A Girsanov Theorem in Manifolds and Projective Maps

Simão Stelmastchuk
Departamento de Matemática, Universidade Estadual de Campinas,
13.081-970 - Campinas - SP, Brazil.

Abstract

Our purpose is to show a version of Girsanov theorem in smooth manifolds. After, we will use it theorem to give stochastic characterization for strongly projective maps. This stochastic characterization yields a proof that projective maps of rank ≥ 2 between Riemannian manifolds, with connected domain, are affine maps. In particular, the groups of affine and projective transformations, in connected Riemannian manifold, are equal.

Key words: Change of probability; Girsanov theorem; projective maps.

MSC2000 subject classification: 53A20, 60A10, 60H30.

1 Introduction

The problem of change of probabilities is well kwnow in Theory of Probability. In stochastic analysis the interest is what this change yields in the process. In $\mathbb{R}^n$, the well know Girsanov theorem shows that martingales and Bronwnian motions turn in other ones when probability is changed.

We wish to investigate the change of martingales in manifolds when probabilities are turned. There are some works about change of probability in manifolds. We cite for instance I. Shigekawa [13], [14] and M. Arnaudon et al. [2], [3].

Our first purpose is to state and to prove the following version of Girsanov theorem in manifolds.

Theorem A: *Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalents probabilities. Let us denote $Z = \frac{dP}{dQ}$. Let M be a smooth manifold equipped whit symmetric connection ∇^M. Let X be a $(\nabla^M, \mathbb{P})$-martingale in M. Then*

$$\int_0^t \theta d^{\nabla^M} X_s - \left[\int_0^t \frac{1}{Z} dZ, \int_0^t \theta d^{\nabla} X_s\right]$$

is $\mathbb{Q}$-local martingale.

In $\mathbb{R}^n$, Girsanov theorem is used in some applications as for example stochastic differential equation. Here, our application is to use Theorem A to study the projective maps.

Projective transformations are of great interrest in Mathematics and Physics. For a fuller tratament we refer the reader to [1]. If one does not consider the specialized studied of projective transformations, first attentions in projective maps were gived by K.Yano and S. Ishihara in [15] and Z.Har'El in [8]. For our study about projective maps we follows the works of T. Nore [11] and J. Hebda [9]. As application of Theorem A we gived the following characterization.

Theorem B: *Let M, N be smooth manifolds equipped with symmetric connections ∇^M and ∇^N, respectively. Let $\phi : M \to N$ be a smooth map of constant rank. Then ϕ is strongly projective if and only if there exist a 1-form α on M such that $\phi(X)$ is $(\nabla^N, \mathbb{Q}^{\nabla^M}_{\alpha,X})$-martingale, for every $(\nabla^M, \mathbb{P})$-martingale X in M, where $\mathbb{Q}^{\nabla^M}_{\alpha,X} = Z\mathbb{P}$ and $Z = \exp(-\int \alpha d^{\nabla^M} X)$.*

As application of Theorem B we show that composition of strongly projective maps is strongly projective map. Finally, we show the surprising result about projective maps.

Theorem C: *Let M and N be Riemannian manifolds endowed with symmetric connections ∇^M and ∇^N, respectively. Suppose that M is connected. Then every projective map $\phi : (M, \nabla^M) \to (N, \nabla^N)$ of rank ≥ 2 is affine map.*

A direct consequence of Theorem C is, when M is a connected Riemannian manifold with $dimM \geq 2$, that the groups of affine and projective transformations of M are equal.

The author is greatly indebted to Professor Pedro Catugno for suggesting the problem and for many stimulating conversations.

2 Preliminaries

We begin by recalling some fundamental facts on stochastic calculus on manifolds, we shall use freely concepts and notations of M. Emery [7], P. Protter [12] and S. Kobayashi and N. Nomizu [10]. For a complete treatment about this section we refer the reader to [5].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a mensuravel space with right continuos filtration.

When we equippe it whit probability $\mathbb{P}$, we ask that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satifies the usual hypothesis (see for example [7]). We also suppose that every stochastic process is continuos.

Definition 2.1 *Let M be a differential manifold. A stochastic process X in M is called semimartingale if $f(X)$ is real semimartingale for all smooth function f on M.*

Let M be a smooth manifold, i.e. C^∞ manifold, endowed with symmetric connection ∇^M. Here, symmetric connection mains that connection is torsion free. Let X be a continuous semimartingale with values in M. Let $(x_1, \ldots, x_n)$ be a system of local coordinates. The Itô integral of an adapted stochastic 1-form θ along X is defined, locally, by

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X) dX_s^i + \frac{1}{2}\int_0^t \theta_i(X_s)\Gamma^i_{jk}(X_s) d[X^j, X^k]_s,$$

where $\theta(x) = \theta_i(x)dx^i$, with θ_i smooth, and Γ^i_{jk} are the Christoffel symbols of ∇^M. Let b an adapted stochastic section of $T^{(2,0)}M$ along X. The quadratic integral of b along X is defined, locally, by

$$\int_0^t b\ (dX_s, dX_s) = \int_0^t b_{ij}(X_s) d[X^i, X^j]_s, \tag{1}$$

where $b(x) = b_{ij}(x)dx^i \otimes dx^j$, whit b_{ij} smooth.

Definition 2.2 *Let M be a smooth manifold with symmetric connection ∇^M. Let $\mathbb{P}$ be a probability. A semimartingale X with values in M is called a $(\nabla^M, \mathbb{P})$-martingale if $\int_0^t \theta\ d^{\nabla^M} X_s$ is a real $\mathbb{P}$-local martingale for all $\theta \in TM^*$.*

Let M and N be smooth manifolds endowed with symmetric connections ∇^M and ∇^N, respectively. Let $\phi : M \to N$ be a smooth map. Let $\phi^{-1}(TN)$ be the induced bundle. We denote by $\nabla^{N'}$ the unique symmetric connection on $\phi^{-1}(TN)$ induced by ∇^N (see for example Proposition I.3.1 in [11]). The bilinear mapping $\beta_\phi : TM \times TM \to TN$ defined by

$$\beta_\phi(X, Y) = \nabla^{N'}_X \phi_*(Y) - \phi_*(\nabla^M_X Y)$$

is called the second fundamental form of ϕ (see for example definition I.4.1.1 in [11]). ϕ is said affine map if β_F is null.

Let X be a semimartingale in M and θ be a 1-form along X. We have the following geometric Itô formula:

$$\int_0^t \theta\ d^{\nabla^N} \phi(X_s) = \int_0^t \phi^*\theta\ d^{\nabla^M} X_s + \frac{1}{2}\int_0^t \beta^*_\phi \theta\ (dX_s, dX_s). \tag{2}$$

3 Girsanov Theorem

We can now formulate our Girsanov Theorem in manifolds.

Theorem 3.1 *Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalents probabilities. Let us denote $Z = \frac{dP}{dQ}$. Let M be a smooth manifold equipped whit symmetric connection ∇^M. Let X be a $(\nabla^M, \mathbb{P})$-martingale in M. Then*

$$\int_0^t \theta d^{\nabla^M} X_s - \left[\int_0^t \frac{1}{Z_s} dZ_s, \int_0^t \theta d^{\nabla} X_s \right] \tag{3}$$

is $\mathbb{Q}$-local martingale.

Proof: Let X be a $(\nabla^M, \mathbb{P})$-martingale in M and $(x_1, \dots, x_n)$ be a local coordinate system in M. By definition of Itô integral, for every 1-form θ on M,

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X_s) dX_s^i + \frac{1}{2} \int_0^t \theta_i(X_s) \Gamma_{jk}^i(X_s) d[X^j, X^k]_s,$$

where $\theta(x) = \theta_i(x) dx^i$. Since X^i is real semimartingale, $X^i = M^i + A^i$, where M^i is $\mathbb{P}$-local martingale and A^i is variation finite process. Substituting these into equation above we obtain

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X_s) dM_s^i + \int_0^t \theta_i(X_s) dA_s^i + \frac{1}{2} \int_0^s \theta^i(X_s) \Gamma_{jk}^i(X_s) d[X^j, X^k]_s.$$

Because X is a $(\nabla^M, \mathbb{P})$-martingale, $\int_0^t \theta d^{\nabla^M} X_s$ is $\mathbb{P}$-local martingale. Therefore by Doob-Meyer decomposition

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X_s) dM_s^i. \tag{4}$$

Girsanov Theorem in $\mathbb{R}$ now shows that $N^i = M^i - \int \frac{1}{Z} d[Z, M^i]$ are $\mathbb{Q}$-martingale. From this and (4) we dedude that

$$\begin{aligned}
\int_0^t \theta_i(X_s) dN_s^i &= \int_0^t \theta_i(X_s) dM_s^i - \int_0^t \theta_i(X_s) \frac{1}{Z_s} d[Z_s, M^i]_s \\
&= \int_0^t \theta_i(X_s) dM_s^i - \left[\int_0^t \frac{1}{Z_s} dZ_s, \int_0^t \theta_i(X_s) dM_s^i \right] \\
&= \int_0^t \theta d^{\nabla^M} X_s - \left[\int_0^t \frac{1}{Z_s} dZ_s, \int_0^t \theta d^{\nabla^M} X_s \right].
\end{aligned}$$

Since $\int_0^t \theta_i(X_s)dN_s^i$ is $\mathbb{Q}$-local martingale, so is

$$\int_0^t \theta d^{\nabla^M} X_s - \left[\int_0^t \frac{1}{Z_s} dZ_s, \int_0^t \theta d^{\nabla^M} X_s \right].$$

□

Corollary 3.2 *Let M be a smooth manifold endowed with two symmetric connections ∇ and ∇'. Let X be a $(\nabla, \mathbb{P})$-martingale in M, Y be a $(\nabla', \mathbb{P})$-martingale in M and α be an 1-form on M. Then*

$$\int_0^t \theta d^{\nabla} X_s + \left[\int_0^t \alpha d^{\nabla'} Y_s, \int_0^t \theta d^{\nabla} X_s \right] \tag{5}$$

is $\mathbb{Q}^{\nabla'}_{\alpha,Y}$-local martingale, where $\mathbb{Q}^{\nabla'}_{\alpha,Y} = Z\mathbb{P}$ and $Z_t = \exp(-\int_0^t \alpha d^{\nabla'} Y_s)$.

Proof: Let X be a $(\nabla, \mathbb{P})$-martingale in M, Y be a $(\nabla', \mathbb{P})$-martingale in M and α be an 1-form on M. Write $Z_t = \exp(-\int_0^t \alpha d^{\nabla'} Y_s)$ and $\mathbb{Q}^{\nabla'}_{\alpha,Y} = Z\mathbb{P}$. As Z is strictly positive we have that $\mathbb{Q}^{\nabla'}_{\alpha,Y}$ and $\mathbb{P}$ are equivalents. It is clear that $dZ = -Z\alpha d^{\nabla'} Y$. Substituting this in equation (3) we get

$$\int_0^t \theta d^{\nabla} X_s - \left[\int_0^t \frac{1}{Z_s} dZ_s, \int_0^t \theta d^{\nabla} X_s \right] = \int_0^t \theta d^{\nabla} X_s + \left[\int_0^t \alpha d^{\nabla'} Y_s, \int_0^t \theta d^{\nabla} X_s \right]$$

From Theorem 3.1 we conclude that

$$\int_0^t \theta d^{\nabla} X_s + \left[\int_0^t \alpha d^{\nabla'} Y_s, \int_0^t \theta d^{\nabla} X_s \right]$$

is $\mathbb{Q}^{\nabla'}_{\alpha,Y}$-local martingale. □

Example 3.1 *A $(2n+1)$-dimensional smooth manifold is a contact manifold if there exist an open covering U_i of M and a 1-form α_i on each U_i such that (1) $\alpha_i \bigwedge (d\alpha_i)^n \neq 0$ everywhere on U_i and (2) if $U_i \cap U_j \neq \emptyset$ then $\alpha_i = \sigma_{ij}\alpha_j$, where σ_{ij} is a function on $U_i \cap U_j$. If M is orientable, there exists a 1-form α (called a contact form) on M such that (1) $\alpha \bigwedge (d\alpha)^n \neq 0$ everywhere on M and (2) $\alpha_i = \sigma_i \alpha$ on U_i, where σ_i is a function on U_i. For a deeper discussion of contact manifolds we refer the reader to [6].*

Let M be an orientable contact manifold and α be a contact form. Suppose that M is endowed with two symmetric connection ∇ and ∇'. Let X be a $(\nabla, \mathbb{P})$-martingale in M and Y be a $(\nabla', \mathbb{P})$-martngale in M. Let us *denote* $Z = exp(-\int \alpha d^{\nabla'} Y)$ *and* $\mathbb{Q}^{\nabla'}_{\alpha,Y} = Z\mathbb{P}$*. It is clear that* $\mathbb{Q}^{\nabla'}_{\alpha,Y}$ *and* $\mathbb{P}$ *are equivalent probabilities. Then by Corollary 3.2*

$$\int_0^t \theta d^{\nabla} X_s + \left[\int_0^t \alpha d^{\nabla'} Y_s, \int_0^t \theta d^{\nabla} X_s \right]$$

is a $\mathbb{Q}^{\nabla'}_{\alpha,Y}$*-local martingale for each* $\theta \in T^*M$. □

4 Projective maps

The following definition is due to J. Hebda [9].

Definition 4.1 *Let* M *be a differential manifold whit symmetric connection* ∇^M*. Let* $\gamma : (a,b) \to M$ *be a smooth curve.*
1. The acceleration bivector field along γ *is the smooth map* $\alpha_\gamma : (a,b) \to TM \wedge TM$ *defined by* $\alpha_\gamma = \dot{\gamma} \wedge \nabla^M_{\dot{\gamma}} \dot{\gamma}$*, where* $\dot{\gamma}$ *is the tangent vector field along* γ*.*
2. The curve γ *is a pregeodesic if* $\alpha_\gamma \equiv 0$*.*
3. A regular pregeodesic is called a geodesic.

J. Hebda and T. Nore in [9, 11] define projective map and strongly projective map, respectively, in the following way.

Definition 4.2 *Let* M *and* N *be differential manifolds endowed with symmetric connections, and let* $\phi : M \to N$ *be a smooth map.*
1. We say that ϕ *is projective map if* $\phi \circ \gamma$ *is a pregeodesic of* N *for every pregeodesic* γ *of* M*.*
2. ϕ *is called strongly projective if, for every geodesic* γ *in* M*,* $f \circ \gamma$ *is either a geodesic or a constant curve in* N*.*

The following remark will be useful for us.

Remark 4.1 *Let* $\phi : M \to N$ *be a projective map. J. Hebda in [9] showed that if* M *is connected then* ϕ_* *is either of rank* ≤ 1 *everywhere or of constant rank* $r \geq 2$*. In the latter case* ϕ *is strongly projective.*

T. Nore in [11] give the following characterization of strongly projective map of constant rank.

Proposition 4.1 *Let* $\phi : (M, \nabla^M) \to (N, \nabla^N)$ *be a smooth map of constant rank between smooth manifolds with symmetric connections. Then* ϕ *is an strongly projective map if and only if there exists a 1-form* α *on* M *such that*

$$\beta_\phi(U,V) = \alpha(U)\phi_* V + \alpha(V)\phi_* U, \quad U, V \in \mathfrak{X}M,$$

where β_ϕ is the second fundamental form of ϕ. We say that α is a 1-form associated to ϕ.

Proposition 4.2 *Let M, N be differential manifolds equipped with symmetric connections ∇^M and ∇^N, respectively. Let $\phi : M \to N$ be a smooth map of constant rank. Then ϕ is strongly projective map if and only if there exists a 1-form α on M such that*

$$\int_0^t \beta_\phi^*\theta(dX_s, dX_s) = 2[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} dX_s], \tag{6}$$

for all semimartingale X in M and for all 1-form θ on N.

Proof: Let $\phi : M \to N$ a smooth map of constant rank. Suppose that ϕ is strongly projective. By Proposition 4.1, there exist a 1-form α on M such that $\beta_\phi(U,V) = \alpha(U)\phi_* V + \alpha(V)\phi_* U$, $U, V \in \mathfrak{X}M$. Let X be a semimartingale in M and θ be a 1-form on N. Let us compute $\int_0^t \beta_\phi^*\theta(dX_s, dX_s)$. Let $(x_1, \ldots, x_n)$ be a system of local coordinates in M. Denote $\partial_i = \frac{d}{dx_i}, i = 1, \ldots, n$. From (1) we see that

$$\int_0^t \beta_\phi^*\theta(dX_s, dX_s) = \int_0^t (\beta_\phi^*\theta)_{ij}(X_s) d[X^i, X^j]_s,$$

where $\beta_\phi^*\theta = (\beta_\phi^*\theta)_{ij} dx^i \otimes dx^j$. Applying β_ϕ in the vectors $\partial_1, \ldots, \partial_n$ yields $\beta_\phi^*\theta(\partial_i, \partial_j) = (\beta_\phi^*\theta)_{ij}$. In the other side,

$$\beta_\phi^*\theta(\partial_i, \partial_j) = \theta(\alpha(\partial_i)\phi_*\partial_j + \alpha(\partial_j)\phi_*\partial_i) = \alpha_i(\phi^*\theta)_j + \alpha_j(\phi^*\theta)_i,$$

where $\alpha = \alpha_i dx^i$ and $(\phi^*\theta) = (\phi^*\theta)_i dx^i$. It follows that

$$\begin{aligned}
\int_0^t \beta_\phi^*\theta(dX_s, dX_s) = & \int_0^t (\alpha_i(\phi^*\theta)_j + \alpha_j(\phi^*\theta)_i)(X_s) d[X^i, X^j]_s \\
= & \int_0^t \alpha_i(\phi^*\theta)_j(X_s) d[X^i, X^j]_s + \int_0^t \alpha_j(\phi^*\theta)_i(X_s) d[X^i, X^j]_s \\
= & [\int_0^t \alpha_i(X_s) dX_s^i, \int_0^t (\phi^*\theta)_j(X_s) dX_s^j] \\
+ & [\int_0^t (\phi^*\theta)_i(X_s) dX_s^i, \int_0^t \alpha_j(X_s) dX_s^j].
\end{aligned}$$

By definition of Itô integral,

$$\int \beta_\phi^*\theta(dX_t, dX_t) = 2[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} dX_s].$$

Conversely, suppose that there exists a 1-form α on M such that the equation (6) is satisfied. In the same manner as above we can see that

$$\int_0^t ((\beta_\phi^*\theta)_{ij} - \alpha_i(\phi^*\theta)_j - \alpha_j(\phi^*\theta)_i)(X_s)d[X^i, X^j]_s = 0,$$

for a system of local coordinates $(x_1, ..., x_n)$ in M. As X is arbitrage semimartingale we have

$$(\beta_\phi^*\theta)_{ij} = \alpha_i(\phi^*\theta)_j + \alpha_j(\phi^*\theta)_i.$$

Therefore

$$\theta(\beta_\phi(\partial_i, \partial_j)) = \theta(\alpha(\partial_i)\phi_*(\partial_j) + \alpha(\partial_j)\phi_*(\partial_i)).$$

Since θ is arbitrage, we see for $U, V \in \mathfrak{X}(M)$ that

$$\beta_\phi(U, V) = \alpha(U)\phi_*(V) + \alpha(U)\phi_*(V).$$

From Proposition (4.1) we conclude that ϕ is projecive map. □

Corollary 4.3 *Let M, N be differential manifolds equipped with symmetric connections ∇^M and ∇^N, respectively. Let $\phi : M \to N$ be an strongly projective map of constant rank. If X is a semimartingale in M, then, for all $\theta \in T^*N$,*

$$\int_0^t \theta d^{\nabla^N}\phi(X_s) = \int_0^t \phi^*\theta d^{\nabla^M} X_s + [\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} dX_s],$$

where α is a 1-form associated to ϕ.

Proof: It follows easily from Itô geometric formula.

Theorem 4.4 *Let M, N be smooth manifolds equipped with symmetric connections ∇^M and ∇^N, respectively. Let $\phi : M \to N$ be a smooth map of constant rank. Then ϕ is strongly projective if and only if there exist a 1-form α on M such that $\phi(X)$ is $(\nabla^N, \mathbb{Q}_{\alpha,X}^{\nabla^M})$-martingale, for every $(\nabla^M, \mathbb{P})$-martingale X in M, where $\mathbb{Q}_{\alpha,X}^{\nabla^M} = Z\mathbb{P}$ and $Z = \exp(-\int \alpha d^{\nabla^M} X)$.*

Proof: Let $\phi : M \to N$ be a smooth map of constant rank. Suppose that ϕ is strongly projective. By proposition 4.1, there exists a 1-form α on M associated to ϕ. Let X be a $(\nabla^M, \mathbb{P})$-martingale in M and θ be a 1-form on N. By Corollary 4.3,

$$\int_0^t \theta d^{\nabla^N}\phi(X_s) = \int_0^t \phi^*\theta d^{\nabla^M} X_s + \left[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} X_s\right].$$

Write $Z_t = \exp(-\int_0^t \alpha d^{\nabla^M} X_s)$ and $\mathbb{Q}^{\nabla^M}_{\alpha,X} = Z\mathbb{P}$. It is clear that $\mathbb{P}$ and $\mathbb{Q}^{\nabla^M}_{\alpha,X}$ are equivalent probabilities. From Corollary 3.2 and equality above we conclude that $\int_0^t \theta d^{\nabla^N} \phi(X_s)$ is $\mathbb{Q}^{\nabla^M}_{\alpha,X}$-local martingale. As θ is arbitrage we have that $\phi(X)$ is $(\nabla^N, \mathbb{Q}^{\nabla^M}_{\alpha,X})$-martingale.

Conversely, let α be a 1-form on M such that $\phi(X)$ is a $(\nabla^N, \mathbb{Q}^{\nabla^M}_{\alpha,X})$-martingale, for every $(\nabla^M, \mathbb{P})$-martingale X in M, where $\mathbb{Q}^{\nabla^M}_{\alpha,X} = Z\mathbb{P}$ and $Z_t = \exp(-\int_0^t \alpha d^{\nabla^M} X_s)$. By geometric Itô formula, for every $\theta \in T^*N$,

$$\int_0^t \theta d^{\nabla^N} \phi(X_s) = \int_0^t \phi^*\theta d^{\nabla^M} X_s + \frac{1}{2}\int_0^t \beta^*_\phi \theta(dX_s, dX_s)$$

Adding and subtrating $\left[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} X_s\right]$ we obtain

$$\begin{aligned}\int_0^t \theta d^{\nabla^N} \phi(X_s) \;=\; & \int_0^t \phi^*\theta d^{\nabla^M} X_s + \left[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} X_s\right] \\ & - \left[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} X_s\right] + \frac{1}{2}\int_0^t \beta^*_\phi \theta(dX_s, dX_s).\end{aligned}$$

As $\phi(X)$ is $(\nabla^N, \mathbb{Q}^{\nabla^M}_{\alpha,X})$-martingale we have that $\int_0^t \theta d^{\nabla^N} \phi(X_s)$ is $\mathbb{Q}^{\nabla^M}_{\alpha,X}$-local martingale. From Corollary 3.2 we see that

$$\int_0^t \phi^*\theta d^{\nabla^M} X_s + \left[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} X_s\right].$$

is $\mathbb{Q}^{\nabla^M}_{\alpha,X}$-local martingale. Doob-Meyer decomposition now gives

$$\int_0^t \beta^*_\phi \theta(dX_s, dX_s) = 2\left[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} X_s\right].$$

From Proposition 4.2 we conclude that ϕ is a projective map. $\square$

J. Hebda shows that composition of projective maps is a projective map, see Theorem 6.4 in [9]. We now prove this result for strongly projective map, using Theorem 4.4.

Proposition 4.5 *Let M, N, P be smooth manifolds equipped with symmetric connections ∇^M, ∇^N and ∇^P, respectively. Let $\phi : M \to N$ and $\psi : N \to P$ be strongly projective maps of constant ranks. If $\psi \circ \phi$ has constant rank, then $\psi \circ \phi$ is strongly projective map.*

Proof: Let X be a $(\nabla^M, \mathbb{P})$-martingale in M. Because ψ is strongly projective map of constant rank, from proposition 4.1 there exist an 1-form α on N associated to ψ. By Corollary 4.3,

$$\int_0^t \theta d^{\nabla^P} \psi\circ\phi(X_s) = \int_0^t \psi^*\theta d^{\nabla^N}\phi(X_s) + \left[\int_0^t \alpha d^{\nabla^N}\phi(X_s), \int_0^t \psi^*\theta d^{\nabla^N}\phi(X_s)\right].$$

Since ϕ is strongly projective map of constant rank, there exist 1-form $\tilde{\alpha}$ on M associated to ϕ. By Corollary 4.3,

$$\begin{aligned}\int_0^t \theta d^{\nabla^P} \phi\circ\psi(X_s) &= \int_0^t \phi^*\psi^*\theta d^{\nabla^M} X_s + \left[\int_0^t \tilde{\alpha} d^{\nabla^M} X_s, \int_0^t \phi^*\psi^*\theta d^{\nabla^M} X_s\right] \\ &+ \left[\int_0^t \alpha d^{\nabla^N}\phi(X_s), \int_0^t \psi^*\theta d^{\nabla^N}\phi(X_s)\right]\end{aligned}$$

From geometric Itô formula we deduce that

$$\int_0^t \theta d^{\nabla^P} \phi\circ\psi(X_s) = \int_0^t \phi^*\psi^*\theta d^{\nabla^M} X_s + \left[\int_0^t \eta d^{\nabla^M} X_s, \int_0^t \phi^*\psi^*\theta d^{\nabla^M} X_s\right],$$

where $\eta = \tilde{\alpha} + \psi^*\alpha$. Write $Z_t = \exp(-\int_0^t \eta d^{\nabla^M} X_s)$ and $\mathbb{Q}_{\eta,X}^{\nabla^M} = Z\mathbb{P}$. It is clear that $\mathbb{Q}_{\eta,X}^{\nabla^M}$ and $\mathbb{P}$ are equivalents probabilities. From Corollary 3.2 we see that

$$\int_0^t \psi^*\phi^*\theta d^{\nabla^M} X_s + \left[\int_0^t \eta d^{\nabla^M} X_s, \int_0^t \phi^*\psi^*\theta d^{\nabla^M} X_s\right]$$

is $\mathbb{Q}_{\eta,X}^{\nabla^M}$-local martingale, so is $\int_0^t \theta d^{\nabla^P}\phi\circ\psi(X_s)$. Therefore $\phi\circ\psi(X)$ is $(\nabla^N, \mathbb{Q}_{\eta,X}^{\nabla^N})$-martingale. From Theorem 4.4 we conclude that $\phi\circ\psi$ is strongly projective map. □

Finally, we use Propostion 4.2 to prove the surprising result about projective maps.

Theorem 4.6 *Let M and N be Riemannian manifolds endowed with symmetric connections ∇^M and ∇^N, respectively. Suppose that M is connected. Then every projective map $\phi : (M, \nabla^M) \to (N, \nabla^N)$ of rank ≥ 2 is affine map.*

Proof: We first observe that dimN ≥ 2. Let $\phi : (M, \nabla^M) \to (N, \nabla^N)$ be a projective map. By remark 4.1, ϕ is strongly projective map. By proposition 4.1, there exists a 1-form α on M such that

$$\beta_\phi(U, V) = \alpha(U)\phi_*V + \alpha(V)\phi_*U, \quad U, V \in TM.$$

It is clear that $TM = \mathrm{Ker}\alpha \oplus (\mathrm{Ker}\alpha)^{\perp}$. For every $V \in TM$ we have $\beta_{\phi}(V,V) = \phi_*(\alpha(V)V)$. Therefore $\beta_{\phi}(V,V) \in \phi_*((\mathrm{Ker}\alpha)^{\perp})$. As dimension of $\mathrm{Ker}\alpha$ is $n-1$ we have that dimension of $\phi_*((\mathrm{Ker}\alpha)^{\perp}) \leq 1$. Let θ be a 1-form on $(\phi_*((\mathrm{Ker}_{\mathrm{x}}\alpha)^{\perp})^{\perp})^*$, that is, $\theta(V) = 0$ for $V \in \phi_*((\mathrm{Ker}\alpha)^{\perp})$. Let X be a $(\nabla, \mathbb{P})$-martingale in M. It follows that

$$\int_0^t \beta_{\phi}^*\theta(dX_s, dX_s) = 0.$$

From Proposition 4.2 we see that

$$[\int_0^t \alpha d^{\nabla^M} X_s, \int_0^t \phi^*\theta d^{\nabla^M} dX_s] = 0. \tag{7}$$

Since X is $(\nabla, \mathbb{P})$-martingale, $\int_0^t \alpha d^{\nabla^M} X_s$ and $\int_0^t \phi^*\theta d^{\nabla^M} dX_s$ are $\mathbb{P}$-local martingale. From (7) we deduce that $\int_0^t \alpha d^{\nabla^M} X_s \cdot \int_0^t \phi^*\theta d^{\nabla^M} dX_s$ is $\mathbb{P}$-local martingale. Taking expectation we obtain

$$\mathbb{E}^{\mathbb{P}}(\int_0^t \alpha d^{\nabla^M} X_s \cdot \int_0^t \phi^*\theta d^{\nabla^M} dX_s) = 0.$$

From this we conclude almost surely that

$$\int_0^t \alpha d^{\nabla^M} X_s \cdot \int_0^t \phi^*\theta d^{\nabla^M} dX_s = 0.$$

Since $\int_0^t \phi^*\theta d^{\nabla^M} dX_s$ is arbitrage, $\int_0^t \alpha d^{\nabla^M} X_s = 0$. Because X is arbitrage, we see that $\alpha = 0$. Hence $\beta_{\phi} = 0$. It follows that ϕ is affine map. $\square$

Let M be a Riemannian manifold. Let us denoted by $\widehat{A}(M)$ the group of affine transformation of M and by $\widehat{P}(M)$ the group of projective transformation of M. A direct consequence of Theorem 4.6 is the following.

Corollary 4.7 *If M is a connected Riemannian manifold such that $dimM \geq 2$, then $\widehat{A}(M)$ is equal to $\widehat{P}(M)$.*

Let M, N be Riemannian manifolds and $\phi : M \to N$ an isometric immersion. We observe that immersions have constant rank. We recall that ϕ is geodesic immersion if $\beta_{\phi} = 0$ (see [4] for more details).

Corollary 4.8 *Let M, N be Riemannian manifolds such that $dimM \geq 2$. Let $\phi : (M, g) \to (N, h)$ be an isometric immersion. If ϕ is projective map, then ϕ is geodesic immersion.*

Other consequence of Theorem 4.6 is to give a new proof of Proposition III.4.5 in [11] due to T. Nore.

Corollary 4.9 *Projective map of rank ≥ 2 between euclidian spaces are affine.*

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