end{aligned}$$

After a simple simplification on the above equality we complete the proof. ∎

Lemma 2.4 *Let* $\mathbf{F} = \sum_{j=r}^{\infty}\mathbf{F}_j$*,* σ*-reversible,* $\sigma = \sum_{j=i_0}^{\infty}\sigma_j$*, where* $\mathbf{F}_j, \sigma_j \in \mathcal{Q}_j^{\mathbf{t}}$*,* $r, i_0 \in \mathbb{Z}$*. Then* $i_0 = 0$*,* σ_0 *is an involution and* $\mathbf{F}_r$ *is* σ_0*-reversible provided* $i_0 \geq 0$*.*

Proof: Taking into account that $\sigma\circ\sigma = Id$ and Lemma 2.1 **b)**, we may write $E\mathbf{y} = \mathbf{x}$, as:

$$\begin{aligned} E\mathbf{y} &= \mathbf{x} = \sigma\left(\sigma(\mathbf{x})\right) = \sigma\left(\sigma(E\mathbf{y})\right) = \sum_{j=i_0}^{\infty}\sigma_j\left(\sum_{i=i_0}^{\infty}\sigma_i\left(E\mathbf{y}\right)\right) \\ &= \sum_{j=i_0}^{\infty}\sigma_j\left(E\sum_{i=i_0}^{\infty}\epsilon^{i}\sigma_i\left(\mathbf{y}\right)\right) = E\epsilon^{i_0}\sum_{j=0}^{\infty}\epsilon^{j+i_0}\sigma_{j+i_0}\left(\epsilon^{i_0}\sum_{i=0}^{\infty}\epsilon^{i+i_0}\sigma_{i+i_0}\left(\mathbf{y}\right)\right). \end{aligned}$$

Thus, if we assume $i_0 \geq 0$ then $i_0 = 0$ and a straightforward computation allows us to obtain $\sigma_0\left(\sigma_0(\mathbf{y})\right) = \mathbf{y}$.

On the other hand, $D\sigma(\mathbf{x})\cdot\mathbf{F}(\mathbf{x}) = -\mathbf{F}(\sigma(\mathbf{x}))$. We apply again the re-scaling $\mathbf{x} = E\mathbf{y}$ to both hands of the above equality and immediately we get $D\sigma_0\cdot\mathbf{F}_r(\mathbf{y}) = -\mathbf{F}_r\left(\sigma_0(\mathbf{y})\right)$. ∎

Lemma 2.5 *Let* σ *be an involution and* $\mathbf{y} = \Psi(\mathbf{x})$ *be a coordinates change, both defined around the origin. If* $\mathbf{F}$ *is* $\sigma-$*reversible then* $\Psi_*^{-1}\mathbf{F}$ *is* $\Psi\circ\sigma\circ\Psi^{-1}-$*reversible.*

Proof: It is obvious that $\Psi\circ\sigma\circ\Psi^{-1}$ is an involution. If $\mathbf{F}$ is $\sigma-$reversible then $D\sigma(\mathbf{x})\mathbf{F}(\mathbf{x}) = -\mathbf{F}(\sigma(\mathbf{x}))$. First of all we show that

$D\hat{\sigma}(\mathbf{y})\mathbf{G}(\mathbf{y}) = -\mathbf{G}(\hat{\sigma}(\mathbf{y}))$, where $\hat{\sigma} = \Psi \circ \sigma \circ \Psi^{-1}$, $\mathbf{y} = \Psi(\mathbf{x})$, $\mathbf{G}(\mathbf{y}) = \Psi_*^{-1}\mathbf{F}(\mathbf{y})$. That is, $\mathbf{G}(\mathbf{y}) = [D\Psi(\Psi^{-1}(\mathbf{y}))]\mathbf{F}(\Psi^{-1}(\mathbf{y}))$. Hence

$$
\begin{aligned}
D\hat{\sigma}(\mathbf{y})\mathbf{G}(\mathbf{y}) &= D(\Psi \circ \sigma \circ \Psi^{-1})(\mathbf{y}) D\Psi(\Psi^{-1}(\mathbf{y}))\mathbf{F}(\Psi^{-1}(\mathbf{y})) \\
&= D\Psi(\sigma \circ \Psi^{-1}(\mathbf{y})) D\sigma(\Psi^{-1}(\mathbf{y})) \overbrace{D\Psi^{-1}(\mathbf{y}) D\Psi(\Psi^{-1}(\mathbf{y}))}^{Id} \mathbf{F}(\mathbf{x}) \\
&= D\Psi(\sigma(\mathbf{x})) D\sigma(\mathbf{x})\mathbf{F}(\mathbf{x}) \quad [\mathbf{F} \text{ is reversible}] \\
&= D\Psi(\sigma(\mathbf{x}))(-\mathbf{F}(\sigma(\mathbf{x}))) = -D\Psi(\sigma \circ \Psi^{-1}(\mathbf{y}))\mathbf{F}(\sigma \circ \Psi^{-1}(\mathbf{y})) \\
&= -D\Psi(\Psi^{-1} \circ (\Psi \circ \sigma \circ \Psi^{-1})(\mathbf{y}))\mathbf{F}(\Psi^{-1} \circ (\Psi \circ \sigma \circ \Psi^{-1})(\mathbf{y})) \\
&= -\mathbf{G}(\Psi \circ \sigma \circ \Psi^{-1}(\mathbf{y})) = -\mathbf{G}(\hat{\sigma}(\mathbf{y})).
\end{aligned}
$$

■

We now use a weak version of Montgomery-Bochner Theorem, (see [11] p. 206).

Theorem 2.6 *If* $\mathbf{F}$ *is* $\sigma-$*reversible, where* $\sigma = \sum_{j=0}^{\infty} \sigma_j$ *and* $\sigma_j \in \mathcal{Q}_j^{\mathbf{t}}$, *then the mapping* $\mathbf{y} = \Psi(\mathbf{x})$, *where*

$$\Psi = \tfrac{1}{2}(Id + \sigma_0 \circ \sigma),$$

satisfies: $\Psi = \sum_{j=0}^{\infty} \Psi_j$, $\Psi_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\Psi_0 = Id$. *Moreover* $\Psi_*^{-1}\mathbf{F}$ *is* σ_0-*reversible.*

Proof: From $\sigma = \sum_{j=0}^{\infty} \sigma_j$ and Lemma 2.1 **b)**, one obtains $\Psi = \sum_{j=0}^{\infty} \Psi_j$ with $\Psi_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\Psi(\mathbf{0}) = \mathbf{0}$ and $\Psi_0 = \frac{1}{2}(Id + \sigma_0 \circ \sigma_0)$. From Lemma 2.4, σ_0 is an involution and $\Psi_0 = \frac{1}{2}(Id + Id) = Id$.

Observe that

$$\Psi \circ \sigma(\mathbf{x}) = \tfrac{1}{2}(\sigma(\mathbf{x}) + \sigma_0(\mathbf{x})) = \tfrac{1}{2}(\sigma_0(\mathbf{x}) + \sigma(\mathbf{x})) = \sigma_0(\Psi(\mathbf{x})) = (\sigma_0 \circ \Psi)(\mathbf{x}).$$

So $\sigma_0 = \Psi \circ \sigma \circ \Psi^{-1}$, and applying Lemma 2.5 one has $\Psi^{-1}\mathbf{F}$ is σ_0-reversible. That is $(\sigma_0)_* (\Psi_*^{-1}\mathbf{F}) = -(\Psi_*^{-1}\mathbf{F})$. This completes the desired proof. ■

Theorem 2.7 *Let* $\mathbf{t} \in \mathbb{N}^n$ *a fixed type. If* $\mathbf{F}$ *is* σ_0-*reversible,* $\sigma_0 \in \mathcal{Q}_0^{\mathbf{t}}$ *and* $\dim(\mathrm{Fix}(\sigma_0)) = k < n$, *then there exists* $\Psi_0 \in \mathcal{Q}_0^{\mathbf{t}}$, $\det(D\Psi_0(\mathbf{0})) \neq 0$, $\mathrm{diag}(D\Psi_0(\mathbf{0})) = \mathrm{diag}(Id)$, *such that* $(\Psi_0)_*\mathbf{F}$ *is* L_k-*reversible, where*

$$L_k(x_1, \cdots, x_n) = ((-1)^{i_1}x_1, \cdots, (-1)^{i_n}x_n),$$

$i_j \in \{0, 1\}$, $j = 1, \cdots, n$ *and* $\sum_{j=1}^{n} i_j = n - k$.

Proof: Consider the decomposition $\sigma_0 = \sum_{i=1}^{s} \sigma_0^{(j_i)}$, $s \geq 0$, where $j_1 < j_2 < \cdots < j_s$ and $\sigma_0^{(j_i)} \in \mathcal{Q}_{j_i}^{(1,\cdots,1)}$. Since the degree of a homogeneous vector field is non-negative and by Lemma 2.4 one derives that $j_1 = 0$ and $\sigma_0^{(0)}$ is an involution.

Let $\hat{\Psi} := \frac{1}{2}(Id + \sigma_0^{(0)} \circ \sigma_0) \in \mathcal{Q}_0^{\mathbf{t}}$, $\hat{\Psi} = \sum_{i=1}^{s} \hat{\Psi}^{(j_i)}$ with $\hat{\Psi}^{(j_i)} \in \mathcal{Q}_{j_i}^{(1,\cdots,1)}$. As $\mathbf{F}$ is σ_0-reversible, applying Theorem 2.6 for the type homogeneous, one has $j_1 = 0$ and $\hat{\Psi}^{(0)} = Id$. Moreover $\hat{\Psi}_*^{-1}\mathbf{F}$ is $\sigma_0^{(0)}$-reversible.

As $\sigma_0^{(0)}$ is homogeneous of $0-$degree one has $\sigma_0^{(0)}(\mathbf{x}) = D\sigma_0^{(0)}(\mathbf{0})\mathbf{x}$. Assume that each one of l coordinates of type $\mathbf{t}$ takes value equal to 1, $0 \leq l \leq n$, and in the case when $l = n$ one has the homogeneous type. For simplicity, assume that $\mathbf{t} = (\overbrace{1, \cdots, 1}^{l}, t_{l+1}, \cdots, t_n)$. So

$$D\sigma_0^{(0)}(\mathbf{0}) = \left(\begin{array}{c|ccc} A & & & \\ \hline & a_1 & & \\ & & \ddots & \\ & & & a_{n-l} \end{array}\right).$$

Since $D\sigma_0^{(0)}(\mathbf{0})\mathbf{x}$ is an involution one has $\left(D\sigma_0^{(0)}(\mathbf{0})\right)^2 = Id$ and we derive that the spectrum of $D\sigma_0^{(0)}(\mathbf{0})$ is $\{\pm 1\}$, A is a square matrix $l \times l$, $A^2 = Id$ and $a_i = \pm 1$, $i = 1, \cdots, n-l$.

On the other hand, if $\dim(\mathrm{Fix}(\sigma_0)) = k$, then $\dim(\mathrm{Fix}(D\sigma_0(\mathbf{0}))) = k$. Hence $\dim\left(\mathrm{Fix}(D\sigma_0^{(0)}(\mathbf{0}))\right) = k$ and so $D\sigma_0^{(0)}(\mathbf{0})$ has 1 as an eigenvalue with algebraic and geometric multiplicity equal to k. Hence $D\sigma_0^{(0)}(\mathbf{0})$ is diagonalizable with eigenvalues 1 and -1 where k and $n-k$ are the algebraic multiplicities of 1 and -1 respectively.

Consider now the matrix P given by: $\mathrm{diag}\left((-1)^{i_1}, \cdots, (-1)^{i_n}\right) = P \cdot D\sigma_0^{(0)}(\mathbf{0}) \cdot P^{-1}$, con $i_1, \cdots, i_n \in \{0, 1\}$, $i_1 + \cdots + i_n = n - k$.

Thus, $L_k(\mathbf{x}) = P \cdot D\sigma_0^{(0)}(\mathbf{0}) \cdot P^{-1}$, where

$$P = \left(\begin{array}{c|ccc} \hat{P} & & & \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right),$$

and $P\mathbf{x} \in \mathcal{Q}_0^{\mathbf{t}} \cap \mathcal{Q}_0^{(1,\cdots,1)}$.

From Lemma 2.5,we get that $(\Psi_0)_*\mathbf{F}$ is L_k-reversible with $\Psi_0 = P \circ \hat{\Psi}^{-1} \in \mathcal{Q}_0^{\mathbf{t}}$. Moreover

$$\det(\Psi_0(\mathbf{0})) \;=\; \det(P)\det\left(\hat{\Psi}(\mathbf{0})\right) = \det(P)\det\left(\hat{\Psi}^{(0)}(\mathbf{0})\right) = \det(P) \neq 0,$$

As $\mathrm{diag}(P) = \left(\mathrm{diag}(\hat{P}), 1, \cdots, 1\right)$ y $\Psi^{(0)} = Id$, we select an appropriate basis of eigenvectors to get $\mathrm{diag}(D\Psi_0(\mathbf{0})) = \mathrm{diag}(Id)$. ■

3 Reversibility of planar vector fields

The first objective in this section is to associate to a given σ-reversible planar vector field $\mathbf{F}$ a type $\mathbf{t}$ such that $\mathbf{F}_{r(\mathbf{t})}$ is σ_0-reversible, where σ_0 is the first quasi-homogeneous term of σ respect to the type $\mathbf{t}$. In this approach Newton Diagram plays an important role. We now need to introduce some definitions and terminology related to the Newton Diagram. (For more details, see Bruno [6], Broer et. al. [5]).

- Write $\mathbf{F} = (P, Q)$ where $P(x, y) = \sum a_{ij} x^i y^{j-1}$, $Q(x, y) = \sum b_{ij} x^{i-1} y^j$. The *support* of $\mathbf{F}$, $\mathrm{supp}(\mathbf{F})$, is the set of all $(i, j)^T$ such that $(a_{ij}, b_{ij})^T \neq (0, 0)$. Such points $(i, j)^T$ in $\mathrm{supp}(\mathbf{F})$ are called *support points*.

- Consider
$$\Upsilon = \bigcup_{(i,j)^T \in \mathrm{supp}(\mathbf{F})} \left((i, j)^T + \mathbb{R}^2_+\right),$$
where $\mathbb{R}^2_+$ is the positive quadrant. The convex envelope of Υ is called the *Newton polygon* of $\mathbf{F}$ and its boundary, $\partial\Upsilon$, consists of two non-bounded rays and a polygonal line $\Gamma_{\mathbf{F}}$. (that can be reduced to a unique point). This polygonal $\Gamma_{\mathbf{F}}$ is called the *Newton diagram* of $\mathbf{F}$. The segments of $\Gamma_{\mathbf{F}}$ are called *edges*. Those points joining the edges of $\Gamma_{\mathbf{F}}$ together with end points are the *vertices* of $\Gamma_{\mathbf{F}}$.

- If $\partial\Upsilon$ contains an unbounded ray that is different from any coordinate axis, then we say that it is an *unbounded edge* of the Newton polygon.

- Let e be an edge of $\Gamma_{\mathbf{F}}$ and $m \in \mathbb{Q}$ be its slope. If $\frac{1}{m} = \frac{t_2}{t_1}$ with irreducible numbers $t_1, t_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $\mathbf{t} = (t_1, t_2)$ and $\mathcal{E}(e) := \frac{t_2}{t_1}$ are the *type* and the *exponent* associated to e, respectively. The exponent of an unbounded horizontal (resp. vertical) edge is defined as ∞ (resp. 0) and its type as $(0, 1)$ (resp. $(1, 0)$).

As an illustration, Figure 1 represents the Newton diagram of the following system:

$$\begin{aligned} \dot{x} &= -xy^2 + 3x^3 + y^4 + 5xy^3 + 2x^3y^2 - x^5, \\ \dot{y} &= 2y^3 + 2x^2y - 7y^4 + x^2y^3 + 4x^4y + 3x^7 \end{aligned} \tag{3.2}$$

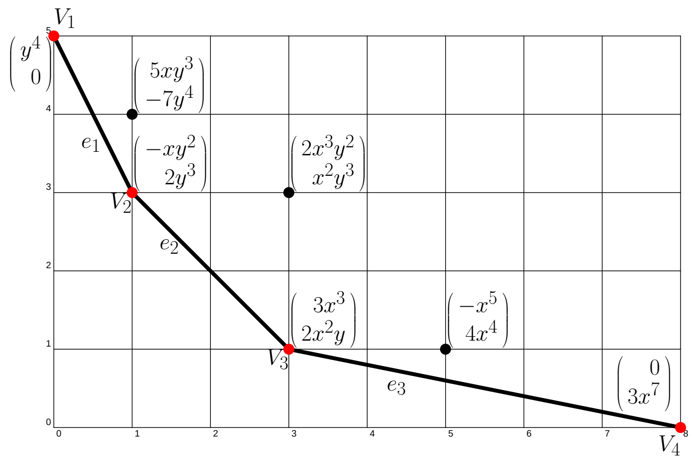


Figure 1: Newton diagram of (3.2)

Observe that this diagram contains four vertices V_1, V_2, V_3, V_4 and three compact edges e_1, e_2, e_3. So $\mathcal{E}(e_1) = 1/2$, $\mathcal{E}(e_2) = 1$ and $\mathcal{E}(e_3) = 5$.

Remark 3.8 *Let $e_1, \cdots, e_n$, be the edges of the diagram $\Gamma_{\mathbf{F}}$. The symbol $i < j$ means that e_i is located at the left side of e_j. In this case $\mathcal{E}(e_i) < \mathcal{E}(e_j)$.*

Lemma 3.9 *If σ is an involution, then the point $(1,1)$ belongs to an edge of the Newton diagram of σ.*

Proof: Observe that $D\sigma(\mathbf{0})^2 = Id$, since σ is an involution satisfying $\sigma(\mathbf{0}) = \mathbf{0}$. So we must distinguish the following two cases:

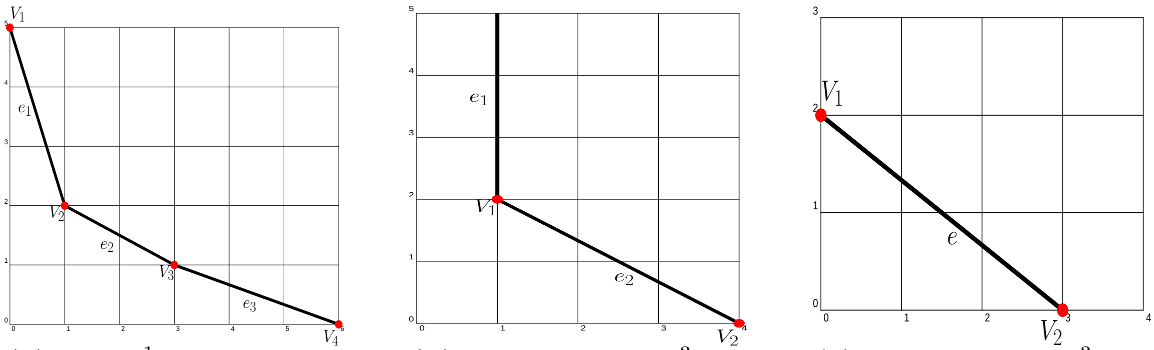
$$\begin{aligned} \sigma(x,y) &= \operatorname{diag}(sx, -sy) + \cdots, \quad \text{with } s = \pm 1, \quad \text{or} \\ \sigma(x,y) &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \cdots, \quad \text{with } a^2 + bc = 1, bc \neq 0. \end{aligned}$$

Concerning the first case, the point $(1,1)$ always is a support point of the Newton diagram of σ. For the second case, we argue as follows. As $bc \neq 0$ the points $(0,2)$ and $(2,0)$ are support points of the diagram of σ and so $(1,1)$ belong to a unique compact edge of such diagram. ∎

Definition 3.10 *Let $\mathbf{F}$ be a vector field. We define the following real numbers associate to $\mathbf{F}$:*

$$\alpha_{\mathbf{F}} := \begin{cases} 1 \text{ if } \{\mathcal{E}(e) : \mathcal{E}(e) \leq 1, e \text{ is an edge of } \Gamma_{\mathbf{F}}\} = \emptyset, \\ \max\{\mathcal{E}(e) : \mathcal{E}(e) \leq 1, e \text{ is an edge of } \Gamma_{\mathbf{F}}\} \text{ otherwise} \end{cases}$$

$$\beta_{\mathbf{F}} := \begin{cases} 1 \text{ if } \{\mathcal{E}(e) : \mathcal{E}(e) \geq 1, e \text{ is an edge of } \Gamma_{\mathbf{F}}\} = \emptyset, \\ \min\{\mathcal{E}(e) : \mathcal{E}(e) \geq 1, e \text{ is an edge of } \Gamma_{\mathbf{F}}\} \text{ otherwise} \end{cases}$$



(a) $\alpha_{\mathbf{F}} = \frac{1}{3}$ y $\beta_{\mathbf{F}} = 2$ **(b)** $\alpha_{\mathbf{F}} = 0$ y $\beta_{\mathbf{F}} = \frac{3}{2}$ **(c)** $\alpha_{\mathbf{F}} = 1$ y $\beta_{\mathbf{F}} = \frac{3}{2}$.

Figure 2: Examples of Newton diagram

Lemma 3.11 *Let σ be an involution. Then for any type $\mathbf{t} = (t_1, t_2)$ with $\alpha_\sigma \leq \frac{t_2}{t_1} \leq \beta_\sigma$, the degree of the first quasi-homogeneous term of σ is zero.*

Proof: From Lemma 3.9 we know that the point $(1,1)$ is in the Newton diagram of σ.

There are two possibilities, $\alpha_\sigma < \beta_\sigma$ or $\alpha_\sigma = \beta_\sigma = 1$.

If $\alpha_\sigma = \beta_\sigma = 1$, then $\mathbf{t} = (1,1)$ and the Newton diagram of σ possesses a unique compact edge of exponent equal to 1 that contains the point $(1,1)$. So the degree of the first quasi-homogeneous term (in this case a homogeneous term) of σ is zero.

If $\alpha_\sigma < \beta_\sigma$, then the Newton diagram of σ possesses two edges, it isn't necessarily compact. If one selects $\mathbf{t} = (t_1, t_2)$ with $\alpha_\sigma \leq \frac{t_2}{t_1} \leq \beta_\sigma$, the point $(1,1)$ is also a support point of the first quasi-homogeneous term of σ. In this way the degree of the first quasi-homogeneous term of σ is zero. ∎

Lemma 3.12 *Let* $\mathbf{F} = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}$, *$\sigma-$reversible and* $\mathbf{F}_{r+j} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$ *(under a type* $\mathbf{t}$*).*

a) *Assume that* $\alpha_\sigma = \beta_\sigma = 1$ *and* $\mathbf{t} = (1,1)$. *Then the Newton diagram of* $\mathbf{F}_r$ *is not reduced to a single support point.*

b) *Assume that* $0 < \alpha_\sigma < \beta_\sigma$ *and* $\mathbf{t} = (t_1, t_2)$ *such that* $\frac{t_2}{t_1} = \alpha_\sigma$. *Then the Newton diagram of* $\mathbf{F}_r$ *is not reduced to a single support point except when the point is on the $y-$axis.*

c) *Assume that* $\alpha_\sigma < \beta_\sigma < +\infty$ *and* $\mathbf{t} = (t_1, t_2)$ *such that* $\frac{t_2}{t_1} = \beta_\sigma$. *Then the Newton diagram of* $\mathbf{F}_r$ *is not reduced to a single support point except when the point is on the $x-$axis.*

Proof: We argue by contradiction. If (m_0, n_0) is the unique support point of $\Gamma_{\mathbf{F}_r}$ with $m_0, n_0 \in \mathbb{N} \cup \{0\}$, $m_0 + n_0 > 0$, then $\mathbf{F}_r = (a_0 \chi_{\{n_0>0\}} x^{m_0} y^{n_0-1}, b_0 \chi_{\{m_0>0\}} x^{m_0-1} y^{n_0})^T$, with $a_0^2 + b_0^2 \neq 0$. Following Lemma 3.9, the point $(1,1)$ always is in a edge of the diagram of σ. There are two possibilities, the point $(1,1)$ is a vertex or not.

If $(1,1)$ is not a vertex then Γ_σ possesses a unique edge e. The exponent of e is 1 and its support points are $(0,2)$ y $(2,0)$. Hence $\alpha_\sigma = \beta_\sigma = 1$. This is exactly the situation given in **a)**. Choosing the type $\mathbf{t} = (1,1)$ associated to the unique compact edge of Γ_σ. So it follows from Lemma 3.11 that the degree of the first homogeneous term of σ is zero. From Lemma 2.4 we conclude that σ_0 is an involution and $\mathbf{F}_r$ is σ_0-reversible. Thus $\sigma_0(x,y) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, with $a^2 + bc = 1$, $bc \neq 0$.

Recall that $D\sigma_0 \cdot \mathbf{F}_r = -\mathbf{F}_r \circ \sigma_0$ and:

$$\begin{aligned} D\sigma_0 \cdot \mathbf{F}_r(x,y) &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} a_0 \chi_{\{n_0>0\}} x^{m_0} y^{n_0-1} \\ b_0 \chi_{\{m_0>0\}} x^{m_0-1} y^{n_0} \end{pmatrix} \\ &= \begin{pmatrix} a a_0 \chi_{\{n_0>0\}} x^{m_0} y^{n_0-1} + b b_0 \chi_{\{m_0>0\}} x^{m_0-1} y^{n_0} \\ c a_0 \chi_{\{n_0>0\}} x^{m_0} y^{n_0-1} - a b_0 \chi_{\{m_0>0\}} x^{m_0-1} y^{n_0} \end{pmatrix}, \\ \mathbf{F}_r \circ \sigma_0(x,y) &= \begin{pmatrix} a_0 \chi_{\{n_0>0\}} (ax+by)^{m_0} (cx-ay)^{n_0-1} \\ b_0 \chi_{\{m_0>0\}} (ax+by)^{m_0-1} (cx-ay)^{n_0} \end{pmatrix}. \end{aligned}$$

Now it is easy to detect that the above equality never happens by virtue the relation $bc \neq 0$, and so we get a contradiction.

On the other hand, if the point $(1,1)$ is a vertex then $\alpha_\sigma < \beta_\sigma$. We now show the claim in **c)**. The proof of **b)** is similar.

As $\beta_\sigma < +\infty$, the inferior edge of Γ_σ is compact. This edge is formed by two vertices: $(1,1)$ and $(m_\sigma+1,0)$ with $m_\sigma \in \mathbb{N}$.

Choosing $\mathbf{t} = (t_1,t_2)$ such that $\frac{t_2}{t_1} = \beta_\sigma$, by Lemma 3.11 the degree of the first homogeneous term of σ is zero and by Lemma 2.4 we conclude that σ_0 is an involution. Moreover $\mathbf{F}_r$ is σ_0-reversible and $\sigma_0(x,y) = (-sx, sy + Ax^{m_\sigma})$ with $A \neq 0$ y $s = \pm 1$. We still recall that m_σ is odd provided that $s = 1$.

Again using the equality $D\sigma_0 \cdot \mathbf{F}_r(x,y) = -\mathbf{F}_r\left(\sigma_0(x,y)\right)$. We have:

$$\begin{aligned}
D\sigma_0 \cdot \mathbf{F}_r(x,y) &= \begin{pmatrix} -s & 0 \\ m_\sigma A x^{m_\sigma - 1} & s \end{pmatrix} \begin{pmatrix} a_0 \chi_{\{n_0>0\}} x^{m_0} y^{n_0-1} \\ b_0 \chi_{\{m_0>0\}} x^{m_0-1} y^{n_0} \end{pmatrix} \\
&= \begin{pmatrix} -s a_0 \chi_{\{n_0>0\}} x^{m_0} y^{n_0-1} \\ s b_0 \chi_{\{m_0>0\}} x^{m_0-1} y^{n_0} + m_\sigma A a_0 \chi_{\{n_0>0\}} x^{m_\sigma+m_0-1} y^{n_0} \end{pmatrix}, \\
-\mathbf{F}_r\left(\sigma_0(x,y)\right) &= -\mathbf{F}_r\left(-sx, sy + Ax^{n_0}\right) = -\begin{pmatrix} a_0 \chi_{\{n_0>0\}} (-sx)^{m_0} (sy + Ax^{m_\sigma})^{n_0-1} \\ b_0 \chi_{\{m_0>0\}} (-sx)^{m_0-1} (sy + Ax^{m_\sigma})^{n_0} \end{pmatrix}.
\end{aligned}$$

We can deduce now that the equality does not happen except when $n_0 = 0$ (i.e. when the unique support point is on the $x-$axis) and $(-s)^{m_0-1} = s$, for every $A \neq 0$.
■

Proposition 3.13 *Let* $\mathbf{F} = \sum_{j=r}^{\infty} \mathbf{F}_j$ *be* $\sigma-$*reversible with* $\sigma = \sum_{j=i_0}^{\infty} \sigma_j$*, where* $\mathbf{F}_j, \sigma_j \in \mathcal{Q}_j^{\mathbf{t}}$*. If* $\mathbf{t} = (t_1,t_2)$ *verifies* $\alpha_{\mathbf{F}} \leq \frac{t_2}{t_1} \leq \beta_{\mathbf{F}}$*, then* $i_0 = 0$*,* σ_0 *is an involution and* $\mathbf{F}_r$ *is* σ_0*-reversible.*

Proof: We are going to prove that $\alpha_\sigma \leq \alpha_{\mathbf{F}}$. In fact if $\alpha_{\mathbf{F}} = 1$, the assertion is immediate. Otherwise if $\alpha_{\mathbf{F}} < \alpha_\sigma \leq 1$, selecting the type $\mathbf{t} = (t_1,t_2)$ with $\frac{t_2}{t_1} = \alpha_\sigma$, it is straightforward to deduce that the Newton polygon of $\mathbf{F}_r$ is reduced to a single support point. So, from Lemma 3.12 **b)**, $\Gamma_{\mathbf{F}_r}$ has a point in the $y-axis$ as the unique support point in the Newton polygon of $\mathbf{F}_r$ and such point is a vertex V of $\Gamma_{\mathbf{F}}$.

If V is the unique vertex of the Newton diagram then $\alpha_{\mathbf{F}} = 1$, which is a contradiction. If there are more vertices by Remark 3.8 we would get $\alpha_\sigma \leq \alpha_{\mathbf{F}}$ what is also a contradiction.

In a similar way we derive that $0 < \beta_{\mathbf{F}} < \beta_\sigma$.

We have shown that $\alpha_\sigma \leq \alpha_{\mathbf{F}} \leq 1 \leq \beta_{\mathbf{F}} \leq \beta_\sigma$. So for every $\mathbf{t} = (t_1,t_2)$ with $\alpha_{\mathbf{F}} \leq \frac{t_2}{t_1} \leq \beta_{\mathbf{F}}$ one has $\alpha_\sigma \leq \frac{t_2}{t_1} \leq \beta_\sigma$. From Lemma 3.11 one obtains $i_0 = 0$. Lemma 2.4 allows us to finish the proof ■

Next result allows us to obtain necessary conditions for the reversibility of $\mathbf{F}$.

Theorem 3.14 *Let* $\mathbf{F}$ *be a reversible vector field and* $\mathbf{t} = (t_1,t_2)$ *be a type such that* $\alpha_{\mathbf{F}} \leq \frac{t_2}{t_1} \leq \beta_{\mathbf{F}}$*. Then there exists a transformation* $\mathbf{y} = \Phi_0(\mathbf{x})$ *with* $\Phi_0 \in \mathcal{Q}_0^{\mathbf{t}}$ *such that* $(\Phi_0)_* \mathbf{F}_{r(\mathbf{t})}$ *is axis-reversible.*

Proof: First of all observe that, under our assumptions, Proposition 3.13 implies that the involution associated to $\mathbf{F}$ is $\sigma = \sum_{j\geq 0}\sigma_j$ with $\sigma_j \in \mathcal{Q}_j^{\mathbf{t}}$. Theorem 2.6 ensures that there is $\mathbf{y} = \Psi(\mathbf{x})$ such that $\Psi_*\mathbf{F}$ is σ_0-reversible, and $\sigma_0 \in \mathcal{Q}_0^{\mathbf{t}}$ is an involution of zero degree. From Theorem 2.7 we deduce immediately the existence of a zero degree transformation $\mathbf{y} = \Theta_0(\mathbf{x})$ that sends the previous vector field to a R_x- or a R_y-reversible field. So, $\Phi_*\mathbf{F}$ is axis-reversible, where $\Phi = \Theta_0 \circ \Psi$, is a change of variables and $\Phi = \sum_{j\geq 0}\Phi_j$, $\Phi_j \in \mathcal{Q}_j^{\mathbf{t}}$. From Lemma 2.3 one obtains that $(\Phi_0)_*\mathbf{F}_{r(\mathbf{t})}$ is axis-reversible. ∎

4 Applications

In this section, we deduce conditions for a vector field to be reversible.

4.1 Reversibility of quasi-homogeneous planar vector field

First of all, consider planar systems expressed by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.3}$$

with $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$, for any fixed type $\mathbf{t} = (t_1, t_2)$.

Lemma 4.15 *Given $\mathbf{t} = (t_1, t_2)$ and $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ there are unique polynomial mappings $\mu \in \mathcal{P}_r^{\mathbf{t}}$ and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ satisfying*

$$\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0. \tag{4.4}$$

Moreover

$$\mu = \tfrac{1}{r+|\mathbf{t}|}\mathrm{div}\,(\mathbf{F}_r) \quad \textit{and} \quad h = \tfrac{1}{r+|\mathbf{t}|}\mathbf{D}_0 \wedge \mathbf{F}_r \tag{4.5}$$

Proof:

First of all we prove the uniqueness. If there are $\mu \in \mathcal{P}_r^{\mathbf{t}}$ and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ satisfying (4.4) then from Euler Theorem (applied to quasi-homogeneous systems) we have

$$\begin{aligned}\mathrm{div}\,(\mathbf{F}_r) &= \mathrm{div}\,(\mathbf{X}_h) + \mathrm{div}\,(\mu\mathbf{D}_0) = \tfrac{\partial\mu}{\partial x}t_1x + \tfrac{\partial\mu}{\partial y}t_2y + \mu\mathrm{div}\,(\mathbf{D}_0)\\ &= \nabla\mu\cdot\mathbf{D}_0 + \mu|\mathbf{t}| = (r+|\mathbf{t}|)\mu.\end{aligned}$$

Also

$$\mathbf{D}_0 \wedge \mathbf{F}_r = \mathbf{D}_0 \wedge \mathbf{X}_h\nabla h\cdot\mathbf{D}_0 = (r+|\mathbf{t}|)h$$

Such objects verify (4.5) and hence the uniqueness follows.

Now we show the existence. Let $\mu \in \mathcal{P}_r^{\mathbf{t}}$ and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ satisfying (4.5).

Again Euler Theorem implies that

$$\begin{aligned}-\tfrac{\partial h}{\partial y} + t_1x\mu &= \tfrac{1}{r+|\mathbf{t}|}\left[-t_1x\tfrac{\partial\mathbf{F}_r\mathbf{e}_2}{\partial y} + t_2\mathbf{F}_r\mathbf{e}_1 + t_2y\tfrac{\partial\mathbf{F}_r\mathbf{e}_1}{\partial y} + t_1x\tfrac{\partial\mathbf{F}_r\mathbf{e}_1}{\partial x} + t_1x\tfrac{\partial\mathbf{F}_r\mathbf{e}_2}{\partial y}\right]\\ &= \tfrac{1}{r+|\mathbf{t}|}\left[t_2\mathbf{F}_r\mathbf{e}_1 + \nabla(\mathbf{F}_r\mathbf{e}_1)\cdot\mathbf{D}_0\right]\\ &= \tfrac{1}{r+|\mathbf{t}|}\left[t_2\mathbf{F}_r\mathbf{e}_1 + (r+t_1)\mathbf{F}_r\mathbf{e}_1\right] = \mathbf{F}_r\mathbf{e}_1\end{aligned}$$

The relationship concerning the other component is similarly deduced. This finishes the proof. ∎

Remark 4.16 *Lemma 4.15 allows us to write* $\mathbf{F}_k = [\mathbf{X}_g + \mu \mathbf{D}_0]$ *with* $\mu = \frac{\operatorname{div}(\mathbf{F}_k)}{k+|\mathbf{t}|}$ *and* $g = \frac{\mathbf{D}_0 \wedge \mathbf{F}_k}{k+|\mathbf{t}|}$.

Throughout this section we assume $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$, with $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ and $\mu \in \mathcal{P}^{\mathbf{t}}_r$.

Lemma 4.17 *Let* $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$ *with* $\mu \not\equiv 0$.

a) $\mathbf{F}_r$ *is* R_x*-reversible if and only if* $h(-x, y) = h(x, y)$ *and* $\mu(-x, y) = -\mu(x, y)$.

b) $\mathbf{F}_r$ *is* R_y-*reversible if and only if* $h(x, -y) = h(x, y)$ *and* $\mu(x, -y) = -\mu(x, y)$.

Proof: We are going to prove only **a)**. The other case is similar.

Observe that $\mathbf{F}_r = (P, Q)^T$ is R_x-reversible if and only if $P(-x, y) = P(x, y)$ y $Q(-x, y) = -Q(x, y)$.

This condition is necessary. If $P(-x, y) = P(x, y)$ and $Q(-x, y) = -Q(x, y)$ then:

$$\begin{aligned} h(-x,y) &= \tfrac{1}{r+|\mathbf{t}}\mathbf{D}_0(-x,y) \wedge \mathbf{F}_r(-x,y) = \tfrac{1}{r+|\mathbf{t}|}(-t_1 x Q(-x,y) - t_2 y P(-x,y)) = h(x,y), \\ \mu(-x,y) &= \tfrac{1}{r+|t|}\left(\tfrac{\partial P(-x,y)}{\partial(-x)} + \tfrac{\partial Q(-x,y)}{\partial y}\right) = \tfrac{1}{r+|t|}\left(-\tfrac{\partial P(x,y)}{\partial x} - \tfrac{\partial Q(x,y)}{\partial y}\right) = -\mu(x,y). \end{aligned}$$

This condition is sufficient. If $h(-x, y) = h(x, y)$ and $\mu(-x, y) = -\mu(x, y)$ then:

$$\begin{aligned} P(-x,y) &= -\tfrac{\partial h(-x,y)}{\partial y} + t_1(-x)\mu(-x,y) - \tfrac{\partial h(x,y)}{\partial y} + t_1 x \mu(x,y) = P(x,y), \\ Q(-x,y) &= \tfrac{\partial h(-x,y)}{\partial(-x)} + t_2 y \mu(-x,y) - \tfrac{\partial h(x,y)}{\partial x} - t_2 y \mu(x,y) = -Q(x,y). \end{aligned}$$

∎

Lemma 4.18 *Let* $\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}^{\mathbf{t}}_r$ *be* σ*-reversible, with* $\sigma = \sum_{i=i_0}^{\infty} \sigma_i$, $\sigma_i \in \mathcal{Q}^{\mathbf{t}}_i$. *Then,* $i_0 = 0$, σ_0 *is an involution and* $\mathbf{F}_r$ *is* σ_0*-reversible.*

Proof: From Proposition 3.13 it is enough to prove the inequality $\alpha_{\mathbf{F}_r} \le \frac{t_2}{t_1} \le \beta_{\mathbf{F}_r}$.

If the Newton diagram $\Gamma_{\mathbf{F}_r}$ of $\mathbf{F}_r$ is reduced to a single point support then $\alpha_{\mathbf{F}_r} = 0$, $\beta_{\mathbf{F}_r} = \infty$, and hence $\alpha_{\mathbf{F}_r} < \frac{t_2}{t_1} < \beta_{\mathbf{F}_r}$.

If $\Gamma_{\mathbf{F}_r}$ is not reduced to a single point support then it possesses a unique compact edge e with $\mathcal{E}(e) = \frac{t_2}{t_1}$. Recall that the diagram may also contain some unbounded edge with exponent 0 or ∞. Therefore, it has $\alpha_{\mathbf{F}_r} \le \frac{t_2}{t_1} \le \beta_{\mathbf{F}_r}$. ∎

Proposition 4.19 *Consider* $\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}^{\mathbf{t}}_r$ *reversible. Then there exists a coordinates change* $\Psi_0 \in \mathcal{Q}^{\mathbf{t}}_0$ *such that* $\operatorname{diag}(D(\Psi_0(\mathbf{0})) = \operatorname{diag}(Id)$ *and* $\Psi_{0*}\mathbf{F}_r$ *is axis-reversible.*

Proof: If $\mathbf{F}_r$ is σ-reversible with $\sigma = \sum_{i=i_0}^{\infty} \sigma_i$, $\sigma_i \in \mathcal{Q}_i^{\mathbf{t}}$ y $\dim(\mathrm{Fix}(\sigma)) = 1$, then from Lemma 4.18 one obtains $i_0 = 0$. Moreover σ_0 is a involution and $\mathbf{F}_r$ is σ_0-reversible. From Theorem 2.7, one has that $\mathbf{F}_r$ is either R_x-reversible or R_y-reversible. ■

Remark 4.20 *The change of variables of zero degree is:* $(x,y)^T = \Phi(u,v)$, $\Phi \in \mathcal{Q}_0^{\mathbf{t}}$, $x = u + b\chi_{\{\mathbf{t}=(t_1,1)\}} v^{t_1}$, $y = v + c\chi_{\{\mathbf{t}=(1,t_2)\}} u^{t_2}$. *Moreover it is verified* $\left(\Phi^{-1}\right)_* \mathbf{F}_r = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$, *where*

$$\begin{aligned} \tilde{h}(u,v) &= \frac{1}{1-bc\chi_{\{\mathbf{t}=(1,1)\}}} h\left(u + b\chi_{\{\mathbf{t}=(t_1,1)\}} v^{t_1}, v + c\chi_{\{\mathbf{t}=(1,t_2)\}} u^{t_2}\right), \\ \tilde{\mu}(u,v) &= \mu\left(u + b\chi_{\{\mathbf{t}=(t_1,1)\}} v^{t_1}, v + c\chi_{\{\mathbf{t}=(1,t_2)\}} u^{t_2}\right). \end{aligned}$$

Next proposition states necessary conditions for the axis-reversibility in the homogeneous case.

Proposition 4.21 *Let* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{(1,1)}$, *with* $\mu(x,y) = \sum_{j=0}^{r} D_j x^j y^{r-j}$ *and* $h(x,y) = \sum_{j=0}^{r+2} C_j x^j y^{r+2-j}$. $\mathbf{F}_r$ *is reversible if only if there are* $b, c \in \mathbb{R}$, $bc \neq 1$ *satisfying:*

a) $$\sum_{i=0}^{2k+1} \sum_{m_1=0}^{i} \frac{(2k+1)!(r+1-2k)!i!(r+2-i)!}{(2k+1-m_1)!m_1!(r-2k+1-i+m_1)!(i-m_1)!} C_i b^{i-m_1} c^{2k+1-m_1} = 0, \text{ for } k = 0,1,\cdots,\left\lfloor \tfrac{r+2}{2} \right\rfloor$$

$$\sum_{i=0}^{2k} \sum_{m_1=0}^{i} \frac{(2k)!(r-2k)!i!(r-i)!}{(2k-m_1)!m_1!(r-2k-i+m_1)!(i-m_1)!} D_i b^{i-m_1} c^{2k-m_1} = 0, \text{ for } k = 0,1,\cdots,\left\lfloor \tfrac{r}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ *represents the integer part of* x.

b) *Same equality as in item* **(a)** *occurs by replacing* b *and* c *by* $\frac{1}{c}$ *and* $\frac{1}{b}$, *respectively.*

Proof: From Theorem 2.7, $\mathbf{F}_r$ is linear-reversible if and only if there is a change of variables $(u,v) = \Phi(x,y) = (x+by, cx+y)$, with $1-bc \neq 0$, such that $\Phi_*\mathbf{F}_r$ is axis-reversible.

If $\mathbf{F}_r^* := \Phi_*\mathbf{F}_r = \mathbf{X}_{h^*} + \mu^*\mathbf{D}_0$, then:

$$\begin{aligned} h^*(u,v) &= (1-bc)h(u+bv, cu+v), \\ \mu^*(u,v) &= \mu(u+bv, cu+v). \end{aligned}$$

So:

$$\begin{aligned} \partial_u h^* &= (\partial_x + c\partial_y)h, \\ \partial_v h^* &= (b\partial_x + \partial_y)h. \end{aligned}$$

Observe now that from Lemma 4.17, $\mathbf{F}_r^*$ is R_u-reversible provided that $h^*(u,v)$ is even in u and $\mu^*(u,v)$ is odd in u.

Moreover, $h^*(u,v)$ is even in u if and only if $\partial_u^{2k+1}\partial_v^{r+2-(2k+1)} h^* = 0$, for $k = 0,1,\cdots,\left\lfloor \frac{r+2}{2} \right\rfloor$ or equivalently:

$$(\partial_x + c\partial_y)^{2k+1}(b\partial_x + \partial_y)^{r+2-(2k+1)} h(x,y) = 0, \quad \text{for} \quad k = 0,1,\cdots,\left\lfloor \tfrac{r+2}{2} \right\rfloor.$$

Observe that

$$\begin{aligned}
\partial_u^M \partial_v^N &= (\partial_x + c\partial_y)^M (b\partial_x + \partial_y)^N \\
&= \left(\sum_{m_1+m_2=M} \begin{pmatrix} M \\ m_1 \end{pmatrix} c^{m_2} \partial_x^{m_1} \partial_y^{m_2} \right) \left(\sum_{n_1+n_2=N} \begin{pmatrix} N \\ n_1 \end{pmatrix} b^{n_1} \partial_x^{n_1} \partial_y^{n_2} \right) \\
&= \sum_{\substack{m_1+m_2=M \\ n_1+n_2=N}} \begin{pmatrix} M \\ m_1 \end{pmatrix} \begin{pmatrix} N \\ n_1 \end{pmatrix} b^{n_1} c^{m_2} \partial_x^{m_1+n_1} \partial_y^{m_2+n_2} \\
&= \sum_{m_1=0}^{M} \sum_{n_1=0}^{N} \begin{pmatrix} M \\ m_1 \end{pmatrix} \begin{pmatrix} N \\ n_1 \end{pmatrix} b^{n_1} c^{M-m_1} \partial_x^{m_1+n_1} \partial_y^{M+N-m_1-n_1}.
\end{aligned}$$

If we denote $i = m_1 + n_1$, one obtains:

$$\partial_u^M \partial_v^N = \sum_{i=0}^{M} \sum_{m_1=0}^{i} \begin{pmatrix} M \\ m_1 \end{pmatrix} \begin{pmatrix} N \\ i - m_1 \end{pmatrix} b^{i-m_1} c^{M-m_1} \partial_x^i \partial_y^{M+N-i}.$$

To complete the proof it is enough to observe that

$$\partial_x^M \partial_y^{r+2-M} h(x,y) = M!(r+2-M)! C_M.$$

The second equation is obtained by imposing that μ^* is odd in u. Replacing u by v and studying the reversibility in u, part **(b)** follows similarly. ∎

4.2 Some examples

The main goal of this subsection is to characterize families of reversible quasi-homogeneous vector fields.

Theorem 4.22 *Let* $\dot{\mathbf{x}} = \mathbf{F}_8(\mathbf{x})$ *with*

$$\mathbf{F}_8(\mathbf{x}) = \begin{pmatrix} a_0 y^3 + a_1 x^3 y^2 + a_2 x^6 y + a_3 x^9 \\ b_0 x^2 y^3 + b_1 x^5 y^2 + b_2 x^8 y + b_3 x^{11} \end{pmatrix} \in \mathcal{Q}_8^{(1,3)}.$$

Then $\mathbf{F}_8$ *is reversible if and only if one of the following conditions is satisfied:*

a) $a_0 = a_1 = b_0 = 0$, $2b_1 a_3 - 3a_2 a_3 - a_2 b_2 = 0$,

b) $a_0 \neq 0$ *y* $a_1 + b_0 = 0$, $-54a_0 a_3 - 6a_0 b_2 - 3a_2 b_0 + 9a_2 a_1 - b_1 b_0 + 3b_1 a_1 = 0$, $(b_0 - 3a_1)^3 + 12a_0 \left[6a_0(b_2 - 3a_3) + b_0(b_1 - 3a_2) - 3a_1(b1 - 3a_2)\right] = 0$.

Proof: Observe that $\mathbf{F}_8 = \mathbf{X}_h + \mu \mathbf{D}_0$ with $h(x,y) = C_0 y^4 + C_1 x^3 y^3 + C_2 x^6 y^2 + C_3 x^9 y + C_4 x^{12}$, $\mu(x,y) = D_0 x^2 y^2 + D_1 x^5 y + D_2 x^8$, where $C_0 = -\frac{3}{12} a_0$, $C_1 = \frac{b_0 - 3a_1}{12}$, $C_2 = \frac{b_1 - 3a_2}{12}$, $C_3 = \frac{b_2 - 3a_3}{12}$, $C_4 = \frac{b_3}{12}$, $D_0 = \frac{3a_1 + 3b_0}{12}$, $D_1 = \frac{6a_2 + 2b_1}{12}$, $D_2 = \frac{9a_3 + b_2}{12}$. From Proposition 4.19, we derive that if $\mathbf{F}_8$ is reversible then there exists a coordinates change $(u,v)^T = \Phi(x,y)$, $u = x$, $v = cx^3 + y$, such that $\Phi_*\mathbf{F}_8 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$ is axis-reversible, where:

$$\begin{array}{rcl} \tilde{h}(u,v) & = & C_0v^4 + (4C_0c + C_1)u^3v^3 + (6C_0c^2 + 3C_1c + C_2)u^6v^2 \\ & & +(4C_0c^3 + 3C_1c^2 + 2C_2c + C_3)u^9v + (C_0c^4 + C_1c^3 + C_2c^2 + C_3c + C_4)u^{12}, \\ \tilde{\mu}(u,v) & = & D_0u^2v^2 + (2D_0c + D_1)u^5v + (D_0c^2 + D_1c + D_2)u^8. \end{array}$$

Observe that $\Phi_*\mathbf{F}_8$ is axis-reversible, provided that there is $c \in \mathbb{R}$ such that:

$$\begin{array}{rcl} 4C_0c + C_1 & = & 0,\ (\mathbf{C1}) \\ 4C_0c^3 + 3C_1c^2 + 2C_2c + C_3 & = & 0,\ (\mathbf{C2}) \\ D_0 & = & 0,\ (\mathbf{D1}) \\ D_0c^2 + D_1c + D_2 & = & 0.\ (\mathbf{D2}). \end{array}$$

If $C_0 = 0$, in (**C1**) one gets that $C_1 = 0$. From (**C2**), (**D2**) one obtains that $2C_2D_2 - D_1C_3 = 0$ and **a)** follows.

If $C_0 \neq 0$, from (**C1**) one gets $c = -\frac{C_1}{4C_0}$. Taking into account (**C2**) and (**D2**) case **b)** follows. ∎

Theorem 4.23 *Consider* $\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x})$ *with*

$$\mathbf{F}_1(\mathbf{x}) \quad = \quad \begin{pmatrix} a_0y^2 + a_1xy + a_2x^2 \\ b_0y^2 + b_1xy + b_2x^2 \end{pmatrix} \in \mathcal{Q}_1^{(1,1)}.$$

Then $\mathbf{F}_1$ *is reversible if and only if one of the following conditions is satisfied:*

(a) $D_0 = D_1 = 0$.

(b) $D_1 \neq 0$, $h(-D_0, D_1) \neq 0$, $h(3h(x,y) - xh_x(x,y), -yh_x(x,y))|_{(x,y)=(-D_0,D_1)} = 0$.

(c) $D_1 \neq 0$, $h(-D_0, D_1) = 0$, $h_x(-D_0, D_1) = 0$, $h_{xx}(-D_0, D_1) \neq 0$.

(d) $D_1 = 0$, $D_0 \neq 0$, $b_2 \neq 0$ *y* $h(a_2 - b_1, 3b_2) = 0$.

(e) $D_1 = 0$, $D_0 \neq 0$, $b_0 - a_1 \neq 0$, $b_2 = b_1 - a_2 = 0$.

(f) $D_0^2 + D_1^2 \neq 0$, $h \equiv 0$.

where $D_1 = \frac{2a_2+b_1}{3}$, $D_0 = \frac{a_1+2b_0}{3}$ *y* $h(x,y) = -\frac{1}{3}a_0y^3 + \frac{b_0-a_1}{3}xy^2 + \frac{b_1-a_2}{3}x^2y + \frac{b_2}{3}x^3$

Proof: First of all, observe that $\mathbf{F}_1 = \mathbf{X}_h + \mu\mathbf{D}_0$, with $h(x,y) = C_0y^3 + C_1xy^2 + C_2x^2y + C_3x^3$, $\mu(x,y) = D_0y + D_1x$, where $C_0 = -\frac{1}{3}a_0$, $C_1 = \frac{b_0-a_1}{3}$, $C_2 = \frac{b_1-a_2}{3}$, $C_3 = \frac{b_2}{3}$, $D_0 = \frac{a_1+2b_0}{3}$, $D_1 = \frac{2a_2+b_1}{3}$. From Proposition 4.19, we deduce that there exists $(x,y)^T = \Phi(u,v)$, $\Phi \in \mathcal{Q}_0^{(1,1)}$, $x = u + bv$, $y = cu + v$ such that $\tilde{\mathbf{F}}_1 :=$

$(\Phi^{-1})_* \mathbf{F}_1 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$ is axis-reversible provided that $\mathbf{F}_1$ es reversible. From Remark 4.20 one obtains that $\tilde{\mathbf{F}}_1$ is R_u-reversible, if and only if:

$$\begin{aligned}(3C_0 + 2C_1 b + C_2 b^2)c + C_1 + 2C_2 b + 3C_3 b^2 &= 0,\\ C_0 c^3 + C_1 c^2 + C_2 c + C_3 &= 0,\\ D_0 + D_1 b &= 0,\end{aligned}$$

and $\tilde{\mathbf{F}}_1$ is R_v-reversible, if and only if:

$$\begin{aligned}C_0 + bC_1 + b^2 C_2 + C_3 b^3 &= 0,\\ 3C_0 c^2 + 2C_1 c + C_2 + (C_1 c^2 + 2C_2 c + 3C_3)b &= 0,\\ D_0 c + D_1 &= 0.\end{aligned}$$

- If $D_0 = D_1 = 0$, then:
  - If $C_0 = C_3 = C_2 = 0$, then $\mu \equiv 0$ y h is even with respect to y. So $\mathbf{F}_1$ is R_y-reversible.
  - If $C_0 = C_3 = 0$, $C_2 \neq 0$, $\mu \equiv 0$ and for $c = 0$ the first group of equations is satisfied. So $\mathbf{F}_1$ is reversible.
  - If $C_0^2 + C_3^2 \neq 0$ we may assume, without loss of generality, that $C_0 \neq 0$ and $C_3 = 0$. If $C_2 \neq 0$, for $c = 0$ there exists b such that the first group of equations is satisfied. If $C_2 = C_1 = 0$ then $\mathbf{F}_1$ is R_x-reversible. If $C_2 = 0$ and $C_1 \neq 0$ we can determine b, c which are solutions of the first group of equations. This is the case **(a)**.
- If $D_1 \neq 0$, $x = u - \frac{D_0}{D_1}v$, $y = v$ takes the original system into

  $$\dot{\mathbf{x}} = \mathbf{X}_{\hat{h}} + \hat{\mu}\mathbf{D}_0 := \hat{\mathbf{F}}_1 \tag{4.6}$$

  where $\hat{\mu}(u,v) = D_1 u$ y $\hat{h}(u,v) = h(u - \frac{D_0}{D_1}v, v) = \hat{C}_0 v^3 + \hat{C}_1 u v^2 + \hat{C}_2 u^2 v + \hat{C}_3 u^3$, with $\hat{C}_0 = \hat{h}(0,1) = h(-\frac{D_0}{D_1}, 1) = D_1^{-3} h(-D_0, D_1)$, $\hat{C}_1 = \hat{h}_u(0,1) = h_x(-\frac{D_0}{D_1}, 1) = D_1^{-2} h_x(-D_0, D_1)$, $\hat{C}_2 = \frac{1}{2}\hat{h}_{uu}(0,1) = \frac{1}{2} h_{xx}(-\frac{D_0}{D_1}, 1) = \frac{D_1^{-1}}{2} h_x(-D_0, D_1)$, $\hat{C}_3 = \frac{1}{6}\hat{h}_{uuu}(1,0) = \frac{1}{6} h_{xxx}(1,0) = C_3$.

  From Proposition 4.19 we deduce that $\hat{\mathbf{F}}_1$ is reversible if and only if there exists $c \in \mathbb{R}$ such that:

  $$\begin{aligned}3\hat{C}_0 c + \hat{C}_1 &= 0,\\ \hat{C}_0 c^3 + \hat{C}_1 c^2 + \hat{C}_2 c + \hat{C}_3 &= 0.\end{aligned}$$

  - If $\hat{C}_0 \neq 0$, then $c = -\frac{\hat{C}_1}{3\hat{C}_0}$. Moreover there exists solution of the system if and only if $\hat{h}(1,c) = 0$ (Case **(c)**).
  - If $\hat{C}_0 = 0$, then $\hat{C}_1 = 0$. If $\hat{C}_2 \neq 0$, then $c = -\frac{\hat{C}_3}{\hat{C}_2}$ (Case **(b)**). If $\hat{C}_2 = 0$ then $\hat{C}_3 = 0$ (Case **(f)**).

- If $D_1 = 0$ and $D_0 \neq 0$, then $\tilde{\mathbf{F}}_1$ is axis-reversible provided that it is R_v-reversible. Hence $c = 0$. So the remaining conditions to be satisfied are: $C_2 + 3C_3 b = 0$, $h(b,1) = C_0 + bC_1 + b^2 C_2 + C_3 b^3 = 0$.
  - If $C_3 \neq 0$, then $b = -\frac{C_2}{3C_3}$ y $h(-\frac{C_2}{3C_3}, 1) = h(\frac{a_2 - b_1}{3b_2}, 1) = 0$. This is equivalent to say that: $h(a_2 - b_1, 3b_2) = 0$ (Case **(d)**).
  - If $C_3 = 0$, then $C_2 = 0$ and $C_0 + bC_1 = 0$. The last relation is achieved if: either $C_1 \neq 0$ and the value of b is determined (Case **(e)**) or $C_1 = C_0 = 0$ (Case **(f)**).

■

Theorem 4.24 *Let*

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \textit{with} \;\; \mathbf{F} = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{Q}_8^{(1,3)}, \tag{4.7}$$

where $h(x,y) = -\frac{1}{12}(x^{12} + y^4)$, $\mu(x,y) = \frac{1}{12}(Ax^8 + Bx^5 y - Cx^2 y^2)$. *Then* $\mathbf{F}$ *is reversible if and only if* $A = C = 0$.

Proof: From Proposition 4.19, $\mathbf{F}$ is reversible if and only if $\Phi_*(\mathbf{F})(u,v)$ is axis-reversible, where $\Phi(u,v) = (x, y + cx^3)^T$ and this happens if and only if $\mathbf{F}$ is axis-reversible or equivalently $A = C = 0$. ■

Remark 4.25 *Recalling the results of Medvedeva in [10], we derive that the origin of the system (4.7) is monodromic. Moreover, Algaba, Garcia and Reyes in [2], have established conditions for the existence of centers and for the analytic integrability of quasi-homogeneous systems. Theorem 3.3 in [2] allows us to deduce that the origin of (4.7) is a center if and only if* $A = C$. *Applying now Theorem 3.2 in [2], one obtains that* $\mathbf{F}$ *is analytically integrable if and only if* $B = 0, A = C, C = \frac{2\sqrt{2}(n_2 - n_1)}{n_1 + n_2 + 2}$, *with* $n_1, n_2 \in \mathbb{N}_0$, $n_1 + n_2 > 0$. *These results together with Theorem 4.24 allow to represent in Figure 3, the family of centers (at the origin) of (4.7). It is worth to point out that there are vector fields in this family that are non-reversible.*

Remark 4.26 *It is fairly known (Poincarè result) that all centers expressed by* $(-y,x)^T + \cdots$ *are analytically integrable and reversible. Berthier and Moussu, in [4], have shown that all nilpotent centers (* $(y,0)^T + \cdots$*) are reversible and Moussu ([12]) proved that not all of them are analytically integrable. In our setting we detect centers in this class that are neither reversible nor analytically integrable.*

Theorem 4.27 *Let* $\dot{\mathbf{x}} = \mathbf{F}_{22}(\mathbf{x})$ *where*

$$\mathbf{F}_{22}(\mathbf{x}) = \begin{pmatrix} a_0 y^5 + a_1 x^5 y^2 \\ b_0 x^4 y^3 + b_1 x^9 \end{pmatrix} \in \mathcal{Q}_{22}^{(3,5)}.$$

Then, $\mathbf{F}_{22}$ *is reversible if and only if* $b_0 = a_1 = 0$.

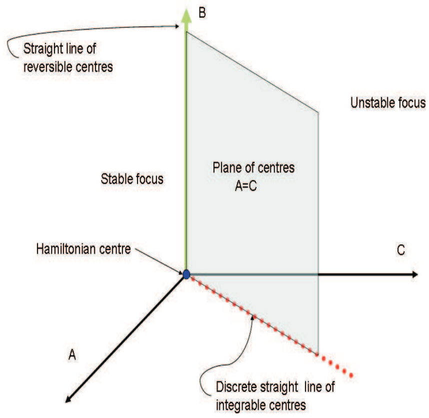


Figure 3: Family of centers of the system (4.7)

Proof: From Proposition 4.19 and Remark 4.20, $\mathbf{F}_{22}$ is reversible if and only if it is axis-reversible or equivalently $b_0 = a_1 = 0$. ■

Remark 4.28 *There are quasi-homogeneous centers that are non-reversible. For example, take the Hamiltonian system* $\dot{\mathbf{x}} = \mathbf{X}_h(\mathbf{x})$ *where* $h \in \mathcal{P}^{\mathbf{t}}_{30}$ *is expressed by:*

$$h(x,y) \quad = \quad (y^3 - ax^5)^2 + x^{10} = y^6 - 2ax^5y^3 + (a^2+1)x^{10}, a \neq 0.$$

So

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \quad = \quad \mathbf{X}_h(\mathbf{x}) = \begin{pmatrix} -6y^5 + 6ax^5y^2 \\ -10ax^4y^3 + 10(a^2+1)x^9 \end{pmatrix},$$

is quasi-homogeneous center with $\mathbf{t} = (3,5)$. *From Theorem 4.27 it cannot be reversible.*

4.3 Reversibility of nilpotent systems

In this subsection we analyze the reversibility of the generic Takens-Bogdanov singularity. Let

$$\begin{array}{rcl} \dot{x} & = & y + x^{n+1}\Psi_1(x) + yf(x,y), \\ \dot{y} & = & x^n y\Phi_1(x) + x^m\Phi_3(x) + y^2 g(x,y), \end{array} \tag{4.8}$$

where $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $f(0,0) = g(0,0) = 0$, $\Psi_1(x) = \sum_{i=0}^{\infty} a_i x^i$, $\Phi_1(x) = \sum_{i=0}^{\infty} b_i x^i$, $\Phi_3(x) = \sum_{i=0}^{\infty} c_i x^i$, with $c_0 \neq 0$ y $a_0^2 + b_0^2 \neq 0$.

Theorem 4.29 *Assume that the system (4.8) is reversible. Then one of the following conditions is satisfied:*

(a) $m < 2n+1$.

(b) *$m \geq 2n+1$ y n is odd.*

(c) *$m \geq 2n+1$, n is even and $(n+1)a_0 + b_0 = 0$.*

Proof: Let $\mathbf{F}$ be the vector field associated to (4.8). We argue by contradiction. Assume for instance that none of the items **(a)**, **(b)** or **(c)** is satisfied. That is, $m \geq 2n+1$, n is even, $(n+1)a_0 + b_0 \neq 0$ and $\mathbf{F}$ is reversible.

If $m \geq 2n+1$ then $\alpha_{\mathbf{F}} = 1$ and $\beta_{\mathbf{F}} = n+1$. Takimg into account that $\mathbf{t} = (1, n+1)$ one has $1 = \alpha_{\mathbf{F}} \leq \frac{t_2}{t_1} = n+1 = \beta_{\mathbf{F}}$. From Corollary 3.14, one deduces that there is $\Phi_0 \in \mathcal{Q}_0^{\mathbf{t}}$ such that $(\Phi_0)_* \mathbf{F}_r$ is axis-reversible.

So $\mathbf{F}_r = \left(y + a_0 x^{n+1}, b_0 x^n y + \chi_{\{m=2n+1\}} c_0 x^{2n+1}\right)^T = \mathbf{X}_h + \mu \mathbf{D}_0$, where

$$\mu \quad = \quad ((n+1)a_0 + b_0)\, x^n.$$

All possible zero degree variable changes are of the form $u = x + \chi_{\{n=0\}} by$, $v = y + cx^{n+1}$. By means of such transformations we cannot get a new system having odd divergence with respect to u or to v. From Lemma 4.17 one arrives to a contradiction since $(n+1)a_0 + b_0 \neq 0$. ■

Now we deal with the vector field $\mathbf{F}$ expressed by the following system:

$$\begin{aligned} \dot{x} &= y^2 + a_0 xy - a_1 x^2 + a_2 x^3 + xy f_1(x) + + x^4 f_2(x) + y^2 f(x,y), \\ \dot{y} &= b_2 x^2 + a_0 y^2 + 2a_1 xy + b_1 x^2 y + 3x^4 + y^2 g_1(x) + x^3 y g_2(x) + y^3 g(x,y), \end{aligned} \tag{4.9}$$

Theorem 4.30 *If the system (4.9) is reversible then one of the following conditions is satisfied:*

(a) *$b_2 \neq 0$, $a_0 = 0$.*

(b) *$b_2 \neq 0$, $a_0 \neq 0$, $a_1^3 + 2a_0 a_1 b_2 + b_2^2 = 0$.*

(c) *$b_2 = 0$, $a_0 = 0$, $a_1 \neq 0$.*

(d) *$b_2 = 0$, $a_1 = 0$, $a_0 \neq 0$, $4a_2 + 3b_1 = 0$.*

(e) *$b_2 = a_1 = a_0 = b_1 = a_2 = 0$.*

Proof: Consider the following steps:

- If $b_2 \neq 0$ then $\alpha_{\mathbf{F}} = \beta_{\mathbf{F}} = 1$. So the reversibility conditions **(a)** and **(b)** coincide with **(d)** and **(e)** of Theorem 4.23.

- When $b_2 = 0$, one has:

  – if $a_1 \neq 0$, then $\alpha_{\mathbf{F}} = 1$, $\beta_{\mathbf{F}} = 3$. So for all $\mathbf{t} = (t_1, t_2)$ with $\frac{t_2}{t_1} < 1$ the system $\mathbf{F}_r = (y^2, 0)^T$ is R_x-reversible. For $\mathbf{t} = (1,1)$, $\mathbf{F}_r$ does not satisfies the cases exhibited in Theorem 4.23, except when $a_0 = 0$. If $\mathbf{t} = (t_1, t_2)$ such that $1 = \alpha_{\mathbf{F}} < \frac{t_2}{t_1} < 3$ then $\mathbf{F}_r = (-a_1 x^2, 2a_1 xy)^T$ is R_x-reversible. If $\mathbf{t} = (1,3)$ then $\mathbf{F}_r = (-a_1 x^2, 2a_1 xy + 3x^4)^T = \mathbf{X}_h + \mu \mathbf{D}_0$ where $\mu = a_1 x$ and $h = x^2(a_1 y + \frac{3}{5} x^3)$. Now, we observe that the system can easily be transformed by a change of variables in a R_x-reversible vector field. This is the case **(c)**.

- If $a_1 = 0$, $a_0 \neq 0$ then $\alpha_{\mathbf{F}} = 1$, $\beta_{\mathbf{F}} = 2$. When for all $\mathbf{t} = (t_1, t_2)$ the relation $\frac{t_2}{t_1} < 1 = \alpha_{\mathbf{F}}$ is satisfied then $\mathbf{F}_r = (y^2, 0)^T$ is R_x-reversible. For $\mathbf{t} = (1,1)$, $\mathbf{F}_r = (y^2 + a_0xy, a_0y^2)^T$ is R_y-reversible. If $\mathbf{t} = (t_1, t_2)$ such that $1 = \alpha_{\mathbf{F}} < \frac{t_2}{t_1} < 2$ then $\mathbf{F}_r = (a_0xy, a_0y^2)^T$ is R_y-reversible. When $\mathbf{t} = (1,2)$ then $\mathbf{F}_r = (a_0xy + a_2x^3, a_0y^2 + b_1x^2y + 3x^4)^T = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mu = \frac{1}{5}(a_0y + (3a_2 + b_1)x^2)$ and $h = \frac{1}{5}x(-a_0y^2 + (b_1 - 2a_1)x^2y + 3x^4)$. By means the mapping $x = u$, $y = v + cu$, the system is transformed in a R_y-reversible vector field, provided that $c = -\frac{3a_2+b_1}{a_0} = \frac{b_1-2a_1}{2a_0}$. Or in another words $4a_2 + 3b_1 = 0$. This is the case **(d)**.
- If $a_1 = a_0 = 0$ then $\alpha_{\mathbf{F}} = 1$ and $\beta_{\mathbf{F}} = \frac{3}{2}$. For $\mathbf{t} = (t_1, t_2)$ such that $\frac{t_2}{t_1} < \frac{3}{2}$, $\mathbf{F}_r = (y^2, 0)^T$ and therefore R_x-reversible, and for $\mathbf{t} = (2,3)$ one has that $\mathbf{F}_r = (y^2 + a_2x^3, b_2x^2y)^T$ that is reversible only when $a_2 = b_1 = 0$. This is the case **(e)**

■

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References

[1] A. Algaba, E. Gamero, C. Garcia, *Integrability problem for a class of planar systems*, Nonlinearity, 22, (2009), pp. 395–420.

[2] A. Algaba, C. Garcia, M. Reyes, *Integrability of two-dimensional quasi-homogeneous polynomial differential systems*, to appear in Rocky Mountain Journal of Mathematics (2009).

[3] M. Berthier, D. Cerveau, A. Lins Neto, *Sur les feuilletages analytiques réels et le problème du centre*, J. Differential Equations, 131,2 (1996), pp. 244–266.

[4] M. Berthier, R. Moussu, *Réversibilité et classification des centres nilpotents*, Ann. Inst. Fourier (Grenoble), Vol. 44, N. 2, (1994), pp. 465–494.

[5] H. W. Broer, F. Dumortier, S. J. van Strien, and F. Takens, *Structures in dynamics. Finite-dimensional deterministic studies*, Studies in Mathematical Physics, 2, North-Holland Publishing Co., Amsterdam, 1991.

[6] A. D. Bruno, *Local Methods in Nonlinear Differential Equations*, Springer Verlag, Berlin, 1989.

[7] J. Chavarriga, H. Giacomin, J. Giné, J. Llibre, *Local analytic integrability for nilpotent centers*, Ergodic Theory and Dynamical Systems, 23, 2, (2003), pp. 417–428.

[8] R.L. Devaney, *Reversibility, homoclinic points, and the Hénon map*, Dynamical systems approaches to nonlinear problems in systems and circuits (Henniker, NH, 1986), 3–14, SIAM, Philadelphia, PA, 1988

[9] J.S.W. Lamb, J.A.G. Roberts, *Time-reversal symmetry in dynamical systems: a survey*, Physica D. Nonlinear Phenomena, 112, 1-2, (1998), pp. 1–39.

[10] N. B. Medvedeva, *A monodromic criterion for a singular point of a vector field on the plane*, St. Peterbourg Math. J., 13 (2002), pp. 253–268.

[11] D. Montgomery and L. Zippin, *Topological transformations groups*, Interscience, New York, 1955.

[12] R. Moussu, *Symetrie et forme normale des centres et foyers degeneres*, Ergodic Theory and Dynamical Systems, (1982), pp. 241–251.

[13] E. Stróżyna, Ewa, H. Żołądek, *The analytic and formal normal form for the nilpotent singularity*, J. Differential Equations, 179, 2, (2002), pp. 479–537.

[14] M.A. Teixeira, J. Yang, *The center-focus problem and reversibility*, J. Differential Equations, 174, (2001), pp. 237–251.

[15] H. Żołądek, *The classification of reversible cubic systems with center*, Topol. Methods Nonlinear Anal., 4, 1, (1994), pp. 79–136.