Martingales on Principal Fiber Bundles

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Abstract

Let $P(M,G)$ be a principal fiber bundle, ω be a connection form on $P(M,G)$ and ∇^P be a projectable connection on $P(M,G)$. The aim of this work is determine the ∇^P-martingales in $P(M,G)$. Our results allow to establish new characterizations of harmonic maps from Riemannian manifolds to principal fiber bundles.

Key words: martingales; principal fiber bundles; harmonic maps.

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1 Introduction

This article is concerned with the characterization of martingales on a principal fiber bundle $P(M,G)$ which is endowed with a connection form ω and a G-invariant symmetric connection ∇^P. Following to N. Abe and K. Hasegawa [1], we says that ∇^P is projectable if there exists an unique symmetric connection ∇^M on M such that for any vector fields X and Y on M

$$\mathbf{h}\nabla^P_{X^h}Y^h = (\nabla^M_X Y)^h,$$

where $\mathbf{h}$ denotes the horizontal projection which is associated to ω and $-^h$ denotes the associated horizontal lift. Let A and T be the fundamental tensors associated to $\pi : P \to M$, see (5) and (6) below. The symmetrized of a tensor R will be denote by R^S.

In this situation, we prove the following results:

1) Let Y be a continuous semimartingale with values in P. Then Y is a ∇^P-martingale if and only if

$$\int \omega\ \delta Y - \frac{1}{2}\int (\nabla^P\omega)(dY, dY) \tag{1}$$

is a local martingale and

$$\int \alpha\, d^{\nabla^M}\pi \circ Y + \frac{1}{2}\int \alpha \circ \pi_* \circ (2A^S + T^S)(dY, dY) \tag{2}$$

is a local martingale for all $\alpha \in \Gamma(T^*M)$.
2) Let N be a Riemannian manifold with metric g and $F : N \to P$ be a smooth map. Then F is a harmonic map if and only if

$$d^*F^*\omega = trF^*(\nabla^P\omega) \tag{3}$$

and

$$\tau_{\pi\circ F} = -\mathrm{tr}\ \pi_* \circ (2A^S + T^S) \circ F_* \otimes F_*. \tag{4}$$

The motivation for this work is to understand the stochastic differential geometry of principal fiber bundles, in particular the martingales of Kaluza-Klein geometries. This paper is organized in the following way: In section 2 we review some fundamental facts on differential geometry of principal fiber bundle and stochastic calculus on manifolds. In section 3 we prove our principal results.

2 Preliminaries

We begin by recalling some fundamental facts on differential geometry of principal fiber bundle and stochastic calculus on manifolds. We shall use freely concepts and notations of S. Kobayashi and N. Nomizu [8], M. Emery [7] and P. Meyer [9].

Let $P(M,G)$ be a principal fiber bundle with projection $\pi : P \to M$. Let us denote the right action of G on P by $R_g(p) = pg$ for $p \in P$ and $g \in G$. A horizontal lift H in $P(M,G)$ is a smooth family of applications $H_p : T_{\pi(p)}M \to T_pP$ such that $\pi_* \circ H_p = Id_{T_{\pi(p)}M}$ for all $p \in P$ and $(R_g)_*H_p = H_{pg}$ for all $p \in P$ and $g \in G$. The horizontal lift H determines a unique decomposition of each tangent space T_pP which is the direct sum of the vertical subspace $V_pP = Ker(\pi_*(p))$ and the horizontal subspace $H_pP = Im(H_p)$ at $p \in P$. This decomposition naturally defines the horizontal lifts of $X \in T_{\pi(p)}M$ as the unique tangent vector $X^h = H_p(X) \in H_pP$ such that $\pi_*(X^h) = X$. We denote by $\mathfrak{g}$ the Lie algebra of G. For $B \in \mathfrak{g}$, the right action of G into P defines a 1-parameter transformation group on P and induces a vector field B^* on P. B^* is the fundamental vector field corresponding to B, which is a vertical vector field. For each $p \in P$, the linear mapping $\sigma_p : \mathfrak{g} \to V_pP$ defined by $\sigma_p(B) = B_p^*$ is an isomorphism.

Let us denote by $\mathbf{h}U_p$ and $\mathbf{v}U_p$ the horizontal and vertical parts of $U \in T_pP$, respectively. The connection form $\omega : TP \to \mathfrak{g}$ is defined by

$$\omega(U_p) = B,$$

where $\mathbf{v}U_p = B_p^*$ at $p \in P$.

We observe that the connection form is a $\mathfrak{g}$-valued 1-form on P satisfying the following conditions:

1. $\omega(B^*) = B$ for $B \in \mathfrak{g}$,
2. $R_g^*\omega = ad_{g^{-1}}\omega$ for $g \in G$,

where $ad_g : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad_g(B) = (I_g)_*B$, I_g being the inner automorphism of G, $I_g(x) = gxg^{-1}$ for all $g \in G$.

Conversely, given a $\mathfrak{g}$-valued 1-form ω on P, which satisfies the above conditions, there is an unique horizontal lift H in P that its connection form is ω. For $p \in P$ and $X \in T_{\pi(p)}M$, we have that $H_p(X) \in T_pP$ is the unique solution of $\pi_*(H_p(X)) = X$ and $\omega_p(H_p(X)) = 0$.

The curvature form Ω of ω is the $\mathfrak{g}$-valued 2-form on P defined by

$$\Omega(U, V) = d\omega(\mathbf{h}U, \mathbf{h}V),$$

where U and V are vector fields on P.

Throughout the paper we adopt the following convention. A connection on a manifold means a torsion free covariant derivative operator on the tangent bundle.

Definition 2.1 *Let ∇^P be a connection on $P(M, G)$. Then we says that ∇^P is G-invariant if the right translations R_g are affine for all $g \in G$.*

Let ω be a connection form on $P(M, G)$. A G-invariant connection ∇^P on $P(M, G)$ is projectable if $\mathbf{h}\nabla^P_{X^h}Y^h$ is projectable for all vector fields X and Y on M.

Proposition 2.1 *Let $P(M, G)$ be a principal fiber bundle and ∇^P be a G-invariant connection on $P(M, G)$. Then ∇^P is projectable if and only if there exist an unique connection ∇^M on M such that*

$$\mathbf{h}\nabla^P_{X^h}Y^h = (\nabla^M_X Y)^h$$

for all vector fields X and Y on M.

Proof: Let ∇^P be a projectable connection. We define $\nabla^M_X Y = \pi_* \nabla^P_{X^h} Y^h$, clearly $\mathbf{h}\nabla^P_{X^h} Y^h = (\nabla^M_X Y)^h$. It remains to prove that ∇^M is a connection. Since $(fX)^h = (f \circ \pi) X^h$, for all $f \in C^\infty(M)$,

$$\begin{aligned} \nabla^M_{gX}(fY) &= \pi_*(\nabla^P_{(g\circ\pi)X^h}(f\circ\pi)Y^h) \\ &= \pi_*((g\circ\pi)X^h(f\circ\pi)Y^h + (g\circ\pi)(f\circ\pi)\nabla^P_{X^h}Y^h) \\ &= gX(f)Y + gf\nabla^M_X Y. \end{aligned}$$

The uniqueness follows from the definition. $\square$

Let ∇^P be a connection on $P(M,G)$ and ω a connection form on $P(M,G)$. Following to B. O´ Neill [10], we describe the geometrical quantities of our interest in terms of the fundamental tensors T and A. They are defined by

$$T_U V = \mathbf{h}\nabla^P_{\mathbf{v}U}\mathbf{v}V + \mathbf{v}\nabla^P_{\mathbf{v}U}\mathbf{h}V \tag{5}$$

and

$$A_U V = \mathbf{v}\nabla^P_{\mathbf{h}U}\mathbf{h}V + \mathbf{h}\nabla^P_{\mathbf{h}U}\mathbf{v}V, \tag{6}$$

for U and V vector fields on P.

We observe that the vanishing of T means that the fibers of $P(M,G)$ are totally geodesic.

Lemma 2.1 *Let $P(M,G)$ be a principal fiber bundle, ω be a connection form on $P(M,G)$ and ∇^P be a projectable connection on $P(M,G)$. If X and Y are vector fields on M and $B, C \in \mathfrak{g}$, then we have the following equations:*

$$\left\{ \begin{array}{rcl} T_{B^*}C^* &=& T_{C^*}B^* \\ T_{B^*}X^h &=& \omega(\nabla^P_{X^h}B^*)^* \\ A_{X^h}Y^h &=& -2\Omega(X^h, Y^h)^* + A_{Y^h}X^h \\ A_{X^h}B^* &=& \nabla^P_{X^h}B^* - \omega(\nabla^P_{X^h}B^*)^* + [X^h, B^*] \end{array} \right. \tag{7}$$

and

$$\left\{ \begin{array}{rcl} \nabla^P_{B^*}C^* &=& \widehat{\nabla}_{B^*}C^* + T_{B^*}C^* \\ \nabla^P_{B^*}X^h &=& \mathbf{h}\nabla^P_{B^*}X^h + T_{B^*}X^h \\ \nabla^P_{X^h}B^* &=& \mathbf{v}\nabla^P_{X^h}B^* + A_{X^h}B^* \\ \nabla^P_{X^h}Y^h &=& \mathbf{h}\nabla^P_{X^h}Y^h + A_{X^h}Y^h, \end{array} \right. \tag{8}$$

where $\widehat{\nabla}$ is the induced connection by ∇^P in the fibers.

Proof: An easy computation shows the equations (8). To prove (7), we recall that $\mathbf{v}U = \omega(U)^*$ for all $U \in TP$ and $\mathbf{v}[X^h, Y^h] = -2\Omega(X^h, Y^h)^*$ for all $X, Y \in TM$. $\square$

From (7) it follows that the horizontal distribution $\{H_pP : p \in P\}$ is integrable if and only if $A_{X^h}Y^h = A_{Y^h}X^h$ for all X and Y vector fields on M.

Example 2.1 *Let M be a differentiable manifold and ∇ be a connection on M. We consider the frame bundle $BM(M, GL(\mathbb{R}^n))$, which is a principal fiber bundle with base M and structure group $GL(\mathbb{R}^n)$.*

The canonical lift ∇^c and horizontal lift ∇^h are projectable connections on BM with the projection ∇. In the book of L. Cordero et al. [6] we find a survey of elementary properties of these connections. Let X and Y be vector fields on M and $B, C \in \mathfrak{gl}(n, \mathbb{R}^n)$. The canonical lift ∇^c and horizontal lift ∇^H are completely defined by the relations:

$$\left\{ \begin{array}{rcl} \nabla^c_{B^*}C^* &=& (BC)^* \\ \nabla^c_{B^*}X^h &=& 0 \\ \nabla^c_{X^h}B^* &=& 0 \\ \nabla^c_{X^h}Y^h &=& (\nabla_X Y)^h + \gamma(R(-, X)Y) \end{array} \right. \tag{9}$$

and

$$\left\{ \begin{array}{rcl} \nabla^H_{B^*}C^* &=& (BC)^* \\ \nabla^h_{B^*}X^h &=& 0 \\ \nabla^H_{X^h}B^* &=& 0 \\ \nabla^H_{X^h}Y^h &=& (\nabla_X Y)^h, \end{array} \right. \tag{10}$$

where R is the curvature tensor of ∇ and γS is the vertical lift defined by $\gamma S(p) = (p^{-1} \circ S \circ p)^*(p)$ for $p \in BM$.

We observe that, in both cases, $T = 0$ and $A_{X^h}B^* = 0$, so π is affine.

Remark 2.1 *In the 1920s T. Kaluza and O. Klein proposed the use of spaces of dimension higher than four in order to unify general relativity and what we now call Yang-Mills theories. From a mathematical view point Kaluza-Klein theory is the differential geometry of a principal fiber bundle with invariant Riemannian metric (see [5] for more details). In this context is fundamental the following reduction theorem: Let $P(M, G)$ be a principal fiber bundle and k be a G-invariant Riemannian metric on P, namely R_g is an isometry for all $g \in G$. Let $\mathcal{M}_{ad}(\mathfrak{g})$ denote the set of metrics on $\mathfrak{g}$ invariant by ad_g for all $g \in G$. Then there exist*

1. *h a Riemannian metric on M,*
2. *ω a connection form on P,*
3. *$F : M \to \mathcal{M}_{ad}(\mathbf{g})$ a smooth function*

such that

$$k(U,V) = h(\pi_*(U), \pi_*(V)) + F \circ \pi(\omega(U), \omega(V)) \tag{11}$$

for all U and V vector fields on P.

Reciprocally, given h a Riemannian metric on M, ω a connection form on P and $F : M \to \mathcal{M}_{ad}(\mathbf{g})$ a smooth function, we have that (11) defines the unique G-invariant Riemannian metric k on P.

It is easy to check that $\pi : P \to M$ is a Riemannian submersion and the connection form ω is associated to the horizontal lift H^k. For $X \in TM$, $H^k(X)$ is completely determined by $\pi_*(H^k(X)) = X$ and $H^k(X)$ is orthogonal to VP.

It is clear that ∇^k, the Riemannian connection associated to k, is a projectable connection with projection ∇^h, the Riemannian connection associated to h.

Example 2.2 *Let $P(M,G)$ be a principal fiber bundle, ω be a connection form on P, k_0 be an $ad(G)$-invariant metric on $\mathfrak{g}$ and h be a Riemannian metric on M. We consider the G-invariant Riemannian metric k on P defined by*

$$k(U,V) = h(\pi_*(U), \pi_*(V)) + k_0(\omega(U), \omega(V)) \tag{12}$$

for all U and V vector fields on P.

An easy computations shows that $\pi : P \to M$ is a Riemannian submersion with $T = 0$ and

$$\begin{aligned} A_{X^h} X^h &= 0 \\ A_{X^h} B^* &= -\frac{1}{2} k_0(B, \Omega(-, X^h))^\sharp \end{aligned}$$

for all X vector field on M and $B \in \mathfrak{g}$ (see for instance [1] and [10]).

Let $(\Omega, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and M a smooth manifold with connection ∇. Let X be a continuous semimartingale with values in M, α a section of TM^* and b a section of $T^{(2,0)}M$. We denote by $\int \alpha\ \delta X$ the Stratonovich integral of α along X, by $\int \alpha\ d^\nabla X$ the Itô integral of α

along X and by $\int b\ d(X,X)$ the quadratic integral of b along X. We recall that X is a ∇-martingale if and only if $\int \alpha\ d^{\nabla}X$ is a local martingale for any $\alpha \in \Gamma(TM^*)$.

Let M be a manifold and α a section of TM^*. The Stratonovich-Itô conversion formula is given by:

$$\int_0^t \alpha\delta X = \int_0^t \alpha d^{\nabla}X + \frac{1}{2}\int_0^t \nabla\alpha\ (dX,dX). \tag{13}$$

Let M and N be manifolds, α a section of TN^*, b a section of $T^{(2,0)}N$ and $F: M \to N$ a smooth map. We have the following Itô formulas for Stratonovich and quadratic integrals:

$$\int_0^t \alpha\ \delta F(X) = \int_0^t F^*\alpha\ \delta X \tag{14}$$

and

$$\int_0^t b\ (dF(X),dF(X)) = \int_0^t F^*b\ (dX,dX). \tag{15}$$

In the case that M and N are endowed with connections ∇ and ∇', respectively, we have the following Itô formula for the Itô integral:

$$\int_0^t \alpha\ d^{\nabla'}F(X) = \int_0^t F^*\alpha\ d^{\nabla}X + \frac{1}{2}\int_0^t \beta_F^*\alpha\ (dX,dX), \tag{16}$$

where β_F is the second fundamental form of F (see [3] for more details).

From the above formula, it follows that F is an affine map if it and only if sends ∇-martingales to ∇'-martingales.

Let M be a Riemannian manifold with metric g. Let B be a continuous semimartingale with values in M, we say that B is a g-Brownian motion in M if B is a martingale with respect to the Levi-Civita connection of g and for any section b of $T^{(2,0)}M$ we have that

$$\int_0^t b(dB,dB) = \int_0^t tr\ b_{B_s}ds. \tag{17}$$

Combining (14) with (17) we obtain the following Manabe's formula:

$$\int_0^t \alpha\delta B = \int_0^t \alpha d^{\nabla}B + \frac{1}{2}\int_0^t d^*\alpha_{B_s}ds. \tag{18}$$

From (16) and (17) we deduce the useful formula:

$$\int_0^t \alpha d^{\nabla'} F(B) = \int_0^t F^* \alpha d^{\nabla} B + \frac{1}{2} \int_0^t \tau_F^* \alpha_{B_s} ds, \tag{19}$$

where τ_F is the tension field of F.

We recall that an application $F : M \to N$ is an harmonic map if

$$\tau_F = 0.$$

Applying the above formula, we obtain the Bismut characterization of harmonic maps: $F : M \to N$ is an harmonic map if and only if it sends Brownian motions to ∇'-martingales.

3 Martingales on principal fiber bundles

In this section we prove our main results.

Theorem 3.1 *Let $P(M, G)$ be a principal fiber bundle, ω be a connection form on P and ∇^P be a projectable connection with projection ∇^M. Let Y be a continuous semimartingale with values in P. Then Y is a ∇^P-martingale if and only if*

$$\int \omega \, \delta Y - \frac{1}{2} \int (\nabla^P \omega)(dY, dY) \tag{20}$$

is a local martingale and

$$\int \alpha \, d^{\nabla^M} \pi \circ Y + \frac{1}{2} \int \alpha \circ \pi_* \circ (2A^S + T^S)(dY, dY) \tag{21}$$

is a local martingale for all $\alpha \in \Gamma(T^*M)$.

Proof: Let Y be a ∇^P-martingale. By the conversion formula (13), we have

$$\int \omega \, \delta Y = \int \omega d^{\nabla^P} Y + \frac{1}{2} \int (\nabla^P \omega)(dY, dY).$$

Since $\int \omega \, d^{\nabla^P} Y$ is a local martingale, so is $\int \omega \, \delta Y - \frac{1}{2} \int (\nabla^P \omega)(dY, dY)$. In order to prove (21), we take $\alpha \in \Gamma(T^*M)$. It is easy to check that

$$\beta_\pi^* \alpha = \alpha \circ \beta_\pi = -\alpha \circ \pi_* \circ (2A^S + T^S). \tag{22}$$

Combining the above identity and Itô formula we conclude that

$$\int \alpha \, d^{\nabla^M} \pi \circ Y + \frac{1}{2} \int \alpha \circ \pi_* \circ (2A^S + T^S)(dY, dY)$$

is the local martingale $\int \pi^* \alpha \, d^{\nabla^P} Y$.

Conversely, take η in $\Gamma(T^*P)$. Since the $\mathcal{C}^\infty$-module $\Gamma(T^*P)$ is generated by ω and by the differential forms $\pi^*\alpha$ with $\alpha \in \Gamma(T^*M)$, we have that η is a linear combination of differential forms $f\pi^*\alpha$ and $h\omega$ with $f, h \in \mathcal{C}^\infty(P)$. It is clear that $\int h\omega \, d^{\nabla^P} Y = \int h(Y) \, d(\int \omega \, d^{\nabla^P} Y)$ is a local martingale and that

$$\int f\pi^*\alpha \, d^{\nabla^P} Y = \int f(Y) \, d(\int \pi^*\alpha \, d^{\nabla^P} Y).$$

Hence, in order to show that $\int \eta \, d^{\nabla^P} Y$ is a local martingale, it is sufficient to show that $\int \pi^*\alpha \, d^{\nabla^P} Y$ is a local martingale. Applying the Itô formula (16) and (22) we deduce that

$$\begin{aligned} \int \pi^*\alpha \, d^{\nabla^P} Y &= \int \alpha \, d^{\nabla^M} \pi \circ Y - \frac{1}{2} \int \beta_\pi^* \alpha(dY, dY) \\ &= \int \alpha \, d^{\nabla^M} \pi \circ Y + \frac{1}{2} \int \alpha \circ \pi_* \circ (2A^S + T^S)(dY, dY). \end{aligned}$$

This completes the proof.

□

Theorem 3.2 *Let $P(M,G)$ be a principal fiber bundle, ω be a connection form on P, ∇^P be a projectable connection with projection ∇^M and N be a Riemannian manifold with metric g. Let $F : N \to P$ be a smooth map. Then F is a harmonic map if and only if*

$$d^* F^* \omega = tr F^*(\nabla^P \omega). \tag{23}$$

and

$$\tau_{\pi \circ F} = -\mathrm{tr}\ \pi_* \circ (2A^S + T^S) \circ F_* \otimes F_*. \tag{24}$$

Proof: Let F be an harmonic map and B be a g-Brownian motion. From the Bismut characterization of harmonic maps and Theorem 3.1 we see that

$$\int \omega \delta F(B) - \frac{1}{2} \int (\nabla^P \omega) \ (dF(B), dF(B)) \tag{25}$$

is a local martingale. Applying (14) and (15) we can rewrite (25) as

$$\int F^*\omega\delta B - \frac{1}{2}\int F^*(\nabla^P\omega)(dB, dB). \tag{26}$$

From the Manabe formula (18) we have

$$\int F^*\omega\delta B = \int F^*\omega d^{\nabla^g} B + \frac{1}{2}\int d^*F^*\omega_{B_s}\, ds, \tag{27}$$

∇^g being the Levi-Civita connection associated to g. Combining (26) and (27) we conclude that

$$\int \omega d^{\nabla^P} F(B) + \frac{1}{2}\int (d^*F^*\omega - trF^*(\nabla^P\omega))_{B_s}\, ds \tag{28}$$

is a local martingale. Doob-Meyer decomposition now yields

$$\int (d^*F^*\omega - trF^*(\nabla^P\omega))_{B_s}\, ds = 0.$$

Since B is arbitrary, it follows that $d^*F^*\omega = trF^*(\nabla^P\omega)$.

It remains to prove that $\tau_{\pi\circ F} = -\mathrm{tr}\ \pi_* \circ (2A^S + T^S) \circ F_* \otimes F_*$. As F is an harmonic map, it follows that $F(B)$ is a ∇^P-martingale. From Theorem 3.1, we see that

$$\int \alpha\ d^{\nabla^M}\pi \circ F(B) + \frac{1}{2}\int \alpha \circ \pi_* \circ (2A^S + T^S)(dF(B), dF(B))$$

is a local martingale for all $\alpha \in \Gamma(T^*M)$. Applying (17) and (19) we conclude that

$$\int (\pi \circ F)^*\alpha\ d^{\nabla^g} B + \frac{1}{2}\int (\tau^*_{\pi\circ F}\alpha + \mathrm{tr}\ \alpha \circ \pi_* \circ (2A^S + T^S) \circ F_* \otimes F_*)_{B_s} ds$$

is a local martingale. Since B and $\alpha \in \Gamma(T^*M)$ are arbitrary, we have

$$\tau_{\pi\circ F} = -\mathrm{tr}\ \pi_* \circ (2A^S + T^S) \circ F_* \otimes F_*.$$

Conversely, suppose that B is a g-Brownian motion. From the Bismut characterization is sufficient to show that $F(B)$ is a ∇^P-martingale. We have that $\int \omega\delta F(B) - \frac{1}{2}\int (\nabla^P\omega)\ (dF(B), dF(B))$ can be written as

$$\int F^*\omega d^{\nabla^g} B + \frac{1}{2}\int (d^*F^*\omega - \mathrm{tr}F^*(\nabla^P\omega))_{B_s}\, ds.$$

Since $d^*F^*\omega = \mathrm{tr}F^*(\nabla^P\omega)$, it follows that

$$\int \omega\delta F(B) - \frac{1}{2}\int (\nabla^P\omega)\ (dF(B), dF(B))$$

is a local martingale. It remains to prove that

$$\int \alpha\ d^{\nabla^M}\pi \circ F(B) + \frac{1}{2}\int \alpha \circ \pi_* \circ (2A^S + T^S)(dF(B), dF(B)) \tag{29}$$

is a local martingale for all $\alpha \in \Gamma(T^*M)$. An straightforward calculation shows that we can rewrite the semimartingale (29) as

$$\int (\pi \circ F)^*\alpha\ d^{\nabla^g} B + \frac{1}{2}\int \alpha \circ (\tau_{\pi\circ F} + \mathrm{tr}\ \pi_* \circ (2A^S + T^S) \circ F_* \otimes F_*)_{B_s} ds,$$

and so (29) is a local martingale. Therefore $F(B)$ is an ∇^P-martingale by Theorem 3.1. □

Example 3.1 *Let M be a differentiable manifold and ∇ be a connection on M. We consider the frame bundle $BM(M, GL(\mathbb{R}^n))$ which is endowed with ∇^c and ∇^h, the canonical lift and horizontal lift of ∇, respectively. Let ω be the connection form on BM which is associated with ∇. The following assertions are true.*

1) $T = 0$ and $\pi_* \circ A = 0$ for ∇^c and ∇^h.

2) The symmetric part of $\nabla^h\omega$ is $-\omega \odot \omega$.

3) The symmetric part of $\nabla^c\omega$ is $-\omega \odot \omega + a^c$ (see [4] for the definition of a^c).

Applying Theorem 3.1 and Theorem 3.2 we recovering the main results of [4]. A BM-valued semimartingale Y is a ∇^h-martingale (∇^c-martingale) if and only if

$\pi \circ Y$ is a ∇-martingale in M and

$$\int \omega\ \delta Y + \frac{1}{2}\int (\omega \odot \omega)(dY, dY)$$

is martingale local.

($\pi \circ Y$ is a ∇^M-martingale in M and

$$\int \omega\ \delta Y + \frac{1}{2}\int \omega \odot \omega(dY, dY) + \frac{1}{2}\int a^c\ (dY, dY)$$

is martingale local.)
Furthermore, $F : N \to BM$ is (g, ∇^h)-harmonic map ((g, ∇^c)-harmonic map) if and only if
$\pi \circ F$ is a (g, ∇)-harmonic map and $d^* F^* \omega + trF^*(\omega \odot \omega) = 0$.
($\pi \circ F$ is a (g, ∇)-harmonic map and $d^* F^* \omega + trF^*(\omega \odot \omega + a^c) = 0$.)

Corollary 3.3 *Let $P(M, G)$ be a principal fiber bundle, ω be a connection form on P, k_0 be the $ad(G)$-invariant metric on $\mathfrak{g}$ and h be a Riemannian metric on M. Let ∇^k be the Riemannian connection associated to k and ∇ be the one associated to h. We consider the G-invariant Riemannian metric k on P defined by*

$$k(U, V) = h(\pi_*(U), \pi_*(V)) + k_0(\omega(U), \omega(V)) \tag{30}$$

for all U and V vector fields on P. We have the following assertions:

1. *A P-valued semimartingale Y is a ∇^k-martingale if and only if*
   1) *$\int \omega\, \delta Y$ is a local martingale,*
   2) *$\int \alpha\, d^\nabla \pi \circ Y - \int \alpha \circ \pi_* \circ A^S(dY, dY)$ is a local martingale.*
2. *Let N be a Riemannian manifold with metric $\bar{g}$. A smooth map $F : N \to P(M, G)$ is a $(\bar{g}, \nabla^P)$-harmonic map if and only if*
   1) *$d^* F^* \omega = 0$,*
   2) *$\tau_{\pi \circ F} = -2\mathrm{tr}\ \pi_* \circ A^S \circ F_* \otimes F_*$.*

Proof: Since $T = 0$, it is sufficient to show that the symmetric part of $\nabla^k \omega$ is zero. From (8) an easy calculations shows that

$$\begin{array}{rcccl}
\nabla^k \omega(B^*, X^h) & = & -\omega(\nabla^k_{B^*} X^h) & = & 0 \\
\nabla^k \omega(X^h, B^*) & = & -\omega(\nabla^k_{X^h} B^*) & = & 0 \\
\nabla^k \omega(B^*, C^*) & = & -\omega(\nabla^k_{B^*} C^*) & = & -\frac{1}{2}[B, C] \\
\nabla^k \omega(X^h, Y^h) & = & -\omega(\nabla^k_{X^h} Y^h) & = & -\omega(A_{X^h} Y^h).
\end{array}$$

According to (30), we have $A_{Z^h} Z^h = 0$ for all Z vector field on M. It follows that the symmetric part of $\nabla^k \omega$ is zero. $\square$

Remark 3.1 *M. Arnaudon and S. Paycha, in [2], shows that semimartingales in a principal fiber bundle $P(M, G)$ with G-invariant Riemannian metric k can be decomposed into G- and M- valued semimartingales. More* *precisely, a semimartingale Y with values in $P(M, G)$ splits in a unique way into a horizontal semimartingale $\widetilde{Y}$ and a semimartingale V with values in G such that*

$$Y = \widetilde{Y} \cdot V.$$

Moreover, V is the stochastic exponential

$$V = \epsilon(\int \omega \delta Y)$$

and $\widetilde{Y}$ is the solution of the Itô equation

$$d^{\nabla^k} \widetilde{Y} = H^k_{\widetilde{Y}} d^{\nabla}(\pi \circ Y).$$

It follows that $\widetilde{Y}$ is a ∇^k-martingale if and only if $\pi \circ Y$ is a ∇-martingale. In the case that k is given by (30), Corollary 3.3 shows that if Y is a ∇^k-martingale then V is a G-martingale.

Finally, we consider Y a solution of the Itô equation

$$d^{\nabla^k} Y = \sum_{i=1}^{n} E_i(Y) dB^i,$$

where $(B^1, ..., B^n)$ is a Brownian motion in $\mathbb{R}^n$ and the E_i are vertical or horizontal vector fields on P. It is clear that Y is a ∇^k-martingale and follows easily that $\widetilde{Y}$ is a ∇^k-martingale and V is a G-martingale.

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