Navier-Stokes equations with fractional dissipation in sum of pseudomeasure spaces.

Lucas C.F. Ferreira [a] & Elder J. Villamizar-Roa [b]

[a] *Universidade Estadual de Campinas, Departamento de Matemática, CEP 50740-540, Campinas-SP, Brazil.*
email: lcff@ime.unicamp.br, lucascatao@gmail.com

[b] *Universidad Industrial de Santander, Escuela de Matemáticas, A.A. 3840, Medellín, Colombia.*
email: ejvillamizarr@unalmed.edu.co

Abstract

In this paper is studied the local well-posedness of the Navier-Stokes system with initial data belonging to a sum of two pseudomeasure-type spaces denoted by $PM^{a,b} := PM^a + PM^b$. New results about local well-posedness and regularity of mild solutions of Navier-Stokes system are obtained. The proof requires to show an interesting Hölder-type inequality in $PM^{a,b}$, as well as to establish estimates of the semigroup generated by fractional power of Laplacian $(-\Delta)^\gamma$ on these spaces.

Key words: Navier-Stokes equations, Singular data, Fractional dissipation

1 Introduction and Statement Results

We consider the following initial value problem for the Navier-Stokes equations, with fractional dissipation, in the whole space $\mathbb{R}^n$:

$$\begin{cases} u_t + (-\Delta)^\gamma u + (u\cdot\nabla)u + \nabla p = f, & x\in\mathbb{R}^n,\ t>0, \\ \operatorname{div} u = 0, & x\in\mathbb{R}^n,\ t>0, \\ u(x,0) = u_0, & x\in\mathbb{R}^n, \end{cases} \tag{1}$$

where $\gamma \in (1/2, 1]$, $n \geq 2(2\gamma - 1)$ and $(-\Delta)^\gamma$ represents the Riesz potential operator which is defined, as usual, through the Fourier transform as $((-\Delta)^\gamma f)\hat{}\,(\xi) = |\xi|^{2\gamma} \widehat{f}(\xi)$, where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$. When $\gamma = 1$, the system (1) reduces to the well known Navier-Stokes equations.

It is well known that an open problem on the Clay Institute's list of prize problems is to know if, in three dimensions, there exists (or not) a smooth global solution of the Navier-Stokes equations. In connection with this problem, a motivation arises to study the Navier-Stokes equation in spaces that contain singular functions. In this paper we are interested in the study of local well-posedness of system (1) in a class of singular functions, denoted by $PM^{a,b} := PM^a + PM^b$, with $a, b \in [n-1, n)$, where PM^a denotes the functional space of pseudomeasure-type defined as

$$PM^a \equiv \left\{ v \in \mathcal{S}' : \hat{v} \in L^1_{loc}(\mathbb{R}^n), \|v\|_a \equiv ess \sup_{\xi \in \mathbb{R}^n} |\xi|^a |\hat{v}(\xi)| < \infty \right\},$$

where $a \geq 0$ is a given parameter. The norm in the $PM^{a,b}$-space is defined by

$$\|u\|_{a,b} = \inf\{\|u_1\|_a + \|u_2\|_b : u = u_1 + u_2, u_1 \in PM^a, u_2 \in PM^b\}.$$

We would like to remark that the pseudomeasure-type PM^a-space is a suitable space of distributions, which satisfies $\mathcal{D}(\mathbb{R}^n) \subset PM^a \subset \mathcal{D}'(\mathbb{R}^n)$, and for all $\lambda > 0$, there exists $K_\lambda > 0$ such that for all $f \in PM^a, f(\lambda x) \in PM^a$ and $\|f(\lambda x)\|_a \leq K_\lambda \|f\|_a$. In particular, the PM^a-space contains homogeneous functions of degree $a - n$.

The pseudomeasure-type PM^a-spaces appeared in the study of the Navier-Stokes system in [10,4]. In particular, in [10], to the best of our knowledge, was proved in first time the bicontinuity of the bilinear term

$$B(u, v) = -\int_0^t G_\gamma(t-s)\mathbb{P}\nabla \cdot (u \otimes v)(s)ds \tag{2}$$

in the PM^2-space when $n = 3, \gamma = 1$. Posteriorly, in [4] was proved the existence of singular solutions of the Navier-Stokes system in $\mathbb{R}^3$, with values in PM^2 and with singular external forces in the pseudomeasure space PM^1. Results of smoothness as well as asymptotic stability of small solutions, including stationary ones, were also proved in [4]. Also, the pseudomeasure-type PM^a-space has been considered in another fluid mechanics models as for instance, dissipative quasi-geostrophic equations, viscous Boussinesq equations and micropolar fluids, and results of existence of global and local mild solutions, regularity and asymptotic stability have been obtained (cf. [5–8]).

In this paper, besides dealing with the fractional dissipation case $\gamma \in (1/2, 1)$, when $\gamma = 1$, the results of [4,10] are improved in the sense that the mild solutions of Navier-Stokes system (1) are constructed with initial data in $PM^{a,b} := PM^a + PM^b$, a more general space than PM^a, obtaining the local well-posedness of mild solutions. For details, cf. Remark 1.4. Moreover, results about the regularity of solutions are also proved. To obtain our claim, as usual, the initial value problem (1) is studied via the following integral equation

$$u(t) = G_\gamma(t)u_0 - \int_0^t G_\gamma(t-s)\mathbb{P}[\nabla \cdot (u \otimes u)(s) - f(s)]ds, \tag{3}$$

where $\{G_\gamma(t)\}_{t\geq 0}$ is the semigroup generated by fractional power of Laplacian $(-\Delta)^\gamma$. The action of $G_\gamma(t)$ is given via convolution with kernel g_γ obtained by means of the Fourier transform $\widehat{g}_\gamma(\xi, t) = e^{-|\xi|^{2\gamma}t}$, the fundamental solution of the linear problem $\partial_t \psi + (-\Delta)^\gamma \psi = 0$. In (3), $\mathbb{P}$ denotes the Leray projector defined as $(\mathbb{P}u)_j = u_j + \Sigma_{k=1}^n R_j R_k u_k$, where the Riesz transforms R_j's are obtained as $\widehat{R_j h}(\xi) = \frac{i\xi_j}{|\xi|}\widehat{h}(\xi)$.

The local well-posedness of solutions $u(t)$ of the integral equation (3) with values in $PM^{a,b}$-spaces, is obtained by using the contraction algorithm. In this sense, the main difficulty in applying such algorithm is to establish the bicontinuity of the bilinear operator $B(u, v)$ in $PM^{a,b}$, which is the content of Proposition 2.6. The proof of Proposition 2.6 is based on a new Hölder-type inequality in $PM^{a,b}$-spaces, which will be proved in Proposition 2.4.

It is appropriate to cite some earlier works based on the Navier-Stokes system with singular initial data and comment about how large are the $PM^{a,b}$-spaces considered in this paper. Firstly, when $\gamma = 1$ (classical Navier-Stokes equations), the existence of solutions with initial data in $\dot{B}_p^{-(1-n/p),\infty}$ and bmo^{-1}, was obtained, respectively, in [2] and [9]. It is worthwhile to recall that $\dot{B}_p^{-s,\infty}$ denotes the well known homogeneous Besov space and bmo^{-1} denotes the space of tempered distributions v such that

$$\|v\|_{bmo^{-1}} := \sup_{x\in\mathbb{R}^n, 0<R<1} \left(|B(x,R)|^{-1} \int_{B(x,R)} \int_0^{R^2} |g(x,t) * v| \, dxdt \right)^{1/2} < \infty,$$

with $g(x, t)$ denoting the heat kernel.

In [4], the authors show that the inclusion $PM^a \subset \dot{B}_p^{-s,\infty}$ is continuous, with $s = n - a - n/p$, $a \in (0, n)$ and $p \in (\frac{n}{n-a}, \infty)$. Indeed, the authors proved the inclusion when $a = 2$ and $n = 3$, but the general case follows analogously. However, the inclusion $PM^a + PM^b \subset \dot{B}_p^{-s,\infty}$ is not verified for any values of $s > 0$, when $a \neq b$. On the other hand, from the analysis of [9], follows that $\dot{B}_p^{-s,\infty} \subset bmo^{-1}$, provided $p \geq n$ and $s \leq 1 - n/p$. Finally, let us remark that $PM^a + PM^b \not\subseteq BMO^{-1}$ when $a \neq b$, where BMO^{-1} denotes the space of distributions $v \in bmo^{-1}$ such that

$$\|v\|_{BMO^{-1}} := \sup_{x\in\mathbb{R}^n, 0<R<\infty} \left(|B(x,R)|^{-1} \int_{B(x,R)} \int_0^{R^2} |g(x,t) * v| \, dxdt \right)^{1/2} < \infty.$$

However, from the inclusion $PM^a \subset \dot{B}_p^{-s,\infty}$, $s = n - a - n/p$, it follows that $PM^a + PM^b \subset bmo^{-1}$ when $a, b \in [n-1, n)$.

As pointed out in [11], so far, bmo^{-1} is the maximal space where can be proved existence of mild solutions. In the context of classical Navier-Stokes equation ($\gamma = 1$), although the $PM^{a,b}$-space is contained in bmo^{-1}, our well-posedness result is useful since it has the advantage of having a straightforward proof and moreover, it states the existence of mild solutions in the same initial data class $PM^{a,b}$. In this sense, we are obtaining a new persistency result. Furthermore, let us stress that the $PM^{a,b}$-space contains some interesting singular functions and, as far as we know, in case of fractional dissipation ($1/2 < \gamma < 1$), the $PM^{a,b}$-space is a new existence class for initial data (cf. Remark 1.4).

We will construct local in time solutions for system (1) in the following time-dependent functional space

$$E^T_{\alpha,\beta,\theta} = \{u : t^{\frac{b(\theta-1)}{2\gamma}} u \in BC((0,T); PM^{\theta\alpha,\theta\beta})\}, \; 1 \le \theta < \infty,$$

with norms defined by

$$\|u\|_{E^T_{\alpha,\beta,\theta}} = \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|u(t)\|_{\theta\alpha,\theta\beta}.$$

Indeed, a more precise notation would be $E^T_{\alpha,\beta,\theta,b}$ instead of $E^T_{\alpha,\beta,\theta}$, however, to simplify the notation we omit the parameter b in the notation $E^T_{\alpha,\beta,\theta}$. When necessary, we emphasize that the space $E^T_{\alpha,\beta,\theta}$, depends of b using the notation $E^T_{\alpha,\beta,\theta,b}$. Note that when $\theta = 1$, $\alpha = a$, $\beta = b$, $E^T_{a,b,1}$ is exactly $BC((0,T); PM^{a,b})$. We recall that if X is a Banach space, $BC((0,T); X)$ denotes the space of bounded vector-value functions which are continuous from $(0,T)$ to X in the norm topology of X. Throughout this work, spaces of scalar-value and vector-value functions will be denoted in the same way. Also, it will be always assumed that $\gamma \in (1/2, 1]$, $n \ge 2(2\gamma - 1)$ and $a_0 = n - (2\gamma - 1)$, the critical index.

The mild solution of (1) in $PM^{a,b}$-spaces is defined as follows:

Definition 1.1 *Let $T \in (0,\infty)$, $\theta \in [1,\infty)$, $h_1 = a - 2\gamma/\theta$ and $h_2 = b - 2\gamma/\theta$. Consider u_0 in the $PM^{a,b}$-space, with div $u_0 = 0$, and $f \in E^T_{h_1,h_2,\theta,b}$. A time-dependent distribution $u \in E^T_{a,b,\theta}$, with div $u = 0$, is called a mild solution of system (1), with initial data u_0, if it satisfies*

$$\hat{u}(\xi,t) = e^{-t|\xi|^{2\gamma}} \hat{u}_0 + \int_0^t e^{-(t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi)[i\xi \cdot (\widehat{u \otimes u})(\xi,s) + \widehat{f}(\xi,s)] ds, \qquad (4)$$

for all $0 < t < T$*, and* $u(t) \rightharpoonup u_0$ *in the distribution sense when* $t \to 0^+$*. In (4),* $\widehat{\mathbb{P}}(\xi)$ *denotes the matrix* $n \times n$ *with components* $(\widehat{\mathbb{P}}(\xi))_{i,j} = \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}$.

We note that although functions that belong to the space $E^T_{a,b,\theta}$ are not necessarily continuous (or even weakly continuous) at $t = 0^+$, in the Definition 1.1, it is requested that the solution of (4), in the space $E^T_{a,b,\theta}$, be weakly continuous in distribution sense when $t = 0^+$. This behavior of the solution is expected. Indeed, as it will be shown later in Lemma 2.2, the family $\{G_\gamma(t)\}_{t\geq 0}$ acting on $PM^{a,b}$-space presents the same behavior.

Before proceeding, let us recall that a_0 denotes the critical index which is given by $a_0 = n - (2\gamma - 1)$. Now, we state our main results:

Theorem 1.2 *Let* $T \in (0, \infty), \theta \in [1, \infty)$, $h_1 = a - 2\gamma/\theta$ *and* $h_2 = b - 2\gamma/\theta$*. Assume* $a_0 + b(\theta - 1) \leq \theta a \leq \theta b < n$*. Then if* $u_0 \in PM^{a,b}$ *and* $f \in E^T_{h_1,h_2,\theta,b}$ *then,*

1. *(Existence). If* $a > (a_0 + b(\theta - 1))/\theta$*, then there exist* $T_0 \in (0, T]$ *and a mild solution* u *in* $E^{T_0}_{a,b,1} \cap E^{T_0}_{a,b,\theta}$*. If* $a = (a_0 + b(\theta - 1))/\theta$*, assuming that* $\limsup_{t\to 0^+} t^{\frac{b(\theta-1)}{2\gamma}} (\|G_\gamma(t)u_0\|_{\theta a,\theta b} + \|f(t)\|_{\theta h_1,\theta h_2})$ *is small enough, there exist* $T_0 \in (0, T]$ *and a mild solution* u *in* $E^{T_0}_{a,b,\theta}$.
2. *(Regularity). For* T_0 *small enough, the mild solutions* u *constructed in item (1) belong to the class* $t^{\frac{r-b}{2\gamma}} u \in BC((0, T_0); PM^r)$*, for* $b \leq r < n$.

Theorem 1.3 *Let* a, b, h_1, h_2, T, T_0 *and* θ *as in Theorem 1.2. Then:*

1. *(Uniqueness). If* $a > (a_0 + b(\theta - 1))/\theta$*, then there exists at most one mild solution in the space* $E^{T_0}_{a,b,1} := BC((0, T); PM^{a,b})$*. Moreover, if* $a = (a_0 + b(\theta - 1))/\theta$*, we have the uniqueness of small mild solution in the class* $E^{T_0}_{a,b,\theta}$*. In particular, taking* $\theta = 1$ *in the last case, we have* $a = a_0$ *and obtain the uniqueness in the class* $E^{T_0}_{a_0,b,1} := BC((0, T); PM^{a_0,b})$.
2. *(Continuous dependence) Let* u, v *be mild solutions with initial data* $u_0, v_0 \in PM^{a,b}$ *and external sources* $f, g \in E^T_{h_1,h_2,\theta,b}$*, respectively. Then there exists a positive constant* C *such that*

$$\|u - v\|_{E^{T_0}_{a,b,\theta}} \leq C(\|u_0 - v_0\|_{a,b} + \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|f(t) - g(t)\|_{\theta h_1,\theta h_2}).$$

Moreover, for $b \leq r < n$*, we have that* $\sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}} \|u(t) - v(t)\|_{PM^r}$ *is bounded by*

$$C(\|u_0 - v_0\|_{a,b} + \max\{\sup_{0<t<T} t^{\frac{(br/a)-b}{2\gamma}} \|f(t) - g(t)\|_{r,\frac{rb}{a}}, \sup_{0<t<T} t^{\frac{r-b}{2\gamma}} \|f(t) - g(t)\|_{\frac{ra}{b},r}\}.$$

Remark 1.4 *Here are some comments related with Theorem 1.2 and Theorem 1.3.*

- *Let us remark that, since $\gamma \in (1/2, 1]$ and $n \geq 2(2\gamma - 1)$, the assumption $a_0 + b(\theta - 1) \leq \theta a \leq \theta b < n$ implies that $a_0 \leq a \leq b < n < 2a$. The condition $n < 2a$ appear in the proof of a Hölder-type inequality in $PM^{a,b}$-space, Proposition 2.4 below.*
- *Considering $b = a = a_0$ and $\theta = 1$ in Theorem 1.2, it can be proved the existence of a mild solution in $E^{T_0}_{a_0,a_0,\theta}$, corresponding to the initial data $u_0 \in PM^{a_0,a_0} = PM^{a_0}$, provided $\limsup_{t\to 0^+} t^{(a_0(\theta-1)/2\gamma)} (\|G_\gamma(t)u_0\|_{\theta a_0} + \|f(t)\|_{\theta a_0 - 2\gamma})$ be small enough, where $\|\cdot\|_r$ denotes the PM^r-norm. In this case, in estimate (11) below, the term $T^\delta \max\{1, T^{\frac{\theta(b-a)}{2\gamma}}\} = 1$ ($\delta = 0$). Thus, since the constant of the bilinear estimate (11) is independent of T, we can take $T_0 = \infty$ provided the smallness assumption be verified. For this, it is sufficient that the norms $\|u_0\|_{PM^{a_0}}$ and $\sup_{0<t<\infty} \|f(t)\|_{PM^{(a_0-2\gamma)}}$ be small enough. In consequence, the existence result given by Theorem 1.2 extends the results given in [4,10] for the case of fractional dissipation $\gamma \in (1/2, 1)$. Moreover, even if $\gamma = 1$(classical Navier-Stokes equations), our results allow to take more general singular initial data than the considered in [4,10], for example, we can take $u_0(x) = |x|^{a-n} + |x|^{b-n}$ with $a \neq b$, that is, we can deal with the sum of homogeneous functions with different degrees.*
- *In the range of fractional dissipation $1/2 < \gamma < 1$, to the best of our knowledge, in [13,14] the author proved the existence of solutions with initial data in some Besov spaces, which do not contain our class $PM^{a,b}$. More precisely, in [14] the author deals with homogeneous and non-homogeneous Besov spaces with positive regularity (which obviously does not contain PM^a and so $PM^{a,b}$), and in [13], with homogeneous Besov spaces with negative regularity $\dot{B}^{-s,q}_p$, where $p \in (2, \infty)$, $q \in [1, \infty]$ and $s = (2\gamma - 1) - \frac{n}{p}$. In the last family, the space $\dot{B}^{-s,\infty}_p$ ($q = \infty$) is larger, and as we said before, $\dot{B}^{-s,\infty}_p$ contains the PM^{a_0}-space ($a_0 = n - (2\gamma - 1)$) when $p \in (\frac{n}{n-a_0}, \infty)$, but it does not contain any $PM^{a,b}$-space when $a \neq a_0$ or $b \neq a_0$.*
- *If $1/2 < \gamma < 1$, we do not know any work dealing with the space bmo^{-1}, and we believe that it is not trivial to obtain a generalization of the results of [9] to the range of fractional dissipation $1/2 < \gamma < 1$.*
- *In Theorem 1.2, we obtain mild solutions in the space $E^{T_0}_{a,b,\theta}$. This is a remarkable point because it permit to show that the mild solutions belong to PM^r-space, $b \leq r < n$. Moreover, due the same property, the mild solutions are continuously dependent in the PM^r-space, on the initial data in $PM^{a,b}$.*

2 Proof of Theorems

We begin by stating some preliminary results which will be of frequent use in the proofs of Theorem 1.2 and Theorem 1.3.

Lemma 2.1 *The Leray Projector $\mathbb{P}$, defined as $(\mathbb{P}u)_j = u_j + \Sigma^n_{k=1} R_j R_k u_k$,* *where the R_j's are the Riesz transform, is continuous on the $PM^{a,b}$-space.*

Proof. Let $u_1 \in PM^a$ and $u_2 \in PM^b$ such that $u = u_1 + u_2 \in PM^{a,b}$. Then

$$\begin{aligned}\|R_j(u_1+u_2)\|_{a,b} &\leq \|R_j u_1\|_a + \|R_j u_2\|_b \\ &\leq ess \sup_{\xi\in\mathbb{R}^n} |\xi|^a |\widehat{R_j u_1}(\xi)| + ess \sup_{\xi\in\mathbb{R}^n} |\xi|^b |\widehat{R_j u_2}(\xi)| \\ &\leq ess \sup_{\xi\in\mathbb{R}^n} |\xi|^a |\hat{u}_1(\xi)| + ess \sup_{\xi\in\mathbb{R}^n} |\xi|^b |\hat{u}_2(\xi)| = \|u_1\|_a + \|u_2\|_b.\end{aligned}$$

Taking the inf over all representations $u = u_1 + u_2$, we obtain the desired claim. ■

From now on, we will use the following notation

$$\widehat{G_\gamma(t)u_0} = e^{-t|\xi|^{2\gamma}} \hat{u}_0, \tag{5}$$

$$\widehat{F(f)} = \int_0^t e^{-(t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi) \widehat{f}(\xi, s) ds, \tag{6}$$

$$\widehat{B(u,v)} = \int_0^t e^{-(t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi) i\xi \cdot (\widehat{u \otimes v})(\xi, s) ds. \tag{7}$$

Lemma 2.2 *Let $n > b - 2\gamma$, $0 < a \leq r, b \leq d$ and $j \in \{0\} \cup \mathbb{N}$. Assume that $u_0 \in PM^{a,b}$. Then, the map $t \to G_\gamma(t)u_0$ is continuous from $(0, \infty)$ to $PM^{a,b}$ in the norm topology and $G_\gamma(t)u_0$ tends to u_0, in the distribution sense, when t tends to 0^+. Moreover,*

$$\|\nabla^j G_\gamma(t)u_0\|_{r,d} \leq C \max\{t^{-\frac{j+r-a}{2\gamma}}, t^{-\frac{j+d-b}{2\gamma}}\} \|u_0\|_{a,b}, \qquad \text{for all } t > 0, \tag{8}$$

where C is a positive constant independent of t and u_0.

Proof. Let us start with the proof of estimate (8). Let $u_{0,1} \in PM^a$, $u_{0,2} \in PM^b$ such that $u_0 = u_{0,1} + u_{0,2} \in PM^{a,b}$. Then

$$\begin{aligned}\|\nabla^j G_\gamma(t)u_0\|_{r,d} &\leq \|\nabla^j G_\gamma(t)u_{0,1}\|_r + \|\nabla^j G_\gamma(t)u_{0,2}\|_d \\ &\leq Ct^{-\frac{j+r-a}{2\gamma}} \sup_{\xi\in\mathbb{R}^n} \left(|t^{\frac{1}{2\gamma}}\xi|^{(j+r-a)} e^{-t|\xi|^{2\gamma}}\right) \sup_{\xi\in\mathbb{R}^n} |\xi|^a |\widehat{u}_{0,1}| \\ &+ Ct^{-\frac{j+d-b}{2\gamma}} \sup_{\xi\in\mathbb{R}^n} \left(|t^{\frac{1}{2\gamma}}\xi|^{(j+d-b)} e^{-t|\xi|^{2\gamma}}\right) \sup_{\xi\in\mathbb{R}^n} |\xi|^b |\widehat{u}_{0,2}| \\ &\leq C \max\{t^{-\frac{j+r-a}{2\gamma}}, t^{-\frac{j+d-b}{2\gamma}}\}(\|u_{0,1}\|_a + \|u_{0,2}\|_b).\end{aligned}$$

Taking the inf over all representations $u_0 = u_{0,1} + u_{0,2}$ in last inequality, we conclude this part of the proof. Next, we deal with the statements about continuity. Note that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and using that $a - 2\gamma \leq b - 2\gamma < n$, we have:

$$
\begin{aligned}
|\langle G_\gamma(t)u_0 - u_0, \varphi\rangle| &= \left|\int (e^{-t|\xi|^{2\gamma}} - 1)\hat{u}_0\hat{\varphi}\, d\xi\right| \\
&\leq t \sup_{x\in\mathbb{R}^n} \frac{|e^{-t|\xi|^{2\gamma}} - 1|}{t|\xi|^{2\gamma}} \|u_{0,1}\|_a \left\|\frac{\hat{\varphi}}{|\xi|^{a-2\gamma}}\right\|_{L^1(\mathbb{R}^n)} + \\
&+ t \sup_{x\in\mathbb{R}^n} \frac{|e^{-t|\xi|^{2\gamma}} - 1|}{t|\xi|^{2\gamma}} \|u_{0,2}\|_b \left\|\frac{\hat{\varphi}}{|\xi|^{b-2\gamma}}\right\|_{L^1(\mathbb{R}^n)} \to 0, \text{ as } t \to 0^+,
\end{aligned}
$$

which shows the weak continuity at $t = 0^+$. In order to prove the continuity in the norm topology, we take $t_0 > 0$, $u_0 = u_{0,1} + u_{0,2}$ as above, and bound the norm $\|(G_\gamma(h+t_0) - G_\gamma(t_0))u_0\|_{a,b}$ by:

$$
\begin{aligned}
&\leq \|(G_\gamma(h+t_0) - G_\gamma(t_0))u_{0,1}\|_a + \|(G_\gamma(h+t_0) - G_\gamma(t_0))u_{0,2}\|_b \\
&\leq C \sup_{\xi\in\mathbb{R}^n} \left(|e^{-(t_0+h)|\xi|^{2\gamma}} - e^{-t_0|\xi|^{2\gamma}}|\right) \left(\sup_{\xi\in\mathbb{R}^n} |\xi|^a|\widehat{u}_{0,1}| + \sup_{\xi\in\mathbb{R}^n} |\xi|^b|\widehat{u}_{0,2}|\right) \\
&\leq C \sup_{\xi\in\mathbb{R}^n} \left(|e^{-(t_0+h)|\xi|^{2\gamma}} - e^{-t_0|\xi|^{2\gamma}}|\right) (\|u_{0,1}\|_a + \|u_{0,2}\|_b) \\
&\leq C\,|h| \sup_{\xi\in\mathbb{R}^n} \left(\sup_{s\in(t_0,t_0+h)} |\xi|^{2\gamma}e^{-s|\xi|^{2\gamma}}\right) (\|u_{0,1}\|_a + \|u_{0,2}\|_b) \\
&\leq C\,|h| \sup_{\xi\in\mathbb{R}^n} \left(|\xi|^{2\gamma}e^{-t_0|\xi|^{2\gamma}}\right) (\|u_{0,1}\|_a + \|u_{0,2}\|_b) \to 0, \text{ as } h \to 0,
\end{aligned}
$$

which concludes the proof of lemma. ∎

Now, we recall a fact about convolution of homogeneous functions which will be useful to perform our estimates (cf. [12]).

Proposition 2.3 *(Convolution of singular kernels) Let* $0 < \alpha < n$, $0 < \beta < n$ *and* $0 < \alpha + \beta < n$. *Then*

$$
(|x|^{\alpha-n} * |x|^{\beta-n})(y) = \int_{\mathbb{R}^n} |z|^{\alpha-n}|y-z|^{\beta-n}dz = C(\alpha,\beta,n)|y|^{\alpha+\beta-n}.
$$

Next, we prove a Hölder-type inequality in space $PM^{a,b}$, which will be a key element in the estimates of the bilinear term (7).

Proposition 2.4 *(Hölder-type inequality) Let* $0 < a_1 \leq b_1 < n$, $0 < a_2 \leq b_2 < n$, *such that* $a_1 + a_2 > n$, *and let* $d_1 = a_1 + a_2 - n$, $d_2 = b_1 + b_2 - n$. *Then, there exists a positive constant* C *such that*

$$
\|u \otimes v\|_{d_1,d_2} \leq C\|u\|_{a_1,b_1}\|v\|_{a_2,b_2}, \tag{9}
$$

for all $u \in PM^{a_1,b_1}$ *and* $v \in PM^{a_2,b_2}$.

Proof. Let $u_1 \in PM^{a_1}, u_2 \in PM^{b_1}$, $v_1 \in PM^{a_2}$ and $v_2 \in PM^{b_2}$ such that $u = u_1 + u_2 \in PM^{a_1,b_1}$, $v = v_1 + v_2 \in PM^{a_2,b_2}$. Then

$$\begin{aligned}|(\widehat{u \otimes v})(\xi)| &= |(\widehat{u_1 + u_2) \otimes (v_1 + v_2})(\xi)| \\ &\leq |\widehat{u_1 \otimes v_1}(\xi)| + |\widehat{u_2 \otimes v_2}(\xi)| + |\widehat{u_1 \otimes v_2}(\xi)| + |\widehat{u_2 \otimes v_1}(\xi)| \\ &= J_1(\xi) + J_2(\xi) + J_3(\xi) + J_4(\xi).\end{aligned}$$

We will work with each quantity $J_i,\ i = 1, 2, 3, 4$. First of all, note that by Proposition 2.3, the term J_1 can be estimated as follows

$$J_1(\xi) \leq C(\int_{\mathbb{R}^n} \frac{1}{|\xi - z|^{a_1}} \frac{1}{|z|^{a_2}} dz \|u_1\|_{a_1} \|v_1\|_{a_2}) \leq C \frac{1}{|\xi|^{a_1+a_2-n}} \|u_1\|_{a_1} \|v_1\|_{a_2}.$$

Consequently, $u_1 \otimes v_1 \in PM^{d_1}$ and the following estimate holds:

$$\|u_1 \otimes v_1\|_{d_1} = \sup_{\xi \in \mathbb{R}^n} |\xi|^{d_1} |\widehat{u_1 \otimes v_1}(\xi)| \leq C \|u_1\|_{a_1} \|v_1\|_{a_2}.$$

Similarly, the quantity J_2 is estimated as

$$J_2(\xi) \leq C(\int_{\mathbb{R}^n} \frac{1}{|\xi - z|^{b_1}} \frac{1}{|z|^{b_2}} dz \|u_1\|_{b_1} \|v_1\|_{b_2}) \leq C \frac{1}{|\xi|^{b_1+b_2-n}} \|u_2\|_{b_1} \|v_2\|_{b_2}.$$

Therefore, $u_2 \otimes v_2 \in PM^{d_2}$ and

$$\|u_2 \otimes v_2\|_{d_2} \leq C \|u_2\|_{b_1} \|v_2\|_{b_2}.$$

In order to estimate the terms J_3, J_4, we define the set $A = \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Then we write $\widehat{u_1 \otimes v_2} = 1_A \left(\widehat{u_1 \otimes v_2}\right) + 1_{A^c} \left(\widehat{u_1 \otimes v_2}\right) := \widehat{h_1}(\xi) + \widehat{h_2}(\xi)$, where 1_M denote the characteristic function of the set M. Using Proposition 2.3 and conditions $a_1 \leq b_1$ and $a_2 \leq b_2$, we obtain

$$\begin{aligned}|\widehat{h_1}(\xi)| = 1_A |\widehat{u_1 \otimes v_2}| \ &\leq C 1_A \int_{\mathbb{R}^n} \frac{1}{|\xi - z|^{a_1}} \frac{1}{|z|^{b_2}} dz \|u_1\|_{a_1} \|v_2\|_{b_2} \\ &\leq C 1_A \frac{1}{|\xi|^{a_1+b_2-n}} \|u_1\|_{a_1} \|v_2\|_{b_2} \leq C 1_A \frac{1}{|\xi|^{b_1+b_2-n}} \|u_1\|_{a_1} \|v_2\|_{b_2} \\ &\leq C \frac{1}{|\xi|^{b_1+b_2-n}} \|u_1\|_{a_1} \|v_2\|_{b_2}\end{aligned}$$

and

$$\begin{aligned}|\widehat{h_2}(\xi)| = 1_{A^c} |\widehat{u_1 \otimes v_2}| \ &\leq C 1_{A^c} \frac{1}{|\xi|^{a_1+b_2-n}} \|u_1\|_{a_1} \|v_2\|_{b_2} \\ &\leq C \frac{1}{|\xi|^{a_1+a_2-n}} \|u_1\|_{a_1} \|v_2\|_{b_2}.\end{aligned}$$

Therefore, $h_1 \in PM^{d_2}$, $h_2 \in PM^{d_1}$, $\|h_1\|_{d_2} + \|h_2\|_{d_1} \leq C\|u_1\|_{a_1}\|v_2\|_{b_2}$ and then

$$u_1 \otimes v_2 \in PM^{d_1,d_2} \text{ with } \|u_1 \otimes v_2\|_{d_1,d_2} \leq C\|u_1\|_{a_1}\|v_2\|_{b_2}.$$

Analogously, for J_4 we have that

$$u_2 \otimes v_1 \in PM^{d_1,d_2} \text{ with } \|u_2 \otimes v_1\|_{d_1,d_2} \leq C\|u_2\|_{b_1}\|v_1\|_{a_2}.$$

Collecting the last inequalities, we obtain $u \otimes v \in PM^{d_1,d_2}$ and

$$\begin{aligned}\|u \otimes v\|_{d_1,d_2} &\leq \|u_1 \otimes v_1\|_{d_1,d_2} + \|u_2 \otimes v_2\|_{d_1,d_2} + \|u_1 \otimes v_2\|_{d_1,d_2} + \|u_2 \otimes v_1\|_{d_1,d_2} \\ &\leq C(\|u_1\|_{a_1}\|v_1\|_{a_2} + \|u_2\|_{b_1}\|v_2\|_{b_2} + \|u_1\|_{a_1}\|v_2\|_{b_2} + \|u_2\|_{b_1}\|v_1\|_{a_2}) \\ &= C(\|u_1\|_{a_1} + \|u_2\|_{b_1})(\|v_1\|_{a_2} + \|v_2\|_{b_2}).\end{aligned}$$

Taking the inf over all representations of $u = u_1 + u_2, v = v_1 + v_2$ in the right-hand of the last inequality, we obtain the inequality (9). ■

The following lemma is technical which will be useful to our ends.

Lemma 2.5 *Let $0 \leq q \leq p$ and $\eta < 1$. Then*

$$\sup_{\xi \in \mathbb{R}^n} \int_0^1 e^{-(1-s)|\xi|^p}|\xi|^q s^{-\eta} ds < \infty. \tag{10}$$

Proof. Splitting the integral in (10) in two parts, we can estimate

$$\begin{aligned}&\int_0^1 e^{-(1-s)|\xi|^p}|\xi|^q s^{-\eta} ds = \int_0^{1/2} e^{-(1-s)|\xi|^p}|\xi|^q s^{-\eta} ds + \int_{1/2}^1 e^{-(1-s)|\xi|^p}|\xi|^q s^{-\eta} ds \\ &\leq e^{-(1-1/2)|\xi|^p}|\xi|^q \int_0^{1/2} s^{-\eta} ds + 2^\eta \int_{1/2}^1 e^{-|(1-s)^{1/p}\xi|^p}|(1-s)^{1/p}\xi|^q (1-s)^{-q/p} ds \\ &\leq \frac{2^{-(1-\eta)}}{1-\eta}\left(\sup_{z>0} z^q e^{-z^p/2}\right) + I(\xi),\end{aligned}$$

where

$$I(\xi) = 2^\eta \int_{1/2}^1 e^{-|(1-s)^{1/p}\xi|^p}|(1-s)^{1/p}\xi|^q (1-s)^{-q/p} ds.$$

Then we bound the integral $I(\xi)$ in the following form

$$\begin{aligned}&I(\xi) \leq 2^\eta \left(\sup_{z>0} z^q e^{-z^p}\right) \int_{1/2}^1 (1-s)^{-q/p} ds < \infty, \text{ if } q < p, \text{ and} \\ &I(\xi) = 2^\eta \int_{1/2}^1 e^{-(1-s)|\xi|^p}|\xi|^p ds \leq 2^\eta(1 - e^{-\frac{1}{2}|\xi|^p}) \leq 2^\eta < \infty, \quad \text{if } q = p.\end{aligned}$$

Hence the proof of lemma is finished. ∎

Proposition 2.6 *Let $B(\cdot,\cdot)$ the bilinear form defined by (2), $\theta \geq 1$ and assume that $a_0 + b(\theta-1) \leq \theta a \leq \theta b < n$. Then there exists a positive constant K_1, independent of T, such that*

$$\|B(u,v)\|_{E^T_{a,b,\theta}} \leq K_1 T^{\delta} \max\{1, T^{\frac{\theta(b-a)}{2\gamma}}\} \|u\|_{E^T_{a,b,\theta}} \|v\|_{E^T_{a,b,\theta}}, \tag{11}$$

for all $u, v \in E^T_{a,b,\theta}$, where $\delta = \frac{b-a_0-\theta(b-a)}{2\gamma} \geq 0$.

Moreover, there exists a positive constant K_2, independent of f and T, such that

$$\|F(f)\|_{E^T_{a,b,\theta}} \leq K_2 \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|f\|_{\theta a-2\gamma, \theta b-2\gamma}. \tag{12}$$

Proof. Let $u(s)$ and $v(s) \in PM^{\theta a, \theta b}$ with $\theta \geq 1$. First of all, since $\gamma \in (1/2, 1]$ and $n \geq 2(2\gamma-1)$, then the assumptions of lemma imply $a_0 \leq a \leq b$ and $n < 2a \leq 2b$. Also, from $a_0 + b(\theta-1) < n$, we have $b(\theta-1) < 2\gamma - 1 < \gamma$. Now, we can apply the Proposition 2.4 with $d_1 = 2\theta a - n$, $d_2 = 2\theta b - n$, $a_1 = a_2 = \theta a$ and $b_1 = b_2 = \theta b$ in order to obtain that $u \otimes v \in PM^{2\theta a-n, 2\theta b-n}$ with the following estimate:

$$\|u(s) \otimes v(s)\|_{2\theta a-n, 2\theta b-n} \leq C\|u(s)\|_{\theta a,\theta b}\|v(s)\|_{\theta a,\theta b},$$

which, in turn, implies the pointwise estimate

$$\begin{aligned}
|\widehat{u\otimes v}(s,\xi)| &\leq (\frac{1}{|\xi|^{2\theta a-n}} + \frac{1}{|\xi|^{2\theta b-n}})\|u(s)\otimes v(s)\|_{2\theta a-n,2\theta b-n} \\
&\leq C(\frac{1}{|\xi|^{2\theta a-n}} + \frac{1}{|\xi|^{2\theta b-n}})\|u(s)\|_{\theta a,\theta b}\|v(s)\|_{\theta a,\theta b}.
\end{aligned}$$

Now, we estimate

$$\begin{aligned}
|\widehat{B(u,v)}(t,\xi)| &\leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi||\widehat{u\otimes v}(s,\xi)|ds \\
&\leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|(\frac{1}{|\xi|^{2\theta a-n}} + \frac{1}{|\xi|^{2\theta b-n}})\|u(s)\|_{\theta a,\theta b}\|v(s)\|_{\theta a,\theta b}ds \\
&\leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{n+1-2\theta a}\|u(s)\|_{\theta a,\theta b}\|v(s)\|_{\theta a,\theta b}ds + \\
&+ C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{n+1-2\theta b}\|u(s)\|_{\theta a,\theta b}\|v(s)\|_{\theta a,\theta b}ds \\
&= I_1(t,\xi) + I_2(t,\xi).
\end{aligned}$$

We have the following pointwise estimates for I_1 and I_2:

$$|\xi|^{\theta a} I_1(t,\xi) \le C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{n+1-\theta a} s^{-\frac{b(\theta-1)}{\gamma}} ds \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right)$$
$$\le Ct^{-\frac{n+1-\theta a}{2\gamma}-\frac{b(\theta-1)}{\gamma}+1}\int_0^1 e^{-(1-s)|t^{1/2\gamma}\xi|^{2\gamma}}|t^{1/2\gamma}\xi|^{n+1-\theta a} s^{-\frac{b(\theta-1)}{\gamma}} ds$$
$$\times \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right)$$
$$\le Ct^{-\frac{b(\theta-1)}{2\gamma}} t^{\frac{b-a_0-\theta(b-a)}{2\gamma}} \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right) \tag{13}$$

and

$$|\xi|^{\theta b} I_2(t,\xi) \le C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{n+1-\theta b} s^{-\frac{b(\theta-1)}{\gamma}} ds \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right)$$
$$\le Ct^{-\frac{n+1-\theta b}{2\gamma}-\frac{b(\theta-1)}{\gamma}+1}\int_0^1 e^{-(1-s)|t^{1/2\gamma}\xi|^{2\gamma}}|t^{1/2\gamma}\xi|^{n+1-\theta b} s^{-\frac{b(\theta-1)}{\gamma}} ds$$
$$\times \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta b,\theta b}\|v(t)\|_{\theta a,\theta b}\right)$$
$$\le Ct^{-\frac{b(\theta-1)}{2\gamma}} t^{\frac{b-a_0}{2\gamma}} \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right). \tag{14}$$

In the above two inequalities, we have used Lemma 2.5 to assure that

$$\int_0^1 e^{-(1-s)|t^{1/2\gamma}\xi|^{2\gamma}}|t^{1/2\gamma}\xi|^{n+1-\theta a} s^{-\frac{b(\theta-1)}{\gamma}} ds \le C,$$
$$\int_0^1 e^{-(1-s)|t^{1/2\gamma}\xi|^{2\gamma}}|t^{1/2\gamma}\xi|^{n+1-\theta b} s^{-\frac{b(\theta-1)}{\gamma}} ds \le C,$$

for all $t>0$ and $\xi \in \mathbb{R}^n$. The inequalities (13) and (14) show that $B(u(t),v(t)) \in PM^{\theta a,\theta b}$ with

$$\|B(u(t),v(t))\|_{\theta a,\theta b} \le$$
$$\le Ct^{-\frac{b(\theta-1)}{2\gamma}} t^{\frac{b-a_0-\theta(b-a)}{2\gamma}} \max\{1, t^{\frac{\theta(b-a)}{2\gamma}}\} \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right)$$
$$\le Ct^{-\frac{b(\theta-1)}{2\gamma}} T^{\frac{b-a_0-\theta(b-a)}{2\gamma}} \max\{1, T^{\frac{\theta(b-a)}{2\gamma}}\} \sup_{0<t<T}\left(t^{\frac{b(\theta-1)}{\gamma}}\|u(t)\|_{\theta a,\theta b}\|v(t)\|_{\theta a,\theta b}\right),$$

which finishes the proof of estimate (11).

Now, let us deal with the estimate (12). Using that $f(s) \in PM^{\theta a-2\gamma,\theta b-2\gamma}$, we get

$$\begin{aligned}|\widehat{F(f)}(t,\xi)| &\leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\widehat{f}(s,\xi)|ds\\ &\leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}(\frac{1}{|\xi|^{\theta a-2\gamma}}+\frac{1}{|\xi|^{\theta b-2\gamma}})\|f(s)\|_{\theta a-2\gamma,\theta b-2\gamma}ds\\ &\leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{2\gamma-\theta a}\|f(s)\|_{\theta a-2\gamma,\theta b-2\gamma}ds+\\ &+C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{2\gamma-\theta b}\|f(s)\|_{\theta a-2\gamma,\theta b-2\gamma}ds\\ &= I_3(t,\xi)+I_4(t,\xi).\end{aligned}$$

We have the following pointwise estimates for $I_3(t,\xi)$ and $I_4(t,\xi)$:

$$\begin{aligned}&|\xi|^{\theta a}I_3(t,\xi) \leq C\int_0^t e^{-(t-s)|\xi|^{2\gamma}}|\xi|^{2\gamma}s^{-\frac{b(\theta-1)}{2\gamma}}ds\sup_{0<t<T}t^{\frac{b(\theta-1)}{2\gamma}}\|f(t)\|_{\theta a-2\gamma,\theta b-2\gamma}\\ &\leq Ct^{-\frac{b(\theta-1)}{2\gamma}}\int_0^1 e^{-(1-s)|t^{1/2\gamma}\xi|^{2\gamma}}|t^{1/2\gamma}\xi|^{2\gamma}s^{-\frac{b(\theta-1)}{2\gamma}}ds\sup_{0<t<T}t^{\frac{b(\theta-1)}{2\gamma}}\|f(t)\|_{\theta a-2\gamma,\theta b-2\gamma}\\ &\leq Ct^{-\frac{b(\theta-1)}{2\gamma}}\sup_{0<t<T}t^{\frac{b(\theta-1)}{2\gamma}}\|f(t)\|_{\theta a-2\gamma,\theta b-2\gamma},\end{aligned}$$

and, in a similar way,

$$\begin{aligned}|\xi|^{\theta b}I_4(t,\xi) &\leq Ct^{-\frac{b(\theta-1)}{2\gamma}}\int_0^1 e^{-(1-s)|t^{1/2\gamma}\xi|^{2\gamma}}|t^{1/2\gamma}\xi|^{2\gamma}s^{-\frac{b(\theta-1)}{2\gamma}}ds\sup_{0<t<T}t^{\frac{b(\theta-1)}{2\gamma}}\|f(t)\|_{\theta a-2\gamma,\theta b-2\gamma}\\ &\leq Ct^{-\frac{b(\theta-1)}{2\gamma}}\sup_{0<t<T}t^{\frac{b(\theta-1)}{2\gamma}}\|f(t)\|_{\theta a-2\gamma,\theta b-2\gamma},\end{aligned}$$

which implies the desired result. ■

2.1 Proof of Theorem 1.2.

(Existence). We construct the mild solution according with the following sequence of successive approximations

$$u^0(t,x)=G(t)u_0(x),\ u^{k+1}(t,x)=u^0(t,x)+B(u^k,u^k)(t,x)+F(f)(t,x),\ k\in\mathbb{N}. \tag{15}$$

Firstly, we show that there exists a field u which satisfies that u^k converges to u in $E^T_{a,b,\theta}$ and verifies the integral equation (4). Using Lemma 2.2 and Proposition 2.6 we obtain the following estimate:

$$M_{k+1}\leq M_0+F_0+K_1T^\delta\max\{1,T^{\frac{\theta(b-a)}{2\gamma}}\}M_k^2,\ k=0,1,2,... \tag{16}$$

where δ is given as in Proposition 2.6, and F_0, M_0 and M_k are given by

$$
\begin{aligned}
F_0 &= \|F(f)\|_{E^T_{a,b,\theta}} \le K_2 \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|f\|_{\theta a-2\gamma,\theta b-2\gamma} = K_2\|f\|_{E^T_{h_1,h_2,\theta,b}},\\
M_0 &= \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|G_\gamma(t)u_0\|_{\theta a,\theta b},\\
M_k &= \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|u^k(t)\|_{\theta a,\theta b},\ \ k = 1,2,...
\end{aligned}
$$

Note that by the assumption on the initial data, Lemma 2.2 and the fact $a \le b$, we have that

$$
\begin{aligned}
M_0 &= \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|G_\gamma(t)u_0\|_{\theta a,\theta b}\\
&\le C \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} [t^{-\frac{b(\theta-1)}{2\gamma}} (t^{\frac{(b-a)(\theta-1)}{2\gamma}} + 1)]\|u_0\|_{a,b}\\
&\le C(T)\|u_0\|_{a,b} < \infty,
\end{aligned}
$$

where $C(T)$ is a continuous function for all $T > 0$ and $\lim_{T\to 0+} C(T) < \infty$. Observe that when $a > (a_0 + b(\theta-1))/\theta$, we have $b > a_0 + \theta(b-a)$, and then $\delta > 0$. Consequently, we can choose $T = T_0$, small enough, such that

$$
\begin{aligned}
M_k &\le \frac{1 - \sqrt{1 - 4(K_1 T^\delta \max\{1, T^{\frac{\theta(b-a)}{2\gamma}}\})(M_0 + K_2\|f\|_{E^T_{h_1,h_2,\theta,b}})}}{2K_1 T^\delta \max\{1, T^{\frac{\theta(b-a)}{2\gamma}}\}}\\
&\le 2(M_0 + K_2\|f\|_{E^T_{h_1,h_2,\theta,b}}) \equiv \Gamma, \qquad (17)
\end{aligned}
$$

provided

$$
4(K_1 T^\delta \max\{1, T^{\frac{\theta(b-a)}{2\gamma}}\})(M_0 + K_2\|f\|_{E^T_{h_1,h_2,\theta,b}}) < 1. \qquad (18)
$$

On the other hand, if $a = (a_0 + b(\theta-1))/\theta$, we have $\delta = 0$. In this case, we use the assumptions on u_0, f, which state that

$$
\limsup_{t\to 0^+} t^{\frac{b(\theta-1)}{2\gamma}} (\|G_\gamma(t)u_0\|_{\theta a,\theta b} + \|f(t)\|_{\theta a-2\gamma,\theta b-2\gamma})
$$

is small enough, such that the inequality (18) is verified.

In order to prove the convergence of the sequence $\{M_k\}$, we consider the successive difference of u^k given by (15):

$$
u^{k+1} - u^k = B(u^k, u^k) - B(u^{k-1}, u^{k-1}).
$$

Therefore, just like deriving (16) we conclude that

$$
\|u^{k+1} - u^k\|_{E^T_{a,b,\theta}} \le 2\Gamma K_1 T_0^\delta \max\{1, T_0^{\frac{\theta(b-a)}{2\gamma}}\}\|u^k - u^{k-1}\|_{E^T_{a,b,\theta}}.
$$

As $2\Gamma K_1 T_0^\delta \max\{1, T_0^{\frac{\theta(b-a)}{2\gamma}}\} < 1$ in accordance with estimate (18), the sequence (u^k) is contractive, and thus, there exists a field u such that $t^{\frac{b(\theta-1)}{2\gamma}} u^k$ converges to $t^{\frac{b(\theta-1)}{2\gamma}} u$ in $L^\infty((0,T_0); PM^{\theta a,\theta b})$.

Finally, from (4) and taking $\eta \in (0,t)$, we obtain that

$$
\begin{aligned}
&\hat{u}(\xi, t+\epsilon) - \hat{u}(\xi,t) = (e^{-(\epsilon+t)|\xi|^{2\gamma}} - e^{-t|\xi|^{2\gamma}})\hat{u}_0(\xi) + \\
&+ \int_t^{t+\epsilon} e^{-(\epsilon+t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi)[i\xi \cdot \widehat{(u\otimes u)}(\xi,s) + \widehat{f}(\xi,s)]ds \\
&+ \int_{t-\eta}^{t} (e^{-(\epsilon+t-s)|\xi|^{2\gamma}} - e^{-(t-s)|\xi|^{2\gamma}})\widehat{\mathbb{P}}(\xi)[i\xi \cdot \widehat{(u\otimes u)}(\xi,s) + \widehat{f}(\xi,s)]ds \\
&+ \int_{0}^{t-\eta} (e^{-(\epsilon+t-s)|\xi|^{2\gamma}} - e^{-(t-s)|\xi|^{2\gamma}})\widehat{\mathbb{P}}(\xi)[i\xi \cdot \widehat{(u\otimes u)}(\xi,s) + \widehat{f}(\xi,s)]ds. \qquad (19)
\end{aligned}
$$

Note that by choosing η small enough, we get to overcome the singularity of the third term of the right hand of (19) (as $\epsilon \to 0$), and then, since $\|u\|_{E^{T_0}_{a,b,\theta}} < \infty$, the equality (19) together the strong continuity of the semigroup $G_\gamma(t)$ at $t > 0$ (cf. Lemma 2.2), imply that $t^{\frac{b(\theta-1)}{2\gamma}} u \in BC((0,T_0); PM^{\theta a,\theta b})$.

Finally, we observe that if a, b satisfy the assumptions of Theorem 1.2 with $\theta > 1$, then they also satisfy the assumptions with $\theta = 1$. Therefore, taking T_0 sufficiently small, we get $u \in E^{T_0}_{a,b,1} \cap E^{T_0}_{a,b,\theta}$.

It remains to show that $(B(u,u) + F(f))(t) \rightharpoonup 0$ as $t \to 0^+$, in the distribution sense, but we omit the proof because, by using the bounds $\|u\|_{E^{T_0}_{a,b,\theta}} < \infty$, it follows in a more or less similar way to the proof of the second part of Lemma 2.2.

(Regularity). From the first part of theorem follows the existence of a function u which verifies $t^{\frac{b(\theta-1)}{2\gamma}} u \in BC((0,T_0]; PM^{\theta a,\theta b})$ for $a_0 + b(\theta-1) \le \theta a \le \theta b < n$. Then we take $\theta = r/a$ and posteriorly $\theta = r/b$ in order to obtain

$$t^{\frac{b((r/a)-1)}{2\gamma}} u \in BC((0,T_0]; PM^{r,\frac{rb}{a}}), \qquad (20)$$

$$t^{\frac{r-b}{2\gamma}} u \in BC((0,T_0]; PM^{\frac{ra}{b},r}). \qquad (21)$$

Let $u_1(t) \in PM^{(ra/b)}, u_2(t) \in PM^r$ and $v_1(t) \in PM^r, v_2(t) \in PM^{(rb/a)}$ such that

$$u = u_1 + u_2 = v_1 + v_2.$$

Let us consider the set $Z = \{\xi \in \mathbb{R}^n : |\xi| > 1\}$. Then,

$$
\begin{aligned}
\widehat{u} &= 1_Z \widehat{u} + 1_{Z^c}\widehat{u} = 1_Z\widehat{v_1} + 1_Z\widehat{v_2} + 1_{Z^c}\widehat{u_1} + 1_{Z^c}\widehat{u_2} \\
&:= \widehat{h_1}(\xi) + \widehat{h_2}(\xi) + \widehat{h_3}(\xi) + \widehat{h_4}(\xi),
\end{aligned}
$$

where 1_Z denotes the characteristic function of Z and the distributions h_i are defined by using the inverse Fourier transform.

We need to show that each term of the above sum h_i belong to the space PM^r. As $u_2, v_1 \in PM^r$, it is straightforward to prove that $h_4, h_1 \in PM^r$, and moreover we have

$$\|h_4\|_r \leq \|u_2\|_r \leq \|u\|_{\frac{ra}{b},r}, \;\; \|h_1\|_r \leq \|v_1\|_r \leq \|u\|_{r,\frac{rb}{a}}.$$

As $b/a \geq 1$ and by definition of the set Z, it follows

$$ess \sup_{\xi\in\mathbb{R}^n} |\xi|^r|\widehat{h_2}| = ess \sup_{\xi\in\mathbb{R}^n} |\xi|^r|1_Z\widehat{v_2}| = ess \sup_{|\xi|>1} |\xi|^r|\widehat{v_2}| \leq ess \sup_{\xi\in\mathbb{R}^n} |\xi|^{rb/a}|\widehat{v_2}| < \infty, \tag{22}$$

and therefore we conclude that $h_2 = (1_Z\widehat{v_2})^\vee \in PM^r$. On the other hand, as $a/b \leq 1$ and $u_1 \in PM^{(ra/b)}$, it follows

$$ess \sup_{\xi\in\mathbb{R}^n} |\xi|^r|\widehat{h_3}| = ess \sup_{\xi\in\mathbb{R}^n} |\xi|^r|1_{Z^c}\widehat{u_1}| = ess \sup_{|\xi|<1} |\xi|^r|\widehat{u_1}| \leq ess \sup_{\xi\in\mathbb{R}^n} |\xi|^{ra/b}|\widehat{u_1}| < \infty, \tag{23}$$

therefore $h_3 \in PM^r$. Furthermore we observe that (22) and (23) imply the following bounds:

$$\|h_2\|_r \leq \|v_2\|_{rb/a} \leq \|u\|_{r,\frac{rb}{a}}, \;\; \|h_3\|_r \leq \|u_1\|_{ra/b} \leq \|u\|_{\frac{ra}{b},r}.$$

Collecting the last facts we conclude that $\sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}} \|u(t)\|_r$ is bounded by

$$\max\{ \sup_{0<t<T_0} t^{\frac{b((r/a)-1)}{2\gamma}} \|u(t)\|_{r,\frac{rb}{a}}, \sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}} \|u(t)\|_{\frac{ra}{b},r}\}.$$

■

2.2 Proof of Theorem 1.3.

Let u, v be two mild solutions of (4) in the class $BC((0,T_0); PM^{a,b})$, with initial data $u_0 \in PM^{a,b}$. As we already said, $a > (a_0 + b(\theta-1))/\theta$ is the same that $b > a_0 + \theta(b-a)$, which implies $b > a_0 + (b-a)$. Then, taking the difference between the respective integral equations (4) satisfied by u and v, we obtain

$$\hat{u}(\xi,t) - \hat{v}(\xi,t) = \int_0^t e^{-(t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi)[i\xi \cdot (\widehat{(u-v)\otimes u}) + (\widehat{v\otimes(u-v)})](\xi,s)ds.$$

Then we compute directly the $\|u-v\|_{E^{T_0}_{a,b,1}}$-norm. So, with the help of Proposition 2.6 we conclude that

$$\|u-v\|_{E^{T_0}_{a,b,1}} \le K_1 T_0^{\delta_1} \max\{1, T_0^{\frac{(b-a)}{2\gamma}}\}(\|u\|_{E^{T_0}_{a,b,1}} + \|v\|_{E^{T_0}_{a,b,1}})\|u-v\|_{E^{T_0}_{a,b,1}}, \quad (24)$$

with $\delta_1 = \frac{b-a_0-(b-a)}{2\gamma}$. Hence, denoting by $R = (\|u\|_{E^{T_0}_{a,b,1}} + \|v\|_{E^{T_0}_{a,b,1}})$, we obtain that

$$\|u-v\|_{E^{T_0}_{a,b,1}} \le K_1 R T_0^{\delta_1} \max\{1, T_0^{\frac{(b-a)}{2\gamma}}\}\|u-v\|_{E^{T_0}_{a,b,1}}. \quad (25)$$

Consequently, as $\delta_1 > 0$, we can take $t_0 \in (0, T_0]$, small enough, such that $CR(t_0)^{\delta_1} \max\{1, (t_0)^{\frac{(b-a)}{2\gamma}}\} < 1$, and hence $u(t) = v(t)$ on the interval $(0, t_0]$ with values in $PM^{a,b}$-space. Now, as the mild solutions u, v belong to the space $C([\epsilon, T_0); PM^{a,b})$, for every $0 < \epsilon < T_0$, we can repeat the last argument considering the initial data as being $u(t_0) = v(t_0)$ and conclude that $u(t) = v(t)$ with $t \in (t_0, 2t_0]$. Continuing this process, we end up with $u(t) = v(t)$ on $(0, T_0)$ with values in $PM^{a,b}$.

Now, we will study the uniqueness of mild solutions in the class $E^{T_0}_{a,b,\theta}$, when $a = (a_0 + b(\theta-1))/\theta$ and $\|u\|_{E^{T}_{a,b,\theta}} = \sup_{0<t<T} t^{\frac{b(\theta-1)}{2\gamma}} \|u(t)\|_{\theta a, \theta b}$ being small enough. In this case we have that $b = a_0 + (b-a)$ and hance $\delta = 0$. In consequence, (24) becomes

$$\begin{aligned}\|u-v\|_{E^{T_0}_{a,b,\theta}} &\le K_1 \max\{1, T_0^{\frac{\theta(b-a)}{2\gamma}}\}(\|u\|_{E^{T_0}_{a,b,\theta}} + \|v\|_{E^{T_0}_{a,b,\theta}})\|u-v\|_{E^{T_0}_{a,b,\theta}} \\ &= K_1 R \max\{1, T_0^{\frac{\theta(b-a)}{2\gamma}}\}\|u-v\|_{E^{T_0}_{a,b,\theta}} < \|u-v\|_{E^{T_0}_{a,b,\theta}}\end{aligned}$$

provided R is small enough, where $R = \|u\|_{E^{T_0}_{a,b,\theta}} + \|v\|_{E^{T_0}_{a,b,\theta}}$. In this case, R is done small enough by using the smallness condition over $E^{T_0}_{a,b,\theta}$. Arguing in an analogous way as in the proof of the first part of uniqueness, we conclude that $u = v$. Finally, taking $\theta = 1$ in the equality $a = (a_0 + b(\theta-1))/\theta$ we have $a = a_0$ and as a consequence of the second part of the uniqueness proof, we obtain the uniqueness in the class $E^{T_0}_{a_0,b,1} := BC((0,T); PM^{a_0,b})$.

(**Continuous dependence**). Subtracting the integral equation (4) satisfied by u from the similar expression satisfied by v we have

$$\begin{aligned}\hat{u}(\xi,t) - \hat{v}(\xi,t) &= e^{-t|\xi|^{2\gamma}}(\hat{u}_0 - \hat{v}_0) \\ &\quad + \int_0^t e^{-(t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi)[i\xi \cdot (\widehat{(u-v)\otimes u} + \widehat{v \otimes (u-v)})](\xi,s)ds \\ &\quad + \int_0^t e^{-(t-s)|\xi|^{2\gamma}} \widehat{\mathbb{P}}(\xi)[\widehat{f}(\xi,s) - \widehat{g}(\xi,s)]ds.\end{aligned}$$

Hence, by Lemma 2.2 and Proposition 2.6 we can bound the $t^{\frac{b(\theta-1)}{2\gamma}}\|u-v\|_{\theta a,\theta b}$-

norm by:

$$\leq C(T_0)\|u_0-v_0\|_{a,b}+K_2\sup_{0<t<T_0} t^{\frac{b(\theta-1)}{2\gamma}}\|f-g\|_{\theta a-2\gamma,\theta b-2\gamma}$$
$$+K_1T_0^{\delta}\max\{1,T_0^{\frac{\theta(b-a)}{2\gamma}}\}\sup_{0<t<T_0}(t^{\frac{b(\theta-1)}{2\gamma}}\|u\|_{\theta a,\theta b}+t^{\frac{b(\theta-1)}{2\gamma}}\|v\|_{\theta a,\theta b})\sup_{0<t<T_0} t^{\frac{b(\theta-1)}{2\gamma}}\|u-v\|_{\theta a,\theta b}$$
$$\leq C(T_0)\|u_0-v_0\|_{a,b}+K_2\|f-g\|_{E^T_{\theta a-2\gamma,\theta b-2\gamma,\theta}}$$
$$+2\Gamma K_1T_0^{\delta}\max\{1,T_0^{\frac{\theta(b-a)}{2\gamma}}\}\sup_{0<t<T_0} t^{\frac{b(\theta-1)}{2\gamma}}\|u-v\|_{\theta a,\theta b}.$$

where Γ is as in (17), $\delta=\frac{b-a_0-\theta(b-a)}{2\gamma}$, $C(T)$ is a continuous function for all $T>0$ and $\lim_{T\to0+}C(T)<\infty$. Since $2\Gamma K_1T_0^{\delta}\max\{1,T_0^{\frac{\theta(b-a)}{2\gamma}}\}<1$, which follows from the existence result, then we conclude that

$$\sup_{0<t<T_0} t^{\frac{b(\theta-1)}{2\gamma}}\|u-v\|_{\theta a,\theta b}\leq C(\|u_0-v_0\|_{a,b}+\|f-g\|_{E^T_{\theta a-2\gamma,\theta b-2\gamma,\theta}}). \tag{26}$$

Finally we will prove the continuous dependence in PM^r. Knowing the inequality (26), valid for all θ as in Theorem 1.2 , we take $\theta=r/a$ and posteriorly $\theta=r/b$, in order to obtain

$$\max\{\sup_{0<t<T_0} t^{\frac{(br/a)-b}{2\gamma}}\|u(t)-v(t)\|_{r,\frac{rb}{a}},\sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}}\|u(t)-v(t)\|_{\frac{ra}{b},r}\}\leq C\|u_0-v_0\|_{a,b}+$$
$$+\max\{\sup_{0<t<T_0} t^{\frac{(br/a)-b}{2\gamma}}\|f(t)-g(t)\|_{r,\frac{rb}{a}},\sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}}\|f(t)-g(t)\|_{\frac{ra}{b},r}\}.$$

From the regularity proof, we know that $\sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}}\|u(t)-v(t)\|_r$ is bounded by

$$\max\{\sup_{0<t<T_0} t^{\frac{b((r/a)-1)}{2\gamma}}\|u(t)-v(t)\|_{r,\frac{rb}{a}},\sup_{0<t<T_0} t^{\frac{r-b}{2\gamma}}\|u(t)-v(t)\|_{\frac{ra}{b},r}\}.$$

Hence we conclude the second part of Theorem. ∎

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