BRANCHING OF PERIODIC ORBITS IN REVERSIBLE HAMILTONIAN SYSTEMS

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Abstract. This paper deals with the dynamics of time-reversible Hamiltonian vector fields with 2 and 3 degrees of freedom around an elliptic equilibrium point in presence of symplectic involutions. The main results discuss the existence of one-parameter families of reversible periodic solutions terminating at the equilibrium. The main techniques used are Birkhoff and Belitskii normal forms combined with the Liapunov-Schmidt reduction.

1. Introduction

The resemblance of dynamics between reversible and Hamiltonian contexts, probably first noticed by Poincaré and Birkhoff, has caught much attention since the sixties of the twentieth century. Since then many important results, e.g. KAM theory, Liapunov center theorems, etc, holding in the Hamiltonian context have been carried over to the reversible one (see [10, 17] and reference therein).

The concept of reversibility is linked with an involution R, i. e., a map $R : \mathbb{R}^N \to \mathbb{R}^N$ such that $R \circ R = Id$. Let X be a smooth vector field on R^N. The vector field is called R–reversible if the following relation is satisfied

$$X(R(x)) = -DR_x.X(x).$$

Reversibility means that $x(t)$ is a solution of X if and only if $Rx(-t)$ is also a solution. The set $Fix(R) = \{x \in \mathbb{R}^N : R(x) = x\}$ plays an important role in the reversible systems. We say that a singular point p is symmetric if $p \in Fix(R)$, and analogously we say that an orbit γ is symmetric if $R(\gamma) = \gamma$.

Many dynamical systems that arise in the context of applications possess robust structural properties, such as for instance symmetries or Hamiltonian structure. In order to understand the typical dynamics of such systems, their structure need to be taken into account, leading one to study phenomena that are generic among dynamical systems with the same structure. In the last decade there has been a surging interest in the study of systems with time-reversal symmetries (see [15] and [8]). Symmetry properties arise naturally and frequently in dynamical systems. In recent years, a lot of

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attention has been devoted to understand and use the interplay between dynamics and symmetry properties. It is worthwhile to mention that one of the characteristic properties of Hamiltonian and reversible systems is that minimal sets appear in one-parameter families. So a number of natural questions can be formulated, such as: (i) how do branches of such minimal sets terminate or originate?; (ii) can one branch of minimal sets bifurcate from another such branch?; (iii) how persistent is such branching process when the original system is slightly perturbed? Recently, there has been a surging interest in the study of systems with time-reversal symmetries and we refer [11] for a survey in reversible systems and related problems.

Our main concern, in this article, is to find conditions for the existence of one-parameter families of periodic orbits terminating at the equilibrium.

We present some relevant historical facts. In 1895 Liapunov published his celebrated center theorem, see Abraham and Marsden [1] p 498. This theorem, for analytic Hamiltonians with n degrees of freedom, states that if the eigenfrequencies of the linearized Hamiltonian are independent over $\mathbb{Z}$, near a stable equilibrium point, then there exists n families of periodic solutions filling up smooth 2-dimensional manifolds going through the equilibrium point. Devaney [5] proved a time-reversible version of the Liapunov center theorem. Recently this center theorem has been generalized to equivariant systems, by Golubitsky, Krupa and Lim [6] in the time-reversible case, and by Montaldi, Roberts and Stewart [13] in the Hamiltonian case. We recall that in [6] the Devaney's theorem was extended and some extra symmetries were considered. Contrasting Devaney's geometrical approach, they used Liapunov-Schmidt reduction, adapting an alternative proof of the reversible Liapunov center theorem given by Vanderbauwhede [16]. In [13] the existence of families of periodic orbits around an elliptic semi-simple equilibrium is analyzed. Systems with symmetry, including time-reversal symmetry, which are anti-symplectic are studied. Their approach is a continuation of the work of Vanderbauwhede, in [16], where the families of periodic solutions correspond bijectively to solutions of a variational problem.

Recently Buzzi and Teixeira in [3] have analyzed the dynamics of time-reversible Hamiltonian vector fields with 2 degrees of freedom around an elliptic equilibrium point in presence of $1:-1$ resonance. Such systems appear generically inside a class of Hamiltonian vector fields in which the symplectic structure is assumed to have some symmetric properties. Roughly speaking, the main result says that under certain conditions the original Hamiltonian H is formally equivalent to another Hamiltonian $\widetilde{H}$ such that the corresponding Hamiltonian vector field $X_{\widetilde{H}}$ has two Liapunov families of symmetric periodic solutions terminating at the equilibrium. It is worth to say that all the systems considered there have been derived from the expression of Birkhoff normal form.

In this paper we address the problem to systems with 2 and 3 degrees of freedom. Physical models of such systems were exhibited in [4, 9]. As usual the main proofs are based on a combined use of normal form theory and the Liapunov-Schmidt Reduction. It is important to mention that our results concerning the existence of Liapunov families generalize those ones in [3]. As a matter of fact we deal with C^∞ Hamiltonian vector fields and not only with systems written in Birkhoff normal form.

We begin in Section 2 with an introduction of the terminology and basic concepts for the formulation of our results. In Section 3 the Belitskii normal form is discussed. In Section 4 the Liapunov-Schmidt reduction is presented. In Section 5 the usefulness of Birkhoff normal form in our approach is pointed. In Section 6 we study the Hamiltonian with 2 degrees of freedom denoted by Ω^0, and we generalize some results presented in [3] by proving Theorem A. Theorem A says that there exists an open set $\mathcal{U}^0 \subset \Omega^0$, in the C^∞–topology, such that (a) $\mathcal{U}^0$ is determined by the 3–jet of the vector fields; and (b) each $X \in \mathcal{U}^0$ possesses two 1–parameter families of periodic solutions terminating at the equilibrium. In section 7 we study the Hamiltonian with 3 degrees of freedom, and we prove Theorems B and C. In Theorem B we consider the involution associated to the system satisfying $dim(Fix(R)) = 2$, and in Theorem C satisfying $dim(Fix(R)) = 4$. We denote these spaces of reversible Hamiltonian vector fields by Ω^1 and Ω^2, respectively. The conclusions are the following: In Theorem B there exists an open set $\mathcal{U}^1 \subset \Omega^1$, in the C^∞–topology, such that (a) $\mathcal{U}^1$ is determined by the 2–jet of the vector fields, and (b) for each $X \in \mathcal{U}^1$ there is no periodic orbit arbitrarily close to the equilibrium. In Theorem C there exists an open set $\mathcal{U}^2 \subset \Omega^2$, in the C^∞–topology, such that (a) $\mathcal{U}^2$ is determined by the 2–jet of the vector fields, and (b) each $X \in \mathcal{U}^2$ has infinitely many one–parameter family of periodic solutions terminating at an equilibrium with the periods tending to $2\pi/\alpha$.

2. Preliminaries

Now we introduce some of the terminology and basic concepts for the formulation of our results.

We consider (germs of) smooth functions $H : \mathbb{R}^{2n}, 0 \to \mathbb{R}$ having the origin as a equilibrium point. The corresponding Hamiltonian vector field, to be denoted by X_H, has the origin as an equilibrium or singular point. We recall that $dH = \omega(X_H, \cdot)$, where $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n$ denotes the standard 2-form on $\mathbb{R}^{2n}$. In coordinates X_H is expressed as:

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \;\; \dot{y}_i = -\frac{\partial H}{\partial x_i}; \;\; i = 1, \cdots, n.$$

In $\mathbb{R}^6$ we have

$$\begin{pmatrix} \dot{x_1} \\ \dot{y_1} \\ \vdots \\ \dot{x_3} \\ \dot{y_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial y_1} \\ \vdots \\ \frac{\partial H}{\partial x_3} \\ \frac{\partial H}{\partial y_3} \end{pmatrix}.$$

Here,

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

is the symplectic structure associated with the 2-form ω given above.

We say that an involution is *symplectic* when it satisfies the equation $\omega(DR_p(v_p), DR_p(w_p)) = \omega(v_p, w_p)$. If the involution R is linear then this definition is equivalent to $JR = R^T J$, where J is the symplectic structure and R^T is the transpose matrix of R.

The next proposition exhibits normal forms for linear symplectic involutions on $\mathbb{R}^6$.

Proposition 2.1. *Fixed the symplectic structure ω and given an involution R there exists a symplectic change of coordinates that transforms R in one of the following normal forms*

(1) $R_0 = Id$,
(2) $R_0(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, x_2, y_2, -x_3, -y_3)$,
(3) $R_0(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, -x_2, -y_2, -x_3, -y_3)$,
(4) $R_0 = -Id$.

Before the proof we observe that the mapping $\psi = (1/2)(R + L)$, where $L = DR(0)$, is a symplectic conjugacy between R and L, i. e., $R \circ \psi = \psi \circ L$. So we may and do assume, without lost of generality, that the involution R is linear.

Lemma 2.2. *If R is a linear symplectic involution, then we have that $\mathbb{R}^6 = \mathrm{Fix}(R) \oplus \mathrm{Fix}(-R)$ and $\omega(\mathrm{Fix}(R), \mathrm{Fix}(-R)) = 0$.*

Proof: For every $u \in \mathbb{R}^6$, we can write $u = ((u + R(u))/2) + ((u - R(u))/2)$. Notice that $(u + R(u))/2 \in \mathrm{Fix}(R)$ and $(u - R(u))/2 \in \mathrm{Fix}(-R)$. Now, let $u \in \mathrm{Fix}(R)$ and $v \in \mathrm{Fix}(-R)$, so we have that $\omega(u, v) = \omega(R(u), -R(v))$. By using that R is symplectic and R is linear, i. e, $\omega(R(u), R(v)) = \omega(u, v)$. So $-\omega(u, v) = \omega(u, v)$, and we have proved that $\omega(\mathrm{Fix}(R), \mathrm{Fix}(-R)) = 0$. $\square$

A linear subspace $U \in \mathbb{R}^6$ is *symplectic* if ω is *non-degenerate* in U, i. e, if $\omega(u, v) = 0$ for all $u \in U$ then $v = 0$.

Lemma 2.3. $\mathrm{Fix}(R)$ *and* $\mathrm{Fix}(-R)$ *are symplectic subspace.*

Proof: Suppose $u \in \mathrm{Fix}(R)$ and $u \neq 0$ such that $\omega(u, R(u)) = 0$. By using Lemma 2.2, we have $\omega(\mathrm{Fix}(R), \mathrm{Fix}(-R)) = 0$, so $\omega(u, \mathrm{Fix}(-R)) = 0$. Again by Lemma 2.2 ($\mathbb{R}^6 = \mathrm{Fix}(R) \oplus \mathrm{Fix}(-R)$) we have $\omega(u, \mathbb{R}^6) = 0$ and so ω is degenerate in $\mathbb{R}^6$ which is not true. Then $\mathrm{Fix}(R)$ is a symplectic subspace. The proof for $\mathrm{Fix}(-R)$ is analogous. $\square$

Proof of Proposition 2.1: Let $R : \mathbb{R}^6 \to \mathbb{R}^6$ be a linear involution and ω be a fixed symplectic structure. From Lemma 2.2, $\mathbb{R}^6 = \mathrm{Fix}(R) \oplus \mathrm{Fix}(-R)$ and as $\mathrm{Fix}(R)$ is a symplectic subspace, then $\dim \mathrm{Fix}(R) = 0, 2, 4$, or 6.

- if $\dim \mathrm{Fix}(R) = 0$, then we can find a coordinate system such that $R_0 = -Id$;
- if $\dim \mathrm{Fix}(R) = 6$, then we can find a coordinate system such that $R_0 = Id$;
- if $\dim \mathrm{Fix}(R) = 4$, consider the bases $\beta_1 = \{e_1, e_2, e_3, e_4\}$ for $\mathrm{Fix}(R)$ and $\beta_2 = \{f_1, f_2\}$ for $\mathrm{Fix}(-R)$. So $\beta = \{e_1, e_2, e_3, e_4, f_1, f_2\}$ is a basis for $\mathbb{R}^6$. Let's show that β can be chosen such that $[\omega]_\beta = J$ and $[R]_\beta = R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$.

  Note that $\omega(e_i, e_i) = 0$ and $\omega(f_j, f_j) = 0$, $i = 1, 2, 3, 4$ and $j = 1, 2$. By the Lemma 2.2 $\omega(e_i, f_j) = 0$, $i = 1, 2, 3, 4$ and $j = 1, 2$. And as ω is alternative, then $\omega(f_1, f_2) = 1$ and $\omega(f_2, f_1) = -1$.

  Define $\omega(e_i, e_j)$ for $i \neq j$. From Darboux's Theorem there exists a coordinate system around 0 such that $\omega|_{\beta_1}$ in this coordinate system is the symplectic structure J.
- if $\dim \mathrm{Fix}(R) = 2$, in the same way as above, we get

$$R_0 = [R]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\square$

Following Proposition 2.1 the following cases will be considered:

i) the associated involution is $R_1(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1,\ y_1,\ -x_2,\ -y_2,\ -x_3,\ -y_3)$ (called the $6:2$–*case*); and

ii) the associated involution is $R_2(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, x_2, y_2,\ -x_3,\ -y_3)$ (called the $6:4$–*case*).

2.1. **Linear part of a R_j–reversible Hamiltonian vector field in $\mathbb{R}^6$.** Denote by Ω^j the space of all R_j–reversible Hamiltonian vector field X_{H_j} in $\mathbb{R}^6$ with 3-degrees freedom where H_j denotes the associate Hamiltonian. Fix the coordinate system $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6, 0$, $j = 1, 2$. We endow Ω^j with the C^∞–topology.

The symplectic structure given by J is:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Observe that the involution R_j is symplectic, i.e, $J.R_j - R_j^T.J = 0$, $j = 1, 2$.

As the involution is symplectic, then the vector field is R_j-reversible if and only if the Hamiltonian function H_j is R_j-anti-invariant, $j = 1, 2$. This is equivalent to say that $H_j \circ R_j = -H_j$. (See [3])

Define $H_j(x_1, y_1, x_2, y_2, x_3, y_3) = a_{01}x_1^2 + a_{02}x_1y_1 + a_{03}x_1x_2 + a_{04}x_1y_2 + a_{05}x_1x_3 + a_{06}x_1y_3 + a_{07}y_1^2 + a_{08}y_1x_2 + a_{09}y_1y_2 + a_{10}y_1x_3 + a_{11}y_1y_3 + a_{12}x_2^2 + a_{13}x_2y_2 + a_{14}x_2x_3 + a_{15}x_2y_3 + a_{16}y_2^2 + a_{17}y_2x_3 + a_{18}y_2y_3 + a_{19}x_3^2 + a_{20}x_3y_3 + a_{21}y_3^2 + h.o.t.$

First of all we impose the R_j–reversibility on our Hamiltonian system, $j = 1, 2$. So:

a) case $6 : 2$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

From $H_1 \circ R_1 = -H_1$, we have

$$\begin{aligned} H_1 = \; & a_{03}x_1x_2 + a_{04}x_1y_2 + a_{05}x_1x_3 + a_{06}x_1y_3 + \\ & a_{08}x_2y_1 + a_{09}y_1y_2 + a_{10}x_3y_1 + a_{11}y_1y_3 + h.o.t., \end{aligned}$$

with $a_{03}, a_{04}, a_{05}, a_{06}, a_{08}, a_{09}, a_{10} \in \mathbb{R}$. Then, the linear part of Hamiltonian vector field X_{H_1} is

$$A_1 = \begin{pmatrix} 0 & 0 & a & b & c & d \\ 0 & 0 & e & f & g & h \\ -f & b & 0 & 0 & 0 & 0 \\ e & -a & 0 & 0 & 0 & 0 \\ -h & d & 0 & 0 & 0 & 0 \\ g & -c & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Just to simplify the notation we replace a_{03}, a_{04}, a_{05}, a_{06}, a_{08}, a_{09}, a_{10}, a_{11} by a, b, c, d, $-e$, $-f$, $-g$, $-h$ respectively. Note that A_1 is R_1-reversible (i. e, $R_1.A_1 + A_1.R_1 = 0$). The eigenvalues of A_1 are $\{0, 0, \pm\sqrt{be - af + dg - ch}, \pm\sqrt{be - af + dg - ch}\}$. We restrict our attention to those systems satisfying the inequality:

$$be - af + dg - ch < 0. \tag{2.1}$$

We shall use the Jordan canonical form from A_1. So we stay, for while, away from the original symplectic structure. We call $\alpha = \sqrt{-be + af - dg + ch}$, and so the transformation matrix is

$$P_1 = \begin{pmatrix} 0 & 0 & \frac{-d}{dg-ch}\alpha & 0 & \frac{-c}{dg-ch}\alpha & 0 \\ 0 & 0 & \frac{-h}{dg-ch}\alpha & 0 & \frac{-g}{dg-ch}\alpha & 0 \\ \frac{df-bh}{be-af} & \frac{cf-bg}{be-af} & 0 & \frac{-df+bh}{dg-ch} & 0 & \frac{-cf+bg}{dg-ch} \\ \frac{-de+ah}{be-af} & \frac{-ce+ag}{be-af} & 0 & \frac{de-ah}{dg-ch} & 0 & \frac{ce-ag}{dg-ch} \\ 0 & 1 & 0 & 0 & & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

So

$$\widehat{A_1} = P_1^{-1}.A_1.P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & -\alpha & 0 \end{pmatrix},$$

where P_1^{-1} is the inverse matrix. Moreover, in this way, $\widehat{R_1} = P_1^{-1}.R_1.P$ takes the form

$$\widehat{R_1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

b) case $6:4$

We proceed in the same way as in the previous case. The involution is

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

and the Hamiltonian function in this case has the form:

$$
\begin{aligned}
H_2 = \ & a_{05}x_1x_3 + a_{14}x_2x_3 + a_{10}x_3y_1 + a_{17}x_3y_2 + \\
& a_{06}x_1y_3 + a_{15}x_2y_3 + a_{11}y_1y_3 + a_{18}y_2y_3 + h.o.t.
\end{aligned}
$$

Then, the linear part of Hamiltonian vector field X_{H_2} is expressed by:

$$
A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & 0 & e & f \\ 0 & 0 & 0 & 0 & g & h \\ -d & b & -h & f & 0 & 0 \\ c & -a & g & -e & 0 & 0 \end{pmatrix}.
$$

Again we change the notation. The eigenvalues of A_2 are given by $\{0, 0, \pm\sqrt{bc - ad + fg - eh}, \pm\sqrt{bc - ad + fg - eh}\}$. We consider the case

$$
bc - ad + fg - eh < 0. \tag{2.2}
$$

We call $\alpha = \sqrt{-bc + ad - fg + eh}$ and consider the transformation matrix

$$
P_2 = \begin{pmatrix} \frac{be-af}{bc-ad} & \frac{-bg+ah}{bc-ad} & 0 & \frac{-b}{\alpha} & 0 & \frac{-a}{\alpha} \\ \frac{de-cf}{bc-ad} & \frac{-dg+ch}{bc-ad} & 0 & \frac{-d}{\alpha} & 0 & \frac{-c}{\alpha} \\ 0 & 1 & 0 & \frac{-f}{\alpha} & 0 & \frac{-e}{\alpha} \\ 1 & 0 & 0 & \frac{-h}{\alpha} & 0 & \frac{-g}{\alpha} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},
$$

and the Jordan canonical form of A_2 is:

$$
\widehat{A_2} = P_2^{-1}.A_2.P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & -\alpha & 0 \end{pmatrix}.
$$

Moreover, in this way, $\widehat{R_2} = P_2^{-1}.R_2.P_2$ takes the form

$$
\widehat{R_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

3. Belitskii normal form

In this section we present the Belitskii Normal Form. When a vector field is in this normal form we can write explicitly the resultant equation of Liapunov–Schmitd reduction.

Consider a formal vector field expressed by

$$\hat{X}(x) = Ax + \sum_{k\geq 2} X^{(k)}(x)$$

where $X^{(k)}$ is the homogeneous part of degree k. Let us look for a "simple" form of the formal vector field $\hat{Y} = \hat{\phi} * \hat{X}$ by means of formal transformation

$$\hat{\phi} = x + \sum_{k}^{\infty} \phi^{(k)}(x).$$

The proof of the next theorem is in [2].

Theorem 3.1. *Given a formal vector field*

$$\hat{X}(x) = Ax + \sum_{k\geq 2} X^{(k)}(x),$$

there is a formal transformation $\hat{\phi}(x) = x + \ldots$ bringing $\hat{X}$ to the form $(\hat{\phi} * X)(x) = Ax + h(x)$ where h is a formal vector field with zero linear part commuting with A^T, i.e

$$A^T h(x) = h'(x) A^T x,$$

where A^T is the transposed matrix.

Here we call the normal form $(\hat{\phi}_* X)(x) = Ax + h(x)$ the Belitskii normal form. By abuse of the terminology, call $X_H = A + h$.

4. Liapunov–Schmidt reduction

In this section we recall the main features of the Liapunov–Schmidt reduction. As a matter of fact, we adapt the setting presented in [18] into our approach. In this way consider the R-reversible system expressed by

$$\dot{x} = X_H(x); \;\; x \in \mathbb{R}^6 \tag{4.3}$$

satisfying $X_H(Rx) = -RX_H(x)$ with R a linear involution in $\mathbb{R}^6$. Assume that $X_H(0) = 0$ and consider

$$A = D_1 X_H(0), \tag{4.4}$$

the Jacobian matrix of X_H in the origin.

In our case the linear part of vector field has the following eigenvalues: 0 with the algebraic and geometric multiplicity 2, and $\pm\alpha i$, also with algebraic and geometric multiplicity 2, $\alpha \in \mathbb{R}$. Performing a time rescaling we may take $\alpha = 1$. We write the real form of the linear part of the vector field X_j:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let $C^0_{2\pi}$ the Banach space of de $2\pi-$periodic continuous mappings $x : \mathbb{R} \to \mathbb{R}^6$ and $C^1_{2\pi}$ the corresponding C^1-subspace. We define an inner product on $C^0_{2\pi}$ by

$$(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} < x_1(t), x_2(t) > dt$$

where $< \cdot, \cdot >$ denotes an inner product in $\mathbb{R}^6$.

The main aim is to find all small periodic solutions of (4.3) with period near 2π.

Define the map $F : C^1_{2\pi} \times \mathbb{R} \to C^0_{2\pi}$ by

$$F(x, \sigma)(t) = (1 + \sigma)\dot{x}(t) - X_H(x(t)).$$

Note that if $(x_0, \sigma_0) \in C^1_{2\pi} \times \mathbb{R}$ is such that

$$F(x_0, \sigma_0) = 0, \tag{4.5}$$

then $\tilde{x}(t) := x_0((1 + \sigma_0)t)$ is a $2\pi/(1 + \sigma_0)-$periodic solution of (4.3).

Our task now is to find the zeroes of F. Clearly, $(x_0, \sigma_0) = (0, 0)$ is one solution of $F(x_0, \sigma_0) = 0$. Let $L := D_x F(0, 0) : C^1_{2\pi} \to C^0_{2\pi}$, explicitly L is given by

$$Lx(t) = \dot{x}(t) - Ax(t).$$

Consider the unique (S-N)-decomposition of A, $A = S + N$. Recall that in our case A is semi-simple, i. e, $A = S$. Define the subspace $\mathcal{N}$ of $C^1_{2\pi}$ as

$$\begin{aligned} \mathcal{N} = \ & \{q; \dot{q}(t) = Sq(t)\} = \\ & \{q;\ q(t) = exp(tS)x;\ x \in \mathbb{R}^6\}. \end{aligned}$$

Observe that $\mathcal{N} \subset C^1_{2\pi}$ and the solution's base of $\dot{q} = Sq$ is given by the set $\{(1, 0, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0, 0)$, $(0, 0, \cos(t), \sin(t), 0, 0)$, $(0, 0, -\sin(t), \cos(t), 0, 0)$, $(0, 0, 0, 0, \cos(t), \sin(t))$, $(0, 0, 0, 0, -\sin(t), \cos(t))\}$.

In order to study certain properties of the operator L we introduce $\mathcal{N} \subset C^1_{2\pi}$ and the following definitions and notations.

We will put the solution of $F(x_0, \sigma_0) = 0$ in one-to-one correspondence with the solutions of an appropriate equation in $\mathcal{N}$. Define the subspaces

$$X_1 = \{x \in C^1_{2\pi} :\ (x, \mathcal{N}) = 0\}$$

and

$$Y_1 = \{y \in C^0_{2\pi} : (y, \mathcal{N}) = 0\}$$

as the orthogonal complements of $\mathcal{N}$ in $C^1_{2\pi}$ and $C^0_{2\pi}$, respectively.

Let $(q_1, q_2, q_3, q_4, q_5, q_6)$ with $q_i = exp(tS)u_i$ where u_i, $i = 1, ..., 6$, is a basis for $\mathbb{R}^6$. Then we define a projection

$$\mathcal{P} : C^0_{2\pi} \to C^0_{2\pi}$$

by

$$\mathcal{P} = \sum_{i=1}^{6} q_i^*(\cdot) q_i \ \in \ \mathcal{L}(C^0_{2\pi})$$

with $q_i^*(x) = (q_i, x)$.

We have $\mathrm{Im}(\mathcal{P}) = \mathcal{N}$ and $\mathrm{Ker}(\mathcal{P}) = Y_1$. Hence,

$$C^1_{2\pi} = X_1 \oplus \mathcal{N}, \ \ C^0_{2\pi} = Y_1 \oplus \mathcal{N}.$$

Now we consider

$$F(x, \sigma) = F(q + x_1, \sigma) =: \hat{F}(q, x_1, \sigma); \ q \in \mathcal{N}, \ x_1 \in X_1.$$

The proof of next result can be found in [7].

Lemma 4.1. *(**Fredholm's Alternative**) Let $A(t)$ be a matrix in C^0_T and let f be in C_T. Here C^0_T is the space of the matrices with entrances continuous and T–periodic, and C_T is the set of T-periodic maps from $\mathbb{R}$ to $\mathbb{R}^n$. Then the equation $\dot{x} = A(t)x + f(t)$ has a solution in C_T if, and only if,*

$$\int_0^T < y(t), g(t) > dt = 0$$

for all solution y of the adjoint equation

$$\dot{y} = -yA(t)$$

such that $y^t \in C_T$

As $L(\mathcal{N}) \subset \mathcal{N}$ this lemma implies the following:

Lemma 4.2. *The mapping $\hat{L} := L|_{X_1} : X_1 \to Y_1$ is bijective.*

Let's study the solutions of $\hat{F}(q, x_1, \sigma) = 0$. These solutions are equivalents to the solutions of the system

$$\begin{aligned} (I - \mathcal{P}) \circ \hat{F}(q, x_1, \sigma) &= 0, \\ \mathcal{P} \circ \hat{F}(q, x_1, \sigma) &= 0. \end{aligned}$$

With the Lemma 4.2 and the Implicit Function Theorem we can solve the first equation as $x_1 = x_1^*(q, \sigma)$. Then, (4.5) is reduced to

$$\tilde{F}(q, \sigma) := \mathcal{P} \circ \hat{F}(q, x_1^*(q, \sigma), \sigma) = 0.$$

This equation is solved if, and only if,

$$q_i^*(\hat{F}(q, x_1^*(q, \sigma), \sigma) = 0, \ i = 1, \cdots, 6.$$

Notice that (u,σ) is a solution of (4.5) provided that

$$B(u,\sigma)=0 \tag{4.6}$$

with $B:\mathcal{N}\times\mathbb{R}\to\mathbb{R}^6$ defined by

$$B(u,\sigma):=\frac{1}{2\pi}\int_0^{2\pi} exp(-tS)F(x^*(u,\sigma),\sigma)dt$$

and

$$x^*(u,\sigma):=exp(tS)u+x_1^*(exp(tS)u,\sigma).$$

Let's present some properties of the mapping B.

The proof of next lemma can be found in [10].

Lemma 4.3. *It holds*

i) $s_\phi B(u,\sigma)=B(s_\phi u,\sigma)$;
ii) $RB(u,\sigma)=-B(Ru,\sigma)$, *where s_ϕ is the S^1-action in $\mathbb{R}^6$ defined by $s_\phi u=exp(-\phi S_0)u$.*

Observe that under the above condition $i)$ the mapping B is S^1- equivariant whereas condition $ii)$ states that the mapping B is $R-$ anti-equivariant, i. e, B inherits the anti-symmetric properties of X_H.

Assume that (4.3) is in Belitskii normal form truncated at the order p. So $X_H(x)=Ax+h(x)+r(x)$ where $r(x)=\mathcal{O}(\| x\|^{p+1})$. The proof of next result is in [18].

Theorem 4.4. *It holds*

i) $x^*(u,\sigma)=exp(tS)u+\mathcal{O}(\| x\|^{p+1})$,
ii) $B(u,\sigma)=(1+\sigma)Su-Au-h(u)+\mathcal{O}(\| x\|^{p+1})$ *for σ near the origin.*

If (u,σ) is a solution of (4.6) then $x=x^*(u,\sigma)$ corresponds to a $2\pi/(1+\sigma)$-periodic solution of (4.5).

Recall that the periodic solution of (4.6) is R-symmetric if and only if it intersects $\mathrm{Fix}(R)$ in exactly two points. In conclusion, we obtain all small symmetric periodic solutions of (4.6) by solving the equation

$$G(u,\sigma)=B(u,\sigma)\mid_{\mathrm{Fix}(R)}=0. \tag{4.7}$$

5. Birkhoff normal form

In this section we briefly discuss some points concerning the Birkhoff normal form that will be useful in the sequel. The Belitskii normal form is useful it preserves the simplectic structure. In our cases if the vector field is in the Birkhoff normal form then it is in the Belitskii normal form, and so we can apply Theorem 4.4.

The function $\{f,g\}=\omega(X_f,X_g)$ is called the *Poisson bracket* of the smooth functions f and g. Let $\mathcal{H}_n$ be the set of all homogeneous polynomials of degree n. The application adjoint $Ad_{H_2}:\mathcal{H}_n\to\mathcal{H}_n$ is defined by

$$Ad_{H_2}(H) = \{H_2, H\} = \omega(X_{H_2}, X_H) = < -X_{H_2}, \nabla H > . \tag{5.8}$$

The *Birkhoff Normal Form Theorem* [14] states that if we have a Hamiltonian $H = H_2 + H_3 + H_4 + \cdots$, where $H_i \in \mathcal{H}_i$ is the homogeneous part of degree i, and $\mathcal{G}_i \subset \mathcal{H}_i$ satisfies $\mathcal{G}_i \oplus Range(Ad_{H_2}) = \mathcal{H}_i$, then there exists a formal symplectic power series transformation Φ such that $H \circ \Phi = H_2 + \widetilde{H_3} + \widetilde{H_4} + \cdots$ where $\widetilde{H_i} \in \mathcal{G}_i$ $(i = 3, 4, \dots)$. In particular, if Ad_{H_2} is semi-simple, as in our case, then $Ker(Ad_{H_2})$ complements $Range(Ad_{H_2})$.

As R_j is symplectic, the change of coordinates Φ can be chosen in such a way that $H \circ \Phi$ satisfies $H \circ \Phi \circ R_j = -H \circ \Phi$. In order to see this, we can split $\mathcal{H}_i = \mathcal{H}_i^+ \oplus \mathcal{H}_i^-$, where $\mathcal{H}_i^+ = \{H \in \mathcal{H}_i : H \circ R_j = H\}$ and $\mathcal{H}_i^- = \{H \in \mathcal{H}_i : H \circ R_j = -H\}$. If R_j is symplectic, then $Ad_{H_2}(\mathcal{H}_i^\pm) = \mathcal{H}_i^\mp$. In this case, if $\mathcal{H}_i = \mathcal{G}_i \oplus Ad_{H_2}(\mathcal{H}_i)$, then $\mathcal{H}_i^- = (\mathcal{G}_i \cap \mathcal{H}_i^-) \oplus Ad_{H_2}(\mathcal{H}_i^+)$. Now we can perform the change of coordinates restricted to $\mathcal{H}_i^-$. It implies that all monomial terms in the image of the adjoint restrict to H_i^- can be removed and it will remain only monomials in the kernel of the adjoint restrict to H_i^-. And so, the normal form is also R_j–reversible.

6. Two degrees of freedom

In [3] a Birkhoff normal form for each $X \in \Omega^0$ is derived and the following result is obtained:

Theorem 6.1. *Assume H is a Hamiltonian that is anti-invariant with respect to the involution and the associated vector field X_H has an elliptical equilibrium point. Then there exists another Hamiltonian $\widetilde{H}$, formally C^k–equivalent to H, such that the vector field $X_{\widetilde{H}}$ has two one–parameter families of symmetric periodic solutions, with period near $2\pi/\sqrt{ad - bc}$, as in the Liapunov's Theorem, going through the equilibrium point.*

Let Ω^0 be the space of the C^∞ R_0–reversible Hamiltonian vector fields with two degrees of freedom in $\mathbb{R}^4$ and fix the coordinate system $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. We endow Ω^0 with the C^∞–topology. In this work we prove the following result, which generalizes the previous one.

Theorem A: *There exists an open set $\mathcal{U}^0 \subset \Omega^0$ such that*

(a) *$\mathcal{U}^0$ is determined by the 3–jet of the vector fields.*
(b) *each $X \in \mathcal{U}^0$ possesses 2–families of symmetric periodic solutions terminating at the equilibrium point.*

Proof: Fix on $\mathbb{R}^4$ a symplectic structure as in the Proposition 2.1. So the normal form of an involution has one of the following form: $Id_{\mathbb{R}^4}$ or $-Id_{\mathbb{R}^4}$ or $R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. We work just with R_0–reversible vector fields.

As in the cases in $\mathbb{R}^6$ we have that by the hypothesis the Hamiltonian H satisfy $H \circ R_0 = -H$, so the linear part of the vector field X_H is given by

$$A = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -d & b & 0 & 0 \\ c & -a & 0 & 0 \end{pmatrix}, \tag{6.9}$$

and their eigenvalues are $\{\pm\sqrt{bc-ad}, \pm\sqrt{bc-ad}\}$. We are interested in the case with $bc-ad < 0$. We call $\alpha = \sqrt{ad-bc}$ and consider the transformation matrix

$$P = \begin{pmatrix} 0 & \frac{-b}{\alpha} & 0 & \frac{-a}{\alpha} \\ 0 & \frac{-d}{\alpha} & 0 & \frac{-c}{\alpha} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

So

$$\widehat{A} = P^{-1}.A.P = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix},$$

where P^{-1} is the inverse matrix. Moreover, in this way, $\widehat{R_0} = P^{-1}.R_0.P$ takes the form

$$\widehat{R_0} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Performing a time rescaling we can assume that $\alpha = 1$. We write the canonical real Jordan form of A as

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

First we obtain the Belitskii normal form of X_H, by considering $h : \mathbb{R}^4 \to \mathbb{R}^4$ until 3rd order, which is given by $X_H(x_1, y_1, x_2, y_2) = A[x_1, y_1, x_2, y_2] + h(x_1, y_1, x_2, y_2)$. And after we impose that the Belitskii normal form is $\widehat{R_0}$–reversible, i. e, $X_H\widehat{R_0} = -\widehat{R_0}X_H$. Then the system obtained is given by

(6.10)

$$
\begin{aligned}
\dot{x_1} = &\quad y_1 + (e_{21}y_1 + e_{23}y_2)(x_1^2 + y_1^2) + e_{30}y_2(x_2^2 + y_2^2) \\
&\quad +(e_{16}x_1 + e_{24}x_2)(y_1x_2 - x_1y_2) + e_{26}y_2(x_1x_2 + y_1y_2), \\
\dot{y_1} = &\quad -x_1 + (-e_{21}x_1 - e_{23}x_2)(x_1^2 + y_1^2) - e_{30}x_2(x_2^2 + y_2^2) \\
&\quad +(e_{16}y_1 + e_{24}y_2)(y_1x_2 - x_1y_2) - e_{26}x_2(x_1x_2 + y_1y_2), \\
\dot{x_2} = &\quad y_2 + (-d_{15}y_1 - d_{22}y_2)(x_1^2 + y_1^2) - (d_{20}y_1 + d_{29}y_2)(x_2^2 + y_2^2) \\
&\quad -(d_{17}y_1 + d_{25}y_2)(x_1x_2 + y_1y_2), \\
\dot{y_2} = &\quad -x_2 + (d_{15}x_1 + d_{22}x_2)(x_1^2 + y_1^2) + (d_{20}x_1 + d_{29}x_2)(x_2^2 + y_2^2) \\
&\quad +(d_{17}x_1 + d_{25}x_2)(x_1x_2 + y_1y_2).
\end{aligned}
$$

Now we apply the Birkhoff normal form. First of all we observe that the canonical symplectic matrix

$$
J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},
$$

after the linear change of coordinates P, is transformed in

$$
\widehat{J} = P^t J P = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
$$

We take a general Hamiltonian function $H : \mathbb{R}^4 \to \mathbb{R}$ of 4^{th} order, compute the kernel of Ad_{H_2} defined on (5.8), where H_2 is the homogeneous part of degree 2 of H, and impose that H satisfies $H \circ \widehat{R_0} = -H$. The Birkhoff normal form until 3^{th} order is given by $h_b(x) = \widehat{J} \cdot \nabla H(x)$ and its expression is

$$
\begin{aligned}
\dot{x_1} = &\quad y_1 + a_1y_1(x_1^2 + y_1^2) + a_2(2x_1x_2y_1 - x_1^2y_2 + y_1^2y_2) \\
&\quad +a_3(3x_2^2y_1 - 2x_1x_2y_2 + y_1y_2^2), \\
\dot{y_1} = &\quad -x_1 + (a_2y_1 + 2a_3y_2)(x_2y_1 - x_1y_2) - x_1(a_1(x_1^2 + y_1^2) \\
&\quad +a_2(x_1x_2 + y_1y_2) + a_3(x_2^2 + y_2^2)), \\
\dot{x_2} = &\quad y_2 + (2a_1x_1 + a_2x_2)(-x_2y_1 + x_1y_2) + y_2(a_1(x_1^2 + y_1^2) \\
&\quad +a_2(x_1x_2 + y_1y_2) + a_3(x_2^2 + y_2^2)), \\
\dot{y_2} = &\quad -x_2 + (2a_1y_1 + a_2y_2)(-x_2y_1 + x_1y_2) - x_2(a_1(x_1^2 + y_1^2) \\
&\quad +a_2(x_1x_2 + y_1y_2) + a_3(x_2^2 + y_2^2)).
\end{aligned}
\tag{6.11}
$$

Observe that the Birkhoff normal form (6.11) is also in Belitskii normal form (6.10) and so we can apply the Theorem 4.4.

The Liapunov-Schmidt reduction gives us all small $\widehat{R_0}$–symmetric periodic solutions by solving the equation

$$
B(x,\sigma)\big|_{x\in \mathrm{Fix}(\widehat{R_0})} = 0,
$$

with

$$B(x,\sigma) = (1+\sigma)Sx - \hat{A}x - h_b(x),\ x \in \mathbb{R}^4,$$

S is the semi-simple part of (unique) $S-N-$decomposition of $\hat{A}$. (See [12]).

In our case, $\hat{A}$ is semi-simple and $\mathrm{Fix}(\widehat{R_0}) = \{(0,y_1,0,y_2);\ y_1,y_2 \in \mathbb{R}\}$. Recall that the reduced equation of the Liapunov-Schmidt, $B(x,\sigma)$, is defined in $\mathcal{N}\times\mathbb{R}$, where $\mathcal{N} = \{exp(\hat{A}t)x; x \in V\} \in C^1_{2\pi}$ and $V = ger\{e_1,e_2,e_3,e_4\} \equiv \mathbb{R}^4$.

$$\begin{aligned} G(y_1,y_2,\sigma) &= B(x,\sigma)|_{x\in\mathrm{Fix}(\widehat{R_0})} = \begin{bmatrix} F_\sigma(y_1,y_2) \\ K_\sigma(y_1,y_2) \end{bmatrix} = \\ &= \begin{bmatrix} -y_1(a_1y_1^2 + a_2y_1y_2 + a_3y_2^2 - \sigma) + \cdots \\ -y_2(a_1y_1^2 + a_2y_1y_2 + a_3y_2^2 - \sigma) + \cdots \end{bmatrix}. \end{aligned} \tag{6.12}$$

Notice that $K_\sigma(0,0) = 0$ and $\dfrac{\partial K_\sigma}{\partial y_2}(0,0) = \sigma \neq 0$, so there exists a neighborhood V of $y_1 = 0$ and an unique function $y_2 = \alpha_\sigma(y_1)$ with $\alpha_\sigma(0) = 0$ such that $K_\sigma(y_1,\alpha_\sigma(y_1)) = 0$. Moreover, we derive directly that $\alpha'_\sigma(0) = \alpha''_\sigma(0) = 0$ and $\alpha_\sigma(y_1) = O(3)$. Now we put $y_2 = \alpha_\sigma(y_1)$ in the first equation of (6.12) and consider $a_1 \neq 0$. So

$$\widetilde{F}(y_1,\sigma) = y_1^3 - \frac{\sigma}{a_1}y_1 + O(4).$$

From the Singularity Theory we get that, if $\dfrac{\sigma}{a_1} > 0$ then the equation $\widetilde{F} = 0$ has two non zero solutions, $\widetilde{y_1}$ and $\widehat{y_1}$. And the equation (6.12) has two non zero solutions $(\widetilde{y_1},\alpha_\sigma(\widetilde{y_1}))$ and $(\widehat{y_1},\alpha_\sigma(\widehat{y_1}))$. Analogously, if $a_3 \neq 0$ and $\dfrac{\sigma}{a_3} > 0$, then equation (6.12) has two non zero solutions.

Given $X \in \Omega^0$ denote by X^* its corresponding Birkhoff normal form at 0.

We define $\mathcal{U}^0 = \mathcal{U}_1^0 \cap \mathcal{U}_2^0$ where

$$\begin{aligned} \mathcal{U}_1^0 = &\ \{\ X \in \Omega^0; \quad \text{the canonical form of } DX(0) \text{ satisfies } ad - bc > 0\ \} \\ \text{and} & \\ \mathcal{U}_2^0 = &\ \{\ X \in \Omega^0; \quad \text{the coefficients of } X^* \text{ satisfies } a_1^2 + a_3^2 \neq 0\ \}. \end{aligned}$$

In $\mathcal{U}^0 = \mathcal{U}_1^0 \cap \mathcal{U}_2^0 \subset \Omega^0$ the equation $G(y_1,y_2,\sigma) = 0$ has two solutions non zero that tend to zero when σ tends to zero. So, in the original problem we have two one parameter families of periodic solutions terminating in the origin (when $\sigma \to 0$). ∎

7. Three degrees of freedom

As in previous Section, let Ω^1 (respec. Ω^2) be the space of the C^∞ R_1-reversible (respec. R_2–reversible) Hamiltonian vector fields with three degrees of freedom in $\mathbb{R}^6$ and fix a coordinate system $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$. We endow Ω^1 and Ω^2 with the C^∞–topology.

7.1. **Case 6:2.**

Theorem B: *There exists an open set $\mathcal{U}^1 \subset \Omega^1$ such that*

(a) *$\mathcal{U}^1$ is determined by the 2–jet of the vector fields.*
(b) *for each $X \in \mathcal{U}^1$ there is no symmetric periodic orbit arbitrarily close to the equilibrium point.*

Proof: First we obtain the Belitskii normal form of X_H, by considering $h : \mathbb{R}^6 \to \mathbb{R}^6$ until 2^{nd} order, and after we impose that the Belitskii normal form is $\widehat{R_1}$–reversible, i. e, $X_H\widehat{R_1} = -\widehat{R_1}X_H$. After that we obtain the Birkhoff normal form. The new symplectic structure is $\widehat{J} = P_1^t J P_1$, where P_1 is the linear matrix that brings the linear part of the vector field to the Jordan canonical form. The Birkhoff normal form is obtained by taking a general Hamiltonian function $H : \mathbb{R}^6 \to \mathbb{R}$ of 3^{rd} order, computing the kernel of Ad_{H_2} and imposing that H satisfies $H \circ \widehat{R_1} = -H$. The Birkhoff normal form until 2^{nd} order is given by $h_b(x) = \widehat{J} \cdot \nabla H(x)$. Finally, the Liapunov-Schmidt reduction gives us all small $\widehat{R_1}$–symmetric periodic solutions by solving the equation

$$B(x,\sigma)|_{x\in \mathrm{Fix}(\widehat{R_1})} = 0,$$

with

$$B(x,\sigma) = (1+\sigma)Sx - \widehat{A_1}x - h_b(x),\ x \in \mathbb{R}^6,$$

and S is the semi-simple part of (unique) $S - N$–decomposition of $\widehat{A_1}$. (See [12]). We observe here that, as in the $\mathbb{R}^4$ case, the Birkhoff normal form is also in the Belitskii normal form and so we can use the expression above for $B(x,\sigma)$. In our case, $\widehat{A_1}$ is semi-simple and $\mathrm{Fix}(\widehat{R_1}) = \{(0,0,x_2,0,x_3,0);\ ,x_2,x_3 \in \mathbb{R}\}$. We recall that the reduced equation of the Liapunov-Schmidt, $B(x,\sigma)$, is defined in $\mathcal{N}\times\mathbb{R}$, where $\mathcal{N} = \{exp(\widehat{A_1}t)x; x \in V\} \in C^1_{2\pi}$ and $V = ger\{e_1, e_2, e_3, e_4, e_5,\ e_6\} \equiv \mathbb{R}^6$.

We derive the following expression

$$G(x_2,x_3,\sigma) = B(x,\sigma)|_{x\in \mathrm{Fix}(\widehat{R_1})} =$$

$$= \begin{bmatrix} b_1x_2^2 + x_3(b_2x_2 + b_3x_3) + \cdots \\ b_4x_2^2 + x_3(b_5x_2 + b_6x_3) + \cdots \\ x_2(-\sigma+\delta) + \cdots \\ x_3(-\sigma+\delta) + \cdots \end{bmatrix}. \tag{7.13}$$

Observe that the equation $b_1x_2^2 + b_2x_2x_3 + b_3x_3^2 = 0$, generically, has the solution $(x_2, x_3) = (0,0)$ or has a pair of straight lines of solutions given by $(c_1x_2+d_1x_3)(c_2x_2+d_2x_3)=0$. The equation $b_4x_2^2+b_5x_2x_3+b_6x_3^2=0$ is analogous. We can conclude that if the two first components of (7.13) has no comom factor of the form cx_2+dx_3 then we have just the solution $(x_2,x_3)=(0,0)$ for the two previous equations.

We define the following open sets:

$$\mathcal{U}_1^1 = \left\{ X\in\Omega^1; \quad \text{the canonical form of } DX(0) \text{ satisfies (2.1)} \right\},$$
$$\mathcal{U}_2^1 = \left\{ \begin{array}{ll} X\in\Omega^1; & \text{the 2–jet of the two first equations of (7.13)} \\ & \text{have no common factor} \end{array} \right\}.$$

Let $\mathcal{U}^1=\mathcal{U}_1^1\cap\mathcal{U}_2^1$ be an open set in Ω^1.

The pair $(x_2,x_3)=(0,0)$ is the unique solution of the equation $G=0$. So, near the origin there are no *symmetric* periodic orbits for this case. ∎

7.2. Case 6:4.

Theorem C: *There exists an open set $\mathcal{U}^2\subset\Omega^2$ such that*

(a) *$\mathcal{U}^2$ is determined by the 2–jet of the vector fields.*
(b) *each $X\in\mathcal{U}^2$ has one 2–parameter family of periodic solutions $\gamma_{\sigma,\lambda}$ with $\sigma\in(-\epsilon,\epsilon)$ and $\lambda\in[0,2\pi]$, such that, for each λ_0, $\lim_{\sigma\to0}\gamma_{\sigma,\lambda_0}=0$ and the periods tend to $2\pi/\alpha$ when $\sigma\to0$.*

Proof: First of all we derive the reversible Belitskii normal form of X_H until 2^{nd} order. We observe that it coincides with the reversible Birkhoff normal form and is given by:

$$X^* = \begin{bmatrix} -\dfrac{b(x_3y_2-x_2y_3)\alpha^2}{\beta} \\ \dfrac{a(x_3y_2-x_2y_3)\alpha^2}{\beta} \\ (-ax_1-by_1)y_2+\alpha y_2 \\ x_2(ax_1+by_1)-x_2\alpha \\ (-ax_1-by_1)y_3+\alpha y_3 \\ x_3(ax_1+by_1)-x_3\alpha \end{bmatrix} \tag{7.14}$$

Finally the Liapunov-Schmidt reduction gives us all small $\widehat{R_2}$–symmetric periodic solutions by solving the equation

$$B(x,\sigma)|_{x\in\mathrm{Fix}(\widehat{R_2})}=0,$$

with

$$B(x,\sigma)=(1+\sigma)Sx-\widehat{A_2}x-h_b(x),\ x\in\mathbb{R}^6.$$

As before S is the semi-simple part of (unique) $S-N-$decomposition of $\widehat{A_2}$. (See [12]). In our case, $\widehat{A_2}$ is semi-simple and $\mathrm{Fix}(\widehat{R_2}) = \{(x_1, y_1, 0, y_2, 0, y_3); x_1, y_1, y_2, y_3 \in \mathbb{R}\}$. We recall that the reduced equation of the Liapunov-Schmidt, $B(x, \sigma)$, is defined in $\mathcal{N} \times \mathbb{R}$, where $\mathcal{N} = \{exp(\widehat{A_2}t)x; x \in V\} \in C^1_{2\pi}$ and $V = ger\{e_1, e_2, e_3, e_4, e_5,\ e_6\} \equiv \mathbb{R}^6$.

We derive the following expression

$$G(x_1, y_1, y_2, y_3, \sigma) = B(x, \sigma)\big|_{x \in \mathrm{Fix}(\widehat{R_2})} =$$

$$\begin{bmatrix} F_\sigma(x_1, y_1, y_2, y_3) \\ K_\sigma(x_1, y_1, y_2, y_3) \end{bmatrix} = \begin{bmatrix} y_2(\sigma + a_1 x_1 + a_2 y_1) + \cdots \\ y_3(\sigma + a_1 x_1 + a_2 y_1) + \cdots \end{bmatrix}. \tag{7.15}$$

Notice that $(F_\sigma, K_\sigma)(0, 0, 0, 0) = 0$ and $\left|\dfrac{\partial(F_\sigma, K_\sigma)}{\partial(y_2, y_3)}\right|_{(0,0,0,0)} = \sigma^2 \neq 0$. Then, there exist a neighborhood V of $(x_1, y_1) = (0, 0)$ and an unique function $y_2 = \xi_\sigma(x_1, y_1)$ and $y_3 = \delta_\sigma(x_1, y_1)$ in V such that $(F_\sigma, K_\sigma)(x_1,\ y_1,$ $\xi_\sigma(x_1, y_1),\ \delta_\sigma(x_1, y_1)) = (0, 0)$ and $\dfrac{\partial(F_\sigma, K_\sigma)}{\partial(x_1, y_1)}\Big|_{(0,0,0,0)} = \mathbf{0}$. This implies that the derivative $\dfrac{\partial(\xi_\sigma, \delta_\sigma)}{\partial(x_1, y_1)}$ is bounded in a neighborhood of the origin. For each σ we consider $\gamma_\sigma : (x_1, y_1) \mapsto (x_1, y_1,\ \xi_\sigma(x_1, y_1),\ \delta_\sigma(x_1, y_1))$. Now we take the parametrization $(x_1, y_1) \mapsto (a\sigma, b\sigma)$ and $\gamma_{\sigma\lambda_0} : (a\sigma, b\sigma) \mapsto (a\sigma, b\sigma, \xi_\sigma(a\sigma, b\sigma),$ $\delta_\sigma(a\sigma, b\sigma))$ where $\lambda_0 = a/b$. Then, there exists a 2–parameter family of periodic orbits $\gamma_{\sigma\lambda}$ such that for each $\lambda_0 \in \mathbb{R}$, the family of periodic orbits $\gamma_{\lambda_0\sigma}$ is a Liapunov family; i. e, $\lim_{\sigma\to 0} \gamma_{\sigma\lambda_0} = 0$ and the period tends to $2\pi/\alpha$.

We define an open set in Ω^2:

$$\mathcal{U}^2 = \ \{\ X \in \Omega^2; \quad \text{the canonical form of } DX(0) \text{ satisfies (2.2) } \}.$$

∎

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References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd edn, Benjamin-Cummings, 1978.

[2] G. Belitskii, C^∞-normal forms of local vector fields. Symmetry and perturbation theory. *Acta Appl. Math.* **70** (2002), 23–41.

[3] C. A. Buzzi and M. A. Teixeira, Time-reversible Hamiltonian Vector Fields with Symplectic Symmetries, *Journal of Dynamics and Differential Equations* **2** (2005), 559-574.

[4] S. N. Chow and Y. Kim, Bifurcation of periodic orbits for non-positive definite Hamiltonian systems. *Appl. Anal.* **31** (1998), 163–199.

[5] R. Devaney, Reversible Diffeomorphism and Flows, *Transactions of the American Mathematical Society* **218** (1976), 89-113.

[6] M. Golubitsky, M. Krupa and C. Lim, Time-reversibility and particle sedimentation, *SIAM J. Appl. Math.* **51** (1981), no. 1, 49–72.

[7] J. Hale, *Ordinary Diferential Equations*, New York, Wilay-Interscience, 1969.

[8] H. Hanβmann, The reversible umbilic bifurcation. In: Time-reversal symmetry in dynamical systems (Coventry, 1996). *Phys. D* **112** (1998), no. 1-2, 81–94.

[9] F. Heinz, Dynamik und Stabilitat eines Hamiltonschen Systems, Diplomarbeit Institut für Reine und Angewandte Mathematik, RWTH Aachen, 1998.

[10] J. Knobloch and A. Vanderbauwhede, A General Reduction Method for Periodic Soluitons in Conservative and Revesible Systems, *J. Dyn. Diff. Equations* **8** (1996), 71-102.

[11] J. S. W. Lamb and J. A. G. Roberts, Time-reversal symmetry in dynamical systems: a survey. *Phys. D*, **112**(1-2):1–39, 1998. Time-reversal symmetry in dynamical systems (Coventry, 1996).

[12] M. Lima and M. A. Teixeira, Degenerate resonances for reversible systems. To appear in *Annali di Matematica Pura ed Applicata.* Electronic version http://www.springerlink.com/content/l5770p5241p2l680/

[13] J. Montaldi, M. Roberts and I. Stewart, Existence of nonlinear normal modes of symmetric Hamiltonian systems, *Nonlinearity* **3** (1990), 695–730.

[14] F. Takens, Singularities of vector fields, *Publ. Math. IHES* **43** (1974), 47-100.

[15] M. A. Teixeira, Singularities of reversible vector fields. *Phys. D* **100** (1997), no. 1-2, 101–118.

[16] A. Vanderbauwhede, Local bifurcation and symmetry, *Res. Notes in Math.* **75**, Pitman, Boston, 1982.

[17] A. Vanderbauwhede and J.-C. van der Meer, A general reduction method for periodic solutions near equilibria in Hamiltonian systems. In: *Normal forms and homoclinic chaos* (Waterloo, ON, 1992), *Fields Inst. Commun.* **4** (1995), 273–294.

[18] T. Wagenknecht, *An analytical study of a two degrees of freedom Hamiltonian System associated the Reversible Hyperbolic Umbilic,* Thesis, University Ilmenau, Germany, 1999.

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