On the stability problem for the Boussinesq equations in weak-L^p spaces

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Abstract

We consider the Boussinesq equations in either an exterior domain in $\mathbb{R}^n$, the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$ or a bounded domain in $\mathbb{R}^n$, where the space dimension n satisfies $n \geq 3$. We give a class of stable steady solutions, which improves and complements the previous stability results. Our results give a complete answer to the stability problem for the Boussinesq equations in weak-L^p spaces, in the sense that we only assume that the stable steady solution belongs to scaling invariant class $L^{(n,\infty)}_\sigma$. Moreover, some considerations about the exponential decay (in bounded domains) and the uniqueness of the disturbance are done.

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1 Introduction

The Boussinesq system of hydrodynamics equations arises from zero order approximation to the coupling between the Navier-Stokes equations and the thermodynamic equation, modeling the fluid movement by the natural convection (cf. [15]). The steady problem for the viscous Boussinesq equations has the following form

$$\begin{cases} -\Delta \bar{u} + \bar{u}\cdot\nabla\bar{u} + \nabla\bar{p} &= \kappa\bar{\theta}f, \text{ in } \Omega, \\ \operatorname{div}\bar{u} &= 0, \text{ in } \Omega, \\ -\Delta\bar{\theta} + \bar{u}\cdot\nabla\bar{\theta} &= h, \text{ in } \Omega, \\ \bar{u},\ \bar{\theta} &= 0, \text{ on } \partial\Omega, \\ (\bar{u},\bar{\theta}) &\to (0,0), \text{ as } |x|\to\infty, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is the spatial domain, $\bar{u}(x) = (\bar{u}_1(x), ..., \bar{u}_n(x))$ denotes the velocity of the fluid at a point $x \in \Omega$, $\bar{p}(x)$ is the hydrostatic pressure and $\bar{\theta}(x)$ is the temperature (cf. [17]). The given field $f(x) = (f_1(x), ..., f_n(x))$ represents the external force by unit of mass, $h(x)$ is the reference temperature and the constant $\kappa > 0$ denotes the coefficient of volume expansion. Without loss of generality we are taking the density and the kinematic viscosity of the fluid equal to one.

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In this paper we consider Ω as either the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$, a bounded domain or an exterior domain with boundary $\partial\Omega$ enough smooth, where the dimension $n \geq 3$. The aim of this paper is to determine a new class of steady solutions $(\bar{u}, \bar{\theta})$ which is stable for nonsmooth initial disturbance. We will start by describing the stability problem for (1.1).

If the pair $(\bar{u}(x), \bar{\theta}(x))$ is initially perturbed by $(u_0(x), \theta_0(x))$, then the perturbed flow $(\tilde{u}, \tilde{p}, \tilde{\theta})$ is given by

$$\left\{\begin{array}{rcl} \partial_t\tilde{u} - \Delta\tilde{u} + \tilde{u}\cdot\nabla\tilde{u} + \nabla\tilde{p} &=& \kappa\tilde{\theta}f, \text{ in } \Omega\times(0,\infty), \\ \text{div } \tilde{u} &=& 0, \text{ in } \Omega\times(0,\infty), \\ \partial_t\tilde{\theta} - \Delta\tilde{\theta} + \tilde{u}\cdot\nabla\tilde{\theta} &=& h, \text{ in } \Omega\times(0,\infty), \\ \tilde{\theta}(x,t),\ \tilde{u}(x,t) &=& 0, \text{ on } \partial\Omega\times(0,\infty), \\ \tilde{u}(x,0) &=& \bar{u}(x) + u_0(x),\ x\in\Omega, \\ \tilde{\theta}(x,0) &=& \bar{\theta}(x) + \theta_0(x),\ x\in\Omega, \\ (\tilde{u},\tilde{\theta}) &\to& (0,0), \text{ as } |x|\to\infty,\ t>0. \end{array}\right. \tag{1.2}$$

Let $(\tilde{u}, \tilde{p}, \tilde{\theta})$ be solution of the problem (1.2) and (u, p, θ) be the disturbance defined by

$$u(x,t) = \tilde{u}(x,t) - \bar{u}(x),\ \theta(x,t) = \tilde{\theta}(x,t) - \bar{\theta}(x),\ p(x,t) = \tilde{p}(x,t) - \bar{p}(x).$$

Then the triple of functions (u, p, θ) satisfies the following system

$$\left\{\begin{array}{rcl} \partial_t u - \Delta u + u\cdot\nabla u + \bar{u}\cdot\nabla u + u\cdot\nabla\bar{u} + \nabla p &=& \kappa\theta f, \text{ in } \Omega\times(0,\infty), \\ \text{div } u &=& 0, \text{ in } \Omega\times(0,\infty), \\ \partial_t\theta - \Delta\theta + u\cdot\nabla\theta + \bar{u}\cdot\nabla\theta + u\cdot\nabla\bar{\theta} &=& 0, \text{ in } \Omega\times(0,\infty), \\ \theta(x,t),\ u(x,t) &=& 0, \text{ on } \partial\Omega\times(0,\infty), \\ u(x,0) &=& u_0(x),\ x\in\Omega, \\ \theta(x,0) &=& \theta_0(x),\ x\in\Omega, \\ (u,\theta) &=& (0,0), \text{ as } |x|\to\infty,\ t>0. \end{array}\right. \tag{1.3}$$

Thus, our stability problem for (1.1) can be reduced to study the existence and large time behavior of global solutions to (1.3). Next, we briefly review the main results concerning the stability of steady solutions for the Boussinesq system, which can be found in [10, 9, 12, 13, 14]. In the exterior of a three-dimensional sphere, the authors of [10, 9, 12] investigate, in the context of L^2-norm, the stability of a particular steady solution given by $\bar{u} = 0$, $\bar{\theta} = \kappa\frac{1}{|x|}$ and $\bar{p} = -\kappa^2\frac{1}{2|x|^2} + constant$, defined as the conduction solution. The authors of [9] studied the stability of weak solutions with restrictions on the Reynolds number range. Through an energy method, the results of [12] improve and supplement those in [10], in the sense that, it is proved a L^∞- decay of disturbance with initial data $(u_0, \theta_0) \in (D(A_2^{1/4}) \times L^2(\Omega))$, where $D(A_q)$ denotes the domain in $W^{2,q}(\Omega)$ of Stokes operator. On the other hand, in [13] the convection problem in a bounded domain of $\mathbb{R}^3$ was considered, and the existence of a global in time strong solution near to the steady state was also proved. In [13], to obtain the global existence and large time behavior of solutions, an analysis of the semigroup (in Lebesgue spaces L^p) generated by the linearized operator around the steady solution is established. This linearized operator is given by

$$\mathcal{L}\begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} Au + \mathbb{P}(\bar{u}\cdot\nabla u + u\cdot\nabla\bar{u} - \theta f) \\ -\Delta\theta + \bar{u}\cdot\nabla\theta + u\cdot\nabla\bar{\theta} \end{bmatrix}. \tag{1.4}$$

It is worthwhile to recall that in [13], the stability of small steady solution $(\bar{u}, \bar{\theta})$ of system (1.1) was obtained in the class $D(A_3) \times D((-\Delta)_m) \subset L^\infty \times L^\infty, m \in (1, \infty)$, where $D((-\Delta)_q)$ denotes the domain of the minus Laplacian operator. Later, in [14] dealing with the case where Ω is an exterior domain of $\mathbb{R}^3$, the stability problem within the framework of $L^{(p,\infty)}$-theory was discussed. The results of [14] were also obtained by considering the linearization of problem (1.3) and establishing the $L^{(p,\infty)} - L^{(q,\infty)}$ estimates for the semigroup $e^{-t\mathcal{L}}$ generated by the linearized operator (1.4). This analysis requires that the steady solution satisfies $\bar{u} \in L^{(3,\infty)}_\sigma(\Omega), \bar{\theta} \in L^{(m',\infty)}(\Omega)$ with $\bar{u}, \nabla\bar{u}, \bar{\theta} \in L^\infty(\Omega)$ and $\nabla\bar{\theta} \in L^d(\Omega)$, $1 < m < 3$, $1/d = (2/3 - 1/m)_+$, where $(\cdot)_+ = \max\{0, \cdot\}$ and m' is conjugate exponent of m. Hence $d = \infty$ for $3 \leq m' < \infty$. These assumptions are used in order to prove that the linearized operator $-\mathcal{L}$ generates a bounded analytic semigroup on the Lorenz spaces $L^{(p,\infty)}_\sigma \times L^{(q,\infty)}$.

The purpose of this paper is to improve the earlier results of [13, 14] and thus to give a complete answer to the stability problem in $L^{(p,\infty)}$-spaces. For this we will show the stability of the steady solutions $(\bar{u}, \bar{\theta})$ of (1.1) in the class $L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega)$ (Theorems 2.5 and 2.8). More precisely, we only assume that $\bar{u} \in L^{(n,\infty)}_\sigma(\Omega), \bar{\theta} \in L^{(n,\infty)}(\Omega)$ (which can verify $\bar{u} \notin L^\infty(\Omega)$ and $\bar{\theta} \notin L^\infty(\Omega)$), with sufficiently small norm and without any assumption on $\nabla\bar{u}, \nabla\bar{\theta}$, opposed to the restrictions of [13, 14]. We remark that in an exterior domain, the conduction solution $\bar{u} = 0$ and $\bar{\theta} = -\kappa\frac{1}{2|x|}$ belongs to $\cap_{p\in(3/2,\infty]} W^{p,\infty}(\Omega) \subset L^\infty(\Omega) \cap L^{(n,\infty)}(\Omega)$. On the other hand, it is important to study the stability problem over spaces which are invariant by the scaling of (1.1), namely $(\bar{u}(x), \bar{\theta}(x)) \to (\lambda\bar{u}(\lambda x), \lambda\bar{\theta}(\lambda x))$. In this spirit, our class covers the relevant case of the scaling invariant $\bar{u}, \bar{\theta} \in L^{(n,\infty)}(\Omega)$ and $\nabla\bar{u}, \nabla\bar{\theta} \in L^{(n/2,\infty)}(\Omega)$, which was not dealt in the previous mentioned works. In order to prove our results, an essential point in our approach is to solve the problem (1.3) by introducing the notion of mild solution through the well known Stokes and heat semigroups, and without making use of the semigroup generated by the linearized operator (1.4) (see Definition 2.4 below). So, we study the existence and uniqueness of global mild solutions in the space of strong decay $E_q = \{(u, \theta) : t^{1/2-n/2q}(u, \theta) \in BC((0, \infty); L^{(q,\infty)}_\sigma(\Omega) \times L^{(q,\infty)}(\Omega))\}, n < q < \infty$, and in the space of persistence $E = BC((0, \infty); L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega))$. Since we only assume that $\bar{u}, \bar{\theta} \in L^{(n,\infty)}(\Omega)$, if we try to prove directly the strong decay (i.e, the existence of solutions in E_q) by using the $L^{(p,\infty)} - L^{(q,\infty)}$ estimates of the Stokes and heat semigroups, then estimating

$$t^{1/2-n/2q} \left\| \int_0^t e^{-(t-s)A}\mathbb{P}(\bar{u} \cdot \nabla u + u \cdot \nabla\bar{u})(s)ds \right\|_{(q,\infty)} \leq C \int_0^t (t-s)^{-1} s^{\frac{\alpha}{2}} \|\bar{u}\|_{(n,\infty)} \|u\|_{E_q} ds, \quad (1.5)$$

a first difficulty arises on the left, since the integral is not finite. The same situation arises when we estimate the norm

$$t^{1/2-n/2q} \left\| \int_0^t e^{(t-s)\Delta}(\bar{u} \cdot \nabla\theta + u \cdot \nabla\bar{\theta})(s)ds \right\|_{(q,\infty)}. \quad (1.6)$$

In order to overcome these difficulties without using the restrictions stated in [13, 14], we need to prove, among other things, the inequalities (3.3) and (3.5) (which will be proved by using the Yamazaki's estimate [23]), and the inequality (3.8) below.

We will also comment about the uniqueness of disturbance (u, θ) in the space of persistence E. Let us stress that in our analysis we are always considering the three kinds of cited unbounded domains Ω. Moreover, in the case of bounded domains, we obtain exponential decay rate towards steady solution (cf. Remark 3.4).

Observing the equation $(1.3)_1$, to deal with the linear term $\int_0^t e^{-tA}\mathbb{P}(\kappa\theta f)(s)ds$ generated by the coupling term $\kappa\theta f$ in the spaces E_q, E, we assume that $\sup_{t>0} t^{1-\frac{n}{2b}}\|f(t,\cdot)\|_b < \infty$ with $b \geq n/2$. In particular, in case $b = n/2$, we can take f as being the gravitational field $f(t,x) = f(x) = G\frac{x}{|x|^3} \in L^{(n/2,\infty)}(\Omega)$, where G is the gravitational constant. From the physical point of view, this particular situation is important and it may be regarded as the Bénard problem (see e.g. [8]). Let us comment that, unlike the references [10, 9, 12, 13, 14], in the proof of our coupling term estimates, the L^∞-norm of the field f does not play any role (cf. Lemma 3.6).

From another point of view, in the case $\Omega = \mathbb{R}^n$, our class of steady solutions allows the existence of self-similar disturbance solutions, under right homogeneity conditions for the steady solution $(\bar{u},\bar{\theta})$, the gravitational field f and the initial disturbance (u_0,θ_0). More precisely, the self-similar solutions correspond to homogeneous steady solutions and initial disturbance of degree -1, and f being a homogeneous field of degree -2.

Furthermore we will also prove that assuming additional conditions on the initial disturbance (u_0,θ_0), a best decay of the disturbance $(u(t),\theta(t))$ can be obtained, complementing the results of convergence for steady solutions provided by Theorems 2.5 and 2.8 (cf. Theorem 2.10). In particular we will show that the disturbance (u,θ) converges to $(0,0)$ as $t \to \infty$, in the L^n-norm and in the norm $t^{\frac{1}{2}-\frac{n}{2q}}\|\cdot\|_q$ $(n < q < \infty)$, provided the initial disturbance lies in $L^n_\sigma(\Omega) \times L^n(\Omega)$.

Finally we mention that the stability problem for Navier-Stokes equations has been largely studied and we refer the reader to the works [3, 16, 4]. Collecting the results of these works, we obtain the space $L^{(n,\infty)}_\sigma(\Omega)$ as a stability class for the steady state with the disturbance $u \in BC((0,\infty); L^{(n,\infty)}_\sigma(\Omega))$ such that $\sup_{t>0} t^{\frac{1}{2}-\frac{n}{2q}}\|u\|_{(q,\infty)} < \infty$. Concerning the non-perturbed problem $(\bar{u},\bar{\theta}) = (0,0)$, results of global existence in some functional spaces, including the weak-L^p space, were obtained in [6, 18, 19] and some references therein.

The outline of this paper is given as follows. In Section 2 we recall some preliminaries, introduce the notion of mild solutions and state our main results. In Section 3 we prove our results. Throughout this paper, some times, spaces of scalar-value and vector-value functions are denoted in same way.

2 Functional spaces and main results

Before stating our results, we introduce some functional spaces. Let $C^\infty_{0,\sigma}(\Omega)$ denote the set of all C^∞- real functions $\varphi = (\varphi_1, ..., \varphi_n)$ with compact support in Ω, such that $div\ \varphi = 0$. The closure of $C^\infty_{0,\sigma}$ with respect to norm $\|\cdot\|_r$ of space $(L^r)^n, 1 < r < \infty$, is denoted by $L^r_\sigma(\Omega)$. Let us recall the Helmholtz decomposition: $(L^r(\Omega))^n = L^r_\sigma(\Omega) \oplus G^r(\Omega)$, $1 < r < \infty$, where $G^r(\Omega) = \{\nabla p \in L^r(\Omega) : p \in L^r_{loc}(\overline{\Omega})\}$ (see [3], [11], for instance). $\mathbb{P}_r$ denotes the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$. The Stokes operator A_r on L^r_σ is then defined by $A_r = -\mathbb{P}_r\Delta$ with domain $D(A_r) = \{u \in (H^{2,r}(\Omega))^n : u|_{\partial\Omega} = 0\} \cap L^r_\sigma$. It is well known that $-A_r$ generates a uniformly bounded analytic semigroup $\{e^{-tA_r}\}_{t\geq 0}$ of class C_0 in L^r_σ. The same result is true for the Laplacian operator Δ_r in $L^r(\Omega)$, that is, Δ_r generates a uniformly bounded analytic semigroup

$\{e^{t\Delta_r}\}_{t\geq 0}$ of class C_0 in L^r.

Now we introduce some preliminaries about the Lorentz spaces. The reader interested in more details on Lorentz spaces $L^{(p,q)}(\Omega)$ and their properties, is refereed to [1]. Let $1 < p \leq \infty$ and $1 \leq q \leq \infty$. A measurable function f defined on a domain $\Omega \subset \mathbb{R}^n$, with smooth boundary $\partial\Omega$, belongs to Lorentz space $L^{(p,q)}(\Omega)$ if the quantity

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{p}{q} \int_0^\infty \left[t^{\frac{1}{p}} f^{**}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{, if} \quad 1 < p < \infty,\, 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \text{, if} \quad 1 < p \leq \infty,\, q = \infty, \end{cases}$$

is finite, where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds, \quad f^*(t) = \inf\{s > 0 : m\{x \in \Omega : |f(x)| > s\} \leq t\}, t > 0.$$

The space $L^{(p,q)}$ with the norm $\|f\|_{(p,q)}$ is a Banach space. Note that $L^p(\Omega) = L^{(p,p)}(\Omega)$. When $q = \infty$, $L^{(p,\infty)}(\Omega)$ are called the Marcinkiewicz spaces or weak-L^p spaces. Moreover, $L^{(p,q_1)}(\Omega) \subset L^p(\Omega) \subset L^{(p,q_2)}(\Omega) \subset L^{(p,\infty)}(\Omega)$ for $1 \leq q_1 \leq p \leq q_2 \leq \infty$. We recall that the space $C_0^\infty(\Omega)$ is not dense in $L^{(p,\infty)}(\Omega)$.

Next, we recall the Hölder's inequality in the framework of Lorentz spaces (cf. [22]).

Proposition 2.1 *(Hölder's inequality). Let $1 < p_1 \leq \infty$, $1 < p_2$, $r < \infty$. Let $f \in L^{(p_1,q_1)}(\Omega)$ and $g \in L^{(p_2,q_2)}(\Omega)$ where $\frac{1}{p_1} + \frac{1}{p_2} < 1$, then the product $h = fg$ belongs to $L^{(r,s)}(\Omega)$ where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, and $s \geq 1$ satisfies $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. Moreover,*

$$\|h\|_{(r,s)} \leq C(r) \|f\|_{(p_1,q_1)} \|g\|_{(p_2,q_2)}. \tag{2.1}$$

Since in this paper we deal with the incompressible Boussinesq equations, we will recall the Helmholtz decomposition in Lorentz spaces. Borchers and Miyakawa [3] established the following Helmholtz decomposition of the Lorentz spaces, extending the operator $\mathbb{P}_r$ to a bounded operator on $\left(L^{(r,d)}(\Omega)\right)^n$, which we denote by $\mathbb{P}_{r,d}$. Setting $L_\sigma^{(r,d)}(\Omega) = Range(\mathbb{P}_{r,d})$ and $G^{(r,d)}(\Omega) = Kernel(\mathbb{P}_{r,d})$, then $\left(L^{(r,d)}(\Omega)\right)^n = L_\sigma^{(r,d)}(\Omega) \oplus G^{(r,d)}(\Omega)$, with $L_\sigma^{(r,d)}(\Omega) = \{u \in \left(L^{(r,d)}(\Omega)\right)^n : \nabla \cdot u = 0,\, u \cdot n|_{\partial\Omega} = 0\}$ and $G^{(r,d)}(\Omega) = \{\nabla v \in \left(L^{(r,d)}(\Omega)\right)^n : v \in L_{loc}^{(r,d)}(\bar{\Omega})\}$. For simplicity, we shall abbreviate the projection operator and the Stokes Operator on Lorentz spaces as $\mathbb{P}$ and A, respectively. In view of [3], the operator $-A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ on $L_\sigma^{(r,d)}(\Omega)$. However, we recall that if $d = \infty$, this semigroup is not strongly continuous at $t = 0$. The Laplacian operator Δ also generates a bounded analytic semigroup $\{e^{\Delta t}\}_{t\geq 0}$ on $L^{(r,d)}(\Omega)$.
Applying the operator projection to the $(1.3)_1$ equation, we can treat the problem (1.3) as the following problem of parabolic type:

$$\begin{cases} u_t + Au + \mathbb{P}\{u \cdot \nabla u + \bar{u} \cdot \nabla u + u \cdot \nabla \bar{u}\} &= \kappa \mathbb{P}(\theta f), \text{ in } \Omega \times (0, \infty), \\ \theta_t - \Delta\theta + u \cdot \nabla\theta + \bar{u} \cdot \nabla\theta + u \cdot \nabla\bar{\theta} &= 0, \text{ in } \Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), \ x \in \Omega, \\ \theta(x, 0) &= \theta_0(x), \ x \in \Omega \end{cases} \tag{2.2}$$

As usual, we use formally the Duhamel Principle in order to introduce the integral formulation associated with the system (2.2):

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}(u\cdot\nabla u)(s)ds - \int_0^t e^{-(t-s)A}\mathbb{P}(\bar{u}\cdot\nabla u + u\cdot\nabla\bar{u} - \kappa\theta f)(s)ds, \quad (2.3)$$

$$\theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-s)\Delta}(u\cdot\nabla\theta)(s)ds - \int_0^t e^{(t-s)\Delta}(\bar{u}\cdot\nabla\theta + u\cdot\nabla\bar{\theta})(s)ds. \quad (2.4)$$

Remark 2.2 *Let us comment about the sense in which is taken the equations (2.3)-(2.4). In general, unlike the case $\Omega = \mathbb{R}^n$, the operators $e^{-(t-s)A}\mathbb{P}$ and $e^{(t-s)\Delta}$ do not commute with derivatives, and consequently, we cannot use $\nabla e^{-(t-s)A}\mathbb{P}$ and $\nabla e^{(t-s)\Delta}$ to derive a notion of solution. Also, under our weak condition over the steady solution $(\bar{u}\,,\bar{\theta})$ and the disturbance (u,θ), the terms within the integrals in the right-hand side of (2.3)-(2.4) are not Bochner integrable. Therefore the integrals must be understood in distributional sense, as in [23, pp. 642] and [3].*

According with the integral equations (2.3)-(2.4) we define the following operators which will be used from now on:

$$B((u_1,\theta_1),(u_2,\theta_2)) := \left(-\int_0^t e^{-(t-s)A}\mathbb{P}(u_1\cdot\nabla u_2)(s)ds, -\int_0^t e^{(t-s)\Delta}(u_2\cdot\nabla\theta_1)(s)ds\right), \quad (2.5)$$

$$T(u,\theta) := (T^1_{\bar{u}}(u) + \mathcal{F}_f(\theta), T^2_{\bar{u},\bar{\theta}}(u,\theta)), \quad (2.6)$$

where

$$T^1_{\bar{u}}(u) = -\int_0^t e^{-(t-s)A}\mathbb{P}(\bar{u}\cdot\nabla u + u\cdot\nabla\bar{u})(s)ds,$$

$$T^2_{\bar{u},\bar{\theta}}(u,\theta) = -\int_0^t e^{(t-s)\Delta}(\bar{u}\cdot\nabla\theta + u\cdot\nabla\bar{\theta})(s)ds,$$

$$\mathcal{F}_f(\theta) = \int_0^t e^{-(t-s)A}\mathbb{P}(\kappa\theta f)(s)ds.$$

We emphasize that the operators within (2.5)-(2.6) are in fact defined by duality, in other words, in distributional sense. More precisely, and analogously to the other ones, $T^1_{\bar{u}}(\cdot)$ is the operator that satisfies

$$\begin{aligned}\left\langle T^1_{\bar{u}}(u),\phi\right\rangle &= \int_0^t \left\langle(\bar{u}\otimes u(s) + u(s)\otimes\bar{u}),\nabla e^{-(t-s)A}\phi\right\rangle ds\\ &= \sum_{j,k=1}^n \int_0^t \left\langle \bar{u}_j(s)u_k(s) + u_j(s)\bar{u}_k(s), D_{x_j}(e^{-(t-s)A}\phi)_k\right\rangle ds, \end{aligned} \quad (2.7)$$

for all vector test $\phi \in L^{(n/(n-1),1)}_\sigma(\Omega)$ and $t>0$, where $D_{x_j} = \frac{\partial}{\partial x_j}$.

In the following, let us introduce suitable time-dependent functional spaces to study the initial value problem (2.2).

Definition 2.3 *Let $n < q < \infty$ and $\alpha = 1 - n/q$. We define the following functional spaces*

$$E = \{(u, \theta) : \ (u, \theta) \in BC((0, \infty); L^{(n,\infty)}_\sigma \times L^{(n,\infty)})\},$$
$$E_q = \{(u, \theta) : \ t^{\alpha/2}(u, \theta) \in BC((0, \infty); L^{(q,\infty)}_\sigma \times L^{(q,\infty)})\},$$

which are Banach spaces with the respective norms defined as:

$$\|(u, \theta)\|_E = \max\{\sup_{t>0} \|u(t)\|_{(n,\infty)}, \sup_{t>0} \|\theta(t)\|_{(n,\infty)}\},$$
$$\|(u, \theta)\|_{E_q} = \max\{\sup_{t>0} t^{\alpha/2}\|u(t)\|_{(q,\infty)}, \sup_{t>0} t^{\alpha/2}\|\theta(t)\|_{(q,\infty)}\}.$$

Now we are in position to give the precise definition of mild solution for (2.2). We assume that the steady solutions $(\bar{u}, \bar{\theta})$ of system (1.1) satisfies

$$(\bar{u}, \bar{\theta}) \in L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega). \tag{2.8}$$

Definition 2.4 *Let $(u_0, \theta_0) \in L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega)$. A pair of functions $(u(t, x), \theta(t, x))$ verifying*

$$\lim_{t\to 0^+} (u(t), \phi) = (u_0, \phi), \ \lim_{t\to 0^+} (\theta(t), \varphi) = (\theta_0, \varphi),$$

for all $\phi \in L^{(\frac{n}{n-1},1)}_\sigma(\Omega)$, $\varphi \in L^{(\frac{n}{n-1},1)}(\Omega)$, is said a global mild solution for the initial value problem (2.2) in the class E_q (or E), if (u, θ) satisfies the integral equations (2.3)-(2.4) in sense of distribution (cf. Remark 2.2), for all $t > 0$.

Our main results are the following:

Theorem 2.5 *Let $n \geq 3$, $n < q < \infty$ and $(u_0, \theta_0) \in L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega)$. Assume the condition (2.8) and f such that $\sup_{t>0} t^{\beta/2}\|f(t)\|_{(b,\infty)} < \infty$, where $\beta = 2 - (n/b), b \geq n/2$. There are constants $C_q, K_1, \delta_q, , \eta_q > 0$, with δ_q , η_q small enough, such that if $\max\{\|u_0\|_{(n,\infty)}, \|\theta_0\|_{(n,\infty)}\} < \delta_q$ and $\max\{\|\bar{u}\|_{(n,\infty)}, \|\bar{\theta}\|_{(n,\infty)}\} + \sup_{t>0} t^{\beta/2}\|f(t)\|_{(b,\infty)} < \eta_q$, then the initial value problem (2.2) has a global mild solution $u(t, x) \in E_q$, which is the unique solution that satisfies $\|u\|_{E_q} \leq \frac{2C_q\delta_q}{1-K_1\eta_q}$.*

In the case $\Omega = \mathbb{R}^n$, since in previous theorem we only assume conditions over scaling invariant norms, we can prove the existence of self-similar disturbance (u, θ) under assumptions of homogeneity for $u_0, \bar{u}, \theta_0, \bar{\theta}$ and f. This is the content of the next corollary.

Corollary 2.6 *(Self-similarity in $\mathbb{R}^n$) Under the assumptions of Theorem 2.5, assuming that $\Omega = \mathbb{R}^n$; $u_0, \bar{u}, \theta_0, \bar{\theta}$ being homogeneous of degree -1, and f satisfying the scale relation $f(t, x) = \lambda^2 f(\lambda^2 t, \lambda x)$, then the disturbance solution (u, θ) obtained through Theorem 2.5 is self-similar, that is, for $\lambda > 0$, $\lambda u(\lambda^2 t, \lambda x) = u(t, x)$, $\lambda\theta(\lambda^2 t, \lambda x) = \theta(t, x)$, almost everywhere $x \in \mathbb{R}^n, t > 0$.*

In the next corollary, we apply Theorem 2.5 in the physical context of Bénard problem.

Corollary 2.7 *(Bénard problem) Assume the same hypothesis of Theorem 2.5 with $b = n/2$ ($\beta = 0$) and $f = G\frac{x}{|x|^3}$ being the Newtonian gravitation field. If κG, is sufficiently small, then the initial value problem (2.2) has a global mild solution $(u(t,x), \theta(t,x)) \in E_q$.*

Theorem 2.5 supplies the existence of solutions with a convergence rate to the steady solution. Using only the classical estimate of the Stokes semigroup (3.1) and the analogous estimate for the heat semigroup, we can bound the norm $\|\cdot\|_E$ of the operators (2.5) and (2.6) by working with the norm $\|\cdot\|_{E_q}$ of the solution obtained, and thus, we guarantee that the solutions also lie in class E. However, proceeding in the same way, we can not assure the uniqueness of disturbance (u, θ) in E, because we have used the norm of space E_q to estimate the norm $\|\cdot\|_E$. Happily, the bilinear operator (2.5) and the coupling term (2.6) can be estimated using only the norm $\|\cdot\|_E$ of (u_1, θ_1) and (u_2, θ_2) (see (3.10), (3.11) below). We have the following theorem:

Theorem 2.8 *Let $n \geq 3$, $(u_0, \theta_0) \in L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega)$, and assume the condition (2.8). There are constants $C_n, K_3, \delta, \eta > 0$, with δ, η small enough, such that if $\max\{\|u_0\|_{(n,\infty)}, \|\theta_0\|_{(n,\infty)}\} < \delta$, $\max\{\|\bar{u}\|_{(n,\infty)}, \|\bar{\theta}\|_{(n,\infty)}\} + \sup_{t>0} t^{\beta/2}\|f(t)\|_{(b,\infty)} < \eta$, $\beta = 2 - (n/b), b > n/2$, then the initial value problem (2.2) has a global mild solution $(u(t,x), \theta(t,x)) \in E$, which is the unique solution that satisfies $\|u\|_E \leq \frac{2C_n\delta}{1-K_3\eta}$.*

Remark 2.9

- *(Uniqueness) Using the arguments found in [20] along with the estimates (3.10), (3.11) below, and assuming that $\lim_{t\to 0} \|e^{-(t-s)A}u_0 - u_0\|_{(n,\infty)} = \lim_{t\to 0} \|e^{(t-s)\Delta}\theta_0 - \theta_0\|_{(n,\infty)} = 0$, we can prove the uniqueness of solution (including large solutions) in $C([0,T); L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega))$. The last class of initial data contains $\overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{(n,\infty)}} \times \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{(n,\infty)}} \supset L^n_\sigma(\Omega) \times L^n(\Omega)$.*

- *(More decay) Firstly, since we already know that $(u, \theta) \in E$, by real interpolation we observe that for $n < r < q < \infty$, the solution obtained in Theorem 2.5 lies in the Lebesgue space version of E_r, that is, $t^{\frac{1}{2}-\frac{n}{2r}}(u, \theta) \in BC((0,\infty), L^r_\sigma(\Omega) \times L^r(\Omega))$. On the other hand, in previous theorems, assuming $(u_0, \theta_0) \in L^{(n,\infty)}_\sigma \cap L^{(p,\infty)}$ with $1 < p' < n$ and considering smallness assumptions on $f, \bar{u}, u_0, \bar{\theta}, \theta_0$, we can prove that the previous solution (u, θ) verifies the additional property $(u, \theta) \in BC((0,\infty), L^{(p,\infty)}_\sigma(\Omega) \times L^{(p,\infty)}(\Omega))$. Moreover, if $p < r < n$ then*
$$t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}(u, \theta) \in BC((0,\infty), L^r_\sigma(\Omega) \times L^r(\Omega)).$$

- *(Bounded domains) If Ω is a bounded domain, the above results hold with a further exponential decay rate. More exactly, the statements still are verified by replacing, respectively, the spaces E and E_q by*
$$E_{\exp} = \{(u, \theta) : \ e^{\mu t}(u, \theta) \in BC((0,\infty); L^{(n,\infty)}_\sigma \times L^{(n,\infty)})\},$$
$$E_{q\exp} = \{(u, \theta) : \ e^{\mu t} t^{\alpha/2}(u, \theta) \in BC((0,\infty); L^{(q,\infty)}_\sigma \times L^{(q,\infty)})\},$$
where $\mu > 0$ is a constant that depends on Ω (cf. Remark 3.4).

Theorems (2.5) and (3.8) imply that $\|u\|_{(q,\infty)}, \|\theta\|_{(q,\infty)} = O(t^{-\alpha/2})$ and $\|u\|_{(n,\infty)}, \|\theta\|_{(n,\infty)} = O(1)$ as $t \to \infty$, respectively. In the next theorem, by assuming additional conditions on the initial disturbance (u_0, θ_0), one will show that $\|u\|_{(q,\infty)}, \|\theta\|_{(q,\infty)} = o(t^{-\alpha/2})$ and $\|u\|_{(n,\infty)}, \|\theta\|_{(n,\infty)} = o(1)$ as $t \to \infty$, which improve the previous results of convergence for steady solutions. Also, if the initial disturbance of the steady solution belongs to Lebesgue space $L^n_\sigma(\Omega) \times L^n(\Omega)$, then the disturbance $(u(t), \theta(t))$ converges to $(0, 0)$, as $t \to \infty$, in the L^n-norm and in the strong decay norm $t^{\frac{\alpha}{2}}\|\cdot\|_q$ (cf. Remark 2.11). Our results now read as below.

Theorem 2.10 *Assume that (u, θ) is a mild solution of (2.2) obtained through Theorem 2.5, corresponding to steady solution $(\bar{u}, \bar{\theta}) \in L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega)$ and the initial data $(u_0, \theta_0) \in L^{(n,\infty)}_\sigma(\Omega) \times L^{(n,\infty)}(\Omega)$. If $\lim_{t\to\infty} t^{\frac{\alpha}{2}} \left\|e^{-tA}u_0\right\|_{(q,\infty)} = 0$ and $\lim_{t\to\infty} t^{\frac{\alpha}{2}} \left\|e^{t\Delta}\theta_0\right\|_{(q,\infty)} = 0$, then*

$$\lim_{t\to\infty} t^{\frac{\alpha}{2}} \|u(t)\|_{(q,\infty)} = \lim_{t\to\infty} t^{\frac{\alpha}{2}} \|\theta(t)\|_{(q,\infty)} = 0. \tag{2.9}$$

Moreover, in Theorems 2.8, if $\lim_{t\to\infty} \left\|e^{-tA}u_0\right\|_{(n,\infty)} = \lim_{t\to\infty} \left\|e^{t\Delta}\theta_0\right\|_{(n,\infty)} = 0$, then

$$\lim_{t\to\infty} \|u(t)\|_{(n,\infty)} = \lim_{t\to\infty} \|\theta(t)\|_{(n,\infty)} = 0. \tag{2.10}$$

Finally, if $(u_0, \theta_0) \in L^n_\sigma(\Omega) \times L^n(\Omega)$, the limits (2.9) and (2.10) hold.

Remark 2.11

- *In the context of the Navier-Stokes equations,results of stability in $L^{(n,\infty)}$ has been studied in [2, 5, 4] and some references therein.*
- *Under additional smallness conditions, only on the norms $\|u_0\|_{L^n}, \|\theta_0\|_{L^n}$, we can prove that the mild solution obtained through Theorem 2.5 and Theorem 2.8 lies in the space $BC\left((0,\infty), L^n_\sigma(\Omega) \times L^n_\sigma(\Omega)\right)$ and $t^{\alpha/2}(u,t) \in BC\left((0,\infty), L^q_\sigma(\Omega) \times L^q_\sigma(\Omega)\right)$. Moreover, analogously to the Theorem 2.10, we can show the decay*

$$\lim_{t\to\infty} \|u(t)\|_{L^n} = \lim_{t\to\infty} t^{\alpha/2} \|u(t)\|_{L^q} = 0 = \lim_{t\to\infty} \|\theta(t)\|_{L^n} = \lim_{t\to\infty} t^{\alpha/2} \|\theta(t)\|_{L^q}.$$

- *(Bounded domains) According Remark 2.9, in the case of bounded domains, Theorem 2.10 can be improvement. In fact, by assuming $\lim_{t\to\infty} e^{\mu t} t^{\frac{\alpha}{2}} \left\|e^{-tA}u_0\right\|_{(q,\infty)} = \lim_{t\to\infty} e^{\mu t} t^{\frac{\alpha}{2}} \left\|e^{t\Delta}\theta_0\right\|_{(q,\infty)} = 0$, we obtain*

$$\lim_{t\to\infty} e^{\mu t} t^{\frac{\alpha}{2}} \|u(t)\|_{(q,\infty)} = \lim_{t\to\infty} e^{\mu t} t^{\frac{\alpha}{2}} \|\theta(t)\|_{(q,\infty)} = 0,$$

and, analogously, the exponential decay version of (2.10) also holds.

3 Proof of Results

In this section we will develop the proofs of the results stated in Section 2. For this, we start with the following lemma in a generic Banach space, (cf. [7]), which generalizes the Theorem 13.2 of [20]. For a proof we also refer the reader to [18]. The proof is based on the Banach fixed point theorem.

Lemma 3.1 *Let X be a Banach space with norm $\|\cdot\|_X$, $T: X \to X$ a linear continuous map with norm $\|\cdot\|_T \le \tau < 1$ and $B: X \times X \to X$ a continuous bilinear map, that is, there is a constant $K > 0$ such that for all x_1 and x_2 in X we have*

$$\|B(x_1, x_2)\|_X \le K \; \| x_1\|_X \; \| x_2\|_X.$$

Then, if $0 < \varepsilon < \frac{(1-\tau)^2}{4K}$ and for any vector $y \in X$, $y \neq 0$, such that $\|y\|_X \le \varepsilon$, there is a solution $x \in X$ for the equation $x = y + B(x, x) + T(x)$ such that $\|x\|_X \le \frac{2\varepsilon}{1-\tau}$. The solution x is unique in the ball $\overline{B}(0, \frac{2\varepsilon}{1-\tau})$. Moreover, the solution depends continuously on y in the following sense: If $\|\tilde{y}\|_X \le \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x}) + T(\tilde{x})$ and $\|\tilde{x}\|_X \le \frac{2\varepsilon}{1-\tau}$, then

$$\| x - \tilde{x}\|_X \le \frac{1-\tau}{(1-\tau)^2 - 4K\varepsilon} \| y - \tilde{y}\|_X.$$

In order to prove the theorems of Section 2, we will need some lemmas.

Lemma 3.2 *Let $\gamma < 2$ and $1 < p \le q < \infty$, with the additional restrictions $q \le n$ if Ω is an exterior domain. Then, there is a constant $C = C(n, p, q) > 0$ such that*

$$\sup_{t>0} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{j}{2}} \left\|\nabla^j e^{-tA}\phi\right\|_{(q,1)} \le C \|\phi\|_{(p,\infty)}, \ \forall \phi \in L^{(p,\infty)}_\sigma(\Omega), \ j = 0, 1, \tag{3.1}$$

$$\int_0^t (t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \left\|e^{-(t-s)A}\phi\right\|_{(q,1)} ds \le C \|\phi\|_{(p,1)}, \ \forall \phi \in L^{(p,1)}_\sigma(\Omega), \ t > 0, \tag{3.2}$$

$$\int_0^t s^{-\frac{\gamma}{2}} (t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \left\|e^{-(t-s)A}\phi\right\|_{(q,1)} ds \le C t^{-\frac{\gamma}{2}} \|\phi\|_{(p,1)}, \ \forall \phi \in L^{(p,1)}_\sigma(\Omega), \ t > 0, \tag{3.3}$$

$$\int_0^t (t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \left\|\nabla e^{-(t-s)A}\phi\right\|_{(q,1)} ds \le C \|\phi\|_{(p,1)}, \ \forall \phi \in L^{(p,1)}_\sigma(\Omega), \ t > 0, \tag{3.4}$$

$$\int_0^t s^{-\frac{\gamma}{2}} (t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \left\|\nabla e^{-(t-s)A}\phi\right\|_{(q,1)} ds \le C t^{-\frac{\gamma}{2}} \|\phi\|_{(p,1)}, \ \forall \phi \in L^{(p,1)}_\sigma(\Omega), \ t > 0. \tag{3.5}$$

Remark 3.3 *When Ω is an exterior domain, in the estimates (3.1) (case $j = 0$), (3.2), (3.3) it is not necessary the additional assumption $q \le n$. In the case of the heat semigroup $\{e^{\Delta t}\}_{t\ge 0}$, the corresponding estimates (3.1)-(3.5) are also true for all $\phi \in L^{(p,1)}(\Omega)$.*

Proof.- The proof of (3.1) is well known and it follows by using the well known $L^p - L^q$ estimates of the Stokes semigroup together real interpolation (cf. [23, pp. 648-649]). On the other hand, the estimates (3.2) and (3.4) are due to Yamazaki and it can be found in [23]. So, we only prove the inequality (3.3) and posteriorly, we comment about the proof of (3.5). For this, we split the integral in the left-side of (3.3) in two parts $\int_0^{t/2} + \int_{t/2}^t = I_1 + I_2$. Then, we use (3.1) with $j = 0$ and the inequality $(t-s)^{-1} \le 2t^{-1}$ for $0 \le s \le \frac{t}{2}$, in order to estimate I_1 as follows:

$$I_1 \le C \int_0^{t/2} s^{-\frac{\gamma}{2}} (t-s)^{-1} \|\phi\|_{(p,1)} ds \le C t^{-1} \int_0^{t/2} s^{-\frac{\gamma}{2}} ds \|\phi\|_{(p,1)} = C t^{-\frac{\gamma}{2}} \|\phi\|_{(p,1)}.$$

Next, we use the inequality $s^{-\frac{\gamma}{2}} \leq 2^{\frac{\gamma}{2}} t^{-\frac{\gamma}{2}}$ for $\frac{t}{2} \leq s \leq t$, and (3.2), in order to estimate I_2 as follows:

$$\begin{aligned} I_2 &\leq 2^{\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \int_{t/2}^{t} (t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \left\| e^{-(t-s)A}\phi \right\|_{(q,1)} ds \\ &\leq 2^{\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \int_{0}^{\infty} s^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \left\| e^{-sA}\phi \right\|_{(q,1)} ds \leq C t^{-\frac{\gamma}{2}} \left\| \phi \right\|_{(p,1)}, \end{aligned}$$

and hence the proof of (3.3) is finished. The inequality (3.5) follows by using analogous arguments to these presented in the proof of (3.3), applying (3.1) with $j = 1$ and (3.4) instead of (3.2) (cf. [4]). $\diamond$

Remark 3.4 *As we have already said, the proof of (3.1) follows by using the well known $L^p - L^q$ estimates of the Stokes semigroup together real interpolation. Moreover, the time decay of $L^p - L^q$ estimates is preserved. However, in a bounded domain, $L^p - L^q$ estimates can be improved by a exponential decay, namely*

$$\sup_{t>0} e^{\mu t} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{j}{2}} \left\| \nabla^j e^{-tA}\phi \right\|_{L^q} \leq C \left\| \phi \right\|_{L^p}, \ \forall \phi \in L^p_\sigma(\Omega),$$

where $\mu > 0$. On the other hand, the proof of estimates (3.2) and (3.4) relies basically in real interpolation and using (3.1). In view of the above comments and proceeding in an entirely parallel way to [23, pp. 648-649], we can prove a sharp version of (3.1), (3.2) and (3.4) with further exponential decay. Finally, (3.3) and (3.5) are improved by using the sharp versions of (3.1), (3.2), (3.4) and similar arguments to proof of Lemma 3.2. For instance, in place of (3.1) and (3.5), we can obtain, respectively

$$\sup_{t>0} e^{\mu t} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{j}{2}} \left\| \nabla^j e^{-tA}\phi \right\|_{(q,1)} \leq C \left\| \phi \right\|_{(p,\infty)} \quad \forall \phi \in L^{(p,1)}_\sigma(\Omega),$$

$$\int_0^t s^{-\frac{\gamma}{2}} e^{\mu(t-s)} (t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \left\| \nabla e^{-(t-s)A}\phi \right\|_{(q,1)} ds \leq C t^{-\frac{\gamma}{2}} \left\| \phi \right\|_{(p,1)}, \ \forall \phi \in L^{(p,1)}_\sigma(\Omega), \ t > 0.$$

Lemma 3.5 *Let $3 \leq n < q < \infty$ and f such that $\sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} < \infty$, where $\beta = 2 - (n/b), b \geq n/2$. Then the following estimate holds*

$$\sup_{t>0} t^{\alpha/2} \|\mathcal{F}_f(\theta)\|_{(q,\infty)} \leq C \sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} \sup_{t>0} t^{\alpha/2} \|\theta(t)\|_{(q,\infty)}. \tag{3.6}$$

Moreover, if we assume either, $n \geq 3$ and $b > n/2$, or, $n \geq 4$ and $b = n/2$, then

$$\sup_{t>0} \|\mathcal{F}_f(\theta)\|_{(n,\infty)} \leq C \sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} \sup_{t>0} \|\theta(t)\|_{(n,\infty)}. \tag{3.7}$$

Proof.- Firstly we prove the estimate (3.6) in case $b = n/2$ ($\beta = 0$). Let l be such that $1 < q' < l$ with $\frac{1}{l} + \frac{1}{q} + \frac{1}{b} = 1$ and thus $\frac{n}{2}(\frac{1}{q'} - \frac{1}{l}) - 1 = \frac{n}{2}(\frac{1}{b}) - 1 = 0$. Applying Hölder's inequality (2.1) and the estimate (3.3) with $\gamma = \alpha$, $q = l$ and $p = q'$, we obtain

$$
\begin{aligned}
|\langle \mathcal{F}_f(\theta), \phi \rangle| &= \left| \int_0^t \left\langle \kappa e^{-(t-s)A}\mathbb{P}(\theta f), \phi \right\rangle ds \right| = \left| \int_0^t \left\langle \kappa\theta f, e^{-(t-s)A}\phi \right\rangle ds \right| \\
&\leq C \int_0^t \|\theta f\|_{(\frac{qb}{q+b},\infty)} \left\| e^{-(t-s)A}\phi \right\|_{(l,1)} ds \\
&\leq C \int_0^t \|\theta(s)\|_{(q,\infty)} \|f(s)\|_{(n/2,\infty)} \left\| e^{-(t-s)A}\phi \right\|_{(l,1)} ds \\
&\leq C \int_0^t s^{-\alpha/2} \left\| e^{-(t-s)A}\phi \right\|_{(l,1)} ds \ \sup_{t>0} \|f(t)\|_{(n/2,\infty)} \sup_{t>0} t^{\alpha/2} \|\theta(t)\|_{(q,\infty)} \\
&\leq C t^{-\alpha/2} \|\phi\|_{(q',1)} \sup_{t>0} \|f(t)\|_{(n/2,\infty)} \ \sup_{t>0} t^{\alpha/2} \|\theta(t)\|_{(q,\infty)}, \quad \forall \phi \in L_\sigma^{(q',1)}(\Omega),
\end{aligned}
$$

which, by duality, implies (3.6). In the case $b > n/2$, we bound the norm $\sup_{t>0} t^{\alpha/2} \|\mathcal{F}_f(\theta)\|_{(q,\infty)}$ by applying directly the inequality (3.1) and Hölder's inequality (2.1). So we have

$$
\begin{aligned}
\|\mathcal{F}_f(\theta)\|_{(q,\infty)} &\leq C \int_0^t (t-s)^{-\frac{n}{2b}} \|f(s)\|_{(b,\infty)} \|\theta(s)\|_{(q,\infty)} ds \\
&\leq C \int_0^t (t-s)^{-\frac{n}{2b}} s^{-\frac{\beta}{2}-\frac{\alpha}{2}} ds \ \sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|\theta(t)\|_{(q,\infty)} \\
&\leq C t^{-\frac{\alpha}{2}} \sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|\theta(t)\|_{(q,\infty)}.
\end{aligned}
$$

The proof of (3.7), when $n \geq 4$ and $b = n/2$, follows in an analogous way to the proof of (3.6) in the case $b = n/2$ ($\beta = 0$). Finally, the proof of (3.7), case $n \geq 3$ and $b > n/2$, follows exactly as the proof of (3.6) when $b > n/2$. Hence the proof of lemma is finished.

$\diamond$

Lemma 3.6 *Let $n \geq 3$, $b \geq n/2$, $n < q < \infty$. Assume $\bar{u} \in L_\sigma^{(n,\infty)}(\Omega), \bar{\theta} \in L^{(n,\infty)}(\Omega)$ and $\sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} < \infty$, and consider the operators $T(\cdot), B(\cdot,\cdot)$ defined by (2.5)-(2.6). Then there are constants $K_1, K_2, K_3, K_4 > 0$ such that for all $(u_1,\theta_1), (u_2,\theta_2) \in E_q$, the following inequalities hold:*

$$\|T(u_1,\theta_1)\|_{E_q} \leq K_1(\ \max\{\|\bar{u}\|_{(n,\infty)}, \|\bar{\theta}\|_{(n,\infty)}\} + \sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} \ ) \|(u_1,\theta_1)\|_{E_q}, \tag{3.8}$$

$$\|B((u_1,\theta_1),(u_2,\theta_2))\|_{E_q} \leq K_2 \|(u_1,\theta_1)\|_{E_q} \|(u_2,\theta_2)\|_{E_q}, \tag{3.9}$$

$$\|T(u_1,\theta_1)\|_{E} \leq K_3(\ \max\{\|\bar{u}\|_{(n,\infty)}, \|\bar{\theta}\|_{(n,\infty)}\} + \sup_{t>0} t^{\beta/2} \|f(t)\|_{(b,\infty)} \ ) \|(u_1,\theta_1)\|_{E}, \tag{3.10}$$

$$\|B((u_1,\theta_1),(u_2,\theta_2))\|_{E} \leq K_4 \|(u_1,\theta_1)\|_{E} \|(u_2,\theta_2)\|_{E}. \tag{3.11}$$

Proof.- When $\theta_1 = \theta_2 = 0$ the estimates (3.9) and (3.11) are reduced to the bi-continuity of the bilinear form of the Navier-Stokes equation on the spaces E_q and E, respectively (see [20, 21, 23]). Thus (3.9) and (3.11) are an extension for the context of Boussinesq system and its proof can be found in [18]. On the order hand, (3.10) elapses of (3.7) and (3.11) by making $(u_2, \theta_2) = (\bar{u}, \bar{\theta})$. Therefore we will only prove (3.8). Let us take l such that $1 < q' < n' < l \leq n$ satisfying $\frac{1}{l} + \frac{1}{q} + \frac{1}{n} = 1$. Observe that $\frac{n}{2}(\frac{1}{q'} - \frac{1}{l}) - \frac{1}{2} = 0$. Since $div(u) = div(\bar{u}) = 0$, one can write $\bar{u} \cdot \nabla u = \nabla(\bar{u} \otimes u)$ and $u \cdot \nabla \bar{u} = \nabla(u \otimes \bar{u})$. In order to deal with the norm $\sup_{t>0} t^{\alpha/2} \|T^1_{\bar{u}}(u)\|_{(q,\infty)}$ we take $\phi \in L^{(q',1)}_\sigma(\Omega)$ and bound

$$
\begin{aligned}
\left|\left\langle T^1_{\bar{u}}(u), \phi\right\rangle\right| &= \left|\int_0^t \left\langle (\bar{u} \otimes u(s) + u(s) \otimes \bar{u}), \nabla e^{-(t-s)A}\phi\right\rangle ds\right| \\
&\leq C \int_0^t \|\bar{u} \otimes u(s)\|_{(\frac{qn}{q+n},\infty)} \left\|\nabla e^{-(t-s)A}\phi\right\|_{(l,1)} ds \\
&\leq C \int_0^t \|\bar{u}\|_{(n,\infty)} \|u(s)\|_{(q,\infty)} \left\|\nabla e^{-(t-s)A}\phi\right\|_{(l,1)} ds \\
&\leq C \|\bar{u}\|_{(n,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|u(t)\|_{(q,\infty)} \int_0^t s^{-\frac{\alpha}{2}} \left\|\nabla e^{-(t-s)A}\phi\right\|_{(l,1)} ds.
\end{aligned}
$$

Now we use the inequality (3.5) to obtain

$$
\left|\left\langle T^1_{\bar{u}}(u), \phi\right\rangle\right| \leq C t^{-\frac{\alpha}{2}} \|\bar{u}\|_{(n,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|u(t)\|_{(q,\infty)} \|\phi\|_{(q',1)},
$$

for all $\phi \in L^{(q',1)}_\sigma(\Omega)$ and all $t > 0$, which implies

$$
\sup_{t>0} t^{\alpha/2} \|T^1_{\bar{u}}(u)\|_{(q,\infty)} \leq C \|\bar{u}\|_{(n,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|u(t)\|_{(q,\infty)}. \tag{3.12}
$$

Analogously we have

$$
\sup_{t>0} t^{\alpha/2} \|T^2_{\bar{u},\bar{\theta}}(u,\theta)\|_{(q,\infty)} \leq C(\|\bar{u}\|_{(n,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|\theta(t)\|_{(q,\infty)} + \left\|\bar{\theta}\right\|_{(n,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|u(t)\|_{(q,\infty)}). \tag{3.13}
$$

From (3.12), (3.13) and Lemma 3.5 we conclude the proof of (3.8).

◇

Proof of Theorem 2.5, Corollary 2.6 and Corollary 2.7

The proof of Theorem 2.5 is a direct application of Lemma 3.1, Lemma 3.6 (inequalities (3.8) and (3.9)) and Lemma 3.2 (estimate (3.1)). In fact, firstly we take $X = E_q$, $y = (e^{-tA}u_0, e^{t\Delta}\theta_0)$. Moreover we take the bilinear operator $B(\cdot,\cdot)$ and the linear operator $T(\cdot)$ defined by (2.5), (2.6), respectively. Let us denote by $\|\cdot\|_{T_q}$ the norm of linear operator $T(\cdot) : E_q \to E_q$. Now, we define $\|\cdot\|_{T_q} \leq \tau_q < 1$, $0 < \eta_q < \frac{1}{K_1}$, $0 < \varepsilon_q < \frac{(1-\tau_q)^2}{4K_2}$ and $0 < \delta_q = \frac{\varepsilon_q}{C_q}$, where K_1,K_2 are as in the Lemma 3.6, and C_q is as in the inequality (3.1) of the Lemma 3.2 when $q = q$ and $p = n$. Hence, we have that

$$
\|y\|_{E_q} = \left\|(e^{-tA}u_0, e^{t\Delta}\theta_0)\right\|_{E_q} \leq \varepsilon_q, \text{ provided } \max\{\|u_0\|_{(n,\infty)}, \|u_0\|_{(n,\infty)}\} \leq \delta_q.
$$

On the other hand,

$$\|\cdot\|_{T_q} \leq \tau_q = \eta_q K_1 < 1, \text{ provided } \max\{\|\bar{u}\|_{(n,\infty)}, \|\bar{\theta}\|_{(n,\infty)}\} + \sup_{t>0} t^{\beta/2}\|f(t)\|_{(b,\infty)} < \eta_q.$$

Now, applying Lemma 3.1, we obtain the existence of a global mild solution $(u,\theta) \in E_q$, which satisfies the integral equations (2.3)-(2.4). Moreover the solution is unique in the ball $\overline{B}(0, \frac{2\varepsilon_q}{1-\tau_q}) = \overline{B}(0, \frac{2C_q\delta_q}{1-\eta_q K_1})$ of E_q. In order to check that solution $(u,\theta) \in E_q$ is the mild solution in the sense of Definition 2.4, it remains to prove that $\lim_{t\to 0^+}(u(t),\phi) = (u_0,\phi)$ and $\lim_{t\to 0^+}(\theta(t),\varphi) = (\theta_0,\varphi)$, but we omit this part because it is standard. Hence, the proof of Theorem 2.5 is finished.

Now we will prove the Corollary 2.6. Let $\Omega = \mathbb{R}^n$ and note that, in this case, $A = -P\Delta = -\Delta$ on $L^{(n,\infty)}_\sigma(\mathbb{R}^n)$ and $e^{-tA}u_0 = G(t,\cdot) * u_0$, where the symbol $*$ denotes the convolution operator with the Gauss kernel $G(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$. Let $u_0, \bar{u}, \theta_0, \bar{\theta}$ be homogeneous functions of degree -1 and f satisfying the scale relation $f(t,x) = \lambda^2 f(\lambda^2 t, \lambda x)$. Theorem 2.5 has been proved by using Lemma 3.1. This method supplies the solution by a successive approximation method. Then, we define the following scheme:

$$\begin{aligned}
u_1 &= e^{-tA}u_0, \ \theta_1 = e^{t\Delta}\theta_0 \\
u_{m+1} &= e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}(u_m \cdot \nabla u_m)ds + \\
&\quad - \int_0^t e^{-(t-s)A}\mathbb{P}(u_m \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_m - \kappa\theta_m f)ds, \\
\theta_{m+1} &= e^{t\Delta}\theta_0 - \int_0^t e^{(t-s)\Delta}(u_m \cdot \nabla\theta_m)ds + \\
&\quad - \int_0^t e^{(t-s)\Delta}(u_m \cdot \nabla\bar{\theta} + \bar{u} \cdot \nabla\theta_m)ds,
\end{aligned}$$

where $m \in \mathbb{N}$. It is easy to verify that $(u_1(t,x), \theta_1(t,x))$ satisfies the scaling property

$$u_1(t,x) = \lambda u_1(\lambda^2 t, \lambda x), \ \theta_1(t,x) = \lambda\theta_1(\lambda^2 t, \lambda x).$$

A simple induction argument proves that (u_m, θ_m) has this property for all m. Therefore, the solution $(u(t,x), \theta(t,x))$ which is the limit of the sequence $\{(u_m, \theta_m)\}$, must verifies

$$u(t,x) = \lambda u(\lambda^2 t, \lambda x), \ \theta(t,x) = \lambda\theta(\lambda^2 t, \lambda x),$$

for almost everywhere $\lambda > 0$, $t > 0$ and $x \in \mathbb{R}^n$.

Finally, in order to prove Corollary 2.7, we need to check the hypothesis of Theorem 2.5. Note that as $b = n/2$ ($\beta = 0$) and $f = G\frac{x}{|x|^3}$, we have that $f \in L^{(n/2,\infty)}$. Moreover, by using Lemma 3.5 (inequality (3.6)) we have that

$$\sup_{t>0} t^{\alpha/2}\|\mathcal{F}_{G\frac{x}{|x|^3}}(\theta)\|_{(q,\infty)} \leq \kappa G C \sup_{t>0} t^{\alpha/2}\|\theta\|_{(q,\infty)}.$$

Therefore, using that κG is sufficiently small along with the smallness of $u_0, \bar{u}, \theta_0, \bar{\theta}$, we obtain the existence of constants K_1, η_q, δ_q verifying the hypothesis of Theorem 2.5 and hence the stability of steady solution for the Bénard problem in E_q is proved. ⋄

Proof of Theorem 2.8

To prove Theorem 2.8 we also apply Lemma 3.1. In this case we take $X = E$, $y = (e^{-tA}u_0, e^{t\Delta}u_0)$, and $B(\cdot,\cdot)$, $T(\cdot)$ defined by (2.5), (2.6). Let us denote by $\|\cdot\|_T$ the norm of linear operator $T(\cdot): E \to E$. Now, we define $\|\cdot\|_T \le \tau < 1$, $0 < \eta < \frac{1}{K_3}$, $0 < \varepsilon < \frac{(1-\tau)^2}{4K_4}$ and $0 < \delta = \frac{\varepsilon}{C_n}$, where K_3 and K_4 are as in the Lemma 3.6, and C_n is as in the inequality (3.1) of the Lemma 3.2 when $q = p = n$. Now, we have that

$$\|y\|_E = \left\|(e^{-tA}u_0, e^{t\Delta}u_0)\right\|_E \le \varepsilon, \text{ provided } \max\{\|u_0\|_{(n,\infty)}, \|\theta_0\|_{(n,\infty)}\} \le \delta.$$

On the other hand,

$$\|\cdot\|_T \le \tau = \eta K_3 < 1, \text{ provided } (\max\{\|\bar{u}\|_{(n,\infty)}, \|\bar{\theta}\|_{(n,\infty)}\} + \sup_{t>0} t^{\beta/2}\|f(t)\|_{(b,\infty)}) < \eta.$$

Hence, Lemma 3.1 guarantees the existence of a global mild solution $u \in E$ which satisfies (2.3)-(2.4). The solution is unique in the ball $\overline{B}(0, \frac{2\varepsilon}{1-\tau}) = \overline{B}(0, \frac{2C_n\delta}{1-\eta K_3})$ of E. ⋄

Proof of Theorem 2.10

We will only prove (2.9) because the proof of (2.10) follows a similar way. Before proceeding, let us remark that in inequalities (3.14),(3.15) below, the integrals represent the corresponding operators defined in a distributional sense (cf. Remark 2.2 and equality (2.7)). Taking the norm $t^{\alpha/2}\|.\|_{(q,\infty)}$ in (2.3)-(2.4) we obtain

$$\begin{aligned} t^{\frac{\alpha}{2}}\|u(t)\|_{(q,\infty)} &\le t^{\frac{\alpha}{2}}\left\|e^{-tA}u_0\right\|_{(q,\infty)} \\ &+ t^{\frac{\alpha}{2}}\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\nabla((u\otimes u)(s))ds\right\|_{(q,\infty)} + t^{\frac{\alpha}{2}}\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\nabla((\bar{u}\otimes u)+(u\otimes\bar{u})(s))ds\right\|_{(q,\infty)} \\ &+ t^{\frac{\alpha}{2}}\left\|\int_0^t e^{-(t-s)A}\mathbb{P}(\kappa(\theta f)(s))ds\right\|_{(q,\infty)} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} t^{\frac{\alpha}{2}}\|\theta(t)\|_{(q,\infty)} &\le t^{\frac{\alpha}{2}}\left\|e^{t\Delta}\theta_0\right\|_{(q,\infty)} \\ &+ t^{\frac{\alpha}{2}}\left\|\int_0^t e^{(t-s)\Delta}\nabla(\theta u(s))ds\right\|_{(q,\infty)} + t^{\frac{\alpha}{2}}\left\|\int_0^t e^{(t-s)\Delta}\nabla(\theta\bar{u}+\bar{\theta}u)(s)ds\right\|_{(q,\infty)}. \end{aligned} \tag{3.15}$$

Making the change of variables $s = tz$ and using that $(u,\theta) \in \overline{B}(0, \frac{2C_q\delta_q}{1-\eta_q K_1})$, we can estimate the second norm on the right-hand side of (3.14) by the expression

$$
\begin{aligned}
&C(\sup_{t>0} t^{\alpha/2} \|u\|_{(q,\infty)}) \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|u(s)\|_{(q,\infty)}\, ds \\
&\leq \frac{2\delta_q C_q}{1-\eta_q K_1} C \int_0^1 (1-z)^{\frac{\alpha}{2}-1} z^{-\frac{\alpha}{2}} (tz)^{\frac{\alpha}{2}} \|u(tz)\|_{(q,\infty)}\, dz,
\end{aligned} \tag{3.16}
$$

for all $t > 0$.

Now we deal with the third norm on the right-hand side of (3.14). It can be bounded by

$$
\begin{aligned}
&t^{\frac{\alpha}{2}} \left\| \int_0^{t\xi} e^{-(t-s)A} \mathbb{P}\nabla(\bar{u}\otimes u + u \otimes \bar{u})(s) ds \right\|_{(q,\infty)} + t^{\frac{\alpha}{2}} \left\| \int_{t\xi}^t e^{-(t-s)A} \mathbb{P}\nabla(\bar{u}\otimes u + u \otimes \bar{u})(s) ds \right\|_{(q,\infty)} \\
&:= I_1 + I_2,
\end{aligned}
$$

where the constant ξ will be chosen later. We estimate I_1 and I_2 as follows:

$$
I_1 \leq C \|\bar{u}\|_{(n,\infty)} t^{\frac{\alpha}{2}} \int_0^{t\xi} (t-s)^{-1} \|u(s)\|_{(q,\infty)}\, ds \leq C \|\bar{u}\|_{(n,\infty)} \int_0^{\xi} (1-z)^{-1} z^{-\frac{\alpha}{2}} (tz)^{\frac{\alpha}{2}} \|u(tz)\|_{(q,\infty)}\, dz, \tag{3.17}
$$

$$
I_2 \leq K_1 \|\bar{u}\|_{(n,\infty)} \sup_{t\xi<s<t} s^{\frac{\alpha}{2}} \|u(s)\|_{(q,\infty)}\,. \tag{3.18}
$$

Now we bound the forth norm on the right-hand side of (3.14). If $b > n/2$, working as in the proof of (3.6) we have

$$
\begin{aligned}
t^{\frac{\alpha}{2}} \left\| \int_0^t e^{-(t-s)A} \mathbb{P}(\kappa(\theta f)(s)) \right\|_{(q,\infty)} &\leq C \int_0^t (t-s)^{-n/2b} \|f(s)\|_{(b,\infty)} \|\theta(s)\|_{(q,\infty)} ds \\
&\leq C \sup_{t>0} t^{\beta/2} \|f(s)\|_{(b,\infty)} \int_0^1 (1-z)^{-n/2b} z^{-\beta/2} \|\theta(tz)\|_{(q,\infty)} dz.
\end{aligned} \tag{3.19}
$$

If $b = n/2$, we get

$$
\begin{aligned}
t^{\frac{\alpha}{2}} \left\| \int_0^t e^{-(t-s)A} \mathbb{P}(\kappa\theta f) ds \right\|_{(q,\infty)} &\leq t^{\frac{\alpha}{2}} \left\| \int_0^{t\xi} e^{-(t-s)A} \mathbb{P}(\kappa\theta f) ds \right\|_{(q,\infty)} + t^{\frac{\alpha}{2}} \left\| \int_{t\xi}^t e^{-(t-s)A} \mathbb{P}(\kappa\theta f) ds \right\|_{(q,\infty)} \\
&:= M_1 + M_2,
\end{aligned} \tag{3.20}
$$

where the constant ξ will be chosen later. Then we have

$$
\begin{aligned}
M_1 &\leq C \sup_{t>0} \|f\|_{(n/2,\infty)} t^{\frac{\alpha}{2}} \int_0^{t\xi} (t-s)^{-1} \|\theta(s)\|_{(q,\infty)}\, ds \\
&\leq C \sup_{t>0} \|f\|_{(n/2,\infty)} \int_0^{\xi} (1-z)^{-1} z^{-\frac{\alpha}{2}} (tz)^{\frac{\alpha}{2}} \|\theta(tz)\|_{(q,\infty)}\, dz,
\end{aligned} \tag{3.21}
$$

$$
M_2 \leq C \sup_{t>0} \|f\|_{(n/2,\infty)} \sup_{t\xi<s<t} s^{\frac{\alpha}{2}} \|\theta(s)\|_{(q,\infty)}\,. \tag{3.22}
$$

Now, we define $\Gamma = \max\{\Gamma_1, \Gamma_2\}$, where

$$\Gamma_1 = \limsup_{t\to\infty} t^{\frac{\alpha}{2}} \left\| u(t) \right\|_{(q,\infty)}, \ \Gamma_2 = \limsup_{t\to\infty} t^{\frac{\alpha}{2}} \left\| \theta(t) \right\|_{(q,\infty)}.$$

Then, computing $\limsup_{t\to\infty}$ in (3.14) and using the inequalities (3.16)-(3.22), we get

$$\begin{aligned}\Gamma_1 &\leq \left(\frac{2\delta_q C_q}{1-\eta_q K_1} C \int_0^1 (1-s)^{\frac{\alpha}{2}-1} s^{-\frac{\alpha}{2}} ds + C\eta_q K_1 \int_0^\xi (1-z)^{-1} z^{-\frac{\alpha}{2}} dz + \eta_q K_1 \right) \Gamma \\ &\leq \left(\frac{2\delta_q C_q K_2}{1-\eta_q K_1} + C\eta_q K_1 (1-\xi)^{-1} \xi^{1-\frac{\alpha}{2}} + \eta_q K_1 \right) \Gamma. \end{aligned} \tag{3.23}$$

On the other hand, making the change of variables $s = tz$ we can estimate the second norm on the right-hand side of (3.15) by the expression

$$\begin{aligned} &C(\sup_{t>0} t^{\alpha/2} \left\| \theta \right\|_{(q,\infty)}) \int_0^t (t-s)^{\frac{\alpha}{2}-1} \left\| \theta(s) \right\|_{(q,\infty)} ds \\ &\leq \frac{2\delta_q C_q}{1-\eta_q K_1} C \int_0^1 (1-z)^{\frac{\alpha}{2}-1} z^{-\frac{\alpha}{2}} (tz)^{\frac{\alpha}{2}} \left\| \theta(tz) \right\|_{(q,\infty)} dz, \end{aligned} \tag{3.24}$$

for all $t > 0$. Now we deal with the third norm on the right-hand side of (3.15). It can be bounded by

$$\begin{aligned} &\leq t^{\frac{\alpha}{2}} \left\| \int_0^{t\xi} e^{-(t-s)\Delta} \nabla(\theta\bar{u} + \bar{\theta}u)(s) ds \right\|_{(q,\infty)} + t^{\frac{\alpha}{2}} \left\| \int_{t\xi}^t e^{-(t-s)\Delta} \nabla(\theta\bar{u} + \bar{\theta}u)(s) ds \right\|_{(q,\infty)} \\ &:= J_1 + J_2, \end{aligned}$$

where the constant ξ will be chosen later. We estimate J_1 and J_2 as follows:

$$\begin{aligned} J_1 &\leq C \left\| \bar{u} \right\|_{(n,\infty)} t^{\frac{\alpha}{2}} \int_0^{t\xi} (t-s)^{-1} \left\| \theta(s) \right\|_{(q,\infty)} ds + C \left\| \bar{\theta} \right\|_{(n,\infty)} t^{\frac{\alpha}{2}} \int_0^{t\xi} (t-s)^{-1} \left\| u(s) \right\|_{(q,\infty)} ds \\ &\leq C \left\| \bar{u} \right\|_{(n,\infty)} \int_0^\xi (1-z)^{-1} z^{-\frac{\alpha}{2}} (tz)^{\frac{\alpha}{2}} \left\| \theta(tz) \right\|_{(q,\infty)} dz \\ &+ C \left\| \bar{\theta} \right\|_{(n,\infty)} \int_0^\xi (1-z)^{-1} z^{-\frac{\alpha}{2}} (tz)^{\frac{\alpha}{2}} \left\| u(tz) \right\|_{(q,\infty)} dz, \end{aligned} \tag{3.25}$$

$$J_2 \leq K_1 \left\| \bar{u} \right\|_{(n,\infty)} \sup_{t\xi<s<t} s^{\frac{\alpha}{2}} \left\| \theta(s) \right\|_{(q,\infty)} + K_1 \left\| \bar{\theta} \right\|_{(n,\infty)} \sup_{t\xi<s<t} s^{\frac{\alpha}{2}} \left\| u(s) \right\|_{(q,\infty)}. \tag{3.26}$$

Computing $\limsup_{t\to\infty}$ in (3.15) we get

$$\begin{aligned} \Gamma_2 &\leq \left(\frac{2\delta_q C_q K_2}{1-\eta_q K_1} + C(\left\| \bar{u} \right\|_{(n,\infty)} + \left\| \bar{\theta} \right\|_{(n,\infty)})(1-\xi)^{-1} \xi^{1-\frac{\alpha}{2}} + \eta_q K_1 \right) \Gamma \\ &\leq \left(\frac{2\delta_q C_q K_2}{1-\eta_q K_1} + C\eta_q K_1 (1-\xi)^{-1} \xi^{1-\frac{\alpha}{2}} + \eta_q K_1 \right) \Gamma. \end{aligned} \tag{3.27}$$

Since $\frac{2\delta_q C_q K_2}{1-\eta_q K_1} + \eta_q K_1 < 1$ (see Lemma 3.1 and the proof of Theorem 2.5), we can choose $\xi > 0$ sufficiently small, such that

$$\frac{2\delta_q C_q K_2}{1-\eta_q K_1} + C\eta_q K_1 (1-\xi)^{-1}\xi^{1-\frac{\alpha}{2}} + \eta_q K_1 < 1.$$

Consequently, from (3.23) and (3.27) we have that the number $\Gamma = 0$. This proves the first part of theorem.

Now, let us deal with the last assertion of Theorem 2.10. It is sufficient to prove that, if the initial data $u_0 \in \overline{L^n_\sigma(\Omega) \cap L^p(\Omega)}^{\|\cdot\|_{(n,\infty)}}, \theta_0 \in \overline{L^n(\Omega) \cap L^p(\Omega)}^{\|\cdot\|_{(n,\infty)}}$, then

$$\lim_{t\to\infty} t^{\alpha/2}\|e^{-tA}u_0\|_{(q,\infty)} = 0, \ \lim_{t\to\infty} t^{\alpha/2}\|e^{t\Delta}\theta_0\|_{(q,\infty)} = 0.$$

Taking $u_{0,k} \in L^n_\sigma(\Omega) \cap L^p(\Omega)$ with $1 < p < n < q$, we have

$$\begin{aligned} t^{\alpha/2}\|e^{-tA}u_0\|_{(q,\infty)} &\leq t^{\frac{\alpha}{2}} \left\|u_0\right\|_p t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \\ &= \left\|u_0\right\|_p t^{-(\frac{n}{2p}-\frac{1}{2})} \to 0 \ \text{ as } t \to \infty. \end{aligned}$$

Therefore, using the density of $L^n_\sigma \cap L^p$ in $\overline{L^n_\sigma(\Omega) \cap L^p(\Omega)}^{\|\cdot\|_{(n,\infty)}}$, we can conclude that $\lim_{t\to\infty} t^{\alpha/2} \left\|e^{-tA}u_0\right\|_{(q,\infty)} = 0$. Likewise we can prove $\lim_{t\to\infty} t^{\alpha/2} \left\|e^{t\Delta}\theta_0\right\|_{(q,\infty)} = 0$. $\diamond$

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