AN EXPLICIT FORMULA FOR A BOUNDARY MAP

MARI SANO

Abstract. We give an explicit formula for the first boundary map of the resolution constructed in [3], by means of writing the aprorpiate map between complexes.

1. Introduction

Determinantal ideals and their resolutions are a central source of examples and inspiration in algebra (see [6] and the more contemporary [5]). We are particularly interested in the representation-theoretic approach pioneered by Lascoux [10] in characteristic zero, but in the characteristic-free setting. Over the integers, a minimal free resolution of the ideal generated by the maximal minors was found by Eagon and Northcott [1], and in the case of submaximal minors by Akin, Buchsbaum and Weyman [2]. However Hashimoto [9] showed that such resolutions cannot be constructed in general. Therefore, it is of great interest to study the structure of explicit resolutions in order to understand to what extent such resolutions exist.

The resolutions in [2] for the submaximal case are built by means of using the mapping cone of appropriate maps between complexes. However, the maps are inexplicit. A formula for the first of these maps –whose structure seems to be key in establishing the formulas for the explicit resolutions– was given by Boffi [4].

The more general representation theoretic approach is studied in [3], where they construct resolutions for Schur modules using Schur complexes. This paper bears the same relation to [3] as Boffi's formulas in [4] to [2]: we will find an explicit formula for the first application of a map between Schur complexes whose mapping cone furnishes a resolution of the relevant Schur module.

The formulas for the two approaches are quite related; more precisely let $\phi : G \to F^*$ be the usual generic map and M denote coker(ϕ), where G,F are free R-modules, $F^* = Hom(F, R)$ with $m = dimF \geq n = dimG$. The determinantal ideal relevant to [2, 4] is the ideal I_{n-1} of submaximal minors of the generic $m \times n$ matrix, whereas in [3] they construct a resolution of the Schur module $\Lambda^{m+n-2}(M)$; we have that the suppport of $\Lambda^{m+n-2}(M)$ equals the support of R/I_{n-p} for all $p = 0, \ldots, n-1$ [7].

It should be remarked that recently letter-place methods have been proven very succesful in building and analyzing resolutions of Schur and Weyl modules with a great simplification of the size of the computations, which allows the tackling of more complicated problems [8, 11, 12].

Date: October 20, 2008.

2000 *Mathematics Subject Classification.* 13D25 .

Key words and phrases. Resolutions, boundary maps.

2. Preliminaries and notation

Let us now briefly review some fundamental definitions and facts.

We begin with the Schur complex, following the basic reference [3] where the reader is referred for details. Let R be a commutative ring and $\phi : G \to F$ be a map of finitely generated free R-modules. We let $\lambda = (\lambda_1, \cdots, \lambda_n)$, $\mu = \mu_1, \cdots, \mu_n)$ be partitions with $\mu \subseteq \lambda$, that is $\mu_i \leq \lambda_i$ for all i. Then λ/μ is the skew shape and the Schur complex, $L_{\lambda/\mu}\phi$ is the cokernel of the map

$$\Box_{\lambda/\mu} : \sum_{i=1}^{n-1} \sum_{v=0}^{\lambda_{i+1}-\mu_i-1} \Lambda^{p_1}\phi \otimes \cdots \Lambda^{p_{i-1}}\phi \otimes \Lambda^{p_i+p_{i+1}-v}\phi \otimes \Lambda^v\phi \otimes \Lambda^{p_{i+2}}\phi \cdots \otimes \Lambda^{p_n}\phi \longrightarrow \Lambda^{p_1}\phi \otimes \cdots \otimes \Lambda^{p_n}\phi$$

where $p_k = \lambda_k - \mu_k$, for all $k = 1, \cdots, n$ and the map

$$\Lambda^{p_1}\phi \otimes \cdots \otimes \Lambda^{p_{i-1}}\phi \otimes \Lambda^{p_i+p_{i+1}-v}\phi \otimes \Lambda^v\phi \otimes \Lambda^{p_{i+2}}\phi \cdots \otimes \Lambda^{p_n}\phi \longrightarrow \Lambda^{p_1}\phi \otimes \cdots \otimes \Lambda^{p_n}\phi$$

is the composition

$$\Lambda^{p_1}\phi \otimes \cdots \otimes \Lambda^{p_{i-1}}\phi \otimes \Lambda^{p_i+p_{i+1}-v}\phi \otimes \Lambda^v\phi \otimes \Lambda^{p_{i+2}}\phi \cdots \otimes \Lambda^{p_n}\phi \xrightarrow{1\otimes\cdots\otimes\triangle\otimes\cdots\otimes 1}$$
$$\Lambda^{p_1}\phi \otimes \cdots \otimes \Lambda^{p_{i-1}}\phi \otimes \Lambda^{p_i}\phi \otimes \Lambda^{p_{i+1}-v}\phi \otimes \Lambda^v\phi \otimes \Lambda^{p_{i+2}}\phi \cdots \otimes \Lambda^{p_n}F \xrightarrow{1\otimes\cdots\otimes m\otimes\cdots\otimes 1}$$
$$\Lambda^{p_1}\phi \otimes \cdots \otimes \Lambda^{p_i}\phi \otimes \Lambda^{p_{i+1}}\phi \otimes \cdots \otimes \Lambda^{p_n}\phi$$

with $\triangle$ standing for the diagonal map $\Lambda^{p_i+p_{i+1}-v}\phi \to \Lambda^{p_i}\phi \otimes \Lambda^{p_{i+1}-v}\phi$ and m standing for the multiplication $\Lambda^{p_{i+1}-v}\phi \otimes \Lambda^v\phi \to \Lambda^{p_{i+1}}\phi$.

The Schur complex generalizes both Schur and Weyl modules: if $G = 0$ then $L_{\lambda/\mu}\phi = L_{\lambda/\mu}F$ and if $F = 0$ then $L_{\lambda/\mu}\phi \approx K_{\lambda/\mu}G$ in degree $|\lambda| - |\mu|$.

When $\phi : G \to F^*$ is the generic map (by generic map we mean that the entries of a matrix $X = (x_{ij})$ of ϕ are variables over a commutative ring K and that R is the polynomial ring $K[X]$ in the variables x_{ij} over K) and M is the coker(ϕ), we have a map $\psi = \{\psi_k\}$ between Schur complexes $L_{(m-n+1,m-n+1)}(\phi) \to \Lambda^{m-n+2}(\phi)$, namely,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & L_{(n,n-1)}F \otimes \Lambda^n G & \longrightarrow & L_{(n,n-1)}F \otimes \Lambda^n G & \longrightarrow & 0 \\
& \downarrow \psi_2 & & \downarrow \psi_1 & & \downarrow \\
\cdots \longrightarrow & \Lambda^n F \otimes D_2 G & \longrightarrow & \Lambda^{n-1}F \otimes G & \xrightarrow{\phi_1} & \Lambda^{n-2}F,
\end{array}$$

where D denotes the divided power. The map ψ_1 is caracterized by it being a lifting of the map $\psi_* : L_{(n-1,n-1)} \otimes \Lambda^n G \to H_1(\Lambda^{m-n+2})$ and $\phi_1 \circ \psi_1 = 0$. The mapping cone of map ψ produces a finite free resolution of the module $\Lambda^{m-n+2}M$ which is minimal when $n \geq 2$ (cf. [3]), here minimal free resolution means the resolutions such that its entries of boundary maps do not have constant terms.

The main result for this article is an explicit formula for the map ψ_1.

The main character in the computation is the bilinear pairing $<,>_{\Lambda^k\phi}$: $\Lambda^k F \otimes \Lambda^k G \to R$, whose image is the ideal I_k generated in R by the minors of order k of the generic matrix, that is, $<,>_{\Lambda^k\phi}$ associates to the element $f_{i_1} \wedge \cdots \wedge f_{i_k} \otimes g_{j_1} \wedge \cdots \wedge g_{j_k}$ of $\Lambda^k F \otimes \Lambda^k G$ the minor $(i_1 \cdots i_k | j_1 \cdots j_k)$ of the generic matrix given by the rows of indices $i_1, \cdots, i_k$ and the columns of indices $j_1, \cdots, j_k$. We'll use the notation $(x|y)$ to indicated $<,>_{\Lambda^k\phi}$, with $x \in \Lambda^k F$ and $x \in \Lambda^k G$.

3. Main Theorem

Let $a = a_1 \wedge a_2 \wedge \cdots \wedge a_{n-1}$ and $b = b_1 \wedge b_2 \wedge \cdots \wedge b_{n-1}$ be in $\Lambda^{n-1}F$ and let $g = g_1 \wedge g_2 \wedge \cdots \wedge g_n$ in $\Lambda^n G$. Write

$$\Delta(a) = \sum_{\alpha} a'_{\alpha k-1} \otimes a_{\alpha n-k} \in \Lambda^{k-1}F \otimes \Lambda^{n-k}F$$

$$\Delta(b) = \sum_{\beta} b'_{\beta k-1} \otimes b_{\beta n-k} \in \Lambda^{k-1}F \otimes \Lambda^{n-k}F$$

Theorem 3.1. *A map $\psi_1 : L_{(n-1,n-1)}F \otimes \Lambda^n G \to \Lambda^{n-1}F \otimes G$ can be defined by*

$$d(a_1 \wedge a_2 \wedge \cdots \wedge a_{n-1} \otimes b_1 \wedge b_2 \wedge \cdots \wedge b_{n-1}) \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_n \mapsto$$

$$\sum_{k=1}^{n} (-1)^{n+k} \big(\sum_{\alpha} \sum_{\beta} (a_{\alpha n-k} | (k+1) \cdots n)(b'_{\beta k-1} | 1 \cdots (k-1)) a'_{\alpha k-1} \wedge b_{\beta n-k} \times g_k \big)$$

satisfying the desired properties: $\phi_1 \circ \psi_1 = 0$ and being a lifting of the map $\psi_* : L_{(n-1,n-1)} \otimes \Lambda^n G \to H_1(\Lambda^{m-n+2})$.

First of all, we will show that the map ψ_1 is a lift of the map α_*. To do that, we just need that $\pi \otimes 1 \circ \psi_1 = \alpha_*$, where $\pi : \Lambda^{n-1}F \to \Lambda^{m-n+1}M$. This reduces to prove the following identity

$$\pi \otimes 1 \big(\sum_{k=1}^{n} (-1)^{n+k} \big(\sum_{\alpha} \sum_{\beta} (a_{\alpha n-k} | (k+1) \cdots n)(b'_{\beta k-1} | 1 \cdots (k-1)) a'_{\alpha k-1} \wedge b_{\beta n-k} \otimes g_k \big) \big) =$$

$$\pi \otimes 1 \big(\sum_{i=1}^{n} (-1)^{i+n} (b_1 \wedge \cdots \wedge b_{n-1} | g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n) a_1 \wedge \cdots \wedge a_{n-1} \otimes g_i \big)$$

To show that ψ_1 is a lift of α_* we'll see that the summand of each sum coincide under the application $\pi \otimes 1$.

For $k = n$, in the first sum we have

$$(b_1 \wedge \cdots \wedge b_{n-1} | 1 \cdots (n-1)) a_1 \wedge \cdots \wedge a_{n-1} \otimes g_n$$

which is exactly the summand $i = n$ in the second sum.

Now for $k = n-1$, we have

$$-\sum_{\alpha} \sum_{\beta} (a_{\alpha 1} | n)(b'_{\beta n-2} | 1 \cdots (n-2)) a'_{\alpha n-2} \wedge b_{\beta 1} \otimes g_{n-1}$$

On the other hand, we know that

$$\gamma\big((b'_{\beta n-2} | 1 \cdots (n-2)) a_1 \wedge \cdots a_{n-1} \wedge b_{\beta 1} \otimes g_n \otimes g_{n-1}\big) =$$

$$(b'_{\beta n-2} | 1 \cdots (n-2))(b_{\beta 1} | n) a_1 \wedge \cdots a_{n-1} \otimes g_{n-1}$$

$$-\sum_{\alpha} (b'_{\beta n-2} | 1 \cdots (n-2))(a_{\alpha 1} | n) a'_{\alpha n-2} \wedge b_{\beta 1} \otimes g_{n-1} = X - Y$$

where $\gamma = \partial_\phi \otimes 1 : \Lambda^n F \otimes G \otimes G \to \Lambda^{n-1}F \otimes G$ and ∂_ϕ is the following composition

$$\Lambda^n F \otimes G \xrightarrow{\Delta \otimes 1} \Lambda^{n-1}F \otimes F \otimes G \xrightarrow{1 \otimes <,>_\phi} \Lambda^{n-1}F$$

Therefore $(\pi \otimes 1) \circ (\partial_\phi \otimes 1) = 0$. Thus

$$\pi \otimes 1 \big(-\sum_{\alpha} \sum_{\beta} (b'_{\beta n-2} | 1 \cdots (n-2))(a_{\alpha 1} | n) a'_{\alpha n-2} \wedge b_{\beta 1} \otimes g_{n-1} \big) =$$

$$\pi\otimes 1\big(-\sum_{\beta}(b'_{\beta n-2}|1\cdots(n-2))(b_{\beta 1}|n)a_1\wedge\cdots\wedge a_{n-1}\otimes g_{n-1}\big)$$

But

$$\sum_{\beta}(b'_{\beta n-2}|1\cdots(n-2))(b_{\beta 1}|n)a_1\wedge\cdots\wedge a_{n-1}\otimes g_{n-1}$$

is precisely $(b_1\wedge\cdots\wedge b_{n-1}|1\cdots(n-2)n)a_1\wedge\cdots\wedge a_{n-1}\otimes g_{n-1}$. Thus we get the summand $i=n-1$ of the the second sum.

For $k=n-2$ one gets the sum

$$\sum_{\alpha}\sum_{\beta}(a_{\alpha 2}|(n-1)n)(b'_{\beta n-3}|1\cdots(n-3))a'_{\alpha n-3}\wedge b_{\beta 2}\otimes g_{n-2}$$

Next, we consider

$$\partial_\phi\otimes 1\big(-\sum_{\alpha}(a_{\alpha 1}|n-1)(b'_{\beta n-3}|1\cdots(n-3))a'_{\alpha n-2}\wedge b_{\beta 2}\otimes g_n\otimes g_{n-2}\big)=$$

$$-\sum_{\alpha,\xi}(a_{\alpha 1}|n-1)(b'_{\beta n-3}|1\cdots(n-3))((b_{\beta 2})'_{\xi 1}|n)a'_{\alpha n-2}\wedge(b_{\beta 2})_{\xi 1}\otimes g_{n-2}\big)$$

$$+\sum_{\alpha,\delta}(a_{\alpha 1}|n-1)(b'_{\beta n-3}|1\cdots(n-3))((a'_{\alpha n-2})_{\delta 1}|n)(a'_{\alpha n-2})'_{\delta n-3}\wedge b_{\beta 2}\otimes g_{n-2}$$

where

$$\Delta(b_{\beta 2})=\sum_{\xi}(b_{\beta 2})'_{\xi 1}\otimes(b_{\beta 2})_{\xi 1}\in F\otimes F$$

$$\Delta(a'_{\alpha n-2})=\sum_{\delta}(a'_{\alpha n-2})'_{\delta n-3}\otimes(a'_{\alpha n-2})'_{\delta 1}\in\Lambda^{n-3}F\otimes F$$

If we sum over β the right side of the last identity, we get

$$-\sum_{\beta,\alpha,\xi}(a_{\alpha 1}|n-1)(b'_{\beta n-3}|1\cdots(n-3))((b_{\beta 2})'_{\xi 1}|n)a'_{\alpha n-2}\wedge(b_{\beta 2})_{\xi 1}\otimes g_{n-2}$$

$$+\sum_{\beta,\alpha,\delta}(a_{\alpha 1}|n-1)(b'_{\beta n-3}|1\cdots(n-3))((a'_{\alpha n-2})_{\delta 1}|n)(a'_{\alpha n-2})'_{\delta n-3}\wedge b_{\beta 2}\otimes g_{n-2}$$

which is equal to

$$-\sum_{\beta,\alpha}(a_{\alpha 1}|n-1)(b'_{\beta n-2}|1\cdots(n-3)n)a'_{\alpha n-2}\wedge b_{\beta 2}\otimes g_{n-2}$$

$$+\sum_{\beta,\alpha}(a_{\alpha 2}|(n-1)n)(b'_{\beta n-3}|1\cdots(n-3))a'_{\alpha n-3}\wedge b_{\beta 2}\otimes g_{n-2}$$

Therefore

$$\pi\otimes 1\big(\sum_{\alpha}\sum_{\beta}(a_{\alpha 2}|(n-1)n)(b'_{\beta n-3}|1\cdots(n-3))a'_{\alpha n-3}\wedge b_{\beta 2}\otimes g_{n-2}\big)=$$

$$\pi\otimes 1\big(\sum_{\beta,\alpha}(a_{\alpha 1}|n-1)(b'_{\beta n-2}|1\cdots(n-3)n)a'_{\alpha n-2}\wedge b_{\beta 2}\otimes g_{n-2}\big)=$$

Using the case $k=n-1$, we end up in

$$\pi\otimes 1\big(\sum_{\beta}(b'_{\beta n-2}|1\cdots(n-3)n)(b_{\beta 1}|n-1)a_1\wedge\cdots\wedge a_{n-1}\otimes g_{n-2}\big)=$$

and this is equal to

$$(b_1\wedge\cdots\wedge b_{n-1}|1\cdots(n-3)(n-1)n)\pi(a_1\wedge\cdots\wedge a_{n-1})\otimes g_{n-2}$$

Finally, for any k we have the double sum

$$(-1)^{n+k}\sum_{\alpha}\sum_{\beta}(a_{\alpha n-k}|(k+1)\cdots n)(b'_{\beta k-1}|1\cdots(k-1))a'_{\alpha k-1}\wedge b_{\beta n-k}\otimes g_k$$

As in the other cases we compute

$$\partial_\phi\otimes 1\big(\pm\sum_{\alpha}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1))(b'_{\beta k-1}|1\cdots(k-1))a'_{\alpha k}\wedge b_{\beta n-k}\otimes g_n\otimes g_k\big)=$$

$$\pm\sum_{\alpha,\xi}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1))(b'_{\beta k-1}|1\cdots(k-1))((b_{\beta n-k})_{\xi 1}|n)a'_{\alpha k}\wedge(b_{\beta n-k})'_{\xi n-k-1}\otimes g_k\big)$$

$$\pm\sum_{\alpha,\delta}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1))(b'_{\beta k-1}|1\cdots(k-1))((a'_{\alpha k})_{\delta 1}|n)(a'_{\alpha k})'_{\delta k-1}\wedge b_{\beta n-k}\otimes g_k$$

where

$$\Delta(b_{\beta n-k})=\sum_{\xi}(b_{\beta n-k})'_{\xi n-k-1}\otimes(b_{\beta n-k})_{\xi 1}\in\Lambda^{n-k-1}F\otimes F$$

$$\Delta(a'_{\alpha k})=\sum_{\delta}(a'_{\alpha k})'_{\delta k-1}\otimes(a'_{\alpha k})_{\delta 1}\in\Lambda^{k-1}F\otimes F$$

Similarly, as in the other cases we add over β the last identity and we obtain

$$\pm\sum_{\beta,\alpha,\xi}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1))(b'_{\beta k-1}|1\cdots(k-1))((b_{\beta n-k})_{\xi 1}|n)a'_{\alpha k}\wedge(b_{\beta n-k})'_{\xi n-k-1}\otimes g_k$$

$$\pm\sum_{\beta,\alpha,\delta}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1)(b'_{\beta k-1}|1\cdots(k-1))((a'_{\alpha k})_{\delta 1}|n)(a'_{\alpha k})'_{\delta k-1}\wedge b_{\beta n-k}\otimes g_k$$

Note that the above sum is the same as the following sum

$$\pm\sum_{\beta,\alpha}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1))(b'_{\beta k}|1\cdots(k-1)n)a'_{\alpha k}\wedge b_{\beta n-k-1}\otimes g_k$$

$$\pm\sum_{\beta,\alpha}(a_{\alpha n-k}|(k+1)\cdots(n-2)(n-1)n)(b'_{\beta k-1}|1\cdots(k-1))a'_{\alpha k-1}\wedge b_{\beta n-k}\otimes g_k$$

In consequence, we have

$$\pi\otimes 1\big(\pm\sum_{\alpha}\sum_{\beta}(a_{\alpha n-k}|(k+1)\cdots n)(b'_{\beta k-1}|1\cdots(k-1))a'_{\alpha k-1}\wedge b_{\beta n-k}\otimes g_k\big)=$$

$$\pi\otimes 1\big(\pm\sum_{\beta,\alpha}(a_{\alpha n-k-1}|(k+1)\cdots(n-2)(n-1))(b'_{\beta k}|1\cdots(k-1)n)a'_{\alpha k}\wedge b_{\beta n-k-1}\otimes g_k\big)=$$

Using the case $k+1$, we have that

$$\pi\otimes 1\big(\pm\sum_{\beta}(b'_{\beta n-2}|1\cdots(k-1)(k+2)\cdots(n-2)(n-1)n)(b_{\beta 1}|k+1)a_1\wedge\cdots\wedge a_{n-1}\otimes g_k\big)=$$

which is equal to

$$\pm(b_1\wedge\cdots\wedge b_{n-1}|1\cdots(k-1)(k+1)(k+2)\cdots(n-1)n)\pi(a_1\wedge\cdots\wedge a_{n-1})\otimes g_k$$

Proof of the theorem

As before let a and b be in $\Lambda^{n-1}F$ and let g be in $\Lambda^n G$, where

$$\Delta(a)=\sum_{\alpha}a'_{\alpha k-1}\otimes a_{\alpha n-k}\in\Lambda^{k-1}F\otimes\Lambda^{n-k}F$$

$$\Delta(b)=\sum_{\beta}b'_{\beta k-1}\otimes b_{\beta n-k}\in\Lambda^{k-1}F\otimes\Lambda^{n-k}F$$

Thus

$$\phi_1\big(\psi_1\big(d(a_1 \wedge a_2 \wedge \cdots \wedge a_{n-1} \otimes b_1 \wedge b_2 \wedge \cdots \wedge b_{n-1}) \otimes g_1 \wedge g_2 \wedge \cdots \wedge g_n\big)\big) =$$

$$\phi_1\big(\sum_{k=1}^{n}(-1)^{n+k}\big(\sum_{\alpha}\sum_{\beta}(a_{\alpha n-k}|(k+1)\cdots n)(b'_{\beta k-1}|1\cdots(k-1))a'_{\alpha k-1}\wedge b_{\beta n-k}\otimes g_k\big)\big) =$$

$$\sum_{k=1}^{n}(-1)^{n+k}\big[\sum_{\beta,\alpha,\xi}(a_{\alpha n-k}|(k+1)\cdots n)(b'_{\beta k-1}|1\cdots(k-1))((b_{\beta n-k})_{\xi 1}|k)a'_{\alpha k-1}\wedge(b_{\beta n-k})'_{\xi n-k-1}+$$
$$\sum_{\beta,\alpha,\delta}(a_{\alpha n-k}|(k+1)\cdots n)(b'_{\beta k-1}|1\cdots(k-1))((a'_{\alpha k-1})_{\delta 1}|k)(a'_{\alpha k-1})'_{\delta k-2}\wedge b_{\beta n-k}\big]$$

where $\Delta(a'_{\alpha k-1}) = \sum_{\delta}(a'_{\alpha k-1})'_{\delta k-2} \otimes (a'_{\alpha k-1})_{\delta 1} \in \Lambda^{n-2}F \otimes F$ and $\Delta(b_{\beta n-k}) = \sum_{\xi}(b_{\beta n-k})'_{\xi n-k-1} \otimes (b_{\beta n-k})_{\xi 1} \in \Lambda^{n-2}F \otimes F$. It is obvious that the above sum is equal to

$$\sum_{k=1}^{n}(-1)^{n+k}\big[\sum_{\beta,\alpha}(a_{\alpha n-k}|(k+1)\cdots n)(b'_{\beta k}|1\cdots k)a'_{\alpha k-1}\wedge b_{\beta n-k-1}$$
$$\sum_{\beta,\alpha}(a_{\alpha n-k+1}|k(k+1)\cdots n)(b'_{\beta k-1}|1\cdots(k-1))a'_{\alpha k-2}\wedge b_{\beta n-k}\big]$$

Writing the terms of the sum we get

$$(-1)^{n+1}(a_{\alpha n-1}|2\cdots n)(b'_{\beta 1}|1)b_{\beta n-2}$$
$$+(-1)^{n+2}\big[(a_{\alpha n-2}|3\cdots n)(b'_{\beta 2}|12)a'_{\alpha 1}\wedge b_{\beta n-3} + (a_{\alpha n-1}|2\cdots n)(b'_{\beta 1}|12)b_{\beta n-2}\big]$$
$$+(-1)^{n+3}\big[(a_{\alpha n-3}|4\cdots n)(b'_{\beta 3}|123)a'_{\alpha 2}\wedge b_{\beta n-4} + (a_{\alpha n-2}|3\cdots n)(b'_{\beta 2}|12)a'_{\alpha 1}\wedge b_{\beta n-3}\big]$$
$$\vdots$$
$$+(-1)^{n+n-1}\big[(a_{\alpha 1}|n)(b'_{\beta n-1}|1\cdots(n-1))a'_{\alpha n-2}+(a_{\alpha 2}|(n-1)n)(b'_{\beta n-2}|1\cdots(n-2))a'_{\alpha n-3}\wedge b_{\beta 1}\big]$$
$$+(-1)^{n+n}\big[(a_{\alpha 1}|n)(b'_{\beta n-1}|1\cdots(n-1))a'_{\alpha n-2}\big] = 0$$

Acknowledgments. The author thanks D. Buchsbaum for introducing her to both the general subject and this particular problem.

References

[1] J.A.Eagon and D.G.Northcott. Ideals defined by matrices and a certain complex associated with them. *Proc. R. Soc. Lond., Ser. A*, 269:188–204, 1962.

[2] K. Akin, D. A. Buchsbaum, and J. Weyman. Resolutions of determinantal ideal:the submaximal minors. *Adv. in Math*, 39:1–30, 1981.

[3] K. Akin, D. A. Buchsbaum, and J. Weyman. Schur functors and Schur complexes. *Adv. in Math*, 44:207–278, 1982.

[4] G. Boffi. The case of the submaximal minors: A finishing touch. *Adv. in Math*, 68:64–84, 1988.

[5] G. Boffi and D. Buchsbaum. *Threading homology through algebra. Selected patterns.* Oxford Science Publications. Oxford Mathematical Monographs. Oxford: Oxford University Press, 2006.

[6] W. Bruns and U. Vetter. *Determinantal rings.* Lecture Notes in Mathematics, 1327, Springer-Verlag, Berlin.

[7] D. Buchsbaum. Generic free resolutions and Schur complexes. Brandeis Lecture notes 3, 1983, Notes by G. Boffi.

[8] D. A. Buchsbaum and G.C. Rota. Approaches to resolution of Weyl modules. *Adv. Appl. Math*, 27(1):82–191, 2001.

[9] M. Hashimoto. Determinantal ideals without minimal free resolutions. *Nagoya Math. J.*, 18:203–216, 1990.

[10] A. Lascoux. Syzygies des variétés déterminantales. *Adv. Math.*, 30:202–237, 1978.

[11] M. Sano. A combinatorial description of the syzygies of certain Weyl modules. *Commun. Algebra*, 31(10):5117–5167, 2003.

[12] M. Sano. Homotopies for the generalized bar complex associated to certain 3-rowed Weyl modules. *Commun. Algebra*, 34(8):3053–3075, 2006.

Mathematics Department, IMECC, CP 6065, State University of Campinas - UNICAMP 13083-859, Campinas, SP, Brazil.

E-mail address: msano@ime.unicamp.br