TEMPERED GENERALIZED FUNCTIONS ALGEBRA, HERMITE EXPANSIONS AND ITÔ FORMULA

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Abstract. The space of tempered distributions $\mathcal{S}'$ can be realized as a sequence spaces by means of the Hermite representation theorems (see [2]). In this work we introduce and study a new tempered generalized functions algebra $\mathcal{H}$, in this algebra the tempered distributions are embedding via its Hermite expansion. We study the Fourier transform, point value of generalized tempered functions and the relation of the product of generalized tempered functions with the Hermite product of tempered distributions (see [6]). Furthermore, we give a generalized Itô formula for elements of $\mathcal{H}$ and finally we show some applications to stochastic analysis.

1. Introduction

The differential algebras of generalized functions of Colombeau type were developed in connection with non linear problems. These algebras are a good frame to solve differential equations with rough initial date or discontinuous coefficients (see [3], [8] and [13]). Recently there are a great interest in develop a stochastic calculus in algebras of generalized functions (see for instance [1], [4], [11], [12] and [14]), in order to solve stochastic differential equations with rough data. A Colombeau algebra $\mathcal{G}$ on an open subset Ω of $\mathbb{R}^m$ is a differential algebra containing $\mathcal{D}'(\Omega)$ as a linear subspace and $\mathcal{C}^\infty(\Omega)$ as a faithful subalgebra. The embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}$ is done via convolution with a mollifier, in the simplified version the embedding depends on the particular mollifier.

The algebra of tempered generalized functions was introduced by J. F. Colombeau in [5] in order to develop a theory of Fourier transform in algebras of generalized functions (see [8] and [7] for applications and references). We observe that in this algebra the most of properties involving Fourier transform and convolution are valid in a weak sense.

In this work we introduced and study a new algebra of tempered generalized functions, this algebra is based in the Fourier-Hermite expansion of tempered distributions. More precisely, the Hermite representation theorem for $\mathcal{S}'$ (see [2], [17], [20], and [21]) which establishes that every $S \in \mathcal{S}'$ can be represented by a Hermite series

$$S = \sum_{n=0}^{\infty} S(h_n) h_n \tag{1}$$

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where $\{h_n\}$ are the Hermite functions and the equality is in the weak sense. The idea is embedded the tempered distributions into the differential subalgebra $\mathcal{H}_{\mathbf{s}'} \subset \mathcal{S}^{\mathbb{N}_0}$ via the sequence of partial sums

$$S_n = \sum_{j=0}^{n} S(h_j)h_j$$

and define the algebra of generalized tempered functions as

$$\mathcal{H} = \mathcal{H}_{\mathbf{s}'}/\mathcal{H}_{\mathbf{s}}$$

where $\mathcal{H}_{\mathbf{s}}$ is a differentiable ideal of $\mathcal{H}_{\mathbf{s}'}$ (see section 3 for precise definitions and details).

The plan of exposition is as follows: Section 2 contains a brief summary without proofs of Hermite functions and the Hermite representation theorems. In section 3, we introduce the Tempered algebra $\mathcal{H}$, this algebra contains to the tempered distributions and extends the product in $\mathcal{S}$. We study its elementary properties and shows that the symmetric product of tempered distributions (see [22] and [6]) is associated with the product in $\mathcal{H}$.

In section 4, we introduce and study the ring of tempered numbers $\mathbf{h}$ and the point value of tempered generalized functions. Section 5, deals with integration, convolutions and Fourier transform of tempered generalized functions. We obtain the Fourier inversion theorem for $\mathcal{H}$, the formula of interchange between the product and the convolution, the rule of integration by parts. The important point to note here is that the identities are in $\mathbf{h}$, this is in a strong sense.

In section 6 it is shown a generalized Itô formula for elements of $\mathcal{H}$. The crucial facts are the existence of point value for generalized tempered distribution, the good definition of Stieljes integral and the Follmer approach to the Itô formula (see [15]). Finally, it should be noted that in [10] and [11] is present a Itô formula for generalized functions, with some mistakes in definitions and proofs how is pointed in [4].

2. N-Representation of tempered distributions.

Let $\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth real valued functions.

For each $m \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, we consider $\| \cdot \|_m$ the norm of $\mathcal{S}$ given by

$$\|\varphi\|_m = \Big(\int_{-\infty}^{\infty} |(N+1)^m \varphi(x)\ |^2\ dx \Big)^{\frac{1}{2}},$$

where $N+1 = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2 + 1)$.

We observe that $\mathcal{S}$ provides with the natural topology given by these norms is a sequentially complete locally convex space and its dual space $\mathcal{S}'$ is the space of tempered distributions. The family of norms $\{\| \cdot \|_m : m \in \mathbb{N}_0\}$ is direct and equivalent to the family of seminorms $\{\| \cdot \|_{\alpha,\beta,\infty} : \alpha, \beta \in \mathbb{N}_0\}$, given by

$$\|\varphi\|_{\alpha,\beta,\infty} = \sup_x |(1+|x|^2)^{\alpha} D^{\beta}\varphi(x)|.$$

We use often the following property of the multiplication on $\mathcal{S}$. For all $m \in \mathbb{N}_0$ there exists $r, s \in \mathbb{N}_0$ and a constant $C_m > 0$ such that

$$\|\varphi\psi\|_m \leq C_m \|\varphi\|_r \|\psi\|_s \qquad (2)$$

for all $\varphi, \psi \in \mathcal{S}$ (see for instance [16] Theorem 2).

The *Hermite polynomials* $H_n(x)$ are defined by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \tag{3}$$

for $n \in \mathbb{N}_0$ or equivalently

$$H_n(x) = 2^{-\frac{n}{2}} \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (\sqrt{2}x)^{n-2k}}{k!(n-2k)!}. \tag{4}$$

The *Hermite functions* $h_n(x)$ are defined by

$$h_n(x) = (\sqrt{2\pi} n!)^{-\frac{1}{2}} e^{-\frac{1}{4}x^2} H_n(x) \tag{5}$$

for $n \in \mathbb{N}_0$. Some properties of the Hermite functions that we will often use follows.

- $h_n \in \mathcal{S}$ for all $n \in \mathbb{N}_0$,
- h_n is an even (odd) function if n is even (odd),
- $\sqrt{n+1} h_{n+1}(x) + 2h_n'(x) = \sqrt{n} h_{n-1}(x)$ for all $n \in \mathbb{N}_0$,
- $\sqrt{n+1} h_{n+1}(x) = x h_n(x) - \sqrt{n} h_{n-1}(x)$ for all $n \in \mathbb{N}_0$,
- $\{h_n : n \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(\mathbb{R})$,
- $(N+1)h_n = (n+1)h_n$ for all $n \in \mathbb{N}_0$.

From the two last properties we have

$$\|\varphi\|_m^2 = \sum_{n=0}^{\infty} (n+1)^{2m} < \varphi, h_n >^2,$$

where $< \varphi, h_n >= \int \varphi(x) h_n(x) dx$ are the Fourier-Hermite coefficients of the expansion of φ.

The Hermite representation theorem for $\mathcal{S}$ ($\mathcal{S}'$) states an topological isomorphism from $\mathcal{S}$ ($\mathcal{S}'$) onto the space of sequences $\mathbf{s}$ ($\mathbf{s}'$).

Let $\mathbf{s}$ be the space of rapidly decreasing sequences

$$\mathbf{s} = \{(a_n) \in \ell^2 : \sum_{n=0}^{\infty} (n+1)^{2m} \mid a_n \mid^2 < \infty, \text{ for all } \ m \in \mathbb{N}_0\}.$$

The space $\mathbf{s}$ is a locally convex space with the sequence of norms

$$\|(a_n)\|_m = (\sum_{n=0}^{\infty} (n+1)^{2m} \mid a_n \mid^2)^{\frac{1}{2}}$$

or with the equivalent sequence of norms

$$\mid (a_n) \mid_{m,\infty} = \sup_n (n+1)^m |a_n|.$$

The topological dual space to $\mathbf{s}$, denoted by $\mathbf{s}'$, is given by

$$\mathbf{s}' = \{(b_n) : \text{for some } \ (C, m) \in \mathbb{R} \times \mathbb{N}_0, \mid b_n \mid \leq C(n+1)^m \text{ for all } n\},$$

and the natural pairing of elements from $\mathbf{s}$ and $\mathbf{s}'$, denoted by $\langle \cdot, \cdot \rangle$, is given by

$$\langle (b_n), (a_n) \rangle = \sum_{n=0}^{\infty} b_n a_n$$

for $(b_n) \in \mathbf{s}'$ and $(a_n) \in \mathbf{s}$.

It is clear that $\mathbf{s}'$ is an algebra with the pointwise operations:

$$\begin{aligned}(b_n) + (b'_n) &= (b_n + b'_n)\\ (b_n)\cdot(b'_n) &= (b_n b'_n),\end{aligned}$$

and $\mathbf{s}$ is an ideal of $\mathbf{s}'$.

The relation between $\mathbf{s}$ ($\mathbf{s}'$) and $\mathcal{S}$ ($\mathcal{S}'$) is induced by the Hermite functions, via Hermite coefficients (evaluation). The following representation theorem is fundamental in our work, for the proof we refer to [17] pp. 143.

Theorem 1 (N-representation theorem for $\mathcal{S}$ and $\mathcal{S}'$). **a**) *Let* $\mathbf{h} : \mathcal{S} \to \mathbf{s}$ *be the application*

$$\mathbf{h}(\varphi) = (<\varphi, h_n>).$$

Then $\mathbf{h}$ *is a topological isomorphism. Moreover,*

$$\|\mathbf{h}(\varphi)\|_m = \|\varphi\|_m$$

for all $\varphi \in \mathcal{S}$.
b) *Let* $\mathbf{H} : \mathcal{S}' \to \mathbf{s}'$ *be the application* $\mathbf{H}(T) = (T(h_n))$. *Then* $\mathbf{H}$ *is a topological isomorphism. Moreover, if* $T \in \mathcal{S}'$ *we have that*

$$T = \sum_{n=0}^{\infty} T(h_n) h_n$$

in the weak sense and for all $\varphi \in \mathcal{S}$,

$$T(\varphi) = \langle \mathbf{H}(T), \mathbf{h}(\varphi)\rangle.$$

We say that the sequences $\mathbf{h}(\varphi)$ and $\mathbf{H}(T)$ are the *Hermite coefficients* of the tempered function φ and the distribution T, respectively.

Now, we show the Hermite coefficients of some tempered distributions.

2.1. The delta distribution. (see [2] pp 191.)

$$\delta(h_n) = h_n(0) = \begin{cases} \frac{(-1)^{\frac{n}{2}}}{\sqrt[4]{2\pi}} \sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}} & \text{for } n \text{ even,}\\ 0 & \text{for } n \text{ odd.}\end{cases} \tag{6}$$

2.2. The constant distribution 1. (see [2] pp 190.)

$$1(h_n) = \int_{-\infty}^{\infty} h_n(x)\, dx = \begin{cases} \sqrt[4]{8\pi}\sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}} & \text{for } n \text{ even,}\\ 0 & \text{for } n \text{ odd.}\end{cases} \tag{7}$$

2.3. The x_+^p distribution. (see [19] pp 162.)

We recall that $< x_+^p, \phi > = \int_0^\infty x^p \phi(x)\, dx$.

$$x_+^p(h_n) = \begin{cases} (\sqrt{2\pi}n!)^{-\frac{1}{2}} 2^p \Gamma(\frac{p+1}{2}) W_n(2p+1) & \text{for } n \text{ even,}\\ (\sqrt{2\pi}n!)^{-\frac{1}{2}} 2^{p+1} \Gamma(\frac{p+2}{2}) W_n(2p+1) & \text{for } n \text{ odd}\end{cases} \tag{8}$$

where $W_n(x)$ are polynomials such that $W_0(x) = W_1(x) = 1$ and

$$W_{n+2}(x) = xW_n(x) + n(n-1)W_{n-2}(x).$$

Note that if $p = 0$, then x_+^p is the Heaviside distribution H.

2.4. The δ' distribution.

$$\delta'(h_n) = -h'_n(0) = \sqrt{n} h_{n-1}(0). \tag{9}$$

3. The Tempered Algebra

In order to introduce the tempered algebra we consider $\mathcal{S}^{\mathbb{N}_0}$ the space of sequences of rapidly decreasing smooth functions. It is clear that $\mathcal{S}^{\mathbb{N}_0}$ has the structure of an associative, commutative differential algebra with the natural operations:

$$\begin{aligned}(f_n) + (g_n) &= (f_n + g_n)\\ a(f_n) &= (af_n)\\ (f_n)\cdot(g_n) &= (f_n g_n)\\ D(f_n) &= (Df_n)\end{aligned}$$

where (f_n) and (g_n) are in $\mathcal{S}$ and $a \in \mathbb{R}$.

Definition 1. Let

$$\mathcal{H}_{\mathbf{s}'} = \{(f_n) \in \mathcal{S}^{\mathbb{N}_0} \ : \ \text{for each } m \in \mathbb{N}_0,\ (\|f_n\|_m) \in \mathbf{s}' \ \} \tag{10}$$

and

$$\mathcal{H}_{\mathbf{s}} = \{(f_n) \in \mathcal{S}^{\mathbb{N}_0} \ : \ \text{for each } m \in \mathbb{N}_0,\ (\|f_n\|_m) \in \mathbf{s} \ \} \tag{11}$$

Lemma 1. *$\mathcal{H}_{\mathbf{s}'}$ is a subalgebra of $\mathcal{S}^{\mathbb{N}_0}$ and $\mathcal{H}_{\mathbf{s}}$ is a differential ideal of $\mathcal{H}_{\mathbf{s}'}$.*

Proof. Let $(f_n), (g_n) \in \mathcal{H}_{\mathbf{s}'}$ and $m \in \mathbb{N}_0$. Applying the inequality (2), there exists $r, s \in \mathbb{N}_0$ and a constant $C_m > 0$ such that

$$\|f_n g_n\|_m \leq C_m \|f_n\|_r \|g_n\|_s.$$

By definition, there exists constants $D, E > 0$ and $p, q \in \mathbb{N}_0$ such that

$$\begin{aligned}\|f_n\|_r &\leq D(n+1)^p\\ \|g_n\|_s &\leq E(n+1)^q.\end{aligned}$$

Combining these inequalities, we obtain

$$\|f_n g_n\|_m \leq C_m DE(n+1)^{p+q}.$$

This proves that $(\|f_n g_n\|_m) \in \mathbf{s}'$, thus $(f_n)\cdot(g_n) \in \mathcal{H}_{\mathbf{s}'}$.
Now, we prove that $\mathcal{H}_{\mathbf{s}}$ is an ideal of $\mathcal{H}_{\mathbf{s}'}$. Let $(f_n) \in \mathcal{H}_{\mathbf{s}'}$, $(g_n) \in \mathcal{H}_{\mathbf{s}}$ and $m \in \mathbb{N}_0$. From (1) we have that for each $r \in \mathbb{N}_0$ there exists a constant $D > 0$ and $p \in \mathbb{N}_0$ such that

$$\|f_n\|_r \leq D(n+1)^p$$

and for all $s, l \in \mathbb{N}_0$,

$$\|(\|g_n\|_s)\|_l^2 = \sum_{n=0}^{\infty}(n+1)^{2l}\|g_n\|_s^2 < \infty.$$

Combining the inequality (2) with the above equations we obtain

$$\begin{aligned}\|(\|f_n g_n\|_m)\|_l^2 &= \sum_{n=0}^{\infty}(n+1)^{2l}\|f_n g_n\|_m^2 \\ &\leq C_m^2\sum_{n=0}^{\infty}(n+1)^{2l}\|f_n\|_r^2\|g_n\|_s^2 \\ &\leq C_m^2 D^2\sum_{n=0}^{\infty}(n+1)^{2(l+p)}\|g_n\|_s^2 \\ &< \infty.\end{aligned}$$

Noted that we have proved that $(\|f_n g_n\|_m) \in \mathbf{s}$, for all $m \in \mathbb{N}_0$. This is $(f_n)\cdot(g_n) \in \mathcal{H}_{\mathbf{s}}$.
Finally, we prove that if $(f_n) \in \mathcal{H}_{\mathbf{s}}$ then $(Df_n) \in \mathcal{H}_{\mathbf{s}}$. In fact, let $m \in \mathbb{N}_0$. Since $\{\|\cdot\|_m : m \in \mathbb{N}_0\}$ is equivalent to $\{\|\cdot\|_{\alpha,\beta,\infty} : \alpha,\beta \in \mathbb{N}_0\}$ we have that there exists $\alpha, \beta, m_{\alpha,\beta} \in \mathbb{N}_0$ and a constants $C_m, C_{\alpha,\beta+1} > 0$ such that

$$\begin{aligned}\|Df_n\|_m &\leq C_m\|Df_n\|_{\alpha,\beta,\infty} \\ &= C_m\|f_n\|_{\alpha,\beta+1,\infty} \\ &\leq C_m C_{\alpha,\beta+1}\|f_n\|_{m_{\alpha,\beta}}.\end{aligned}$$

As $(f_n) \in \mathcal{H}_{\mathbf{s}}$ we have $(\|Df_n\|_m) \in \mathbf{s}$. This implies that $(Df_n) \in \mathcal{H}_{\mathbf{s}}$. □

Proposition 1. *Let $T \in \mathcal{S}'$. Then $(T_n) \in \mathcal{H}_{\mathbf{s}'}$, where $T_n = \sum_{j=0}^{n} T(h_j)h_j$.*

Proof. From Theorem 1, there exists a constant $C > 0$ and $p \in \mathbb{N}_0$ such that

$$|T(h_j)| \leq C(j+1)^p$$

for all $j \in \mathbb{N}_0$. Then

$$\begin{aligned}\|T_n\|_m^2 &= \sum_{j=0}^{n}(j+1)^{2m}|T(h_j)|^2 \\ &\leq (n+1)^{2m}\sum_{j=0}^{n}|T(h_j)|^2 \\ &\leq C(n+1)^{2(m+p+1)}.\end{aligned}$$

This completes the proof. □

Definition 2. The tempered algebra is defined as

$$\mathcal{H} = \mathcal{H}_{\mathbf{s}'}/\mathcal{H}_{\mathbf{s}}.$$

The elements of $\mathcal{H}$ are called tempered generalized functions.

Let $(f_n) \in \mathcal{H}_{s'}$ we will use $[f_n]$ by denoted the equivalent class $(f_n) + \mathcal{H}_s$.

Proposition 2. *Let $\iota : \mathcal{S} \to \mathcal{H}$ be the application*

$$\iota(T) = [T_n].$$

Then ι is a linear embedding. Moreover,we have that
a) *For all $\varphi \in \mathcal{S}$,*

$$\iota(\varphi) = [\varphi].$$

b) *For all $T \in \mathcal{S}'$*

$$\iota(DT) = D\iota(T).$$

Proof. It is clear from the above Proposition, that ι is well defined and a linear application. We claim that $\iota(T) = 0$ implies $T = 0$. Since $(T_n) \in \mathcal{H}_{\mathbf{s}}$, we have

$$\lim_{n\to\infty} \|T_n\|_m = 0$$

for all $m \in \mathbb{N}_0$. This is the sequence (T_n) converge weakly to 0, which proves the claim.
a) Let $\varphi \in \mathcal{S}$. We have that $\iota(\varphi) = [\varphi_n]$ where $\varphi_n = \sum_{j=0}^{n} < \varphi, h_j > h_j$. Then for all $m, s \in \mathbb{N}_0$ we have that

$$\begin{aligned} \lim_{n\to\infty} (n+1)^{2s} \|\varphi - \varphi_n\|_m^2 &= \lim_{n\to\infty} (n+1)^{2s} \sum_{j=n+1}^{\infty} (j+1)^{2m} | < \varphi, h_j > |^2 \\ &\leq \lim_{n\to\infty} \sum_{j=n+1}^{\infty} (j+1)^{2(m+s)} | < \varphi, h_j > |^2 \\ &= 0 \end{aligned}$$

where the last equality follows from $\|\varphi\|_{m+s} < \infty$. Therefore, we conclude that $(\|\varphi - \varphi_n\|_m) \in \mathbf{s}$. Since $(\varphi - \varphi_n) \in \mathcal{H}_{\mathbf{s}}$, it follows that $\iota(\varphi) = [\varphi]$.
b) Let $T \in \mathcal{S}'$. By definitions and properties of Hermite functions,

$$\begin{aligned} DT_n &= D(\sum_{j=0}^{n} T(h_j) h_j) \\ &= \sum_{j=0}^{n} T(h_j) D h_j \\ &= \sum_{j=0}^{n} T(h_j) \frac{1}{2} (\sqrt{j} h_{j-1} - \sqrt{j+1} h_{j+1}) \\ &= -\sum_{j=0}^{n} \frac{1}{2} (\sqrt{j} T(h_{j-1}) - \sqrt{j+1} T(h_{j+1})) h_j \\ &= \sum_{j=0}^{n} DT(h_j) h_j \\ &= (DT)_n. \end{aligned}$$

Therefore $D\iota(T) = [DT_n] = [(DT)_n] = \iota(DT)$. □

Corollary 1. *Let $\varphi, \psi \in \mathcal{S}$. Then*

$$\iota(\varphi\psi) = \iota(\varphi) \cdot \iota(\psi).$$

Proof. We first observe that

$$(\varphi\psi) - (\varphi_n) \cdot (\psi_n) = (\varphi) \cdot (\psi - \psi_n) + (\varphi - \varphi_n) \cdot (\psi).$$

Applying Proposition 2 and Lemma 1 we obtain $(\varphi\psi) - (\varphi_n) \cdot (\psi_n) \in \mathcal{H}_{\mathbf{s}}$. Therefore $\iota(\varphi\psi) = \iota(\varphi) \cdot \iota(\psi)$. □

Remark 1. *Let $\mathcal{O}_M$ be the ring of multipliers of $\mathcal{S}$ (see [17]). We have a natural multiplication from $\mathcal{O}_M$ by $\mathcal{H}$ into $\mathcal{H}$, defined by*

$$g[f_n] := [gf_n]$$

where $g \in \mathcal{O}_M$ and $[f_n] \in \mathcal{H}$.
It is easy to check that the product is well defined and that $\mathcal{H}$ is a $\mathcal{O}_M$-module.

We give two examples of tempered generalized functions.

Example 1. *The distribution δ. We have that $\iota(\delta) = [\delta_n]$, where $\delta_n = \sum_{j=0}^{n} h_j(0)h_j$. Applying the formula (6) and the following equality*

$$\sum_{j=0}^{n} h_j(x)h_j(y) = \frac{\sqrt{n+1}}{x-y}\Big(h_{n+1}(x)h_n(y) - h_{n+1}(y)h_n(x)\Big),$$

we see that

$$\delta_n(x) = \begin{cases} \sqrt{n+1}\frac{(-1)^{\frac{n}{2}}}{\sqrt[4]{2\pi}}\sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}}\ \frac{h_{n+1}(x)}{x} & \textit{for } n \textit{ even,} \\ \sqrt{n+1}\frac{(-1)^{\frac{n+3}{2}}}{\sqrt[4]{2\pi}}\sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n}{n+1}}\ \frac{h_n(x)}{x} & \textit{for } n \textit{ odd.} \end{cases} \tag{12}$$

Example 2. *The element δ^2. We have that $\delta^2 \equiv \iota(\delta)\cdot\iota(\delta) = [\delta_n^2]$. From (12) it follows that*

$$\delta_n^2(x) = \begin{cases} (n+1)\frac{1}{\sqrt{2\pi}}(\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n})\ \frac{h_{n+1}^2(x)}{x^2} & \textit{for } n \textit{ even,} \\ (n+1)\frac{1}{\sqrt{2\pi}}(\frac{1}{2}\frac{3}{4}\cdots\frac{n}{n+1})\ \frac{h_n^2(x)}{x^2} & \textit{for } n \textit{ odd.} \end{cases} \tag{13}$$

We introduce the concept of association for tempered generalized functions.

Definition 3. Let $[f_n]$ and $[g_n]$ be tempered generalized functions. We say that $[f_n]$ and $[g_n]$ are associated, denoted by $[f_n] \approx [g_n]$, if for all $\varphi \in \mathcal{S}$

$$\lim_{n\to\infty} < f_n - g_n, \varphi >= 0.$$

We observe that the relation $\approx$ is well defined, because $(l_n) \in \mathcal{H}_{\mathbf{s}}$ and $\varphi \in \mathcal{S}$ we have that $\lim_{n\to\infty} < l_n, \varphi >= 0$. It follows immediately that $\approx$ is an equivalence relation on $\mathcal{H}$.

Proposition 3. **a**) *Let $[f_n], [g_n] \in \mathcal{H}$ such that $[f_n] \approx [g_n]$. Then $D^\alpha[f_n] \approx D^\alpha[g_n]$ for all $\alpha \in \mathbb{N}$.*
b) *Let $[f_n], [g_n] \in \mathcal{H}$ such that $[f_n] \approx [g_n]$ and $l \in \mathcal{O}_M$. Then $l[f_n] \approx l[g_n]$.*
c) *Let $T, S \in \mathcal{S}'$ such that $\iota(T) \approx \iota(S)$. Then $T = S$.*

Proof. **a**) By integration by parts and hypothesis,

$$\begin{aligned} \lim_{n\to\infty} < D^\alpha f_n - D^\alpha g_n, \varphi > &= \lim_{n\to\infty} < f_n - g_n, (-1)^\alpha D^\alpha \varphi > \\ &= 0 \end{aligned}$$

for all $\varphi \in \mathcal{S}$. This is $D^\alpha[f_n] \approx D^\alpha[g_n]$.
b) Let $\varphi \in \mathcal{S}$. As $l \in \mathcal{O}_M$ we have $l\varphi \in \mathcal{S}$. By assumption,

$$\begin{aligned} \lim_{n\to\infty} < lf_n - lg_n, \varphi > &= \lim_{n\to\infty} < f_n - g_n, l\varphi > \\ &= 0. \end{aligned}$$

We conclude that $l[f_n] \approx l[g_n]$.
c) By definition, $\lim_{n\to\infty} \int (T-S)_n(x)\ h_k(x)\ dx = (T-S)(h_k)$ for all $k \in \mathbf{N}_0$. But $\lim_{n\to\infty} \int (T-S)_n(x)\ h_k(x)\ dx = 0$ since $\iota(T) \approx \iota(S)$. Applying the N-representation theorem we conclude that $T = S$. □

Example 3. $x\iota(\delta) \approx 0$. *In fact,*

$$\lim_{n\to\infty} < x\delta_n, h_k >= (xh_k)(0) = 0.$$

Finally, we study the relation between the symmetric product of tempered distributions via hermite expansions and association for tempered generalized functions. The symmetric product of tempered distribution was introduced by Shen, C. and Sun, M. in [22] based on ideas of [6].

Definition 4. Let S and T be tempered distributions. Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$c_k = \lim_{n\to\infty} < T_n S_n, h_k >$$

and that $(c_k) \in \mathbf{s}'$. The symmetric Hermite product of S and T, denoted by $S \bullet T$, is defined to be the tempered distribution

$$\sum_{k=0}^{\infty} c_k h_k. \tag{14}$$

Lemma 2. **a**) *The symmetric Hermite product is commutative, distributive.*
b) *The symmetric Hermite product verifies the Leibnitz rule: Let S and T be in $\mathcal{S}'$, then*

$$D(S \bullet T) = DS \bullet T + S \bullet DT.$$

c) *Let S and T be in $\mathcal{S}'$ such that there exists $S \bullet T$. Then*

$$\iota(S) \cdot \iota(T) \approx \iota(S \bullet T).$$

Proof. **a**), **b**) The proof are straightforward (see [6]).
c) It is immediate from the definitions.

□

Remark 2. *In order to work with ordinary differential equations in the generalized tempered functions setting, we introduce the algebra $\mathcal{H}_T$ of time depended tempered generalized functions. We can proceed in a similar way to the construction of the algebra $\mathcal{H}$, the details are left to the reader. Let $\mathcal{S}_T$ be the set of functions $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that for each $t \in [0,T]$, $f(t,\cdot) \in \mathcal{S}$ and for each $x \in \mathbb{R}$, $f(\cdot,x) \in C^1([0,T])$. The set $\mathcal{H}^T_{\mathbf{s}'}$ is given by*

$$\{(f_n) \in \mathcal{S}_T^{\mathbb{N}_0} \ : \ \textit{for each } m \in \mathbb{N}_0,\ (\sup_{t\in[0,T]} \|f_n(t,\cdot)\|_m),\ (\sup_{t\in[0,T]} \|\frac{\partial f_n}{\partial t}(t,\cdot)\|_m) \in \mathbf{s}'\ \}$$

and the set $\mathcal{H}^T_{\mathbf{s}}$ given by

$$\{(f_n) \in \mathcal{S}_T^{\mathbb{N}_0} \ : \ \textit{for each } m \in \mathbb{N}_0,\ (\sup_{t\in[0,T]} \|f_n(t,\cdot)\|_m),\ (\sup_{t\in[0,T]} \|\frac{\partial f_n}{\partial t}(t,\cdot)\|_m) \in \mathbf{s}\ \}.$$

It is clear that $\mathcal{H}^T_{\mathbf{s}}$ is a differentiable ideal of the algebra $\mathcal{H}^T_{\mathbf{s}'}$. We define the algebra $\mathcal{H}_T$ as $\mathcal{H}^T_{\mathbf{s}'}/\mathcal{H}^T_{\mathbf{s}}$. The elements of $\mathcal{H}_T$ are called time depended tempered generalized functions. It follows immediately that for $[f_n] \in \mathcal{H}_T$ we have that $\frac{\partial}{\partial t}[f_n(t,\cdot)]$ define by $[\frac{\partial}{\partial t} f_n(t,\cdot)] \in \mathcal{H}$.

4. Tempered numbers and point values

Definition 5. The ring of tempered numbers is defined as

$$\mathbf{h} = \mathbf{s}'/\mathbf{s}. \tag{15}$$

The elements of $\mathbf{h}$ are called tempered numbers.

Let $(b_n) \in \mathbf{s}'$ we will use $[b_n]$ by denoted the equivalent class $(b_n) + \mathbf{s}$.

Lemma 3. **a**) *Let $\iota : \mathbb{R} \to \mathbf{h}$ be the application*

$$\iota(a) = [a].$$

Then ι is a embedding.

b) *$\mathcal{H}$ is a $\mathbf{h}$-module with the natural operations.*

c) *Let $[f_n] \in \mathcal{H}$ and $a \in \mathbb{R}$. Then $[f_n(a)] \in \mathbf{h}$.*

Proof. **a**) It is clear that $(a) \in \mathbf{s}'$, then $\iota(a) = [a]$ is well defined. Assuming that $\iota(a) = 0$, we have that $(a) \in \mathbf{s}$. In particular $\lim_{n\to\infty} na = 0$, it follows that $a = 0$.
b) We have divided the proof into two parts. We first prove that for $(b_n) \in \mathbf{s}'$ and $(f_n) \in \mathcal{H}_{\mathbf{s}'}$ we have $(b_n f_n) \in \mathcal{H}_{\mathbf{s}'}$. In fact, by definition there exists constants $E, F > 0$ and $p, q \in \mathbb{N}_0$ such that

$$\begin{aligned} \|f_n\|_m &\leq E(n+1)^p \\ |b_n| &\leq F(n+1)^q. \end{aligned}$$

Combining these inequalities, we obtain

$$\|b_n f_n\|_m \leq EF(n+1)^{p+q}.$$

This proves that $(\|b_n f_n\|_m) \in \mathbf{s}'$, thus $(b_n f_n) \in \mathcal{H}_{\mathbf{s}'}$.
Finally, the proof is completed by showing that for $(a_n) \in \mathbf{s}$ and $(f_n) \in \mathcal{H}_{\mathbf{s}'}$ or $(a_n) \in \mathbf{s}'$ and $(f_n) \in \mathcal{H}_{\mathbf{s}}$ we have $(a_n f_n) \in \mathcal{H}_{\mathbf{s}}$.
c) Since $\delta_a \in \mathcal{S}'$, there exists a constant $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|f_n(a)| = |\delta_a(f_n)| \leq C\|f_n\|_m,$$

for all $n \in \mathbb{N}_0$.
Combining the above inequality with $(\|f_n\|_m) \in \mathbf{s}'$ we conclude that $(f_n(a)) \in \mathbf{s}'$. ☐

Remark 3. *We observe that $\mathbf{h}$ is not a field, since there exist zero divisors in $\mathbf{h}$. In fact, $[1 + (-1)^n], [1 + (-1)^{n+1}] \in \mathbf{h}$ are non zero and its product is zero.*

Definition 6. The point value of $[f_n] \in \mathcal{H}$ in $a \in \mathbb{R}$, denoted by $[f_n](a)$, is defined to be $[f_n(a)]$.

Example 4. *The point value of δ in $a \in \mathbb{R}$. From (12) we have that $\iota(\delta)(a) = [a_n]$, where*

$$a_n = \begin{cases} \frac{\sqrt{n+1}}{a} h_{n+1}(a) h_n(0) & \text{for } n \text{ even,} \\ -\frac{\sqrt{n+1}}{a} h_n(a) h_{n+1}(0) & \text{for } n \text{ odd.} \end{cases}$$

Example 5. *The point value of x_+ in 0. It is easy to check that $\iota(x_+)(0) = [a_n]$, where*

$$a_n = \begin{cases} \sqrt{n+1}\, h_n(0) \int_0^\infty h_{n+1}(x)\, dx & \text{for } n \text{ even,} \\ \sqrt{n}\, h_{n-1}(0) \int_0^\infty h_n(x)\, dx & \text{for } n \text{ odd.} \end{cases}$$

We introduce the concept of association for tempered numbers.

Definition 7. Let $[a_n]$ and $[b_n]$ be tempered numbers. We say that $[a_n]$ and $[b_n]$ are associated, denoted by $[a_n] \approx [b_n]$, if

$$\lim_{n\to\infty}(a_n - b_n) = 0.$$

We observe that the relation $\approx$ is well defined and that $\approx$ is an equivalence relation on $\mathbf{h}$.

Example 6. $\iota(x_+)(0) \approx 0$.

5. Integration and Fourier transform

In this section we present the integration theory of tempered generalized functions and the Fourier transform.

Definition 8. Let $[f_n] \in \mathcal{H}$ and A be a Lebesgue measurable set. The integral of $[f_n]$ on A, denoted by $\int_A [f_n](x)\, dx$ is defined to be

$$[\int_A f_n(x)\, dx]. \tag{16}$$

We observe that the integral is well defined as an element of $\mathbf{h}$. In fact, as 1_A is a tempered distribution there exists a constant $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|\int_A g(x)\, dx| \le C\|g\|_m,$$

for all $g \in \mathcal{S}$. In particular, for $[f_n] \in \mathcal{H}$ we have that

$$|\int_A f_n(x)\, dx| \le C\|f_n\|_m.$$

As $(\|f_n\|_m) \in \mathbf{s}'$, we conclude that $\int_A [f_n](x)\, dx \in \mathbf{h}$.

In the next Lemma we collect some fundamental properties of the integral of tempered generalized functions.

Lemma 4. *Let $[f_n]$ and $[g_n]$ be tempered generalized functions, $a = [a_n] \in \mathbf{h}$ and $\alpha \in \mathbb{N}$. Then*

a) *Let A and B disjoint Lebesgue measurable sets. Then*

$$\int_{A\cup B} [f_n](x)\, dx = \int_A [f_n](x)\, dx + \int_B [f_n](x)\, dx.$$

b)

$$\int_A ([f_n] + a[g_n])(x)\, dx = \int_A [f_n](x)\, dx + a\int_A [g_n](x)\, dx.$$

c) *Let $\varphi \in \mathcal{S}$. Then*

$$\iota(\int_A \varphi\, dx) = \int_A \iota(\varphi)\, dx.$$

d) *"Rule of integration by parts"*

$$\int_{\mathbb{R}} [f_n] D^\alpha [g_n](x)\, dx = (-1)^\alpha \int_{\mathbb{R}} (D^\alpha [f_n])[g_n](x)\, dx.$$

e) *Let $\varphi \in \mathcal{S}$ and $T \in \mathcal{S}'$. Then*

$$\int_{\mathbb{R}} \iota(T)\cdot\iota(\varphi)(x)\, dx = \iota(T(\varphi)).$$

Proof. The proof of **a**), **b**) and **c**) are immediate.
d) We have that

$$\begin{aligned}\int_{\mathbb{R}}[f_n]D^{\alpha}[g_n](x)\ dx &= [\int_{\mathbb{R}} f_n D^{\alpha} g_n(x)\ dx] \\ &= [\int_{\mathbb{R}}(-1)^{\alpha} D^{\alpha} f_n(x) g_n(x)\ dx] \\ &= (-1)^{\alpha}\int_{\mathbb{R}}(D^{\alpha}[f_n])[g_n](x)\ dx.\end{aligned}$$

e) By definitions

$$\int_{\mathbb{R}} \iota(T)\cdot\iota(\varphi)(x)\ dx = [\int_{\mathbb{R}} T_n(x)\varphi(x)\ dx]$$

Let us prove that $(T(\varphi)-\int T_n(x)\varphi(x)\ dx)\in \mathbf{s}$. Combining definitions, N-representation theorem and $\lim_{n\to\infty} < \varphi, h_j >= 0$, we obtain that there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$,

$$\begin{aligned}|T(\varphi)-\int_{\mathbb{R}} T_n(x)\varphi(x)\ dx| &= |T - T_n(\varphi)| \\ &\leq \sum_{j=n+1}^{\infty} |T(h_j)||<\varphi, h_j>| \\ &\leq \sum_{j=n+1}^{\infty} C(j+1)^p|<\varphi, h_j>| \\ &\leq \sum_{j=n+1}^{\infty} C(j+1)^{2p}|<\varphi, h_j>|^2.\end{aligned}$$

Since $\varphi \in \mathcal{S}$, it follows that

$$\lim_{n\to\infty}\sum_{j=n+1}^{\infty}(j+1)^q|<\varphi, h_j>|^2 = 0$$

for all $q \in \mathbb{N}_0$. Combining the above inequalities we see that

$$\lim_{n\to\infty}(n+1)^r|T(\varphi)-\int_{\mathbb{R}} T_n(x)\varphi(x)\ dx| = 0,$$

for all $r \in \mathbb{N}_0$. This proves that $(T(\varphi)-\int T_n(x)\varphi(x)\ dx)\in \mathbf{s}$, which completes the proof. □

Example 7. *Let $T \in \mathcal{S}'$. Then*

$$\begin{aligned}\int_{\mathbb{R}}\iota(T)(x)\ dx &= [\int_{\mathbb{R}} T_n(x)\ dx] \\ &= [1(T_n)] \\ &= [\sum_{j\ even}^{n} T(h_j)h_j(0)(-1)^{\frac{j}{2}}],\end{aligned}$$

where the last equality follows from formula (7).

Example 8. δ^2. *From formula (13) we have that*

$$\begin{aligned}\int_{\mathbb{R}} \delta^2(x)\, dx &= [\int_{\mathbb{R}} \delta_n^2(x)\, dx] \\ &= [\frac{(n+1)}{\sqrt{2\pi}} \frac{1}{2}\frac{3}{4} \cdots \frac{n-1}{n}].\end{aligned}$$

The Fourier transform and convolution are very important tools of classical and modern analysis, our aim is introduce these operations in the context of tempered generalized functions. We recall that the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is defined by

$$\mathcal{F}(\varphi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} \varphi(x)\, dx$$

and the convolution $* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is defined by

$$\varphi * \psi(t) = \int_{\mathbb{R}} \varphi(t-x)\psi(x)\, dx.$$

For a fuller treatment about these issues we refer the reader to [18].

Definition 9. The Fourier transform of a generalized tempered function $[f_n]$, denoted by $\mathcal{F}([f_n])$, is defined to be $[\mathcal{F}(f_n)]$.

We observe that the above definition is independent of the representatives, because for all $m \in \mathbb{N}_0$ and $\varphi \in \mathcal{S}$ we have that $\|\mathcal{F}(\varphi)\|_m = \|\varphi\|_m$.
Here are some elementary properties of the Fourier transform and convolution.

Theorem 2. a) *The Fourier transform $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is a linear isomorphism and its inverse is given by*

$$\mathcal{F}^{-1}([f_n]) = [\mathcal{F}^{-1}(f_n)].$$

b) *Let $[f_n] \in \mathcal{H}$ and $\alpha \in \mathbb{N}_0$. Then*

$$\begin{aligned}\mathcal{F}(D^\alpha[f_n]) &= (ix)^\alpha \mathcal{F}([f_n]) \\ \mathcal{F}(x^\alpha[f_n]) &= i^\alpha D^\alpha \mathcal{F}([f_n]).\end{aligned}$$

c) *Let T be a tempered distribution. Then $\iota(\mathcal{F}(T)) = \mathcal{F}(\iota(T))$.*

Proof. **a**) Define $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{G}([f_n]) = [\mathcal{F}^{-1}(f_n)]$. We observe that $\mathcal{G}$ is well defined, because for any $m \in \mathbb{N}_0$ and $\varphi \in \mathcal{S}$ we have $\|\varphi\|_m = \|\mathcal{F}^{-1}\varphi\|_m$. It is clear that $\mathcal{F} \circ \mathcal{G} = I_{\mathcal{H}}$ and $\mathcal{G} \circ \mathcal{F} = I_{\mathcal{H}}$.
b) and **c**). The proofs follows from the definitions and properties of the Fourier transform in $\mathcal{S}$. □

Definition 10. Let $[f_n]$ and $[g_n]$ be generalized tempered functions. The convolution of $[f_n]$ and $[g_n]$, denoted by $[f_n]*[g_n]$, is defined to be $\mathcal{F}^{-1}(\sqrt{2\pi}\mathcal{F}([f_n])\cdot\mathcal{F}([g_n]))$.

Theorem 3. a) *Let $[f_n], [g_n] \in \mathcal{H}$. Then*

$$[f_n] * [g_n] = [f_n * g_n].$$

b) *Let $[f_n], [g_n], [h_n] \in \mathcal{H}$ and $\alpha \in \mathbb{N}_0$. Then*

$$\begin{aligned}[f_n] * [g_n] &= [g_n] * [f_n] \\ D^\alpha([f_n] * [g_n]) &= (D^\alpha[f_n]) * [g_n] \\ ([f_n] * [g_n]) * [h_n] &= [f_n] * ([g_n] * [h_n])\end{aligned}$$

c) *Let* $[f_n], [g_n] \in \mathcal{H}$. *Then*

$$\begin{aligned}\mathcal{F}([f_n]\cdot[g_n]) &= \frac{1}{\sqrt{2\pi}}\mathcal{F}([f_n]) * \mathcal{F}([g_n])\\ \mathcal{F}([f_n]*[g_n]) &= \sqrt{2\pi}\mathcal{F}([f_n])\cdot\mathcal{F}([g_n])\end{aligned}$$

d) *Let* $T \in \mathcal{S}'$ *and* $\varphi \in \mathcal{S}$. *Then*

$$\iota(T) * \iota(\varphi) \approx \iota(T * \varphi).$$

Proof. **a**) The proof is a consequence of the above theorem and definitions.
b) and **c**) The proofs follows from the definitions and properties of the convolution in $\mathcal{S}$.
d) We have that for all $\psi \in \mathcal{S}$,

$$\begin{aligned}\lim_{n\to\infty} < (T_n * \varphi), \psi > &= T * \varphi(\psi)\\ &= \lim_{n\to\infty} < (T*\varphi)_n, \psi > .\end{aligned}$$

This shows that $\iota(T) * \iota(\varphi) \approx \iota(T * \varphi)$. □

Example 9. *The Fourier transform of* δ. *By formula (6) we have that*

$$\mathcal{F}(\iota(\delta)) = [\sum_{k=0}^{n}(-i)^k h_k(0) h_k(x)].$$

6. Generalized Stochastic Calculus

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P})$ be a filtered probability space, which satisfies the usual hypotheses. For a recent account of stochastic calculus we refer the reader to the book of Ph. Protter [15].

Definition 11. Let $[f_n] \in \mathcal{H}$, X be a continuous jointly measurable process and V be an finite variation process. We define the integral of $[f_n](X)$ in relation to V from 0 to t, denote by $\int_0^t [f_n](X_s)dV_s$, to be

$$[\int_0^t f_n(X_s)dV_s].$$

It is clear that for each $\omega \in \Omega$ and $t \in [0,T]$ we have that $[\int_0^t f_n(X_s)dV_s(\omega)] \in \mathbf{h}$, because

$$|\int_0^t f_n(X_s)dV_s(\omega)| \leq \sup_x |f_n(x)||V|_t(\omega)$$

where $|V|_t(\omega)$ is the total variation of V in $[0,t]$.

Definition 12. **a**) Let $[f_n] \in \mathcal{H}$ and X be a random variable. We define the expectation of $[f_n](X)$, denote by $\mathbb{E}([f_n](X))$, to be $[\mathbb{E}(f_n(X))]$.

b) Let $[f_n] \in \mathcal{H}$, X be a continuous jointly measurable process and V be an finite variation process such that $|V|_t$ is integrable. We define the expectation of $\int_0^t [f_n](X_s)dV_s$, denote by $\mathbb{E}(\int_0^t [f_n](X_s)dV_s)$, to be $[\mathbb{E}(\int_0^t f_n(X_s)dV_s)]$.

It is easily to check that the above definition is a well definition. We observe that the natural definition of expectation doesn't work. In fact, let $Y_n : \Omega \to \mathbb{R}$ be random variables such that $(Y_n(\omega)) \in \mathbf{s}'$ for all $\omega \in \Omega$. We have that $[(\mathbb{E}(Y_n))]$ is dependent of the representatives (Y_n), because if $\{A_n : n \in \mathbb{N}\}$ is a partition measurable of Ω such that $\mathbb{P}(A_n) = \frac{1}{2^n}$ for all $n \in \mathbb{N}$ and $(b_n) \in \mathbf{s}'$, $[Y_n(\omega)] = [Y_n(\omega) + b_n 2^n 1_{A_n}(\omega)]$ and $(\mathbb{E}(b_n 2^n 1_{A_n})) = (b_n) \notin \mathbf{s}$.

We can now prove the Itô formula for generalized tempered functions. Clearly, this formula is an extension of the classical Itô formula via infinite dimensional methods.

Theorem 4. *Let $[f_n] \in \mathcal{H}$ and X be a continuous semimartingale. Then*

$$[f_n](X_t) = [f_n](X_0) + \int_0^t D[f_n](X_s)dX_s + \frac{1}{2}\int_0^t D^2[f_n](X_s)d < X >_s \tag{17}$$

where $\int_0^t D[f_n](X_s)dX_s(\omega)$ defined by $[\int_0^t Df_n(X_s)dX_s(\omega)]$ is the Itô integral of $[f_n](X)$ in relation to X from 0 to t.

Proof. We first show that $[\int_0^t Df_n(X_s)dX_s(\omega)]$ is well defined. In fact, let $(g_n) \in \mathcal{H}_\mathbf{s}$. Since $(Dg_n) \in \mathcal{H}_\mathbf{s}$, we have $(\int_0^t Dg_n(X_s)d < X >_s (\omega)) \in \mathbf{s}$. We see that $(g_n(X_t(\omega)))$ and $(g_n(X_0(\omega)))$ are in $\mathbf{s}$, which is clear from the definition of point value. Combining this facts with the Itô formula we have

$$(\int_0^t Dg_n(X_s)dX_s(\omega)) = (g_n(X_t(\omega))) - (g_n(X_0(\omega))) - (\frac{1}{2}\int_0^t D^2 g_n(X_s)d < X >_s (\omega))$$

are in $\mathbf{s}$. Finally we see that $[\int_0^t Df_n(X_s)dX_s(\omega)] \in \mathbf{h}$ and the formula (17) holds, this is clear from the Itô formula applied to f_n,

$$f_n((X_t(\omega))) = f_n((X_0(\omega))) + \int_0^t Df_n(X_s)dX_s(\omega) + \frac{1}{2}\int_0^t D^2 f_n(X_s)d < X >_s (\omega).$$

□

Remark 4. *Let $f \in \mathcal{S}$. By Proposition 2, $[f] = [f_n]$. Then it is clear that*

$$[f_n](X_t) = [f(X_t)]$$

and

$$\int_0^t D^2[f_n](X_s)d < X >_s = [\int_0^t D^2 f(X_s)d < X >_s].$$

Consequently,

$$\int_0^t D[f_n](X_s)dX_s = [\int_0^t Df(X_s)dX_s].$$

In particular, the members of the Itô formula for f as function are the same that the members of the Itô formula for f as tempered generalized function.

Remark 5. *We observe that the members of the Itô formula for C^4 functions with appropriated decreasing at infinite are associated with the corresponding members of the Itô formula as generalized tempered functions. In fact, we have that (f_n) converge uniformly over compacts whenever f is twice continuously differentiable and $O(e^{-cx^2})$ for some $c > 1$ as $x \to \infty$ (see [23] for more details). In particular,*

$$[f_n](x) \approx [f(x)]$$

for all $x \in \mathbb{R}$. Thus

$$[f_n](X_t) \approx [f(X_t)] \quad and \quad [f_n](X_0) \approx [f(X_0)].$$

If $D^2 f \in \mathcal{C}^2$ and $D^2 f$ is $O(e^{-cx^2})$ for some $c > 1$ as $x \to \infty$ we have that

$$\int_0^t D^2[f_n](X_s) d < X >_s \approx [\int_0^t D^2 f(X_s) d < X >_s].$$

Combining the above identities with the classical Itô formula for f we conclude that

$$\int_0^t D[f_n](X_s) dX_s \approx [\int_0^t Df(X_s) dX_s].$$

We can now state the Meyer-Tanaka formula (see for instance [15] for the classical Meyer-Tanaka formula).

Corollary 2. *Let X be a semimartingale. Then*

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + \int_0^t \delta_a(X_s) d < X >_s . \tag{18}$$

If $L^a(X)$ is the local time of X at point a, we have that

$$\iota(L^a(X)_t) \approx \int_0^t \delta_a(X_s) d < X >_s . \tag{19}$$

Proof. Applying the Itô formula (17) to the tempered distribution $|\cdot - a|$ we obtain (18). The formula (19) is a consequence of (18) and the classical Meyer-Tanaka formula,

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L^a(X)_t.$$

□

Corollary 3. *Let $[f_n] \in \mathcal{H}$ and B be a Brownian motion such that $B_0 = 0$. Then*

$$\mathbb{E}([f_n](B_t + x)) = [f_n](x) + \frac{1}{2} \int_0^t \mathbb{E}(D^2[f_n](B_s + x)) ds \tag{20}$$

Proof. We have

$$\mathbb{E}([\int_0^t D[f_n](B_s + x) dB_s]) = [\mathbb{E} \int_0^t Df_n(B_s + x) dB_s] = 0,$$

because $\int Df_n(B_s + x) dB_s$ is a martingale. □

Corollary 4. *Let $[f_n] \in \mathcal{H}$. Then $g_t = [\mathbb{E}(f_n(B_t + \cdot))] \in \mathcal{H}_T$ solves the Cauchy problem*

$$\begin{aligned} D_t g &= \frac{1}{2} D_x^2 g \\ g_0 &= [f_n]. \end{aligned}$$

Proof. We observe that

$$\mathbb{E}(f_n(B_t + x)) = \int_{\mathbb{R}} f_n(y) p_t(x - y)\, dy = f_n * p_t(x)$$

where $p_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$ is the heat kernel. As $*$ is a continuous operation in $\mathcal{S}$ and $\lim_{n\to\infty} f_n * p_t = f_n$ in $\mathcal{S}$ we conclude that $[g_n] \in \mathcal{H}_T$ where $g_n : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given by $g_n(t, x) = \mathbb{E}(f_n(B_t + x))$. Applying the formula (20) we completes the proof. □

Theorem 5. *Let* $[f_n] \in \mathcal{H}_T$ *and* X *be a continuous semimartingale. Then*

$$\begin{aligned}[f_n](X_t) &= [f_n](X_0) + \int_0^t D_t[f_n](s, X_s)ds + \int_0^t D_x[f_n](s, X_s)dX_s \\ &\quad + \frac{1}{2}\int_0^t D_x^2[f_n](s, X_s)d < X >_s .\end{aligned}$$

Proof. We observe that $\int_0^t D_t[f_n](s, X_s)ds$ is well defined and proceed analogously to the proof of the extension of Itô formula. □

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