Self-similarity and uniqueness of solutions for semilinear reaction-diffusion systems

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Abstract

We study the well-posedness of the initial value problem for a coupled semilinear reaction-diffusion system in Marcinkiewicz spaces $L^{(p_1,\infty)}(\Omega)\times L^{(p_2,\infty)}(\Omega)$. The exponents p_1, p_2 of the initial value space are chosen to allow the existence of self-similar solutions (when $\Omega=\mathbb{R}^n$). As a nontrivial consequence of our coupling-term estimates, we prove the uniqueness of solutions in the scaling invariant class $C([0,\infty); L^{p_1}(\Omega)\times L^{p_2}(\Omega))$ regardless of their size and sign. We also analyze the asymptotic stability of the solutions, show the existence of a basin of attraction for each self-similar solution and that solutions in $L^{p_1}\times L^{p_2}$ present a simple long time behavior.

Keywords: Reaction-diffusion system; Self-similarity; Uniqueness; Marcinkiewicz spaces.
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1 Introduction

In this paper we are interested in the Cauchy problem for the following semilinear parabolic system

$$\begin{cases} u_t - \Delta u = g_1(u,v), & x \in \Omega,\ t>0 \\ v_t - \Delta v = g_2(u,v), & x \in \Omega,\ t>0 \\ u(0,x) = u_0, \quad v(0,x) = v_0 & x \in \Omega \end{cases} \tag{1.1}$$

where

$$g_1(u,v) = |u|^{(\rho_1-1)}u|v|^{(\rho_2-1)}v \text{ and } g_2(u,v) = |u|^{(r_1-1)}u|v|^{(r_2-1)}v, \tag{1.2}$$

$1<\rho_i, r_i<\infty$, $i=1,2$. Here, we consider Ω being either $\mathbb{R}^n$, $\mathbb{R}^n_+$, a bounded domain or an exterior domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$. Also, we assume homogeneous Dirichlet boundary conditions.

The equations (1.1) provide an example of reaction-diffusion system which arise naturally as physical and chemical models. For instance, they can be used to describe heat propagation of two interactive substances and model the behavior of chemical reactions, where the unknown u and v represent the chemical concentrations. For more details, we refer the reader to [9] and the references therein. (1.1) generalizes the well-known semilinear heat equation, which has been studied by several authors (cf. e.g. [5, 7, 23, 14, 22, 13]).

The purpose of this paper is to show existence and uniqueness of global mild solutions, and asymptotic stability results for the Cauchy problem (1.1) in Marcinkiewicz spaces. Mild solutions will be obtained in the time-dependent space $E = BC((0,\infty); L^{(p_1,\infty)} \times L^{(p_2,\infty)})$ with the right homogeneity to allow the existence of self-similar solutions. These solutions may not be radially symmetric, and they correspond, for instance, to homogeneous initial data u_0, v_0 of degree $-k_1, -k_2$, respectively, where k_1, k_2 are determined by scaling of (1.1). It is worth pointing out that such approach in finding self-similar solution was introduced by Giga and Miyakawa [15] in the framework of Morrey spaces for Navier-Stokes equations and posteriorly by several authors in other spaces (cf. [17]). Since L^p contains only trivial homogeneous functions, the advantage in considering $L^{(p,\infty)}$, in view of Chebyshev's inequality, is that it can be regarded as a natural extensions of L^p which contains homogeneous functions of degree $-n/p$.

In our estimates of the coupling terms $g_1(u,v)$ and $g_2(u,v)$ (cf. Lemma 4.3), we do not use Kato-Fujita's approach, which uses two norms to prove the continuity of the nonlinear term, namely the natural norm in E and an "auxiliary norm". So, since $L^p \subset L^{(p,\infty)}$, as an important product of these estimates, we obtain the uniqueness of solutions in the class $C([0,\infty); L^{p_1} \times L^{p_2})$ without smallness assumption and regardless of the sign of solutions (cf. Theorem 3.4).

Moreover, with further work, this time by employing Kato-Fujita's approach, we prove some time-decay properties in the norm $\|\cdot\|_{(q_i,\infty)}$ with $q_i > p_i$, $i=1,2$, and L^p-persistence for the obtained solutions, namely $(u,v) \in BC([0,\infty); L^{p_1} \times L^{p_2})$ provided $(u_0,v_0) \in L^{p_1} \times L^{p_2}$. Also, we stress that our results do not require a sign assumption and they work well for the completely coupled case, i.e., $g_1(u,v) = v\,|v|^{\rho_2-1}$ and $g_2(u,v) = u\,|u|^{r_1-1}$ (cf. Remark 3.6).

On the other hand, using arguments related to those of [6], we discuss the asymptotic stability of solutions and, as a consequence, a criterium for vanishing small perturbations of initial data at large time will be obtained (cf. Theorem 5.1). Applying this result to the particular case in which the initial conditions u_0, v_0 are homogeneous of degree $-k_1, -k_2$, one obtains the existence of a basin of attraction for each self-similar solution. Also, by assuming $(u_0, v_0) \in L^{p_1} \times L^{p_2}$, one shows that the solution (u, v) presents a simple long-time diffusive behavior, i.e., $(u(t), v(t)) \to (0, 0)$ as $t \to \infty$ in $L^{p_1} \times L^{p_2}$ (cf. Remark 5.2) .

Let us finally review some works of the literature concerning the Cauchy problem (1.1). For $\rho_1, r_2 > 1$ and on a bounded domain Ω, Chen [8] found positive bounded classical solution under smallness assumption on $L^\infty-$norm of the initial data. In whole space, for the completely coupled system $\rho_2 = r_1 = 0$, among other results about $L^\infty-$blow-up, Escobedo and Herrero [10, Theorem 3] proved that there exists at most one nonnegative bounded classical solution by assuming initial data in L^∞ with small norm in $L^{p_1} \times L^{p_2}$. Late on, in [11], they showed uniqueness of nonnegative classical solutions in $L^\infty(\mathbb{R}^n)$. In these last two works, the techniques rely in maximum principle, positivity and boundedness. The results of [11] were extended to the system of $n-$ equations by Bokes [4]. By applying invariant solution method and reducing the problem to a system of ODE's, Qi [20] obtained positive radial self-similar solutions for (1.1). In the paper [21], by employing Kato-Fujita's approach and with $\Omega = \mathbb{R}^n$, Snoussi and Tayachi proved existence of small self-similar solutions in the framework of Besov spaces. There, the authors also studied the asymptotic behavior of the solutions for a specific type of initial data: $(u_0, v_0) = \eta(\phi_1, \phi_2)$ where ϕ_i homogeneous of degree $-k_i$ and η is a cut-off function with $\eta \equiv 1$ near $x = 0$.

Comparing to the above works, this manuscript presents, among other things, the following improvements and novelties :

- The existence of self-similar solutions, which may not be radially symmetric, in a new framework.
- The employed approach, without using two norms, enables us to obtain a new class of uniqueness.
- Our results yield an orbit solution in space $L^{(p_1,\infty)} \times L^{(p_2,\infty)}$ (persistence), which lies in $L^{p_1} \times L^{p_2}$ when the initial data is also there.
- New large time behavior results which describe the behavior of the solutions within same initial data class considered. In general, we obtain a basin of attraction for each self-similar solution and, in case of $L^{p_1} \times L^{p_2}$, the ω-limit is precisely a point, namely $\omega(\{(u(t), v(t))\}_{t>0}) = \{(0, 0)\}$.
- The results work well on the four types of domain mentioned above. Besides the unbounded domains $\mathbb{R}^n$ and $\mathbb{R}^n_+$, we point out the exterior domain case.

We conclude the Introduction by describing the plan of this paper. The basic properties of $L^{(p,\infty)}$- spaces and some estimates of the heat semigroup will be reviewed in the next section. In Section 3 we shall state our results of existence and uniqueness which will be proved in Section 4. Finally, in Section 5 we shall analyze the asymptotic stability of solutions. Throughout this paper the letter $C > 0$ will denote generic positive constants which may change from line to line or even within a same line.

2 Preliminaries

Lorentz spaces: We begin the section by recalling some facts about the Lorentz spaces. For a deeper discussion about these spaces the reader is referred to [3, 2] and the references given there.

A measurable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ belongs to Lorentz space $L^{(p,q)}(\Omega)$, for short $L^{(p,q)}$, if the quantity

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{p}{q} \int_0^\infty \left[t^{\frac{1}{p}} f^{**}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{, if} \quad 1 < p < \infty,\ 1 \le q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \text{, if} \quad 1 < p \le \infty,\ q = \infty, \end{cases}$$

is finite, where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\ ds, \quad f^*(t) = \inf\{s > 0 : m\{x \in \Omega : |f(x)| > s\} \le t\}.$$

The space $L^{(p,q)}$ endowed with the norm $\|\cdot\|_{(p,q)}$ is a Banach space and $L^{(p,p)} = L^p$ is the usual Lebesgue space. In case $q = \infty$, $L^{(p,\infty)}$ is called the Marcinkiewicz space or weak-L^p space. While $C_0^\infty(\Omega)$ is dense in $L^{(p,q)}(\Omega)$ with $1 \le q < \infty$, the same one is not verified for $L^{(p,\infty)}(\Omega)$. Also, one has the following continuous inclusions $L^{(p,q_1)} \subset L^p \subset L^{(p,q_2)} \subset L^{(p,\infty)}$ with $1 \le q_1 \le p \le q_2 \le \infty$.

For $1 < p, q < \infty$, the dual space of $L^{(p,q)}$ is $L^{(p',q')}$, and so, these spaces are reflexive. Moreover $L^{(p,\infty)}$ is the dual space of $L^{(p',1)}$ provided $1/p + 1/p' = 1$, and $L^{(p,1)}$ is not the dual space of $L^{(p',\infty)}$.

By real interpolation, the Lorentz spaces can be alternatively constructed as (cf. [3, 2])

$$L^{(p,q)} = (L^1, L^\infty)_{1-\frac{1}{p},q},\ 1 < p < \infty,$$

and the following interpolation property holds:

$$(L^{(p_0,q_0)}, L^{(p_1,q_1)})_{\theta,q} = L^{(p,q)},$$

provided $0 < p_0 < p_1 < \infty,\ 0 < \theta < 1,\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \le q_0, q_1, q \le \infty$.

In the sequel we recall the Holder inequality in the framework of Lorentz spaces (cf. [19])

Proposition 2.1 *(Generalized Holder's inequality). Let $1 < p_1, p_2 < \infty$. Let $f \in L^{(p_1,d_1)}(\Omega)$ and $g \in L^{(p_2,d_2)}(\Omega)$ where $\frac{1}{p_1} + \frac{1}{p_2} < 1$, then the product $h = fg$ belongs to $L^{(r,d_3)}(\Omega)$ where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, and $d_3 \ge 1$ satisfies $\frac{1}{d_1} + \frac{1}{d_2} \ge \frac{1}{d_3}$. Moreover,*

$$\|h\|_{(r,d_3)} \le C(r) \|f\|_{(p_1,d_1)} \|g\|_{(p_2,d_2)}. \tag{2.1}$$

Heat semigroup: One defines the heat semigroup $\{G(t)\}_{t\ge 0}$ as the family of convolution operators with corresponding kernels $g(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$, that is $G(t)f = g(t,\cdot) * f$.

In the following lemma we recall two estimates for $\{G(t)\}_{t\ge 0}$ on Lorentz spaces. The first one is the well known smoothing effect of the heat semigroup in Lorentz spaces and the second one is Yamazaki's estimate (cf. [24]).

Lemma 2.2

- *(Smoothing effect) Let* $1 < p \leq q < \infty$ *and* $1 \leq d \leq \infty$*. There exists a constant* $C > 0$ *such that*
$$\sup_{t>0} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|G(t)\phi\|_{(q,d)} \leq C\|\phi\|_{(p,d)} \tag{2.2}$$
for all $\phi \in L^{(p,d)}(\Omega)$.

- *(Yamazaki's estimate) Let* $1 < p < q < \infty$*. There exists a constant* $C > 0$ *such that,*
$$\int_0^\infty t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\ \frac{1}{2}}\|G(t)\phi\|_{(q,1)}\, ds \leq C\|\phi\|_{(p,1)} \tag{2.3}$$
for all $\phi \in L^{(p,1)}(\Omega)$.

3 Results

The purpose of this section is to state the results of well-posedness for the Cauchy problem (1.1) in the space $L^{(p_1,\infty)} \times L^{(p_2,\infty)}$. In order to find suitable functional spaces to study (1.1), we start by performing a scaling analysis. Just for a moment, let us take $\Omega = \mathbb{R}^n$. Assuming that the pair (u, v) is a classical solutions of (1.1), we look for the values of k_1 and k_2, such that the pair (u_λ, v_λ) given by
$$u_\lambda(t, x) = \lambda^{k_1}u(\lambda^2 t, \lambda x) \text{ and } v_\lambda(t, x) = \lambda^{k_2}v(\lambda^2 t, \lambda x)$$
is also a solution of (1.1). To this end, one substitutes $u_\lambda(t, x)$ and $v_\lambda(t, x)$ in (1.1) to obtain
$$\lambda^{k_1+2}u_t(\lambda^2 t, \lambda x) - \lambda^{k_1+2}\Delta u(\lambda^2 t, \lambda x) = \lambda^{k_1\rho_1+k_2\rho_2}|u|^{\rho_1-1}u(\lambda^2 t, \lambda x)|v|^{\rho_2-1}v(\lambda^2 t, \lambda x) \tag{3.1}$$
$$\lambda^{k_2+2}v_t(\lambda^2 t, \lambda x) - \lambda^{k_2+2}\Delta v(\lambda^2 t, \lambda x) = \lambda^{k_1 r_1+k_2 r_2}|u|^{r_1-1}u(\lambda^2 t, \lambda x)|v|^{r_2-1}v(\lambda^2 t, \lambda x), \tag{3.2}$$
for all $\lambda > 0$, $t > 0$ and $x \in \mathbb{R}^n$. It is easy to check that the equalities (3.1) and (3.2) yield
$$(\rho_1 - 1)k_1 + \rho_2 k_2 = 2 \text{ and } r_1 k_1 + (r_2 - 1)k_2 = 2, \tag{3.3}$$
and so a simple computation gives us
$$k_1 = \frac{2(\rho_2 - r_2 + 1)}{r_1\rho_2 - (\rho_1 - 1)(r_2 - 1)} \quad\text{and}\quad k_2 = \frac{2(r_1 - \rho_1 + 1)}{r_1\rho_2 - (\rho_1 - 1)(r_2 - 1)} \tag{3.4}$$
provided
$$r_1\rho_2 - (\rho_1 - 1)(r_2 - 1) \neq 0. \tag{3.5}$$
The map $(u, v) \to (u_\lambda, v_\lambda)$ is named scaling transformation of (1.1). At this point a question arises naturally: Are there scaling-invariant solutions of (1.1), in other words, solutions satisfying
$$(u, v)(t, x) = (u_\lambda, v_\lambda)(t, x), \tag{3.6}$$

for all $t > 0$, $x \in \mathbb{R}^n$ and $\lambda > 0$? These type of solutions are called self-similar solutions of the problem (1.1). At least formally, taking $t \to 0^+$ in (3.6), $u_0 = u(0,x)$ and $v_0 = v(0,x)$ should be homogeneous functions of degrees $-k_1$ and $-k_2$, respectively. This remark suggests that suitable initial-condition space to find self-similar solutions should be one containing homogeneous functions with that exponent. For instance,

$$(u_0, v_0) \in L^{(p_1,\infty)} \times L^{(p_2,\infty)} \text{ with } p_i = \frac{n}{k_i} \text{ and } k_i \text{ given by (3.4).} \tag{3.7}$$

On the other hand, since we are interested in self-similar solutions, we shall study the existence of solutions in time-dependent spaces, in which the norm is invariant to the scaling of (1.1). In the next definition we denote by BC the class of bounded and continuous functions from the corresponding interval onto a Banach space.

Definition 3.1 *Let k_i be given by (3.4), $p_i = \frac{n}{k_i} > 1$, $1 < q_i \leq \infty$ and $\alpha_i = \frac{n}{2}(\frac{1}{p_i} - \frac{1}{q_i})$ with $i = 1, 2$. We define the following Banach spaces*

$$E \equiv BC((0,\infty), L^{(p_1,\infty)} \times L^{(p_2,\infty)}\}$$

$$E_{q_1q_2} \equiv \{(u,v) \in E : (t^{\alpha_1}u, t^{\alpha_2}v) \in BC((0,\infty); L^{(q_1,\infty)} \times L^{(q_2,\infty)})\},$$

with respective norms given by

$$\|(u,v)\|_E = \max\{\sup_{t>0} \|u\|_{(p_1,\infty)}, \sup_{t>0} \|v\|_{(p_2,\infty)}\} \tag{3.8}$$

$$\|(u,v)\|_{E_{q_1q_2}} = \|(u,v)\|_E + \max\{\sup_{t>0} t^{\alpha_1} \|u(t)\|_{(q_1,\infty)}, \sup_{t>0} t^{\alpha_2} \|v(t)\|_{(q_2,\infty)}\} \tag{3.9}$$

Next, according to Duhamel's principle, we introduce the notion of solution for the initial value problem (1.1).

Definition 3.2 *A global mild solution of the initial value problem (1.1) in E is a pair $\omega = (u(t), v(t))$ satisfying*

$$(u(t), v(t)) = (G(t)u_0, G(t)v_0) + B(u,v)(t), \tag{3.10}$$

where

$$B(u,v)(t) = \left(\int_0^t G(t-s)|u|^{\rho_1 - 1}u|v|^{\rho_2 - 1}v ds\ ,\ \int_0^t G(t-s)|u|^{r_1-1}u|v|^{r_2-1}v ds\right).$$

In what follows, we state our well-posedness result of mild solutions for (1.1). Before proceeding we recall that p_i is given by (3.7).

Theorem 3.3 *Let $n \geq 3$, $1 < r_i, \rho_i < p_i < \infty$ and $p_i \geq \frac{n}{n-2}$, $i = 1, 2$. Assume that $(u_0, v_0) \in L^{(p_1,\infty)} \times L^{(p_2,\infty)}$.*

(i) *(Well-posedness) There exist $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ such that if $\|u_0\|_{(p_1,\infty)} < \delta, \|v_0\|_{(p_2,\infty)} < \delta$, then the initial value problem (1.1) has a global mild solution $(u(t,x), v(t,x)) \in E$, with initial data (u_0, v_0), which is the unique one satisfying $\|(u,v)\|_E \leq 2\varepsilon$*

(ii) *(Decay) Let $p_i < q_i \leq \infty$, $i = 1, 2$, satisfying $\frac{1}{p_1} < \frac{\rho_1}{q_1} + \frac{\rho_2}{q_2}$ and $\frac{1}{p_2} < \frac{r_1}{q_1} + \frac{r_2}{q_2}$. There exists $0 < \delta_{q_1,q_2} \leq \delta$ such that if $\|u_0\|_{(p_1,\infty)}, \|v_0\|_{(p_2,\infty)} < \delta_{q_1,q_2}$, then the previous solution $(u(t,x), v(t,x))$ belongs to $E_{q_1 q_2}$.*

(iii) *(L^p−persistence) If $(u_0, v_0) \in L^{p_1} \times L^{p_2}$, with $\|u_0\|_{L^{p_1}}$ and $\|v_0\|_{L^{p_2}}$ small enough, then the solutions obtained through item (i) belongs to $BC((0,\infty); L^{p_1} \times L^{p_2})$.*

In view of the continuous inclusion $L^p \subset L^{(p,\infty)}$, Theorem 3.3 supplies uniqueness of solutions in the class $BC([0,\infty); L^{p_1} \times L^{p_2})$, under smallness assumptions. However, adapting an argument found in [17], one can remove the smallness conditions. More precisely, one has the following statement:

Theorem 3.4 *(Uniqueness) Assume $n \geq 3$, $1 < r_i, \rho_i < p_i < \infty$ and $p_i \geq \frac{n}{n-2}$, $i = 1, 2$. Let (u, v) and $(\widetilde{u}, \widetilde{v})$ be two mild solutions of (1.1) in the class $C([0,\infty); L^{p_1} \times L^{p_2})$ with initial data $(u_0, v_0) \in L^{p_1} \times L^{p_2}$. Then $u = \widetilde{u}$ and $v = \widetilde{v}$.*

As pointed out in the Introduction, since Marcinkiewicz spaces contain homogeneous functions, a consequence of Theorem 3.3 is the existence of self-similar solutions. This is the content of the next corollary.

Corollary 3.5 *Let $\Omega = \mathbb{R}^n$, k_1 and k_2 be given by (3.4) and $(u_0, v_0) \in L^{(p_1,\infty)} \times L^{(p_2,\infty)}$. Assume that u_0 and v_0 are homogeneous functions of degrees $-k_1$ and $-k_2$, respectively. If $\|u_0\|_{(p_1,\infty)} < \delta, \|v_0\|_{(p_2,\infty)} < \delta$, then the solution $(u(t,x), v(t,x))$ provided by Theorem 3.3 is self-similar, i.e.,*

$$(u(t,x), v(t,x)) = (\lambda^{k_1} u(\lambda^2 t, \lambda x), \lambda^{k_2} v(\lambda^2 t, \lambda x))$$

almost everywhere $x \in \mathbb{R}^n$, $t > 0$ and all $\lambda > 0$.

We finish this section by making further remarks which complement our results.

Remark 3.6

- *(Convergence towards initial data) Leaving the details to the reader, we comment in what sense the initial condition is taken. Adapting arguments found in [1, 24] we can prove that the solution $(u(t), v(t)) \rightharpoonup (u_0, v_0)$ as $t \to 0^+$ in weak-star topology of $L^{(p_1,\infty)} \times L^{(p_2,\infty)}$. This behavior of the solution is expected, and no more, because the heat semigroup $\{G(t)\}_{t \geq 0}$ presents the same one. Indeed, letting $f \in L^{(p,\infty)}$ and $\varphi \in L^{(p',1)}$, one has*

$$\begin{aligned} |\langle G(t)f - f, \varphi \rangle| &= |\langle f, G(t)\varphi - \varphi \rangle| \\ &\leq \|f\|_{(p,\infty)} \|G(t)f - f\|_{(p',1)} \to 0, \text{ as } t \to 0^+ \end{aligned}$$

On the other hand, assuming $(u_0, v_0) \in L^{p_1} \times L^{p_2}$ *and proceeding closely to [16, pp. 5], one can show* $(u(t), v(t)) \to (u_0, v_0)$ *in topology of the norm of* $L^{p_1} \times L^{p_2}$*. So, the solution obtained through item (iii) is actually continuous at* $t = 0^+$*.*

- *Observe that the conditions of Theorem 3.3 are not empty. For example, taking* $r_2 - 1 \cong \rho_2$ *and* $\rho_1 - 1 \cong r_1$*, with* $r_2 - 1 < \rho_2$ *and* $\rho_1 - 1 < r_1$*, one obtains* $0 < k_i < n$, $i = 1, 2$*. Moreover, by continuity, one can choose* $q_i \cong p_i$ *so that* $\frac{1}{p_1} < \frac{\rho_1}{q_1} + \frac{\rho_2}{q_2}$ *and* $\frac{1}{p_2} < \frac{r_1}{q_1} + \frac{r_2}{q_2}$*.*

- *In Theorem 2.1, we can deal with the case* $n = 2$ *by assuming*

$$\frac{1}{p_1} < 1 - \frac{\rho_1 - 1}{q_1} - \frac{\rho_2}{q_2} \text{ and } \frac{1}{p_2} < 1 - \frac{r_1 - 1}{q_1} - \frac{r_2}{q_2}$$

instead of $p_i > \frac{n}{n-2}$, $i = 1, 2$*. This latest condition comes from estimates (4.5)-(4.6) where, in duality argument, one needs* $1 < p_i' < \frac{n}{2}$*; Therefore, to include the case* $n = 2$*, one should avoid the duality argument and perform the estimates by using the auxiliary norms* $\sup_{t>0} t^{\alpha_i} \|\cdot\|_{(q_i,\infty)}$ *of* $E_{q_1 q_2}, i = 1, 2$*. However, in this case, we could not prove the uniqueness of solutions in* $C([0, \infty); L^{p_1} \times L^{p_2})$ *because we should lose the bounds (4.5)-(4.6) which are essential to showing Theorem 3.4 (cf. estimate (4.27)).*

- *(Other coupling terms) Above resuls still hold if we consider more general couple terms* g_i *satisfying*

$$|g_1(u, v) - g_1(\widetilde{u}, v)| \leq C\, |u - \widetilde{u}|\, (|u|^{(\rho_1 - 1)} + |\widetilde{u}|^{(\rho_1 - 1)}) |v|^{\rho_2},$$
$$|g_1(u, v) - g_1(u, \widetilde{v})| \leq C\, |v - \widetilde{v}|\, (|v|^{(\rho_2 - 1)} + |\widetilde{v}|^{(\rho_2 - 1)}) |u|^{\rho_1},$$

and the same inequalities for g_2*, replacing* ρ_i *by* r_i, $i = 1, 2$*. Observe that* g_i *given by (1.2) satisfies these conditions and the need of them arises in deriving (4.8) below.*

Moreover, by taking $\rho_1 = r_2 = 0$ *and removing the superfluous conditions* $\rho_1, r_2 > 1$ *in the statement, all our results work well for the completely coupled system, i.e.* $g_1(u, v) = v\,|v|^{\rho_2 - 1}$ *and* $g_2(u, v) = u\,|u|^{r_1 - 1}$*, with a slight adaptation of the proofs. For instance, in place of estimates (4.5) and (4.12), we would have (4.10) (dual version of the needed estimate) and (4.18)-(4.19) with* $\rho_1 = 0$*, respectively. In this case, by (3.4) the scaling exponents would be* $k_1 = 2(\rho_2 + 1)/(\rho_2 r_1 - 1)$ *and* $k_2 = 2(r_1 + 1)/(\rho_2 r_1 - 1)$*.*

- *(Positive solutions) Let* u_0, v_0 *be non-zero functions with* $u_0, v_0 \geq 0$ *. It is a simple matter to check that the elements* (u_m, v_m) *of the interactive sequence (4.24) are positives. Then, as* $(u_m, v_m) \to (u, v)$ *in* E*, the solution satisfies* $u(t, x), v(t, x) > 0$ *a.e* $x \in \Omega$ *and* $t > 0$*.*

- *(Local theory) In proof of Theorem 3.4, indeed we have proved the uniqueness of solutions in the class* $C([0, T); L^{p_1} \times L^{p_2})$ *with* $0 < T \leq \infty$*.*

  Assuming that

$$\limsup_{t \to 0^+} t^{\alpha_1} \|G(t) u_0\|_{(q_1,\infty)} \text{ and } \limsup_{t \to 0^+} t^{\alpha_2} \|G(t) v_0\|_{(q_2,\infty)} \tag{3.11}$$

are small enough, one can prove a local-in-time version of Teorema 3.3 by replacing the smallness of the initial data by the smallness of the existence time. In case $(u_0, v_0) \in L^{p_1} \times L^{p_2}$*, the condition (3.11) holds (cf. [16, pp. 4]) because*

$$\limsup_{t\to 0^+} t^{\alpha_1}\|G(t)u_0\|_{L^{q_1}} = 0 \text{ and } \limsup_{t\to 0^+} t^{\alpha_2}\|G(t)v_0\|_{L^{q_2}} = 0.$$

Also, if $(u_0, v_0) \in L^{(b_1,\infty)} \times L^{(b_2,\infty)}$ *with* $b_i > p_i$ $i = 1, 2$*, then one can solve the initial value problem (1.1) in* $BC((0,T); L^{(b_1,\infty)} \times L^{(b_2,\infty)})$ *for some value of* $T > 0$ *small enough. The proof, in this case, is simpler. Applying the semigroup estimate (2.2) and Holder's inequality, we can estimate directly the norm of nonlinear coupling term in* $BC((0,T); L^{(b_1,\infty)} \times L^{(b_2,\infty)})$*, without using duality and Yamazaki's estimate (2.3) or the approach with two norms.*

4 Proofs

In this section we shall develop the proofs of the results stated in Section 2.2. We start with a lemma about existence of solutions for an abstract equation, which will be useful to our ends.

Lemma 4.1 *Let* $1 < \rho_1, \rho_2, r_1, r_2 < \infty$ *and* X *be a Banach space with norm* $\|\cdot\|$*, and* $B : X \to X$ *be a map satisfying* $B(0) = 0$ *and*

$$\|B(x) - B(z)\| \le K\|x - z\| \left[\left(\|x\|^{\rho_1 - 1} + \|z\|^{\rho_1 - 1}\right)\|x\|^{\rho_2} + \left(\|x\|^{\rho_2 - 1} + \|z\|^{\rho_2 - 1}\right)\|z\|^{\rho_1} + \right.$$
$$\left. + \left(\|x\|^{r_1 - 1} + \|z\|^{r_1 - 1}\right)\|x\|^{r_2} + \left(\|x\|^{r_2 - 1} + \|z\|^{r_2 - 1}\right)\|z\|^{r_1} \right] \quad (4.1)$$

Let $\varepsilon > 0$ *satisfy*

$$2^{\rho_1+\rho_2+1}\varepsilon^{\rho_1+\rho_2-1} + 2^{r_1+r_2+1}\varepsilon^{r_1+r_2-1} < \frac{1}{K}.$$

If $y \in X$ *is such that* $\|y\| \le \varepsilon$*, then there exists a solution* $x \in X$ *for the equation* $x = y + B(x)$ *satisfying* $\|x\| \le 2\varepsilon$*. The solution* x *is unique in the closed ball* $\overline{B}(0, 2\varepsilon)$*. Moreover, the solution depends continuously on* y *in the following sense: If* $\|\tilde{y}\| \le \varepsilon$*,* $\tilde{x} = \tilde{y} + B(\tilde{x})$*, and* $\|\tilde{x}\| \le 2\varepsilon$*, then*

$$\|x - \tilde{x}\| \le \frac{1}{1 - (2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1} + 2^{r_1+r_2+1}K\varepsilon^{r_1+r_2-1})}\|y - \tilde{y}\|. \quad (4.2)$$

Proof. The proof relies in the Banach fixed point theorem. For each $y \in X$, let us define a map $F_y : X \to X$ so that $F_y(x) = y + B(x)$. We claim that F is a contraction on $\overline{B}(0, 2\varepsilon)$ when $\|y\| \le \varepsilon$. Thus the Banach contraction principle implies the existence and uniqueness of solution in $\overline{B}(0, 2\varepsilon)$. In the following we prove the claim. Since

$$2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1} + 2^{r_1+r_2+1}K\varepsilon^{r_1+r_2-1} - 1 < 0, \quad (4.3)$$

we apply (4.1) with $x = x$ and $z = 0$ to estimate

$$\|F(x)\| \le \|y\| + \|B(x)\| \le \|y\| + K\left(\|x\|^{\rho_1+\rho_2} + \|x\|^{r_1+r_2}\right)$$
$$\le \varepsilon + 2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2} + 2^{r_1+r_2+1}K\varepsilon^{r_1+r_2} < \varepsilon + \varepsilon = 2\varepsilon$$

and so, one obtains that $F(\overline{B}(0,2\varepsilon))\subset\overline{B}(0,2\varepsilon)$. Futhermore, taking $x,z\in\overline{B}(0,2\varepsilon)$, we get

$$\begin{aligned}\|F(x)-F(z)\| \le \|B(x)-B(z)\| &\le K\|x-z\|[\left(\|x\|^{\rho_1-1}+\|z\|^{\rho_1-1}\right)\|x\|^{\rho_2}\\ &+\left(\|x\|^{\rho_2-1}+\|z\|^{\rho_2-1}\right)\|y\|^{\rho_1}+\left(\|x\|^{r_1-1}+\|z\|^{r_1-1}\right)\|x\|^{r_2}\\ &+\left(\|x\|^{r_2-1}+\|z\|^{r_2-1}\right)\|y\|^{r_1}]\\ &\le \left(2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1}+2^{r_1+r_2+1}K\varepsilon^{r_1+r_2-1}\right)\|x-z\|\end{aligned}$$

which, together with (4.3), give us the desired claim.

In order to conclude the proof it remains to prove (4.2). To this end, we take $\tilde{x}$ and x as in the statement of the lemma and estimate

$$\begin{aligned}\|x-\tilde{x}\|\le\|y-\tilde{y}\|+\|B(x)-B(\tilde{x})\|\le\|y-\tilde{y}\|+(2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1}+\\ +2^{r_1+r_2+1}K\varepsilon^{r_1+r_2-1})\|x-\tilde{x}\|,\end{aligned}$$

which implies the inequality (4.2). ■

Remark 4.2 *It is useful to recall that the solution obtained through Lemma (4.1) is the limit in* X *of the following Picard sequence* $\{x_n\}_{n\in\mathbb{N}}$,

$$x_1=y \quad \text{and} \quad x_{n+1}=F(x_n),\ n\in\mathbb{N}.$$

Nonlinear estimates

In this part of manuscript we will derive the needed estimates of the nonlinear term $B(u.v)$ in the spaces E and $E_{q_1q_2}$. These will be used to show that B satisfies the property (4.1) with $X=E$ and $X=E_{q_1q_2}$

In order to perform the estimates in space E we consider the following nonlinear operator

$$F(h_1,h_2)(x)=(F_1(h_1,h_2),F_2(h_1,h_2))$$

where

$$\begin{aligned}F_1(h_1,h_2)&=\int_0^\infty G(s)(|h_1|^{\rho_1-1}h_1|h_2|^{\rho_2-1}h_2)(s)ds\\ F_2(h_1,h_2)&=\int_0^\infty G(s)(|h_1|^{r_1-1}h_1|h_2|^{r_2-1}h_2)(s)ds,\end{aligned}$$

and we recall that $\{G(t)\}_{t>0}$ denotes the heat semigroup.

Before proceeding to the next lemma, let us remind the reader an useful real number inequality: If $r>1$, then

$$\left|b\,|b|^{r-1}-a\,|a|^{r-1}\right|\le C\,|b-a|\,(|b|^{r-1}+|a|^{r-1}), \tag{4.4}$$

for all $a,b\in\mathbb{R}$ with the constant $C>0$ being independent on a,b.

Lemma 4.3 *Let $n \geq 3$, p_i be given by (3.7), $1 < \rho_i, r_i < p_i$ with $p_i > \frac{n}{n-2}$, $i = 1, 2$. There exist positive constants K_1 and K_2 such that*

$$\|F_1(h_1,h_2) - F_1(\widetilde{h}_1,\widetilde{h}_2)\|_{(p_1,\infty)} \leq K_1 \left[\sup_{t>0} \|(h_1 - \widetilde{h}_1)\|_{(p_1,\infty)} \sup_{t>0} (\|h_1\|^{\rho_1-1}_{(p_1,\infty)} + \|\widetilde{h}_1\|^{\rho_1-1}_{(p_1,\infty)}) \sup_{t>0} \|h_2\|^{\rho_2}_{(p_2,\infty)} + \right.$$
$$\left. + \sup_{t>0} \|(h_2 - \widetilde{h}_2)\|_{(p_2,\infty)} \sup_{t>0} (\|h_2\|^{\rho_2-1}_{(p_2,\infty)} + \|\widetilde{h}_2\|^{\rho_2-1}_{(p_2,\infty)}) \sup_{t>0} \|\widetilde{h}_1\|^{\rho_1}_{(p_1,\infty)} \right], \tag{4.5}$$

$$\|F_2(h_1,h_2) - F_2(\widetilde{h}_1,\widetilde{h}_2)\|_{(p_2,\infty)} \leq K_2 \left[\sup_{t>0} \|(h_2 - \widetilde{h}_2)\|_{(p_2,\infty)} \sup_{t>0} (\|h_2\|^{r_2-1}_{(p_2,\infty)} + \|\widetilde{h}_2\|^{r_2-1}_{(p_2,\infty)}) \sup_{t>0} \|h_1\|^{r_1}_{(p_1,\infty)} + \right.$$
$$\left. + \sup_{t>0} \|(h_1 - \widetilde{h}_1)\|_{(p_1,\infty)} \sup_{t>0} (\|h_1\|^{r_1-1}_{(p_1,\infty)} + \|\widetilde{h}_1\|^{r_1-1}_{(p_1,\infty)}) \sup_{t>0} \|\widetilde{h}_2\|^{r_2}_{(p_2,\infty)} \right], \tag{4.6}$$

for all measurable pairs (h_1, h_2) and $(\widetilde{h}_1, \widetilde{h}_2)$.

Proof.- We only will prove the inequality (4.5) since the proof of (4.6) is entirely parallel by substituting F_1, p_1, ρ_1 and ρ_2 by F_2, p_2, r_1 and r_2, respectively. Let $\tau_1 > 1$ such that $\frac{1}{\tau_1} = \frac{\rho_1}{p_1} + \frac{\rho_2}{p_2}$. By definition of p_i (3.7) and the relation between k_i's (3.3), note that

$$\frac{n}{2}(\frac{1}{p_1'} - \frac{1}{\tau_1'}) - 1 = \frac{n}{2}(\frac{\rho_2}{p_2} + \frac{\rho_1 - 1}{p_1}) - 1 = \frac{1}{2}(\rho_2 k_2 + (\rho_1 - 1)k_1) - 1 = 0. \tag{4.7}$$

In the following, one takes $\phi \in L^{(p_1',1)}$. After recalling that $G(s) = e^{s\Delta}$ with Δ self-adjoint, we use the bound (4.4) and Hölder's inequality to estimate

$$\begin{aligned} \left| < F_1(h_1,h_2) - F_1(\widetilde{h}_1,h_2), \phi > \right| &= \left| < \int_0^\infty G(s)((|h_1|^{\rho_1-1}h_1 - |\widetilde{h}_1|^{\rho_1-1}\widetilde{h}_1)|h_2|^{\rho_2-1}h_2)ds, \phi > \right| \\ &= \left| < \int_0^\infty (|h_1|^{\rho_1-1}h_1 - |\widetilde{h}_1|^{\rho_1-1}\widetilde{h}_1)|h_2|^{\rho_2-1}h_2 \, ds, G(s)\phi > \right| \\ &\leq C \int_0^\infty \|(h_1 - \widetilde{h}_1)(|h_1|^{\rho_1-1} + |\widetilde{h}_1|^{\rho_1-1})|h_2|^{\rho_2}\|_{(\tau_1,\infty)} \|(G(s)\phi)\|_{(\tau_1',1)} ds \end{aligned} \tag{4.8}$$

Applying again Hölder's inequality and Yamazaki's estimate (2.3) with $q = \tau_1'$ and $p = p_1'$, one bounds the right hand side of (4.8) by

$$\begin{aligned} &\leq C \sup_{t>0} \|(h_1 - \widetilde{h}_1)\|_{(p_1,\infty)} \sup_{t>0} (\|h_1\|^{\rho_1-1}_{(p_1,\infty)} + \|\widetilde{h}_1\|^{\rho_1-1}_{(p_1,\infty)}) \sup_{t>0} \|h_2\|^{\rho_2}_{(p_2,\infty)} \times \int_0^\infty \|(G(s)\phi)\|_{(\tau_1',1)} ds \\ &\leq K_1 \sup_{t>0} \|(h_1 - \widetilde{h}_1)\|_{(p_1,\infty)} \sup_{t>0} (\|h_1\|^{\rho_1-1}_{(p_1,\infty)} + \|\widetilde{h}_1\|^{\rho_1-1}_{(p_1,\infty)}) \sup_{t>0} \|h_2\|^{\rho_2}_{(p_2,\infty)} \|\phi\|_{(p_1',1)}. \end{aligned} \tag{4.9}$$

Also, just like the proof of (4.9), we can bound

$$\left| < F_1(\widetilde{h}_1, h_2) - F_1(\widetilde{h}_1, \widetilde{h}_2), \phi > \right| \leq K_1 \sup_{t>0} \|(h_2 - \widetilde{h}_2)\|_{(p_2,\infty)} \sup_{t>0} (\|h_2\|^{\rho_2 - 1}_{(p_2,\infty)} + \|\widetilde{h}_2\|^{\rho_2-1}_{(p_2,\infty)}) \times \\ \times \sup_{t>0} \|\widetilde{h}_1\|^{\rho_1}_{(p_1,\infty)} \|\phi\|_{(p_1',1)}. \quad (4.10)$$

Finally, collecting the last two inequalities, one gets

$$\left| < F_1(h_1, h_2) - F_1(\widetilde{h}_1, \widetilde{h}_2), \phi > \right| \leq \left| < F_1(h_1, h_2) - F_1(\widetilde{h}_1, h_2), \phi > \right| + \left| < F_1(\widetilde{h}_1, h_2) - F_1(\widetilde{h}_1, \widetilde{h}_2), \phi > \right|$$

$$\leq K_1 \left[\sup_{t>0} \|(h_1 - \widetilde{h}_1)\|_{(p_1,\infty)} \sup_{t>0} (\|h_1\|^{\rho_1-1}_{(p_1,\infty)} + \|\widetilde{h}_1\|^{\rho_1-1}_{(p_1,\infty)}) \sup_{t>0} \|h_2\|^{\rho_2}_{(p_2,\infty)} + \right. \\ \left. + \sup_{t>0} \|(h_2 - \widetilde{h}_2)\|_{(p_2,\infty)} \sup_{t>0} (\|h_2\|^{\rho_2-1}_{(p_2,\infty)} + \|\widetilde{h}_2\|^{\rho_2-1}_{(p_2,\infty)}) \sup_{t>0} \|\widetilde{h}_1\|^{\rho_1}_{(p_1,\infty)} \right] \|\phi\|_{(p_1',1)}$$

which, by duality $L^{(p_1,\infty)} = (L^{(p_1',1)})^*$, yields the estimate (4.5).

■

In the next lemma we deal with the estimates of $B(u, v)$ in the time-decay part of the norm $\|(u, v)\|_{E_{q_1 q_2}}$, namely

$$\|(u, v)\|_{q_1 - q_2} = \max\{\sup_{t>0} t^{\alpha_1} \|u(t)\|_{(q_1,\infty)}, \sup_{t>0} t^{\alpha_2} \|v(t)\|_{(q_2,\infty)}\}. \quad (4.11)$$

Below, one also uses the convention

$$\|(u, v)\|_{l_1 - l_2} = \max\{\sup_{t>0} t^{\beta_1} \|u(t)\|_{(l_1,\infty)}, \sup_{t>0} t^{\beta_2} \|v(t)\|_{(l_2,\infty)}\},$$

where $\beta_1 = \frac{n}{2}(\frac{1}{p_1} - \frac{1}{l_1})$ and $\beta_2 = \frac{n}{2}(\frac{1}{p_2} - \frac{1}{l_2})$. In case $l_i = q_i$, $\|(u, v)\|_{l_1 - l_2} = \|(u, v)\|_{q_1 - q_2}$ and $\|(u, v)\|_{p_1 - p_2} = \max\{\sup_{t>0} \|u(t)\|_{(p_1,\infty)}, \sup_{t>0} \|v(t)\|_{(p_2,\infty)}\}$.

Lemma 4.4 *Let p_i be given by (3.7), $p_i < q_i \leq \infty$ and $p_i \leq l_i \leq \infty$, $i = 1, 2$, satisfying $\frac{1}{p_1} < \frac{1}{l_1} + \frac{\rho_1 - 1}{q_1} + \frac{\rho_2}{q_2}$ and $\frac{1}{p_2} < \frac{1}{l_2} + \frac{r_1}{q_1} + \frac{r_2 - 1}{q_2}$. There exists a positive constant $K_{q_1 q_2}$ such that*

$$\sup_{t>0} t^{\beta_1} \|B_1(u, v) - B_1(\widetilde{u}, \widetilde{v})\|_{(l_1,\infty)} \leq K_{q_1 q_2} \|(u - \widetilde{u}, v - \widetilde{v})\|_{l_1 - l_2} \times \\ \times \left[(\|(u, v)\|^{\rho_1 - 1}_{q_1 - q_2} + \|(\widetilde{u}, \widetilde{v})\|^{\rho_1 - 1}_{q_1 - q_2}) \|(u, v)\|^{\rho_2}_{q_1 - q_2} + (\|(u, v)\|^{\rho_2 - 1}_{q_1 - q_2} + \|(\widetilde{u}, \widetilde{v})\|^{\rho_2 - 1}_{q_1 - q_2}) \|(\widetilde{u}, \widetilde{v})\|^{\rho_1}_{q_1 - q_2} \right], \quad (4.12)$$

$$\sup_{t>0} t^{\beta_2}\|B_2(u,v)-B_2(\widetilde{u},\widetilde{v})\|_{(l_2,\infty)} \leq K_{q_1q_2}\left\|(u-\widetilde{u},v-\widetilde{v})\right\|_{l_1-l_2} \times$$
$$\times\Big[(\|(u,v)\|^{r_1-1}_{q_1-q_2}+\|(\widetilde{u},\widetilde{v})\|^{r_1-1}_{q_1-q_2})\,\|(u,v)\|^{r_2}_{q_1-q_2}+(\|(u,v)\|^{r_2-1}_{q_1-q_2}+\|(\widetilde{u},\widetilde{v})\|^{r_2-1}_{q_1-q_2})\,\|(\widetilde{u},\widetilde{v})\|^{r_1}_{q_1-q_2}\Big], \tag{4.13}$$

for all measurable pairs (u,v) *and* $(\widetilde{u},\widetilde{v})$.

Proof. Likewise the proof of the previous lemma, we only show (4.12) and skip the proof of (4.13) which follows in an analogous way. We start by writting

$$\begin{aligned}\|B_1(u,v)-B_1(\widetilde{u},\widetilde{v})\|_{(q_1,\infty)} &= \|B_1(u,v)-B_1(\widetilde{u},v)+B_1(\widetilde{u},v)-B_1(\widetilde{u},\widetilde{v})\|_{(q_1,\infty)}\\ &\leq \|B_1(u,v)-B_1(\widetilde{u},v)\|_{(q_1,\infty)}+\|B_1(\widetilde{u},v)-B_1(\widetilde{u},\widetilde{v})\|_{(q_1,\infty)}\end{aligned} \tag{4.14}$$

Firstly we deal with the first term in the right hand side of (4.14). Take $\widetilde{r}>1$ such that $\frac{1}{\widetilde{r}}=\frac{1}{l_1}+\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2}>\frac{1}{p_1}>\frac{1}{l_1}$. Applying (2.2) with $q=l_1$, $p=\widetilde{r}$, $d=\infty$, the bound (4.4) and afterwards Hölder´s inequality (2.1), one obtains

$$\begin{aligned}\|B_1(u,v)-B_1(\widetilde{u},v)\|_{(l_1,\infty)} &= \int_0^t \|G(t-s)(|u|^{\rho_1-1}u-|\widetilde{u}|^{\rho_1-1}\widetilde{u})|v|^{\rho_2-1}v\|_{(l_1,\infty)}ds\\ &\leq C\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{\widetilde{r}}-\frac{1}{l_1})}\|(|u|^{\rho_1-1}u-|\widetilde{u}|^{\rho_1-1}\widetilde{u})|v|^{\rho_2-1}v\|_{(\widetilde{r},\infty)}ds\\ &\leq C\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{\widetilde{r}}-\frac{1}{l_1})}\|(u-\widetilde{u})\|_{(l_1,\infty)}\ (\|u\|^{\rho_1-1}_{(q_1,\infty)}+\|\widetilde{u}\|^{\rho_1-1}_{(q_1,\infty)})\|v\|^{\rho_2}_{(q_2,\infty)}ds.\end{aligned} \tag{4.15}$$

Now multiplying and dividing by $s^{\beta_1+\alpha_1(\rho_1-1)+\alpha_2\rho_2}$ inside the integral and taking the supremum over $s>0$, one bounds (4.15) by

$$\begin{aligned}&= C\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{\widetilde{r}}-\frac{1}{q_1})}s^{\beta_1}\|(u-\widetilde{u})\|_{(l_1,\infty)}\ s^{\alpha_1(\rho_1-1)}(\|u\|^{\rho_1-1}_{(q_1,\infty)}+\|\widetilde{u}\|^{\rho_1-1}_{(q_1,\infty)})s^{\alpha_2\rho_2}\|v\|^{\rho_2}_{(q_2,\infty)}s^{-(\alpha_1\rho_1+\alpha_2\rho_2)}ds\\ &\leq C\int_0^t (t-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2})}s^{-(\beta_1+\alpha_1(\rho_1-1)+\alpha_2\rho_2)}ds\ \times\Lambda_1,\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}\Lambda_1 &= \sup_{t>0} t^{\beta_1}\|(u-\widetilde{u})\|_{(l_1,\infty)}\sup_{t>0} t^{\alpha_1(\rho_1-1)}(\|u\|^{\rho_1-1}_{(q_1,\infty)}+\|\widetilde{u}\|^{\rho_1-1}_{(q_1,\infty)})\sup_{t>0} t^{\alpha_2\rho_2}\|v\|^{\rho_2}_{(q_2,\infty)}\\ &\leq \left\|(u-\widetilde{u},v-\widetilde{v})\right\|_{l_1-l_2}\ (\|(u,v)\|^{\rho_1-1}_{q_1-q_2}+\|(\widetilde{u},\widetilde{v})\|^{\rho_1-1}_{q_1-q_2})\,\|(u,v)\|^{\rho_2}_{q_1-q_2}\end{aligned}$$

Since $\frac{1}{p_1}<\frac{1}{l_1}+\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2}$ and $p_i<q_i$, $i=1,2$, the equality (4.7) implies $\beta_1+\alpha_1(\rho_1-1)+\alpha_2\rho_2<1$ and $\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2})<1$, respectively. Thus, the integral in (4.16) is finite, for all $t>0$, and the change of variable $s \to st$ yields

$$\int_0^t (t-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2})} s^{-(\beta_1+\alpha_1(\rho_1-1)+\alpha_2\rho_2)} ds \ = t^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2})-(\beta_1+\alpha_1(\rho_1-1)+\alpha_2\rho_2)+1} \times$$
$$\times \int_0^1 (1-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2})} s^{-(\beta_1+\alpha_1(\rho_1-1)+\alpha_2\rho_2)} ds \ = Ct^{-\beta_1},$$

and so (4.16) gives us

$$\sup_{t>0} t^{\beta_1} \|B_1(u,v) - B_1(\widetilde{u},v)\|_{(l_1,\infty)} \leq C\Lambda_1. \tag{4.17}$$

Similarly, one can prove

$$\sup_{t>0} t^{\beta_1} \|B_1(\widetilde{u},v) - B_1(\widetilde{u},\widetilde{v})\|_{(l_1,\infty)} \leq C\Lambda_2, \tag{4.18}$$

where

$$\begin{aligned} \Lambda_2 &= \sup_{t>0} t^{\beta_2} \|(v-\widetilde{v})\|_{(l_2,\infty)} \sup_{t>0} t^{\alpha_2(\rho_2-1)} (\|v\|^{\rho_2-1}_{(q_2,\infty)} + \|\widetilde{v}\|^{\rho_2-1}_{(q_2,\infty)}) \sup_{t>0} t^{\alpha_1\rho_1} \|\widetilde{u}\|^{\rho_1}_{(q_1,\infty)} \\ &\leq \|(u-\widetilde{u}, v-\widetilde{v})\|_{l_1-l_2} (\|(u,v)\|^{\rho_2-1}_{q_1-q_2} + \|(\widetilde{u},\widetilde{v})\|^{\rho_2-1}_{q_1-q_2}) \, \|(\widetilde{u},\widetilde{v})\|^{\rho_1}_{q_1-q_2}. \end{aligned} \tag{4.19}$$

We finish the proof by observing that (4.14), (4.17) and (4.18) imply (4.12).

■

Remark 4.5 *The bounds (4.12) and (4.13) still hold by replacing the norms* $\|\cdot\|_{(l_1,\infty)}$, $\|*\|_{(l_2,\infty)}$ *and* $\|(\cdot,*)\|_{l_1-l_2}$ *by their respective versions in Lebesgue spaces, namely*

$$\sup_{t>0} \|\cdot\|_{L^{l_1}}, \ \sup_{t>0} \|*\|_{L^{l_2}} \ \textit{and} \ \max\{\sup_{t>0} \|\cdot\|_{L^{l_1}}, \sup_{t>0} \|*\|_{L^{l_2}}\}.$$

In order to prove it, since $L^{l_i} = L^{(l_i,l_i)}$, *in step to get (4.15) one must apply (2.2) with* $d = l_1$ *instead of* $d = \infty$, *and Hölder inequality (2.1) with exponents of the first factor* $p_1 = d_1 = l_1$ *in place of* $p_1 = l_1$ *and* $d_1 = \infty$.

Proof of Theorem 3.3

The proof of Theorem 3.3 is an application of Lemmas 4.1, 4.3 and 4.4.

Proof of (i). We wish to apply Lemma 4.1 to the integral equation (3.10) with $X = E$, $y = (G(t)u_0, G(t)v_0)$, $x = (u,v)$ and

$$B(x) := B(u,v) = (B_1(u,v), B_2(u,v)). \tag{4.20}$$

Clearly B defined by (4.20) satisfies $B(0) = B(0,0) = 0$, and the estimate (4.1) will be obtained by using the Lemma (4.3). For this, we express $B(u,v)$ as

$$B_1(u,v) = F_1(h_1,h_2) \text{ and } B_2(u,v) = F_2(h_1,h_2) \tag{4.21}$$

with

$$\begin{aligned} h_1(s,\cdot) &= u(t-s,\cdot), \text{ if } 0 \le s < t \quad \text{and} \quad h_1(s,\cdot) = 0 \text{ otherwise}, \\ h_2(s,\cdot) &= v(t-s,\cdot), \text{ if } 0 \le s < t \quad \text{and} \quad h_2(s,\cdot) = 0 \text{ otherwise}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{h}_1(s,\cdot) &= \widetilde{u}(t-s,\cdot), \text{ if } 0 \le s < t \quad \text{and} \quad \widetilde{h}_1(s,\cdot) = 0 \text{ otherwise}, \\ \widetilde{h}_2(s,\cdot) &= \widetilde{v}(t-s,\cdot), \text{ if } 0 \le s < t \quad \text{and} \quad \widetilde{h}_2(s,\cdot) = 0 \text{ otherwise}. \end{aligned}$$

Now let K_1 and K_2 be as in Lemma 4.3 and take $K_E = \max\{K_1, K_2\}$. Handling the norm $\|(\cdot,*)\|_E$ on inequalities (4.5) and (4.6), and denoting $x = (u,v)$, $z = (\widetilde{u}, \widetilde{v})$, one obtains

$$\begin{aligned} \|B(x) - B(z)\|_E \le K_E\|x-z\|_E[&\left(\|x\|_E^{\rho_1-1} + \|z\|_E^{\rho_1-1}\right)\|x\|_E^{\rho_2} + \left(\|x\|_E^{\rho_2-1} + \|z\|_E^{\rho_2-1}\right)\|z\|_E^{\rho_1} \\ &+ \left(\|x\|_E^{r_1-1} + \|z\|_E^{r_1-1}\right)\|x\|_E^{r_2} + \left(\|x\|_E^{r_2-1} + \|z\|_E^{r_2-1}\right)\|z\|_E^{r_1}]. \end{aligned} \tag{4.22}$$

Futhermore, using the semigroup estimate (2.2), we bound the linear part of (3.10) as

$$\begin{aligned} \|y\|_E = \|(G(t)u_0, G(t)v_0)\|_E &= \max\{\sup_{t>0}\|G(t)u_0\|_{(p_1,\infty)}, \sup_{t>0}\|G(t)v_0\|_{(p_2,\infty)}\} \\ &\le C_E \max\{\|u_0\|_{(p_1,\infty)}, \|v_0\|_{(p_2,\infty)}\} < C_E\delta = \varepsilon, \end{aligned}$$

provided $\|u_0\|_{(p_1,\infty)}, \|v_0\|_{(p_2,\infty)} < \delta$. Thus if $\varepsilon > 0$ is small enough, in a way that

$$2^{\rho_1+\rho_2+1}\varepsilon^{\rho_1+\rho_2-1} + 2^{r_1+r_2+1}\varepsilon^{r_1+r_2-1} < \frac{1}{K_E}, \tag{4.23}$$

then a direct application of Lemma 4.1 concludes the proof of *(i)*.

Proof of (ii). In this time we apply Lemma 4.1 with $X = E_{q_1q_2}$ instead of $X = E$. Taking $l_1 = q_1$ and $l_2 = q_2$, it is easy to see that the inequalities (4.12) and (4.13) imply that B satisfies the property (4.1) with the norm

$$\|(\cdot,*)\|_{q_1-q_2} = \max\{\sup_{t>0} t^{\alpha_1}\|\cdot\|_{(q_1,\infty)}, \sup_{t>0} t^{\alpha_2}\|*\|_{(q_2,\infty)}\}.$$

Since (4.1) is also verified in the norm $\|(\cdot,*)\|_E$, one obtains that the same property is also verified in the norm $\|(\cdot,*)\|_{E_{q_1q_2}} = \|(\cdot,*)\|_E + \|(\cdot,*)\|_{q_1-q_2}$ with $K_{E_{q_1q_2}} = K_E + K_{q_1q_2}$. Moreover, by estimate (2.2), we have

$$\begin{aligned} \|(G(t)u_0, G(t)v_0)\|_{E_{q_1q_2}} &= \|(G(t)u_0, G(t)v_0)\|_E + \|(G(t)u_0, G(t)v_0)\|_{q_1-q_2} \\ &\le C_{E_{q_1q_2}} \max\{\|u_0\|_{(p_1,\infty)}, \|v_0\|_{(p_2,\infty)}\}. \end{aligned}$$

Now, proceeding analogously to the proof of item *(i)*, one obtains a solution (u,v) belonging to closed ball $\overline{B}_{E_{q_1q_2}}(0,2\varepsilon_q)$ with $\varepsilon_q = C_{E_{q_1q_2}}\delta_{q_1q_2}$. Futhermore, as $C_E < C_{E_{q_1q_2}}$ and $K_E < K_{E_{q_1q_2}}$ then $\delta_{q_1q_2} \leq \delta$ and $\varepsilon_q \leq \varepsilon$. Therefore $(u,v) \in \overline{B}_E(0,2\varepsilon)$ and so, by uniqueness of solutions in $\overline{B}_E(0,2\varepsilon)$, the solution (u,v) coincides with the obtained previous one.

Proof of (iii). Let $\widetilde{E}_{q_1q_2} = E_{q_1q_2} \cap BC((0,\infty); L^{p_1} \times L^{p_2})$. From Remark 4.5 and Lemma 4.4, one observes that B satisfies the bound (4.1) with $X = \widetilde{E}_{q_1q_2}$and the norm

$$\|(\cdot,*)\|_{\widetilde{E}_{q_1q_2}} = \max\{\sup_{t>0}\|\cdot\|_{L^{p_1}}, \sup_{t>0}\|*\|_{L^{p_2}}\} + \|(\cdot,*)\|_{q_1-q_2}.$$

So, leaving the details to the reader, one can now proceed as in the proof of (ii) and (iii) to obtain the desired result.

■

Proof of Corollary 3.5

Proof: Theorem 3.3 was proved by ultimately applying Lemma 4.1. Thus, according to Remark 4.2, the solution (u,v) is obtained as the limit of the following Picard interaction:

$$\begin{aligned} u_1(t,x) &= G(t)u_0\ ,\quad v_1(t,x) = G(t)v_0, \\ u_{m+1}(t,x) &= u_1 + B_1(u_m,v_m) \text{ and } v_{m+1}(t,x) = v_1 + B_2(u_m,v_m),\ m \in \mathbb{N}. \end{aligned} \tag{4.24}$$

Let u_0 and v_0 be homogeneous functions of degree $-k_1$ and k_2, respectively. Without difficulties one verifies that $(u_1(t,x), v_1(t,x))$ is invariant to the scaling of (1.1), i.e.

$$u_1(t,x) = \lambda^{k_1}u_1(\lambda^2 t, \lambda x) \text{ and } \quad v_1(t,x) = \lambda^{k_2}v_1(\lambda^2 t, \lambda x).$$

By an induction argument, one proves that u_m and v_m also verify

$$u_m(t,x) = \lambda^{k_1}u_m(\lambda^2 t, \lambda x) \text{ and } v_m(t,x) = \lambda^{k_2}v_m(\lambda^2 t, \lambda x), \text{ for all } m.$$

Since the limit $(u_m,v_m) \to (u,v)$ is taken with respect to scaling invariant norm $\|\cdot\|_E$, by uniqueness of the limit, the solution (u,v) must also satisfy

$$u(t,x) = \lambda^{k_1}u(\lambda^2 t, \lambda x),\ v(t,x) = \lambda^{k_2}v(\lambda^2 t, \lambda x),$$

for all $\lambda > 0$, $t > 0$ and a.e. $x \in \mathbb{R}^n$.

■

Proof of Theorem 3.4

Let $W=(u,v)$ and $\widetilde{W}=(\widetilde{u},\widetilde{v})$ be two mild solutions of (1.1) in $C([0,\infty);L^{p_1}\times L^{p_2})$, corresponding to initial condition $W_0=(u_0,v_0)\in L^{p_1}\times L^{p_2}$. By covering the interval $(0,\infty)$ with $I_i=[a_i,b_i]$ and $b_i-a_i=T$, observe that it is sufficient to show that $W=\widetilde{W}$ in $[0,T]$, with $T>0$ small enough.

Let us denote $\Theta=W-\widetilde{W}=(u-\widetilde{u},v-\widetilde{v})=(\theta_1,\theta_2)$, $\Psi=(\psi_1,\psi_2)=(G(t)u_0-u,G(t)v_0-v)$ and $\widetilde{\Psi}=(\widetilde{\psi}_1,\widetilde{\psi}_2)=(G(t)u_0-\widetilde{u},G(t)v_0-\widetilde{v})$. Firstly, we rewrite the norm $\|\cdot\|_{(p_1,\infty)}$ of $\theta_1=B_1(u,v)-B_1(\widetilde{u},\widetilde{v})$, the first coordinate of Θ, as

$$\begin{aligned}\|B_1(u,v)-B_1(\widetilde{u},\widetilde{v})\|_{(p_1,\infty)}&=\|B_1(u,v)-B_1(\widetilde{u},v)+B_1(\widetilde{u},v)-B_1(\widetilde{u},\widetilde{v})\|_{(p_1,\infty)}\\&\le\|B_1(u,v)-B_1(\widetilde{u},v)\|_{(p_1,\infty)}+\|B_1(\widetilde{u},v)-B_1(\widetilde{u},\widetilde{v})\|_{(p_1,\infty)}.\end{aligned}\tag{4.25}$$

In view of (4.4), one bounds the r.h.s of (4.25) (up to a constant $C>0$) by

$$\begin{aligned}&\le\left\|\int_0^t G(t-s)|\theta_1|(|u|^{\rho_1-1}+|\widetilde{u}|^{\rho_1-1})|v|^{\rho_2}ds\right\|_{(p_1,\infty)}\\&+\left\|\int_0^t G(t-s)|\theta_2|(|v|^{\rho_2-1}+|\widetilde{v}|^{\rho_2-1})|\widetilde{u}|^{\rho_1}ds\right\|_{(p_1,\infty)}\\&:=I+J\end{aligned}\tag{4.26}$$

Since $|u|\le|\psi_1|+|G(t)u_0|$ and $|\widetilde{u}|\le|\widetilde{\psi}_1|+|G(t)u_0|$, one estimates I (up to a constant $C>0$) as

$$\begin{aligned}I&\le\left\|\int_0^t G(t-s)|\theta_1|(|\psi_1|^{\rho_1-1}+|\widetilde{\psi}_1|^{\rho_1-1})|v|^{\rho_2}ds\right\|_{(p_1,\infty)}+\left\|\int_0^t G(t-s)|\theta_1||G(s)u_0|^{\rho_1-1}||v|^{\rho_2}ds\right\|_{(p_1,\infty)}\\&=I_1+I_2.\end{aligned}$$

Working as in the proof of the bound (4.5) (or (4.9)) and using $L^r\subset L^{(r,\infty)}$, one gets

$$I_1\le C\sup_{0<t<T}\|\theta_1\|_{(p_1,\infty)}(\sup_{0<t<T}\|\psi_1\|_{L^{p_1}}^{\rho_1-1}+\sup_{0<t<T}\|\widetilde{\psi}_1\|_{L^{p_1}}^{\rho_1-1})\sup_{0<t<T}\|v\|_{(p_2,\infty)}^{\rho_2}.\tag{4.27}$$

Let us take $\frac{1}{l}=\frac{1}{p_1}+\frac{\rho_2}{p_2}+\frac{\rho_1-1}{q_1}$. In order to estimate I_2, we first apply (2.2) with $q=p_1$ and $p=l$ followed by Hölder inequality (2.1) to obtain

$$\begin{aligned}I_2&\le\int_0^t(t-s)^{\frac{n}{2}(\frac{1}{l}-\frac{1}{p_1})}\||\theta_1||G(s)u_0|^{\rho_1-1}||v|^{\rho_2}\|_{(l,\infty)}ds\\&\le C\sup_{0<t<T}\|\theta_1\|_{(p_1,\infty)}(\sup_{0<t<T}t^{\alpha_1}\|G(t)u_0\|_{L^{q_1}})^{\rho_1-1}\sup_{0<t<T}\|v\|_{(p_2,\infty)}^{\rho_2}\int_0^t(t-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{p_2})}s^{-\alpha_1(\rho_1-1)}ds\\&\le C\sup_{0<t<T}\|\theta_1\|_{(p_1,\infty)}(\sup_{0<t<T}t^{\alpha_1}\|G(t)u_0\|_{L^{q_1}})^{\rho_1-1}\sup_{0<t<T}\|v\|_{(p_2,\infty)}^{\rho_2},\end{aligned}\tag{4.28}$$

since $-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{p_2})-\alpha_1(\rho_1-1)+1=0$ and

$$\int_0^t(t-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{p_2})}s^{-\alpha_1(\rho_1-1)}ds=t^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{p_2})-\alpha_1(\rho_1-1)+1}\int_0^1(1-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{p_2})}s^{-\alpha_1(\rho_1-1)}ds=C.$$

Now, adding the inequalities (4.27) and (4.28) one gets

$$I \leq I_1 + I_2 \leq A(T) \sup_{0<t<T} \|\theta_1\|_{(p_1,\infty)} \sup_{0<t<T} \|v\|^{\rho_2}_{(p_2,\infty)},$$

with $A(T)$ given by

$$A(T) = C[\sup_{0<t<T} \|\psi_1\|^{\rho_1 - 1}_{L^{p_1}} + \sup_{0<t<T} \|\widetilde{\psi}_1\|^{\rho_1 - 1}_{L^{p_1}} + (\sup_{0<t<T} t^{\alpha_1} \|G(t)u_0\|_{L^{q_1}})^{\rho_1 - 1}].$$

Analogously, one can prove the following estimate for J:

$$J \leq B(T) \sup_{0<t<T} \|\theta_2\|_{(p_2,\infty)} \sup_{0<t<T} \|\widetilde{u}\|^{\rho_1}_{(p_1,\infty)},$$

where

$$B(T) = C[(\sup_{0<t<T} \|\psi_2\|^{\rho_2 - 1}_{L^{p_2}} + \sup_{0<t<T} \|\widetilde{\psi}_2\|^{\rho_2 - 1}_{L^{p_2}}) + (\sup_{0<t<T} t^{\alpha_2} \|G(t)v_0\|_{L^{q_2}})^{\rho_2 - 1})].$$

In an entirely parallel way, the norm $\|\cdot\|_{(p_2,\infty)}$ of $\theta_2 = B_2(u,v) - B_2(\widetilde{u},\widetilde{v})$, the second coordinate of Θ, can be bounded by $\widetilde{I} + \widetilde{J}$ with

$$\widetilde{I} \leq \widetilde{A}(T) \sup_{0<t<T} \|\theta_1\|_{(p_1,\infty)} \sup_{0<t<T} \|v\|^{r_2}_{(p_2,\infty)} \quad \text{and} \quad \widetilde{J} \leq \widetilde{B}(T) \sup_{0<t<T} \|\theta_2\|_{(p_2,\infty)} \sup_{0<t<T} \|\widetilde{u}\|^{r_1}_{(p_1,\infty)},$$

where

$$\widetilde{A}(T) = [(\sup_{0<t<T} \|\psi_1\|^{r_1 - 1}_{L^{p_1}} + \sup_{0<t<T} \|\widetilde{\psi}_1\|^{r_1 - 1}_{L^{p_1}}) + (\sup_{0<t<T} t^{\alpha_1} \|G(t)u_0\|_{L^{q_1}})^{r_1 - 1})]$$

$$\widetilde{B}(T) = [(\sup_{0<t<T} \|\psi_2\|^{r_2 - 1}_{L^{p_2}} + \sup_{0<t<T} \|\widetilde{\psi}_2\|^{r_2 - 1}_{L^{p_2}}) + (\sup_{0<t<T} t^{\alpha_2} \|G(t)v_0\|_{L^{q_2}})^{r_2 - 1})].$$

So, collecting the inequalities above, we have

$$\begin{aligned}
\sup_{0<t<T} \|\theta_1\|_{(p_1,\infty)} + \sup_{0<t<T} \|\theta_2\|_{(p_2,\infty)} \leq& \\
&\leq C\left(A(T) \sup_{0<t<T} \|v\|^{\rho_2}_{(p_2,\infty)} + \widetilde{A}(T) \sup_{0<t<T} \|v\|^{r_2}_{(p_2,\infty)}\right) \sup_{0<t<T} \|\theta_1\|_{(p_1,\infty)} + \\
&+ C\left(B(T) \sup_{0<t<T} \|\widetilde{u}\|^{\rho_1}_{(p_1,\infty)} + \widetilde{B}(T) \sup_{0<t<T} \|\widetilde{u}\|^{r_1}_{(p_1,\infty)}\right) \sup_{0<t<T} \|\theta_2\|_{(p_2,\infty)}.
\end{aligned}$$

Next, we remind that $\limsup_{t\to 0} t^{\alpha_1} \|G(t)u_0\|_{L^{q_1}} = 0$ and $\limsup_{t\to 0} t^{\alpha_2} \|G(t)v_0\|_{L^{q_2}} = 0$ when $u_0 \in L^{p_1}$ and $v_0 \in L^{p_2}$, respectively. On the other hand, since W and $\widetilde{W}$ satisfy the same initial condition and are continuous from $[0,T)$ to $L^{p_1} \times L^{p_2}$, the norms $\|\psi_1(t)\|_{L^{p_1}}, \|\widetilde{\psi}_1\|_{L^{p_1}}, \|\psi_2\|_{L^{p_2}}$ and $\|\widetilde{\psi}_2\|_{L^{p_2}}$ goes to zero as $t \to \infty$. Hence we can choose $T > 0$ small enough in a such way that

$$C\left(A(T) \sup_{0<t<T} \|v\|^{\rho_2}_{(p_2,\infty)} + \widetilde{A}(T) \sup_{0<t<T} \|\widetilde{v}\|^{r_2}_{(p_2,\infty)}\right) < 1,$$

$$C\left(B(T) \sup_{0<t<T} \|\widetilde{u}\|^{\rho_1}_{(p_1,\infty)} + \widetilde{B}(T) \sup_{0<t<T} \|u\|^{r_1}_{(p_1,\infty)}\right) < 1,$$

which yield $\Theta = (\theta_1, \theta_2) = 0$.

■

5 Asymptotic Stability in $L^{(p_1,\infty)} \times L^{(p_2,\infty)}$

In this section, we study the asymptotic stability of the solutions obtained through Section 3. The corresponding results provide two interesting consequences: the first one is the existence of a basin of attraction for each self-similar solution and the second one is that mild solutions, with small initial data in Lebesgue space $L^{p_1} \times L^{p_2}$, have a simple long-time diffusive behavior since all the solutions decay to 0 as $t \to \infty$ (cf. Remark 5.2). Our results now read as below.

Theorem 5.1 *Assume that (u,v) and $(\widetilde{u},\widetilde{v})$ are mild solutions of (1.1) obtained through Theorem 3.3, corresponding to respective initial data (u_0,v_0) and $(\widetilde{u}_0,\widetilde{v}_0) \in L^{(p_1,\infty)} \times L^{(p_2,\infty)}$. If*

$$\lim_{t\to\infty} \|G(t)(u_0-\widetilde{u}_0)\|_{(p_1,\infty)} = 0 \text{ and } \lim_{t\to\infty} \|G(t)(v_0-\widetilde{v}_0)\|_{(p_2,\infty)} = 0, \tag{5.1}$$

then

$$\lim_{t\to\infty} \|(u(t)-\widetilde{u}(t)\|_{(p_1,\infty)} \text{ and } \lim_{t\to\infty} \|v(t)-\widetilde{v}(t)\|_{(p_2,\infty)} = 0. \tag{5.2}$$

Futhermore, if, instead of (5.1), we assume

$$\lim_{t\to\infty} t^{\alpha_1} \|G(t)(u_0-\widetilde{u}_0)\|_{(q_1,\infty)} = 0 \text{ and } \lim_{t\to\infty} t^{\alpha_2} \|G(t)(v_0-\widetilde{v}_0)\|_{(q_2,\infty)} = 0,$$

then

$$\lim_{t\to\infty} t^{\alpha_1} \|u(t)-\widetilde{u}(t)\|_{(q_1,\infty)} = 0 \text{ and } \lim_{t\to\infty} t^{\alpha_2} \|v(t)-\widetilde{v}(t)\|_{(q_2,\infty)} = 0. \tag{5.3}$$

As consequence, if $(u_0,v_0) \in L^{p_1} \times L^{p_2}$, then the solution satisfies

$$\lim_{t\to\infty} \|u(t)\|_{(p_1,\infty)} = \lim_{t\to\infty} \|v(t)\|_{(p_2,\infty)} = 0. \tag{5.4}$$

Remark 5.2

- *(Basin of atraction) Let (u,v) be the self-similar solution with initial conditon (u_0,v_0). Take $\psi_0 = u_0 + \omega_1$ and $\varphi_0 = v_0 + \omega_2$ small enough such that $\omega_i \in C_C^\infty$, $i = 1,2$, and consider the mild solution (ψ,φ) with initial data (ψ_0,φ_0). Since $\omega_1 = \psi_0 - u_0$ and $\omega_2 = \varphi_0 - v_0$ satisfy (5.1) then*

$$\lim_{t\to\infty} \|\psi(t)-u(t)\|_{(p_1,\infty)} \text{ and } \lim_{t\to\infty} \|\varphi(t)-v(t)\|_{(p_2,\infty)} = 0.$$

  Therefore, considering compact suport small perturbations, one obtains a basin of atraction to each self-similar solutions. Indeed, this basin is characterized by all perturbations (ω_1,ω_2) that satisfy (5.1).

- *(Simple long-time diffusive behavior) Let* $(u_0, v_0) \in L^{p_1} \times L^{p_2}$. *By using the Lebesgue version of (4.12),(4.13) with* $l_i = p_i$, $i = 1, 2$ *(cf. Remark 4.5), and working as in the proof of (5.2), we can prove*

$$\lim_{t\to\infty} \|u(t)\|_{L^{p_1}} = \lim_{t\to\infty} \|v(t)\|_{L^{p_2}} = 0, \tag{5.5}$$

which clearly is stronger than (5.4). Also, by interpolation between (5.5) and $\|(u, v)\|_{q_1-q_2} < \infty$ *(cf. notation (4.11)), one obtains*

$$\lim_{t\to\infty} t^{\alpha_1} \|u(t)\|_{L^{q_1}} = \lim_{t\to\infty} t^{\alpha_2} \|v(t)\|_{L^{q_2}} = 0.$$

The proof of (5.5) is simpler than the one of (5.2). The reason is that, as well as in the proof of (5.3), we do not deal with non-integrable factors as $(t - s)^{-1}$ *when we bound the norms* $\sup_{t>0} \|\cdot\|_{L^{p_i}}$, $i = 1, 2$ *(cf. Remark 4.5 and inequality (4.16)).*

Proof of Thereom 5.1. Let (u, v) and $(\widetilde{u}, \widetilde{v})$ be two mild solutions. We first handle the norm $\|u(t) - \widetilde{u}(t)\|_{(p_1,\infty)}$ and afterwards, without giving details, write the corresponding estimate for the second component $\|v(t) - \widetilde{v}(t)\|_{(p_2,\infty)}$. To this end, we subtract the integral equations satisfied by u and $\widetilde{u}$ (cf. Definition (3.2)) and then take the norm $\|\cdot\|_{(p_1,\infty)}$ to get

$$\begin{aligned}
\|u(t) - \widetilde{u}(t)\|_{(p_1,\infty)} &\leq \|G(t)(u_0 - \widetilde{u}_0)\|_{(p_1,\infty)} \\
&+ \left\| \int_0^{\delta t} G(t-s)(|u|^{\rho_1-1}u|v|^{\rho_2-1}v - |\widetilde{u}|^{\rho_1-1}\widetilde{u}|\widetilde{v}|^{\rho_2-1}\widetilde{v})ds \right\|_{(p_1,\infty)} \\
&+ \left\| \int_{\delta t}^{t} G(t-s)(|u|^{\rho_1-1}u|v|^{\rho_2-1}v - |\widetilde{u}|^{\rho_1-1}\widetilde{u}|\widetilde{v}|^{\rho_2-1}\widetilde{v})ds \right\|_{(p_1,\infty)} \\
&= I_0 + I_1 + I_2,
\end{aligned}$$

where the small constant δ will be chosen later. Recalling that $\|(u, v)\|_E, \|(\widetilde{u}, \widetilde{v})\|_E \leq 2\varepsilon$, one makes the change variable $s \to ts$ and bounds I_1 (up to a constant $C > 0$) as

$$\begin{aligned}
I_1 &\leq \int_0^{\delta t} (t-s)^{-1} \|u - \widetilde{u}\|_{(p_1,\infty)} [(\|u\|^{\rho_1-1}_{(p_1,\infty)} + \|\widetilde{u}\|^{\rho_1-1}_{(p_1,\infty)}) \|v\|^{\rho_2}_{(p_2,\infty)}] ds \\
&+ \int_0^{\delta t} (t-s)^{-1} \|v - \widetilde{v}\|_{(p_2,\infty)} [(\|v\|^{\rho_2-1}_{(p_2,\infty)} + \|\widetilde{v}\|^{\rho_2-1}_{(p_2,\infty)}) \|u\|^{\rho_1}_{(p_1,\infty)}] ds \\
&\leq 2^{\rho_1+\rho_2} \varepsilon^{\rho_1+\rho_2-1} \int_0^{\delta} (1-s)^{-1} (\|u(ts) - \widetilde{u}(ts)\|_{(p_1,\infty)} + \|v(ts) - \widetilde{v}(ts)\|_{(p_2,\infty)}) ds
\end{aligned} \tag{5.6}$$

Applying Lemma 4.3 with

$$(h_1(s,\cdot), h_2(s,\cdot)) = (u(s,\cdot), v(s,\cdot))\chi_{(\delta t,t)}(s), \quad (\widetilde{h}_1(s,\cdot), \widetilde{h}_2(s,\cdot)) = (\widetilde{u}(s,\cdot), \widetilde{v}(s,\cdot))\chi_{(\delta t,t)}(s) \quad \text{and}$$

χ_M denoting the characteristic function of the set M, we obtain the following bound for I_2:

$$\begin{aligned} I_2 &\le K\left(\sup_{\delta t<s<t}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)}\right)\sup_{\delta t<s<t}[(\|u\|^{\rho_1-1}_{(p_1,\infty)}+\|\widetilde{u}\|^{\rho_1-1}_{(p_1,\infty)})\|v(s)\|^{\rho_2}_{(p_1,\infty)}+ \\ &+K\left(\sup_{\delta t<s<t}\|v(s)-\widetilde{v}(s)\|_{(p_1,\infty)}\right)(\|v\|^{\rho_2-1}_{(p_2,\infty)}+\|\widetilde{v}\|^{\rho_2-1}_{(p_2,\infty)})\|\widetilde{u}\|^{\rho_1}_{(p_1,\infty)}] \\ &\le 2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1}(\sup_{\delta t<s<t}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)}+\sup_{\delta t<s<t}\|v(s)-\widetilde{v}(s)\|_{(p_2,\infty)}) \end{aligned} \tag{5.7}$$

So, in light of (5.6) and (5.7), one bounds $\|u(t)-\widetilde{u}(t)\|_{(p_1,\infty)}$ by

$$\begin{aligned} &\le \|G(t)(u_0-\widetilde{u}_0)\|_{(p_1,\infty)}+ \\ &+C2^{\rho_1+\rho_2}\varepsilon^{\rho_1+\rho_2-1}\int_0^\delta(1-s)^{-1}(\|u(ts)-\widetilde{u}(ts)\|_{(p_1,\infty)}+\|v(ts)-\widetilde{v}(ts)\|_{(p_2,\infty)})ds+ \\ &+2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1}(\sup_{\delta t<s<t}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)}+\sup_{\delta t<s<t}\|v(s)-\widetilde{v}(s)\|_{(p_2,\infty)}), \end{aligned} \tag{5.8}$$

for all $t>0$. Letting the details to the reader, we can estimate $\|v(t)-\widetilde{v}(t)\|_{(p_2,\infty)}$ by

$$\begin{aligned} &\le \|G(t)(v_0-\widetilde{v}_0)\|_{(p_2,\infty)}+ \\ &+C2^{r_1+r_2}\varepsilon^{r_1+r_2-1}\int_0^\delta(1-s)^{-1}(\|u(ts)-\widetilde{u}(ts)\|_{(p_1,\infty)}+\|v(ts)-\widetilde{v}(ts)\|_{(p_2,\infty)})ds \\ &+2^{r_1+r_2}K\varepsilon^{r_1+r_2-1}(\sup_{\delta t<s<t}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)}+\sup_{\delta t<s<t}\|v(s)-\widetilde{v}(s)\|_{(p_2,\infty)}), \end{aligned} \tag{5.9}$$

for all $t>0$. In the following, we remember the notation $\limsup_{t\to\infty}f(t)\equiv\lim_{k\to\infty}\sup_{t>k}f(t)$ and define

$$\Gamma_1=\limsup_{t\to\infty}\|u(t)-\widetilde{u}(t)\|_{(p_1,\infty)} \text{ and } \Gamma_2=\limsup_{t\to\infty}\|v(t)-\widetilde{v}(t)\|_{(p_2,\infty)}.$$

We shall prove that $\Gamma_1=0$ and $\Gamma_2=0$. By the dominated convergence theorem, we obtain

$$\limsup_{t\to\infty}\int_0^\delta(1-s)^{-1}\|u(ts)-\widetilde{u}(ts)\|_{(p_1,\infty)}ds\le\Gamma_1\log\left(\frac{1}{1-\delta}\right), \tag{5.10}$$

$$\limsup_{t\to\infty}\int_0^\delta(1-s)^{-1}\|v(ts)-\widetilde{v}(ts)\|_{(p_2,\infty)}ds\le\Gamma_2\log\left(\frac{1}{1-\delta}\right). \tag{5.11}$$

The following easy inequalities

$$\sup_{t>k}\sup_{\delta t<s<t}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)}\le\sup_{\delta k<s<\infty}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)},$$
$$\sup_{t>k}\sup_{\delta t<s<t}\|v(s)-\widetilde{v}(s)\|_{(p_2,\infty)}\le\sup_{\delta k<s<\infty}\|v(s)-\widetilde{v}(s)\|_{(p_2,\infty)},$$

imply

$$\limsup_{t\to\infty}\sup_{\delta t<s<t}\|u(s)-\widetilde{u}(s)\|_{(p_1,\infty)}\le\Gamma_1 \text{ and } \limsup_{t\to\infty}\sup_{\delta t<s<t}\|v(s)-\widetilde{v}(s)\|_{(p_2,\infty)}\le\Gamma_2. \tag{5.12}$$

Summing (5.8) and (5.9) and afterwards calculating $\limsup_{t\to\infty}$ on the result, the condition (5.1) together with (5.10)-(5.12) yields

$$\Gamma_1+\Gamma_2 \leq (2^{\rho_1+\rho_2}K\varepsilon^{\rho_1+\rho_2-1}+2^{r_1+r_2}K\varepsilon^{r_1+r_2-1})\left(C\log\left(\frac{1}{1-\delta}\right)+K\right)(\Gamma_1+\Gamma_2).$$

From the proof of Theorem 3.3 (cf. (4.23)), ε is taken in a such way that $(2^{\rho_1+\rho_2+1}K\varepsilon^{\rho_1+\rho_2-1})+2^{r_1+r_2+1}K\varepsilon^{r_1+r_2-1}<1$. So, since Γ_1 and Γ_2 are nonnegative numbers, we choose $\delta>0$ small enough and conclude that $\Gamma_1=\Gamma_2=0$.

We omit the proof of (5.3) because it follows in the same spirit that the above proof. Indeed, in this case, as one does not need to split $\int_0^t=\int_0^{\delta t}+\int_{\delta t}^t$, the proof is easier than the previous one. The reason is that the estimates in the norms $\sup_{t>0} t^{\alpha_i}\left\|\cdot\right\|_{(q_i,\infty)}$, $i=1,2$ are more regular, in the sense that they bring either the factor $(t-s)^{-\frac{n}{2}(\frac{\rho_1-1}{q_1}+\frac{\rho_2}{q_2})}$ or $(t-s)^{-\frac{n}{2}(\frac{r_1}{q_1}+\frac{r_2-1}{q_2})}$ within the corresponding integral instead of the non-integrable factor $(t-s)^{-1}$ (cf. Lemma 4.4 and inequality (4.16)).

On the other hand, (5.4) is an immediate consequence of the well known fact $\lim_{t\to\infty}\left\|G(t)f\right\|_{L^r}=0$ provided $f\in L^r$ (cf. e.g. [16, pp. 4]) and the continuous inclusion $L^r\subset L^{(r,\infty)}$. ■

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