FRACTIONAL TERM STRUCTURE MODELS: NO-ARBITRAGE AND CONSISTENCY

ALBERTO OHASHI

ABSTRACT. In this work we introduce Heath-Jarrow-Morton (HJM) interest rate models driven by fractional Brownian motions. By using support arguments we prove that the resulting model is arbitrage-free under proportional transaction costs in the same spirit of Guasoni et al [20, 21, 22]. In particular, we obtain a drift condition which is similar in nature to the classical HJM no-arbitrage drift restriction.

The second part of this paper deals with consistency problems related to the fractional HJM dynamics. We give a fairly complete characterization of finite-dimensional invariant manifolds for HJM models with fractional Brownian motion by means of Nagumo-type conditions. As an application, we investigate consistency of Nelson-Siegel family with respect to Ho-Lee and Hull-White models. It turns out that similar to the Brownian case such family does not go well with the fractional HJM dynamics with deterministic volatility. In fact, there is no nontrivial fractional interest rate model consistent with the Nelson-Siegel family.

1. Introduction

Financial models driven by semimartingales and Markov noises have been intensively studied over the last years by many authors. In general, absence of arbitrage is the basic equilibrium condition which fulfills the minimum requirement for any sensible pricing model. On the other hand, empirical studies propose models which are not consistent with this basic assumption. In particular, some evidence of non-trivial long-memory behavior in bond markets has been recently suggested by many authors [4, 7, 39, 28]. In most cases, the presence of long-range dependence in short-rate interest rates seems to be common and it is originated by the fundamentals of the economy. In this regard, it is important to study bond markets with extrinsic memory driven by non-Markovian noises which allow nontrivial long-range dependence over time.

Recall that in the classical Musiela parametrization ([29]) the forward rate r_t satisfies a stochastic partial differential equation (henceforth abbreviated by SPDE) of the following form

$$dr_t(x) = \Big(\frac{\partial}{\partial x} r_t(x) + \alpha_{HJM}(t,x)\Big)dt + \sum_{j\geq 1} \sigma^j_t(x) dB^j_t, \tag{1.1}$$

Date: February 11, 2008.

Key words and phrases. Fractional Brownian motion. Interest rate models. Stochastic PDEs. Invariant manifolds. Nelson-Siegel family.

The author gratefully acknowledges the support from the FAPESP grant 05/57064-4. The author would like to thank Pedro Catuogno and David Elworthy for fruitful discussions and suggestions.

where α_{HJM} is the so-called Heath-Jarrow-Morton (henceforth abbreviated by HJM) drift condition which is completely determined by the volatilities $(\sigma^j)_{j\geq 1}$ under a risk-neutral measure, $(B^j)_{j\geq 1}$ is a sequence of stochastic noises and x is the *time to maturity*. The forward rate r_t is considered as a Hilbert space-valued stochastic process. Due to this infinite-dimensional intrinsic nature, it is important to understand the relation between forward curves $x \mapsto r_t(x)$ at time $t > 0$ and finite-dimensional parametrized families of smooth forward curves, frequently used in estimating the term structure of interest rates (e.g Nelson-Siegel and Svensson families).

Originally proposed by Björk and Christensen ([6]) and recently studied by Filipovic and Teichmann in a series of papers ([15, 18, 19]), the so-called *consistency problems* regards to the characterization and existence of finite-dimensional invariant manifolds with respect to $t \mapsto r_t$. In fact, the stochastic invariance is essentially equivalent to a deterministic tangency condition on the coefficients $\frac{\partial}{\partial x}, \alpha_{HJM}, \sigma$ in equation (1.1). In particular, if short-rate interest rates exhibit long-range dependence then standard statistical procedures may be misspecified, since in this case the classical HJM no-arbitrage drift restriction may not be the correct one. Moreover, by fixing (non-semimartingale) long-memory stochastic noises $(B_j)_{j\geq 1}$ in (1.1), one has to obtain new tangency conditions on the coefficients of (1.1) to get appropriate arbitrage-free invariant parametrized families of smooth forward curves. This is the program that we start to carry out in this work.

We have chosen the driving noise in equation (1.1) given by the fractional Brownian motion (henceforth abbreviated by fBm) with Hurst parameter $H \in (1/2, 1)$. For many reasons (see e.g. [37]), the fBm appears naturally as the canonical process with nontrivial time correlations inserting memory into system under consideration. Indeed, the main difficulty in dealing with fBm is the fact that such process is a semimartingale if and only if it is a standard Brownian motion (H $= 1/2$). This lack of semimartingale property immediately implies from [10] that fBm allows arbitrage opportunities (for any $H \neq 1/2$) in the absence of transaction costs.

The main goal of this work is to introduce arbitrage-free HJM interest rate models driven by a cilindrical fBm under arbitrary small proportional transaction costs in the bond market. In this paper, the forward rate is considered as the solution of a SPDE (in Skorohod sense) of type (1.1) under the Musiela parametrization. In this work, we only treat the case of deterministic volatilities, leaving open the general stochastic volatility case for future research. In particular, there is a gaping lack of results for fractional SPDEs in Skorohod sense with general multiplicative noise. See Section 5.5 for more details.

Under deterministic volatility assumption, we obtain a drift condition which is similar in nature to the classical HJM no-arbitrage drift restriction. Although such condition is not sufficient to ensure no-arbitrage in the market, when combined with an additional mild condition on the volatilities it results in absence of arbitrage in the same spirit of the works [20, 21, 22], where the support of the driving noise plays a key rule in the no-arbitrage characterization for markets with transactions costs.

In the second part of this paper, we characterize finite-dimensional invariant submanifolds for HJM models driven by fBm by means of Nagumo-type conditions. Such characterization is the key ingredient to tackle the *consistency* problems related to the model. As an application of these abstract results, we investigate consistency of the Nelson-Siegel family with respect to Ho-Lee and Hull-White models driven by fBm. Similar to the Brownian case, such family is not consistent with respect to these models. In general, we arrive at the same classical result of the Brownian case: No nontrivial interest rate model with deterministic volatility structure is consistent with Nelson-Siegel family.

This work is organized as follows. In Section 2, we give some general results regarding portfolios and absence of arbitrage in fractional bond markets. In Section 3, we characterize finite-dimensional invariant forward manifolds with respect to HJM models driven by fBm. In Section 4, we examine consistency of the Nelson-Siegel family with respect to concrete interest rate models.

2. The bond market: Portfolios and no-arbitrage

In what follows, we are given a stochastic basis $\big(\Omega,(\mathcal{F}_t)_{t\ge 0},\mathbb{P}\big)$ satisfying the usual conditions. Let us consider a sequence of fBm $(\beta^j_H)_{j\ge 1}$ with the same parameter $1/2<H<1$ and adapted to $(\mathcal{F}_t)_{t\ge0}$. We fix $1/2<H<1$ once and for all. In other words, for each $j\ge 1$, β^j_H is a centered Gaussian process with continuous sample paths, $\beta^j_H(0)=0$ and covariance

$$\mathbb{E}(\beta^i_H(t)-\beta^i_H(s))(\beta^j_H(t)-\beta^j_H(s))=\delta_{ij}|t-s|^{2H-2}.$$

Throughout this paper we omit the subscript H and we write β^j_t instead of $\beta^j_H(t)$. For each $j\ge 1$ there exists a unique Brownian motion W^j such that

$$\beta^j_t=\int_0^t K(t,s)dW^j_s,$$

where $K(t,s)$ is given by

$$K(t,s):=c_H s^{1/2-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}du. \tag{2.1}$$

for some $c_H>0$ (see [25] for more details). For a detailed discussion on the stochastic analysis of the fBm, the reader may refer to [25].

In what follows, we consider the following subset of $\mathbb{R}^2$

$$\Delta^2:=\{(t,T)\in\mathbb{R}^2|0\le t\le T<\infty\}.$$

Let us consider a term structure of bond prices $\{P(t,T);(t,T)\in\Delta^2\}$ where $P(t,T)$ is the price of a zero coupon bond at time t maturing at time T. We assume the usual normalization condition

$$P(t,t)=1,\quad \forall t>0,$$

and $P(t,T)$ is a.s continuously differentiable in the variable T. In this way, we introduce the term structure of interest rates $\{f(t,T);(t,T)\in\Delta^2\}$ given by

$$f(t,T)=-\frac{\partial logP(t,T)}{\partial T};\quad (t,T)\in\Delta^2. \tag{2.2}$$

Then the following relation holds

$$P(t,T)=\exp\Big(-\int_t^T f(t,u)du\Big);\quad (t,T)\in\Delta^2.$$

In this paper, we adopt the Musiela parametrization where $x := T - t$ is the time to maturity. Then we shall write

$$P(t,T) = \exp\Big(-\int_0^{T-t} r_t(x)dx\Big),$$

where $r_t(x) := f(t, t+x)$ for $(t,x) \in \mathbb{R}^2_+$. We adopt the Heath-Jarrow-Morton framework ([24]) in the fBm setting. In particular, we seek the forward curve $x \mapsto r_t(x)$ as a Hilbert space-valued stochastic process described by a SPDE

$$dr_t = \Big(\frac{d}{dx}r_t + \alpha_t\Big)dt + \sigma_t dB_t, r_0(\cdot) = \xi \in E \tag{2.3}$$

in a separable Hilbert space E to be defined. The first-order derivative operator $\frac{d}{dx}$ is the infinitesimal generator of the right-shift family of operators $\{S(t); t \geq 0\}$ acting on E. We seek the drift α as a function of the volatility σ in such way that the resulting bond market is arbitrage-free under arbitrary small proportional transaction costs. Here B_t is a cilindrical fBm with parameter $1/2 < H < 1$ taking values in a separable Hilbert space U. Formally we shall write

$$B_t = \sum_{j=1}^{\infty} \beta^j_t e_j, \tag{2.4}$$

for some orthonormal basis $(e_j)_{j\geq 1}$ in U. Of course, the SPDE (2.3) must be interpreted in the integral form. Moreover, in this work the stochastic integral is considered in the Skorohod sense [12] as a Paley-Wiener integral.

2.1. **The specification of the model.** Initially, let us fix a $d-$dimensional fBm $(\beta^1, \ldots, \beta^d)$ on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let us assume for the moment that the forward rate is given by the following system of stochastic differential equations

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{d}\int_0^t \sigma^i(s,T)d\beta^i_s, \quad 1 \leq d < +\infty. \tag{2.5}$$

From now on the coefficients $(\sigma^1, \ldots, \sigma^d)$ and α are deterministic functions. Equation (2.5) is well-defined if for each $i = 1, \ldots, d$

$$\int_0^T |\alpha(s,T)|ds + \int_0^T\int_0^T |\sigma^i(s,T)||\sigma^i(t,T)|\phi_H(t-s)dsdt < \infty,$$

for all $0 < T < \infty$, where $\phi_H(u) := H(2H-1)|u|^{2H-2}$, $u \in \mathbb{R}$.

Let $\{S(t); t \geq 0\}$ be the semigroup of right-shifts defined by $S(t)g(x) := g(t+x)$ for any function $g : \mathbb{R}_+ \to \mathbb{R}$. Fix $(t,x) \in \mathbb{R}^2_+$. Then (2.5) can be written as

(2.6)
$$f(t,t+x) = S(t)f(0,x) + \int_0^t S(t-s)\alpha(s,s+x)ds + \sum_{i=1}^{d}\int_0^t S(t-s)\sigma^i(s,s+x)d\beta^i_s.$$

In (2.6) we deal with the parametrization $T = t + x$. The operator $S(t)$ acts on $f(0,x)$, $\alpha(s,x+s)$ and $\sigma^j(s,x+s)$ as functions of x. By setting

$$r_t(x) := f(t, t+x),$$

it follows that

$$P(t,T) = \exp\Big\{ -\int_0^{T-t} r_t(x)dx \Big\}; \quad (t,T) \in \Delta^2.$$

We can work out in an axiomatic way the minimal requirements on a Hilbert space E such that (2.6) can be given a meaning when

$$r_t(\cdot) = f(t, t+\cdot); \quad t \in \mathbb{R}_+ \tag{2.7}$$

is considered as an $E-$valued stochastic process in such way that $\{S(t); t \geq 0\}$ is a strongly continuous semigroup in E with infinitesimal generator given by the first order derivative $\frac{d}{dx}$. The strategy follows very similar to Filipovic [15]. So we omit the details and the reader may refer to this work. We just want to mention that the minimal requirements on the state-space E are the following:

(H1) The point-wise evaluation $h \mapsto h(x)$ is a continuous linear functional on E for every $x \in \mathbb{R}_+$. Moreover, we assume that for every element $h \in E$ there exists a well-defined continuous representative, still denoted by h.
(H2) $\{S(t); t \geq 0\}$ is a strongly continuous semigroup on E with infinitesimal generator $(A, \text{dom }(A))$, where $A := \frac{d}{dx}$.

Remark 2.1. *We choose the state-space E as defined in Filipovic* [15]. *He proposed a family of suitable Hilbert spaces to study HJM models in the semimartingale case. One should notice that even in the fBm case, such spaces are regular enough to attend our needs since they fulfill conditions (H1-H2). Moreover, they are coherent with realistic economic assumptions on the forward rate.*

At this point, we relax the hypothesis on the noise and we allow from now on the cilindrical fBm $B = (\beta^j)_{j=1}^\infty$ defined in (2.4) on a separable Hilbert space U. In the sequel, we denote $\mathcal{L}_{(2)}(U, E)$ the space of Hilbert-Schmidt linear operators from U into E with the usual norm $\|\cdot\|_{(2)}$.

We make use of the following notation: We set $\alpha_t(\cdot) := \alpha(t, t+\cdot)$ and $\sigma = (\sigma^j)_{j=1}^\infty$, where

$$\sigma_t^j := \sigma_t e_j := \sigma^j(t, t+\cdot),$$
$$\sigma_t^j(x) := \sigma_t e_j(x); \quad (t,x) \in \mathbb{R}_+^2, \ j \geq 1.$$

In this paper, we are interested in Gaussian interest rate models where we assume that the coefficients $\alpha : \mathbb{R}_+ \to E$ and $\sigma : \mathbb{R}_+ \to \mathcal{L}_{(2)}(U, E)$ satisfy the following set of assumptions:

$$\int_0^T \|\alpha_s\|_E ds + \int_0^T \|\sigma_s\|_{(2)}^2 ds < \infty, \text{for every } 0 < T < \infty. \tag{2.8}$$

To ensure existence of a continuous version for the mild solution of equation (2.3) we assume there exists $\gamma \in (0, 1/2)$ such that

(2.9)
$$\int_0^T\int_0^T u^{-\gamma}v^{-\gamma}\|S(u)\sigma_u\|_{(2)}\|S(v)\sigma_v\|_{(2)}\phi_H(u-v)dudv<\infty, \quad \text{for every } 0<T<\infty.$$

In order to get a well-defined expression for the bond prices $\{P(t,T);(t,T)\in\Delta^2\}$ we also assume the following growth conditions: In the sequel, we denote l^2 the usual Hilbert space of real sequences $(a_i)_{i\geq 1}$ such that $\|a\|^2_{l^2}:=\sum_{i=1}^\infty |a_i|^2<\infty$. We assume that

$$\int_{[0,T]^4}\|\sigma_u(s)\|_{l^2}\|\sigma_v(r)\|_{l^2}\phi_H(u-v)dudvdsdr<\infty, \text{for every } 0<T<\infty; \tag{2.10}$$

$$\int_{[0,T]^3}\|\sigma_u(t)\|_{l^2}\|\sigma_v(t)\|_{l^2}\phi_H(u-v)dvdudt<\infty, \text{for every } 0<T<\infty. \tag{2.11}$$

One should note that (2.8) yields $\int_0^T\|S(t)\sigma_t\|^2_{(2)}dt<\infty$ for every $0<T<\infty$, and therefore we can write the stochastic convolution as an E-valued Gaussian random variable given by

$$\int_0^t S(t-s)\sigma_s dB_s=\sum_{j=1}^\infty\int_0^t S(t-s)\sigma^j_s d\beta^j_s,\quad t>0.$$

Under the above assumptions one can easily show the following lemma.

Lemma 2.1. *Assume that the coefficients α and σ satisfy assumptions (2.8), (2.9), (2.10) and (2.11). Then the forward rate r_t is the continuous mild solution of equation (2.3). Moreover, the term structure of bond prices is given by the continuous process*

(2.12)
$$P(t,T)=P(0,T)\exp\bigg\{\int_0^t\big[r_s(0)-\mathcal{I}_\alpha(s,T)\big]ds+\sum_{j=1}^\infty\int_0^t-\mathcal{I}_{\sigma^j}(s,T)d\beta^j_s\bigg\};\quad (t,T)\in\Delta^2,$$

where $\mathcal{I}_\alpha(s,T):=\int_0^{T-s}\alpha_s(x)dx$ and $\mathcal{I}_{\sigma^j}(s,T):=\int_0^{T-s}\sigma^j_s(x)dx$.

Proof. This is a straightforward application of stochastic Fubini theorem (see [27]) in the fBm setting by using conditions (2.10) and (2.11). Condition (2.9) allows the existence of a continuous mild solution of (2.3) as in [12]. □

We assume the existence of a traded asset that pays interest. In other words, the unit of money invested at time zero in this asset gives at time t the amount

$$S_0(t):=\exp\bigg\{\int_0^t r_s(0)ds\bigg\},$$

where $r_t(0)=f(t,t)$ for $t>0$. By considering S_0 as a *numéraire*, the discounted prices are then expressed by

$$Z_t(T):=\frac{P(t,T)}{S_0(t)},\quad (t,T)\in\Delta^2. \tag{2.13}$$

Up to now the bond price $P(t,T)$ has been defined only for $(t,T)\in\Delta^2$. It will be convenient to work with $P(t,T)$ when $t>T$. For this, we make use of the same trick as in [5]. We set $P(t,T)=S_0(t)S_0^{-1}(T)$ for $t\geq T$.

Following the arguments in [38] and [5] we now introduce the notions of admissible self-financing portfolios in our context. Let us denote $\mathcal{M}(\mathbb{R}_+)$ the space of (finite) signed measures on $\mathbb{R}_+$ endowed with the total variation norm $\|\cdot\|_{TV}$. Let μ be a measure-valued elementary process of the form

$$\mu_t(\omega,\cdot):=\sum_{i=0}^{N-1}\chi_{F_i}(\omega)\chi_{(t_i,t_{i+1}]}(t)m_i, \tag{2.14}$$

where $m_i\in\mathcal{M}(\mathbb{R}_+)$, $0=t_0<\ldots<T_N<\infty$ and $F_i\in\mathcal{F}_{t_i}$. We assume the support of μ_t is concentrated on $[t,\infty)$ for every $(t,\omega)\in\mathbb{R}_+\times\Omega$.

We denote by $\mathcal{S}_b$ the set of elementary processes of the form (2.14), endowed with the following norm

$$\|\mu\|^2_{\mathrm{V}}:=\mathbb{E}\sup_{0\leq t<\infty}\|\mu_t\|^2_{TV}. \tag{2.15}$$

From now on all economic activity will be assumed to take place on the bounded set $[0,T^*]^2$. So we assume that $Z(t,T)=0$ if $(t,T)\notin[0,T^*]^2$. Under the hypotheses (2.8), (2.9), (2.10) and (2.11), the discounted price process $Z_t(T)$ satisfies the following condition:

(A1) $\{Z_t(T);(t,T)\in[0,T^*]^2\}$ is a jointly continuous real-valued stochastic process such that

$$\mathbb{E}\sup_{(t,T)\in\mathbb{R}^2}|Z_t(T)|^2<\infty.$$

If $\mu\in\mathcal{S}_b$ is given by (2.14) then we define

$$\int_0^t\mu_s dZ_s:=\sum_{i=0}^{N-1}\chi_{F_i}\Big(Z_{t_{i+1}\wedge t}-Z_{t_i\wedge t}\Big)m_i,$$

where $Z_{t_i}m_{t_i}$ is the usual dual action. By Hölder inequality it follows that

$$\mathbb{E}\sup_{0\leq t<\infty}\Big|\int_o^t\mu_s dZ_s\Big|\leq\|\mu\|_{\mathrm{V}}\mathbb{E}^{1/2}\sup_{0\leq s,t<\infty}\|Z_s-Z_t\|^2_\infty<\infty \tag{2.16}$$

where $\|\cdot\|_\infty$ denotes the usual (uniform topology) norm on the space of real-valued bounded functions defined on $\mathbb{R}_+$. Let V be the completion of $\mathcal{S}_b$ with respect to (2.15). By the estimate (2.16) and the definition of V we may easily define $\int_0^\cdot\mu_s dZ_s$ for every $\mu\in\mathrm{V}$.

In the sequel, we denote $\mathcal{P}_{T^*}$ the set of all partitions of $[0,T^*]$. We also need the following assumption:

(A2) $\Pi_{T^*}(\mu):=\sup_{\pi\in\mathcal{P}_{T^*}}\sum_{t_i\in\pi}\|\mu_{t_{i+1}}-\mu_{t_i}\|_{TV}$ is square integrable.

By taking into account proportional transaction costs in the bond market, the liquidation value of a portfolio with zero initial capital is

$$
\begin{aligned}
V_t^k(\mu) \quad &:= \quad \sum_{t_i<t} \chi_{F_i}\Big(Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}\Big) m_i \\
&- \quad k\sum_{t_i<t} Z_{t_i}|\mu_{t_{i+1}} - \mu_{t_i}| - kZ_t|\mu_t|,
\end{aligned}
$$

where k is an arbitrary positive number and $|\cdot|$ denotes the total variation measure. The first term accounts for the capital gain of holding an elementary strategy μ of the form (2.14) (without transaction costs) during the interval $[0,t]$. The second and third term account for the transaction costs incurred in various transactions and the eventual liquidation value of the portfolio, respectively.

By passing from a finite number of transactions to continuous trading one can easily show that if $\mu \in \mathrm{V}$ satisfies assumption **(A2)** then the above quantities converge to the following

$$
V_t^k(\mu) := \int_0^t \mu_s dZ_s - k\int_0^t Z_s d|\mu_s| - kZ_t|\mu_t|. \tag{2.17}
$$

See Appendix for more details, including the definition of the second integral in (2.17). Now we are able to introduce the following notions:

Definition 2.1. *We say that $\mu \in \mathrm{V}$ is an **admissible trading strategy** if it satisfies **(A2)**, it is $\mathcal{F}_t-$adapted and there exists a constant $M > 0$ such that $V_t^k(\mu) \geq -M$ a.s for every $t \leq T^*$. An admissible trading strategy is an **arbitrage opportunity with transaction costs** $k > 0$ on $[0,T^*]$, if $V_{T^*}^k(\mu) \geq 0$ a.s and $\mathbb{P}\big\{V_{T^*}^k(\mu) > 0\big\} > 0$. Therefore, the bond market is **k-arbitrage-free** on $[0,T^*]$ with transaction costs k if for every admissible strategy μ, $V_{T^*}^k(\mu) \geq 0$ a.s only if $V_{T^*}^k(\mu) = 0$ a.s.*

Remark 2.2. *Since the main dynamics takes place on Δ^2 we do assume that all admissible strategies μ are Markovian in the sense that the support of μ_s is concentrated on $[s,+\infty)$.*

It is straightforward to prove the following result in the same spirit of Guasoni ([20] - Proposition 2.1). Thus we omit the details.

Proposition 2.1. *Let us fix $k > 0$. If for every $(\mathcal{F}_t)_{t\geq 0}-$stopping time τ such that $\mathbb{P}\{\tau < T^*\} > 0$ we have*

$$
\mathbb{P}\Bigg\{ \sup_{\tau \leq t \leq T \leq T^*} \Big|\frac{Z_\tau(\tau)}{Z_t(T)} - 1\Big| < k, \tau < T^* \Bigg\} > 0, \tag{2.18}
$$

then the bond market is arbitrage-free on $[0,T^*]$ with transaction costs k.

One can also show by using similar arguments from Guasoni et al ([21]) that the k-arbitrage-free property in Definition 2.1 is essentially equivalent to the existence of a k-consistent price system. Thus (2.18) is also a sufficient condition for it. In fact, a sufficient condition for no-arbitrage is the conditional full support property for Z which is equivalent to full support only if Z is Markovian. Hence in the non-Markovian setting it is more natural to find mild conditions on the volatility in such way that $\log Z$ has only full support. See Lemmas 3.1 and 3.2 for details.

In the sequel, we consider $\mathcal{K}^*$ the usual isometry between the reproducing kernel Hilbert space of the fBm and $L^2(0,T^*;\mathbb{R})$. See [2, 3] for this notation.

Lemma 2.2. *Assume that assumptions (2.8), (2.9), (2.10) and (2.11) hold. Then the discounted bond price process satisfies the following stochastic differential equation*

$$dZ_t(T) = \Big[-\mathcal{I}_\alpha(t,T) + \Sigma_\sigma(t,T)\Big]Z_t(T)dt + \sum_{j=1}^{\infty} -\mathcal{I}_{\sigma^j}(t,T)Z_t(T)d\beta^j_t, \tag{2.19}$$

where $\Sigma_\sigma(t,T) := \frac{1}{2}\sum_{j=1}^{\infty}\frac{\partial}{\partial t}\int_0^t \Big[\mathcal{K}^*_t(\mathcal{I}_{\sigma^j}(\cdot,T))_r\Big]^2 dr$.

Proof. Consider $1 \le d < +\infty$ and notice that Itô formula [2, 3] applied to (2.13) yields

$$dZ^d_t(T) = \Big[-\mathcal{I}_\alpha(t,T) + \Sigma^d_\sigma(t,T)\Big]Z^d_t(T)dt + \sum_{j=1}^{d} -\mathcal{I}_{\sigma^j}(t,T)Z^d_t(T)d\beta^j_t,$$

where $\Sigma^d_\sigma(t,T) := \frac{1}{2}\sum_{j=1}^{d}\frac{\partial}{\partial t}\int_0^t \Big[\mathcal{K}^*_t(\mathcal{I}_{\sigma^j}(\cdot,T))_r\Big]^2 dr$ and $Z^d_t(T)$ is given in (2.13) with $1 \le d < \infty$. Here $\mathcal{K}^*_t(\cdot) := \mathcal{K}^*(\chi_{[0,t]}\cdot)$ in the notation of [2].

By applying again Itô formula with respect to the cilindrical fBm and considering the mapping $\Pi(\sigma) : [0,T] \to \mathcal{L}_{(2)}(U;\mathbb{R})$ defined by

$$\Pi_s(\sigma)e_i := \int_0^{T-s} \sigma_s e_i(y)dy; \quad s \in [0,T],$$

we may conclude the proof. ☐

3. Absence of Arbitrage

For simplicity, we assume from now on that the driving noise in equation (2.3) is given by a d-dimensional fBm with $d < \infty$. We prove that under suitable conditions on the volatility $\sigma = (\sigma^j)_{j\ge 1}$, the bond market model is k-arbitrage-free for every $k > 0$. The main ingredient in the no-arbitrage argument consists in the full support property on $\mathcal{C}(\Delta^2_{T^*};\mathbb{R})$, the space of real-valued continuous functions on $\Delta^2_{T^*} := \{(t,T); 0 \le t \le T \le T^*\}$ endowed with the sup norm. This property together with a suitable choice on the drift will result in k-no-arbitrage for every $k > 0$. Recall that if $\mathcal{X}$ is a Polish space then a random element $\xi : \Omega \to \mathcal{X}$ has $\mathbb{P}-$full support when $\mathbb{P}_\xi := \mathbb{P}\circ\xi^{-1}(\mathcal{U}) > 0$ for every non-empty open set $\mathcal{U}$ in $\mathcal{X}$.

Lemma 3.1. *Let* $\mathbb{Y} : \Omega \to \mathcal{C}(\Delta^2_{T^*};\mathbb{R})$ *be a measurable map such that* $\mathbb{X} := \log \mathbb{Y}$ *has* $\mathbb{P}-$ *full support. Then* $\mathbb{Y}$ *satisfies assumption in Proposition 2.1.*

Proof. Given $\varepsilon > 0$ and τ a $\mathcal{F}_t-$stopping time such that $\mathbb{P}\{\tau < T^*\} > 0$, it is sufficient to check that

$$\mathbb{P}\Big\{\sup_{\tau\le t\le T\le T^*} |\mathbb{X}(t,T) - \mathbb{X}(\tau,\tau)| < \varepsilon, \tau < T^*\Big\} > 0.$$

If $p \in \mathcal{C}(\Delta^2_{T*};\mathbb{R})$ then triangle inequality yields

$$\Big\{\sup_{(t,T)\in\Delta^2_{T^*}}|\mathbb{X}(t,T)-p(t,T)|<\varepsilon/2,\tau<T^*\Big\}$$
$$\subset\quad\Big\{\sup_{\tau\le t\le T\le T^*}|\mathbb{X}(t,T)-\mathbb{X}(\tau,\tau)|<\varepsilon,\tau<T^*\Big\}.$$

Let us consider $\mathcal{P}$ the set of polynomials p on $\Delta^2_{T^*}$ with rational coefficients such that $p(0,0)=0$. We claim that there exists $p\in\mathcal{P}$ such that

$$\mathbb{P}\Big\{\sup_{(t,T)\in\Delta^2_{T^*}}|\mathbb{X}(t,T)-p(t,T)|<\varepsilon/2,\tau<T^*\Big\}>0. \tag{3.1}$$

Suppose that (3.1) is violated for every $p\in\mathcal{P}$. Then we obtain

$$\Big\{\sup_{(t,T)\in\Delta^2_{T^*}}|\mathbb{X}(t,T)-p(t,T)|<\varepsilon/2,\tau<T^*\Big\}$$
$$\subset\quad\Big\{\tau\ge T^*\Big\}\quad\mathbb{P}-a.s,\quad\forall p\in\mathcal{P}.$$

Therefore

$$\bigcup_{p\in\mathcal{P}}\Big\{\sup_{(t,T)\in\Delta^2_{T^*}}|\mathbb{X}(t,T)-p(t,T)<\varepsilon/2\Big\}\subset\{\tau\ge T^*\}\quad\mathbb{P}-a.s. \tag{3.2}$$

By the density of $\mathcal{P}$ in $\mathcal{C}(\Delta^2_{T^*};\mathbb{R})$ and the full support of $\mathbb{X}$ it follows that

$$\mathbb{P}\Bigg\{\bigcup_{p\in\mathcal{P}}\Big\{\sup_{(t,T)\in\Delta^2_{T^*}}|\mathbb{X}(t,T)-p(t,T)<\varepsilon/2\Big\}\Bigg\}=1$$

and therefore $\mathbb{P}\{\tau<T^*\}=0$ which is a contradiction. □

Remark 3.1. *Recall that the fBm has γ- Hölder continuous paths a.s for any $\gamma<H$. Moreover, one can prove the existence of the fBm Wiener measure on a separable Banach space $\mathbb{W}$ continuously imbedded on the space $\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ such that the elements of $\mathbb{W}$ are γ- Hölder continuous functions on any compact interval. See* [23] *for the proof of this fact.*

The following remark turns out to be very useful for the approach taken in this work.

Lemma 3.2. *Assume that $\mathcal{I}_{\sigma^j}(t,T)$ is λ-Hölder continuous on $\Delta^2_{T^*}$ for every $j\ge1$ where $1/2<\lambda<1$. Then the process $\sum_{j=1}^d\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\beta^j_s$ has $\mathbb{P}$-full support on $\mathcal{C}(\Delta^2_{T^*};\mathbb{R})$.*

Proof. Fix $(\xi^j)_{j=1}^d$ a sequence of γ- Hölder continuous functions on $[0,T^*]$ where $1/2<\gamma<H$. We recall that if $\mathcal{I}_{\sigma^j}(t,T)$ is λ-Hölder continuous on $\Delta^2_{T^*}$ then the pathwise Young integral $\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\xi^j_s$ is well-defined and there exists a constant $C>0$ which depends only on T^*, γ and λ such that

$$\Bigg\|\int_0^\cdot\mathcal{I}_{\sigma^j}(s,\cdot\cdot)d\xi^j_s\Bigg\|_\gamma\le C\|\mathcal{I}_{\sigma^j}\|_\lambda\|\xi^j\|_\gamma, j=1,\ldots,d, \tag{3.3}$$

where $\|\cdot\|_\eta$ denotes the usual η-Hölder norm.

Moreover, the pathwise Young integral coincides with the symmetric integral in Russo and Vallois [35]. Recall that we are assuming that the volatilities are deterministic functions and therefore the Gross-Sobolev derivative of $\mathcal{I}_{\sigma^j}(t,T)$ vanishes for each $j\ge 1$ and $(t,T)\in\Delta^2_{T^*}$. Since the fBm has γ- Hölder continuous paths a.s, Proposition 3 in [2] tells that the Skorohod integral

$$\int_0^t \mathcal{I}_{\sigma^j}(s,T)d\beta^j_s;\quad j\ge 1,$$

can be interpreted as a pathwise Young integral. By the estimate (3.3) and Remark 3.1 it follows that each $\int_0^\cdot \mathcal{I}_{\sigma^j}(s,\cdot\cdot)d\beta^j_s$ has $\mathbb{P}$-full support on $\mathcal{C}(\Delta^2_{T^*};\mathbb{R})$. Moreover, since $(\beta^j)_{j\ge1}$ is a sequence of real-valued independent fBm we then conclude that

$$(t,T)\mapsto \sum_{j=1}^d \int_0^t \mathcal{I}_{\sigma^j}(s,T)d\beta^j_s$$

has $\mathbb{P}$-full support as well. □

By Lemma 3.1 and Proposition 2.1 we know that if $\log Z$ has $\mathbb{P}-$full support then the bond market is k-arbitrage-free for every $k>0$. One should notice that assuming that the volatility $\sigma=(\sigma^j)_{j\ge1}$ satisfies the assumptions in Lemma 3.2, there are infinitely many choices of α which give the full support property for $\log Z$ and therefore absence of arbitrage in the fractional bond market. But there is a canonical choice for the drift which gives the desirable property:

$$\mathbb{E}Z_t(T)=P(0,T)\quad\forall(t,T)\in\Delta^2. \tag{3.4}$$

As a direct consequence of Lemma 2.2 we have the following basic result.

Corollary 3.1. *Condition (3.4) holds if and only if the drift α satisfies the following equality*

$$\alpha_t(\cdot)=\sum_{j=1}^d\left\{\sigma^j_t(\cdot)\int_0^t \mathcal{I}_{\sigma^j}(\theta,\cdot{+}t)\phi_H(t{-}\theta)d\theta+\int_0^\cdot \sigma^j_t(y)dy\int_0^t\sigma^j_\theta(\cdot{+}t{-}\theta)\phi_H(t{-}\theta)d\theta\right\}. \tag{3.5}$$

Proof. By Lemma 2.2 we know that $Z_t(T)$ satisfies the stochastic differential equation (2.19). Since Skorohod integrals has zero expectation, we choose α in such way that

$$\mathcal{I}_\alpha(t,T)=\Sigma^d_\sigma(t,T),$$

for each $(t,T)\in\Delta^2$. Therefore,

$$\alpha(t,t+T-t)=\frac{\partial}{\partial T}\Sigma^d_\sigma(t,T). \tag{3.6}$$

Differentiating expression (3.6) and taking into account that

$$|t-\theta|^{2H-2} = \frac{(t\theta)^{H-\frac{1}{2}}}{\beta(2-2H, H-\frac{1}{2})} \times \int_0^{t\wedge\theta} v^{1-2H}(t-v)^{H-\frac{3}{2}}(\theta-v)^{H-\frac{3}{2}}dv,$$

and

$$\frac{\partial K}{\partial r}(r,s) = c_H\left(\frac{r}{s}\right)^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}},$$

where $\beta(\cdot,\cdot)$ denotes the beta function, we then arrive at the expression (3.5) by considering the parametrization $x = T-t$. By observing that $Z_t(T) > 0$ a.s for every $(t,T)\in\Delta^2$ we conclude the proof. □

Remark 3.2. *We notice that if $H = 1/2$ then the operator $\mathcal{K}^*$ in Lemma 2.2 is just the identity and we therefore arrive at the classical HJM drift condition in Corollary 3.1*

$$\alpha_t(\cdot) = \sum_{j=1}^d \sigma^j_t(\cdot)\int_0^{\cdot}\sigma^j_t(y)dy.$$

Let us consider

$$\mathcal{S}_H\sigma_t(\cdot) := \sum_{j=1}^d \left\{\sigma^j_t(\cdot)\int_0^t \mathcal{I}_{\sigma_j}(\theta,\cdot+t)\phi_H(t-\theta)d\theta + \int_0^{\cdot}\sigma^j_t(y)dy\int_0^t\sigma^j_\theta(\cdot+t-\theta)\phi_H(t-\theta)d\theta\right\}.$$

We assume that the volatilities are regular enough in such way that

$$\int_0^T \|\mathcal{S}_H\sigma_t\|_E dt < \infty \tag{3.7}$$

for every $0 < T < \infty$. Indeed, it is not very restrictive to assume that the volatility σ_t satisfies such integrability condition on the forward curve space E given in Remark 2.1. See Section 3.2 in [16] for more details.

3.1. **Drift condition and quasi-martingale measure.**

Similar to the semimartingale case the measure $\mathbb{P}$ is considered as physical measure. This motivates the following definition.

Definition 3.1. *We say that an equivalent probability measure $\mathcal{Q}\sim\mathbb{P}$ is a **quasi-martingale measure** if the discounted bond price process $Z_t(T)$ has $\mathcal{Q}$-constant expectation, that is,*

$$\mathbb{E}_{\mathcal{Q}}Z_t(T) = P(0,T) \quad \forall (t,T)\in\Delta^2. \tag{3.8}$$

We now state the main result of this section. Before this, we present some elementary results concerning Girsanov change of measures in the fBm setting. Without any loss of generality we take $U = l^2$. It is well-known (see e.g [9]) that the following operator

$$\mathcal{K}h(t) := \int_0^t K(t,s)h(s)ds; \quad h \in L^2(0,T^*;l^2);\ 0 < T^* < \infty, \tag{3.9}$$

is a bijection between $L^2(0,T^*;l^2)$ and the Sobolev space $I_{0+}^{H+1/2}(L^2(0,T^*;l^2))$ which is the image of the fractional integral $I_{0+}^{H+1/2}$ of order $H+1/2$ defined on $L^2(0,T^*;l^2)$. In general, if $\alpha \in (0,1)$ then I_{0+}^{α} admits an inverse given by the Marchaud fractional derivative D_{0+}^{α}. See [36] for these notations and a complete review on fractional calculus. Furthermore, one can show that $\mathcal{K}^{-1}$ is given by

$$\mathcal{K}^{-1}v(t) = c_H^{-1} t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} \big(u^{\frac{1}{2}-H} Dv\big)(t), \tag{3.10}$$

where D is the usual derivative operator. The next result is a straightforward consequence of the representation of fBm in terms of the standard Brownian motion.

Lemma 3.3. *Let $\{\gamma(t); 0 \le t \le T^*\}$ be an l^2-valued measurable function such that $\int_0^{T^*} \|\gamma(t)\|_{l^2} dt < \infty$ and $R(\cdot) := \int_0^{\cdot} \gamma(s)ds \in I_{0+}^{H+1/2}(L^2(0,T^*;l^2))$. Then $\tilde{B}_t := B_t - \int_0^t \gamma(s)ds$ is a $\mathcal{Q}_{T^*}$-cilindrical fBm on $[0,T^*]$ such that*

$$\frac{d\mathcal{Q}_{T^*}}{d\mathbb{P}} = \mathcal{E}\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*}, \tag{3.11}$$

where

$$\mathcal{E}\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*} := \exp\left[\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*} - \frac{1}{2}\int_0^{T^*} \|\mathcal{K}^{-1}R(t)\|_{l^2}^2 dt\right],$$

and $\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*}$ is the usual Itô stochastic integral with respect to the cilindrical Brownian motion W associated to B. In this case, we may formally write

$$\tilde{B}_t = \sum_{j=1}^{\infty} \tilde{\beta}_t^j e_j,$$

where $\tilde{\beta}_t^j := \beta_t^j - \int_0^t \gamma_s^j ds$ is a sequence of $\mathcal{Q}_{T^*}$-real valued independent fBm.

Recall that all economic activity is assumed to take place on the finite horizon $[0,T^*]$. Let us fix $k > 0$ which corresponds to arbitrary small proportional transaction costs in the bond market. The main result of this section is then the following.

Theorem 3.1. *Assume that the volatility satisfies assumptions in Lemma 3.2 and there exists an l^2-valued measurable function γ_t satisfying assumptions in Lemma 3.3 in such way that*

$$\sigma_t \gamma_t = \mathcal{S}_H \sigma_t - \alpha_t; \quad t \ge 0. \tag{3.12}$$

Then there exists a quasi-martingale measure for the bond market. In addition, the market is arbitrage-free on $[0,T^*]$ with transaction costs k.

Proof. The forward rate is the continuous mild solution of

$$r_t = (Ar_t + \alpha_t)dt + \sum_{j=1}^{d} \sigma_t^j d\beta_t^j,$$

under the measure $\mathbb{P}$. Assuming assumptions in Lemma 3.3 and (3.12), we may write

$$r_t = \big(Ar_t + \mathcal{S}_H \sigma_t\big)dt + \sum_{j=1}^{d} \sigma_t^j d\tilde{\beta}_t^j$$

under the equivalent probability measure $\mathcal{Q}$ with respect to $\mathbb{P}$ given in (3.11). By (3.7) one should notice that the above equation is well defined under $\mathcal{Q}$. By changing the measure $\mathbb{P}$ to $\mathcal{Q}$ in Corollary 3.1 it follows that

$$\mathbb{E}_{\mathcal{Q}} Z_t(T) = P(0,T); \quad \forall (t,T) \in \Delta^2_{T^*},$$

and therefore $\mathcal{Q}$ is a quasi-martingale measure. By Lemma 3.2 the two parameter process $(t,T) \mapsto \sum_{j=1}^{d} \int_0^t \mathcal{I}_{\sigma^j}(s,T) d\tilde{\beta}_s^j$ has $\mathcal{Q}$-full support and therefore $\log Z$ has $\mathcal{Q}$-full support as well. By Lemma 3.1 and Proposition 2.1 we conclude the proof. □

Remark 3.3. *One should notice that if there exists a quasi-martingale measure then it should be of the form (3.11), (3.12).*

The next result gives an explicit formula for the term structure of bond prices in terms of a conditional expectation. In the sequel, it will be convenient to express the fBm through the following representations (See [34]):

$$\tilde{\beta}_t^j = \alpha_H \int_0^t \int_r^t s^{H-1/2}(s-r)^{H-3/2} ds dM_r^j \tag{3.13}$$

$$= \alpha_H \int_0^t s^{H-1/2} \int_0^s (s-r)^{H-3/2} dM_r^j ds, \quad j = 1, \ldots, d \tag{3.14}$$

where the processes $M^j = (M_t^j)_{t\ge 0}$ are continuous martingales given by

$$M_t^j = c_1 \int_0^t s^{1/2-H}(t-s)^{1/2-H} d\tilde{\beta}_s^j,$$

where c_1 and α_H are normalizing constants. In the sequel, we denote

$$\theta^j(r,t) := \alpha_H \int_r^t \sigma^j(s,t) s^{H-1/2}(s-r)^{H-3/2} ds, \quad j = 1 \ldots, d$$

for $0 < r < t < \infty$. We also write $[M^j]$ to denote the usual quadratic variation of the martingale M^j.

Proposition 3.1. *Assume that $\mathcal{Q}$ is a quasi-martingale measure. Then the bond price can be expressed by*

$$P(t,T) = e^{\xi(t,T)} \mathbb{E}_{\mathcal{Q}} \left[\exp\left(- \int_t^T r_s(0) ds \right) \middle| \mathcal{F}_t \right],$$

where the kernel $\xi(t,T)$ is given by

$$\xi(t,T) := \int_0^t \Sigma_\sigma^d(s,T)ds - \sum_{j=1}^d \int_0^t \mathcal{I}_{\sigma^j}(s,T)d\tilde{\beta}_s^j - G(t,T),$$

where $G(t,T) := \sum_{j=1}^d \int_0^t \int_r^T \theta^j(r,u)dudM_r^j - \frac{1}{2}\sum_{j=1}^d \int_t^T \left(\int_r^T \theta^j(r,u)du\right)^2 d[M^j]_r$.

Proof. By the very definition

$$P(t,T) = e^{\xi(t,T)}\mathbb{E}_\mathcal{Q}\left[\exp\left(-\int_t^T r_s(0)ds\right)\middle|\mathcal{F}_t\right],$$

where

$$\xi(t,T) = \int_0^t \Sigma_\sigma^d(s,T)ds - \sum_{j=1}^d \int_0^t \mathcal{I}_{\sigma^j}(s,T)d\tilde{\beta}_s^j - \ln \mathbb{E}_\mathcal{Q}\left[\exp\left(-\sum_{j=1}^d \int_0^T \mathcal{I}_{\sigma^j}(s,T)d\tilde{\beta}_s^j\right)\middle|\mathcal{F}_t\right].$$

We only need to compute the above conditional expectation. By using representation (3.13) it follows that

$$\int_0^t \sigma^j(s,t)d\tilde{\beta}_s^j = \int_0^t \theta^j(r,t)dM_r^j.$$

By using conditions (2.8), (2.10) and (2.11) and changing the order of integration, we obtain

$$\int_0^T \mathcal{I}_{\sigma^j}(s,T)d\tilde{\beta}_s^j = \int_0^T \int_r^t \theta^j(r,u)dudM_r^j, \quad j=1,\dots,d$$

and therefore the conditional expectation can be written as

$$\sum_{j=1}^d \int_0^t \int_r^T \theta^j(r,u)dudM_r^j - \frac{1}{2}\sum_{j=1}^d \int_t^T \left(\int_r^T \theta^j(r,u)du\right)^2 d[M^j]_r.$$

□

We end this section by showing some examples of familiar short-rate models in the fBm setting as developed in this section under the quasi-martingale $\mathcal{Q}$.

Example 1 (Ho-Lee) Let us assume that $d=1$ and $\sigma_t(x)=\sigma$, a constant, for all $(t,x)\in\mathbb{R}_+^2$. Then in this case the model is k-arbitrage-free for every $k>0$ and the short-rate dynamics under $\mathcal{Q}$ is given by

$$r_t(0) = r_0(t) + \sigma^2 \int_0^t \int_0^s [2t-(s+\theta)]\phi_H(s-\theta)d\theta ds + \sigma\tilde{\beta}_t$$

and we recognize this as the Ho-Lee model with a deterministic time-varying drift.

Example 2 (Hull-White) Again assume $d = 1$ but now take $\sigma_t(x) = \sigma \exp(-\alpha x)$, where σ and α are positive constants. Straightforward integration imply

$$\begin{aligned} r_t(0) &= r_0(t) + \frac{\sigma^2}{\alpha} \int_0^t e^{-\alpha(t-s)} \int_0^s [1 - e^{-\alpha(t-\theta)}] \phi_H(s-\theta) d\theta ds \\ &+ \frac{\sigma^2}{\alpha} \int_0^t [1 - e^{-\alpha(t-s)}] \int_0^s e^{-\alpha(t-\theta)} \phi_H(s-\theta) d\theta ds + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{\beta}_s, \end{aligned}$$

which is consistent with the Hull-White model or the Vasicek model with time-varying drift parameters. Furthermore, in the fBm setting this model is k-arbitrage-free for every $k > 0$.

4. Consistency for Fractional HJM Models

In this section, we study consistency problems related to HJM models introduced in the previous section. We now assume that the following dynamics

$$dr^{x_0}(t) = \left(\frac{d}{dx} r(t) + \mathcal{S}_H \sigma(t) \right) dt + \sum_{j=1}^{\infty} \sigma^j d\tilde{\beta}^j(t), \quad r_0 = x_0 \in E, \tag{4.1}$$

(under a quasi-martingale measure $\mathcal{Q}$) induces an arbitrage-free bond market as described in the last section. We study the important case of time-homogeneous HJM models, in the sense that the volatility σ does not depend on time $t \in [0, T]$ where $0 < T < \infty$ is a fixed terminal time.

Let $\mathcal{P}$ be the interest rate model produced by r_t and let $\mathcal{M}$ be a parametrized family of smooth forward curves (e.g Nelson - Siegel or Svensson families). We recall that a pair $(\mathcal{P}, \mathcal{M})$ is *consistent* if all forward curves which may be produced by the interest rate model $\mathcal{P}$ are contained within the family $\mathcal{M}$, provided that the initial curve is in $\mathcal{M}$. There are several reasons, why in practice, one is interested in consistent pairs $(\mathcal{P}, \mathcal{M})$ with respect to HJM dynamics (see [6, 15]). In particular, the following questions are of great importance in calibrating interest rate models:

Given an interest rate model $\mathcal{P}$ and a family of forward rate curves $\mathcal{M}$, what are the necessary and sufficient conditions for consistency? Let $\mathcal{M}$ be an exponential-polynomial family of smooth forward curves. Is there a nontrivial $\mathcal{P}$ which is consistent to $\mathcal{M}$?

The remainder of this paper will be devoted to answer the above problems in the fBm setting.

4.1. Consistent pairs $(\mathcal{P}, \mathcal{M})$.

In this section, we give a fairly complete characterization of a given $\mathcal{M}$ to be consistent with respect to $\mathcal{P}$. We adopt an abstract framework by considering $\mathcal{M}$ as a finite dimensional smooth submanifold of E. The concept of invariance used in this work is the following:

Definition 4.1. *A closed set $K \subset E$ is said to be* ***invariant*** *for the forward rate $(r(t))_{0 \le t \le T}$ when*

$$\mathcal{Q}(r^{x_0}(t) \in K, \forall t \in [0, T]) = 1, \quad \textit{for every } x_0 \in K.$$

Hence the pair $(\mathcal{P},\mathcal{M})$ is consistent if and only if $\mathcal{M}$ is invariant for the correspondent forward rate r_t. The main difficulty in characterizing consistent pairs $(\mathcal{P},\mathcal{M})$ is the obtention of the topological support for the law of $\{r(t); 0\le t\le T\}$ on the space $\mathcal{C}(0,T;E)$ of the E-valued continuous functions on $[0,T]$. We recall that the topological support supp μ of a probability measure μ on a Polish space $\mathcal{X}$ is the smallest closed set in $\mathcal{X}$ with total mass. Recall that the forward rate satisfies the following equation

$$r(t) = S(t)x_0 + \int_0^t S(t-s)\mathcal{S}_H\sigma(s)ds + \int_0^t S(t-s)\sigma d\tilde{B}(s)$$

where $\{S(t), t\ge 0\}$ is the right-shift semigroup acting on the Hilbert space E and $\tilde{B}$ is a cilindrical fBm (under $\mathcal{Q}$) on a Hilbert space U. Let us denote

$$Z(t) := \int_0^t S(t-s)\sigma d\tilde{B}(s), \tag{4.2}$$

and

$$J_d(t) := \sum_{j=1}^{d}\int_0^t S(t-s)\sigma^j d\tilde{\beta}^j(s), \quad 1\le d<\infty. \tag{4.3}$$

One should notice that $\mathbb{E}\sup_{0\le t\le T}\|Z(t)-J_d(t)\|_E \to 0$ as $d\to\infty$.

Our main task is to characterize supp $\mathcal{Q}_Z$ on the space $\mathcal{C}(0,T;E)$. For this purpose, we take advantage of the fact that fBm is a centered Gaussian process. The theory of Gaussian processes provides a sharp characterization for the support of the measure $\mathcal{Q}_Z$. A direct (but lengthy) calculation shows that the law of $Z(\cdot)$ in $L^2(0,T;E)$ is a symmetric Gaussian measure whose covariance operator is given by

$$\Lambda_H\varphi(t) := \int_0^T g_H(t,s)\varphi(s)ds,$$

where

$$g_H(t,s) := \int_0^{s\wedge t}\int_0^{s\wedge t} S(t-v)\sigma\sigma^* S^*(s-u)\phi_H(u-v)dudv.$$

By condition (2.9) it follows that $\mathcal{Q}_Z$ is concentrated on $\mathcal{C}_0=\{u\in\mathcal{C}([0,T];E): u(0)=0\}$. Therefore, the closure of Image $\Lambda_H^{1/2}$ in the $\mathcal{C}_0-$topology is the support of $\mathcal{Q}_Z$. This fact would lead to a straightforward characterization of supp $\mathcal{Q}_Z$ as long as we know how to calculate the square root of the covariance operator Λ_H. In fact, a direct calculation proves to be very hard. Moreover, it is not trivial to find a bounded linear operator $\mathcal{A}$ such that $\Lambda_H=\mathcal{A}\mathcal{A}^*$. See Corollary B.4 in [8]. Therefore other non-direct techniques should be applied.

In the sequel, we consider the Wiener space $(\mathcal{W},\mathbb{H},\mathbf{P})$ of the $\mathbb{R}^d$-valued fBm, where $\mathcal{W}$ is the space of the $\mathbb{R}^d$-valued continuous functions f on $[0,T]$ such that $f(0)=0$, $\mathbb{H}$ is the correspondent Cameron-Martin space and $\mathbf{P}$ is the Wiener measure on $\mathcal{W}$. The set $\mathbb{H}$ is equal to Image $\mathcal{K}$ as a vector space, where $\mathcal{K}$ is the operator defined in (3.9), and the respective inner product is given by

$$\langle \mathcal{K}h, \mathcal{K}g\rangle_{\mathbb{H}} := \langle h, g\rangle_{L^2}; \quad h, g \in L^2(0,T;\mathbb{R}^d).$$

We have the following sufficient conditions for inclusions of the support of the law of an abstract Wiener functional $\mathcal{V} : \mathcal{W} \rightarrow \mathcal{X}$. See Aida et al [1] for the details.

Proposition 4.1. *Let $\mathcal{V} : \mathcal{W} \rightarrow \mathcal{X}$ be a measurable map, where $\mathcal{X}$ is a separable Banach space.*

(i) Let $\zeta_1 : \mathbb{H} \rightarrow \mathcal{X}$ be a measurable map, and let $\mathcal{J}_n : \mathcal{W} \rightarrow \mathbb{H}$ be a sequence of random elements such that for any $\varepsilon > 0$,

$$\lim_n \mathbf{P}\Bigg(\|\mathcal{V} - \zeta_1 \circ \mathcal{J}_n\|_{\mathcal{X}} > \varepsilon\Bigg) = 0. \tag{4.4}$$

Then

$$supp\ \mathbf{P}_{\mathcal{V}} \subset \overline{\zeta_1(\mathbb{H})}.$$

(ii) Let $\zeta_2 : \mathbb{H} \rightarrow \mathcal{X}$ be a map, and for each fixed $h \in \mathbb{H}$ let $\mathcal{T}^h_n : \mathcal{W} \rightarrow \mathcal{W}$ be a sequence of measurable transformations such that $\mathbf{P}_{\mathcal{T}^h_n} \ll \mathbf{P}$ for every n, and for any $\varepsilon > 0$,

$$\limsup_n \mathbf{P}\Bigg(\|\mathcal{V} \circ \mathcal{T}^h_n - \zeta_2(h)\|_{\mathcal{X}} < \varepsilon\Bigg) > 0. \tag{4.5}$$

Then supp $\mathbf{P}_{\mathcal{V}} \supset \overline{\zeta_2(\mathbb{H})}$.

The remainder of this section will be devoted to characterize the topological support of the forward rate $r : \Omega \rightarrow \mathcal{C}(0,T;E)$ by using conditions (4.4) and (4.5). Clearly, we only need to analyze the support of the probability measure $\mathcal{Q}_Z$. The strategy is to characterize invariant sets for HJM models via a controlled deterministic equation associated to (4.1). In the sequel, we write $\mathbf{E}$ to denote the expectation with respect to $\mathbf{P}$.

4.2. **Invariance for HJM Models.** We now introduce a polygonal approximation for the fBm. Let us recall the Volterra representation of the fBm

$$\beta(t) = \int_0^t K(t,s)dW(s), \tag{4.6}$$

where W is the unique Wiener process that provides the integral representation (4.6) and $K(t,s)$ is the kernel defined in (2.1).

Remark 4.1. *From the above representation we notice that W is adapted to the filtration generated by the fBm β and both processes generate the same filtration.*

Let $\Pi = 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$ where $t_k := k\frac{T}{n}$ and $|\Pi| := \max_{0\le j\le n-1}(t_{j+1} - t_j) = \frac{T}{n}$. Let us consider the following polygonal approximations

$$\beta_\Pi(t) := \int_0^t K(t,s)dW_\Pi(s) = \int_0^t K(t,s)\dot{W}_\Pi(s)ds \tag{4.7}$$

where

$$W_\Pi(t) := W(t_j) + \frac{W(t_{j+1}) - W(t_j)}{(t_{j+1} - t_j)}(t - t_j),$$

for $t_j \le t \le t_{j+1}$; $j = 0, 1, \dots n-1$.

One can check (see [25]) that $\forall \gamma < 1-H$ there exists a constant $C_{H,\gamma}$ independent of Π such that

$$\mathbf{E} \sup_{0\le t\le T} |\beta_\Pi(t) - \beta(t)| \le C_{H,\gamma} |\Pi|^\gamma. \tag{4.8}$$

If $\omega \in \mathcal{W}$ and $|\Pi| = T/n$ then we define $\omega^{(n)}(t) = \big(\omega_1^{(n)}(t), \dots, \omega_d^{(n)}(t)\big)$ where

$$\omega_i^{(n)}(t) := \int_0^t K(t,s)\dot{W}_{\Pi,i}(\omega)(s)ds, \quad 1 \le i \le d.$$

Obviously $\omega^{(n)} \in \mathbb{H}$ for all $n \ge 1$ and $\omega \in \mathcal{W}$. For each $h \in \mathbb{H}$ we define

$$\mathcal{T}_n^h \omega := \omega + (h - \omega^{(n)}). \tag{4.9}$$

Lemma 4.1. *If $h \in \mathbb{H}$ then $\mathbf{P}_{\mathcal{T}_n^h} \ll \mathbf{P}$ for all $n \ge 1$.*

Proof. Let us consider $h = \mathcal{K}\gamma$ and $J_h^{(n)}(\omega) := \mathcal{K}\gamma - \omega^{(n)}$ for $\omega \in \mathcal{W}$ and $\gamma \in \mathrm{L}^2(0,T;\mathbb{R}^d)$. By definition of W_Π it follows that

$$\int_0^t K(t,s)\dot{W}_\Pi(s)ds = \sum_{i=0}^{n-1} \int_{t_i\wedge t}^{t_{i+1}\wedge t} K(t,s)\dot{W}_\Pi(s)ds$$

and therefore $J_h^{(n)}$ is adapted to the internal filtration generated by the Brownian motion W. By the Novikov condition

$$\mathbf{E}\left[1/2 \exp\left(\int_0^t |\gamma(s) - \dot{W}(s)|^2 ds\right)\right] < \infty$$

and the representation (4.6), Girsanov theorem (see [9]) for the fBm yields

$$\mathbf{P}_{\mathcal{T}_n^h} \sim \mathbf{P}.$$

□

The following result is crucial to get (4.4) in Proposition 4.1. In the sequel, we write $(\Psi \cdot \beta)$ and $(\Psi \cdot \beta_\Pi)$ to denote the Paley-Wiener integrals with respect to β and β_Π, respectively.

Proposition 4.2. *Let β_Π be the polygonal approximation of the real-valued fBm. If $\Psi \in L^2(0,T;E)$ then*

$$\lim_{\|\Pi|\to 0} \mathbf{E} \sup_{0\le t\le T} \|(\Psi\cdot\beta)(t) - (\Psi\cdot\beta_\Pi)(t)\|_E = 0$$

Proof. We proceed by approximating Ψ by step functions f. Assume that

$$f(s) = \sum_{i=0}^{n-1} \alpha_i \chi_{[s_i, s_{i+1})(s)}; \quad 0 = s_0 < s_1 < \ldots s_n = T,$$

and consider the operator $\theta_H := I_{0+}^{H-1/2} \circ D_{0+}^{H+1/2}$ defined on $I_{0+}^{H+1/2}\big(L^2(0,T;\mathbb{R})\big)$.

By the semigroup property of fractional integrals and taking into account that $D_{0+}^{H+1/2}$ is the inverse $I_{0+}^{H+1/2}$ it follows that

$$\begin{aligned}
\|(f\cdot\beta)(t) - (f\cdot\beta_\Pi)(t)\|_E &= \left\| \sum_{i=0}^{n-1} \alpha_i \Big[\big(\beta(t_{i+1\wedge t}) - \beta(t_i \wedge t)\big) - \int_{t_i\wedge t}^{t_{i+1}\wedge t} \theta_H \beta_\Pi(s) ds \Big] \right\|_E \\
&\le \sum_{i=0}^{n-1} \|\alpha_i\|_E \Big| \big(\beta(t_{i+1\wedge t}) - \beta(t_i \wedge t)\big) - \big(\beta_\Pi(t_{i+1}\wedge t) - \beta_\Pi(t_i \wedge t)\big) \Big|
\end{aligned}$$

By the estimate (4.8) we conclude that the assertion is true for step functions. Now let us consider $\Psi \in L^2(0,T;E)$ and a sequence $(f_n)_{n\ge 1}$ of step functions which converges to Ψ in $L^2(0,T;E)$. We have

$$\begin{aligned}
\sup_{0\le t\le T} \|(\Psi\cdot\beta)(t) - (\Psi\cdot\beta_\Pi)(t)\|_E &\le \sup_{0\le t\le T} \|(\Psi\cdot\beta)(t) - (f_n\cdot\beta)(t)\|_E \\
&+ \sup_{0\le t\le T} \|(f_n\cdot\beta)(t) - (f_n\cdot\beta_\Pi)(t)\|_E \\
&+ \sup_{0\le t\le T} \|(f_n\cdot\beta_\Pi)(t) - (\Psi\cdot\beta_\Pi)(t)\|_E \\
&= T_1(n) + T_2(n,\Pi) + T_3(n,\Pi).
\end{aligned}$$

By the first step we only need to estimate T_1 and T_3. Hölder inequality yields

$$T_3(n,\Pi) \le \|f_n - \Psi\|_{L^2(0,T;E)} \|\theta_H \beta_\Pi\|_{L^2(0,T;\mathbb{R})} < \infty \quad a.s, \tag{4.10}$$

where we observe that $\|\theta_H\beta_\Pi\|_{L^2(0,T;\mathbb{R})}$ is square integrable for all partition Π. In fact, we can rewrite (4.7) in the following way

$$\beta_\Pi(t) = c_H I_{0+}^1 \left(u^{H-1/2} I_{0+}^{H-1/2}\big(u^{1/2-H}\dot{W}_\Pi\big)\right)(t),$$

and therefore

$$\theta_H\beta_\Pi(t) = I_{0+}^{H-1/2} D_{0+}^{H+1/2} \beta_\Pi(t) = c_H t^{H-1/2} I_{0+}^{H-1/2}(u^{1/2-H}\dot{W}_\Pi)(t), \quad 0 \le t \le T.$$

Then $\mathbf{E}\|\theta_H\beta_\Pi\|^2_{L^2(0,T;\mathbb{R})} < \infty$ and therefore we can conclude that for each partition Π

$$\lim_{n\to\infty} \mathbf{E} T_3(n,\Pi) = 0.$$

It remains to estimate T_1. For this we shall use the factorization method on the fractional Wiener integral. Recall the identity

$$\frac{\pi}{\sin \pi\alpha} = \int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha}ds; \quad \sigma \le s \le t, \quad 0<\alpha<1. \tag{4.11}$$

Fix $0<\alpha<1/2$ and $p>1/2\alpha$. By using (4.11) and the stochastic Fubini theorem for fractional Wiener integrals ([27]) we may write

$$((\Psi - f_n)\cdot\beta)(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t (t-s)^{\alpha-1} y_m(s)ds,$$

where $y_m(s) := \int_0^s (\Psi - f_m)(\sigma)(s-\sigma)^{-\alpha}d\beta(\sigma)$. Hölder inequality yields

$$\sup_{0\le t\le T}\|((\Psi - f_n)\cdot\beta)(t)\|_E^{2p} \le C_1\int_0^T \|y_m(s)\|_E^{2p}ds$$

where the constant C_1 depends only on p,α and T. We now choose $p=1$. The ordinary Fubini theorem and the isometry of the fractional Wiener integral with the reproducing kernel Hilbert space of the fBm (see [12, 3]) yields the following estimate

$$\begin{aligned}
\mathbf{E}T_1^2(n) &\le C_1\int_0^T \mathbf{E}\|y_m(s)\|_E^2 ds\\
&= C_1\int_0^T\int_0^s\int_0^s \left\langle (\Psi - f_n)(u)(s-u)^{-\alpha}, (\Psi - f_n)(v)(s-v)^{-\alpha}\right\rangle_E\\
&\times \phi_H(u-v)dudvds.
\end{aligned}$$

By estimate (11) in [3] we can find positive constants C_2 and C_3 such that

$$\begin{aligned}
\mathbf{E}T_1^2(n) &\le C_2\int_0^T\int_0^s \|(\Psi - f_n)(u)(s-u)^{-\alpha}\|_E^2 duds\\
&\le C_3\|\Psi - f_n\|_{L^2(0,T;E)}.
\end{aligned}$$

Summing up all the estimates we complete the proof of the proposition. □

In the sequel, with a slight abuse of notation we write $\theta_H = I_{0+}^{H-1/2}\circ D_{0+}^{H+1/2}$ defined on $I_{0+}^{H+1/2}\big(L^2(0,T;\mathcal{X})\big)$ where $\mathcal{X}$ can be $\mathbb{R}$ or the Hilbert space U, depending on the context. In accordance with Proposition 4.1, we are now in position to define the following mappings

$$\zeta_1^d h(t) := \sum_{i=1}^d \int_0^t S(t-s)\sigma^i\theta_H h_i(s)ds, \quad h\in\mathbb{H}, \tag{4.12}$$

$$\mathcal{J}_n(\omega) := \omega^{(n)}, \quad \omega\in\mathcal{W}, \tag{4.13}$$

$$\zeta_1(t)g := \int_0^t S(t-s)\sigma\theta_H g(s)ds, \quad g \in I_{0+}^{H+1/2}(L^2(0,T;U)). \tag{4.14}$$

Proposition 4.3. *The support of the Wiener functional Z in (4.2) is given by*

$$\overline{\zeta_1(I_{0+}^{H+1/2}(L^2(0,T;U)))}$$

Proof. For each fixed $d \geq 1$ we apply Proposition 4.1 to the Wiener functional J_d defined in (4.3) with the correspondent transformations ζ_1^d, $\mathcal{J}_n$ and $\mathcal{T}_n^h$, defined in (4.12), (4.13) and (4.9), respectively. Conditions (4.4) and (4.5) in Proposition 4.1 are direct consequences of Proposition 4.2 and Lemma 4.1. We then have the following characterization

$$\text{supp } \mathcal{Q}_{J_d} = \zeta_1^d\overline{(I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R}^d))},$$

where $I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R}^d)))$ is equal (as a vector space) to the Cameron-Martin space $\mathbb{H}$.

We now consider the full sequence of independent fBm $\{\tilde{\beta}_n; n \geq 1\}$. At first, since U is separable one should note that we have the following orthogonal Hilbertian sum

$$I_{0+}^{H+1/2}(L^2(0,T;U)) \equiv \bigoplus_{i=1}^{\infty} I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R})). \tag{4.15}$$

To shorten notation we set $\mathcal{O}_d := \zeta_1^d(I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R}^d)))$. Obviously the following inclusions hold

$$\mathcal{O}_d \subset \mathcal{O}_{d+1}, \quad \text{for all } d \geq 1,$$

and therefore $\lim_{d\to\infty} \text{supp } \mathcal{Q}_{J_d} = \bigcup_{i=1}^{\infty} \overline{\mathcal{O}}_d$. On the other hand, we can approximate the stochastic convolution Z in probability uniformly in $[0,T]$ as follows

$$Z(t) = \int_0^t S(t-s)\sigma d\tilde{B}(s) = \lim_{d\to\infty} J_d(t).$$

Therefore we have

$$\text{supp } \mathcal{Q}_Z = \overline{\bigcup_{n=1}^{\infty}} \overline{\mathcal{O}}_n.$$

By the relation (4.15) we can conclude that supp $\mathcal{Q}_Z = \overline{\zeta_1(I_{0+}^{H+1/2}(L^2(0,T;U)))}$. □

We say that a closed set K in E is invariant for the evolution equation

(4.16)
$$\frac{d}{dt}y^{(x_0,u)}(t) = Ay^{(x_0,u)}(t) + \mathcal{S}_H\sigma(t) + \sigma I_{0+}^{H-1/2}u(t), \quad y(0) = x_0 \in E, \ u \in L^2([0,T];U)$$

if for each initial condition $x_0 \in K$ and a control $u \in L^2([0,T];U)$ we have

$$y^{(x_0,u)}(t) \in K; \quad \text{for all } t \in [0,T].$$

Theorem 4.1. *A closed set is invariant for the differential equation (4.16) if and only if it is invariant for the HJM equation (4.1). In particular,*

$$supp\ \mathcal{Q}_{r^{x_0}} = \overline{\{y^{(x_0,u)}; u \in L^2(0,T;U)\}}.$$

Proof. The hard part of the proof is the obtention of the support of the stochastic convolution $Z(t) = \int_0^t S(t-s)\sigma d\tilde{B}(s)$. We know from Proposition 4.3 that the law of Z is concentrated on the set of continuous functions of the form

$$\int_0^t S(t-s)\sigma I_{0+}^{H-1/2} h(s) ds, \quad h \in L^2([0,T];U).$$

Then the proof follows the same lines of [30] and therefore we omit the details. □

4.3. **Nagumo conditions and finite-dimensional invariant manifolds.** In this section we prove the main result of this section. We recall that if $\mathcal{M}$ is a C^1-manifold in E then the associated tangent space at $x \in \mathcal{M}$ may be written as

$$T_x\mathcal{M} = \{g \in E; \liminf_{t\downarrow 0} \frac{1}{t} dist[x+tg, \mathcal{M}] = 0\}; \quad \text{if } x \in \mathcal{M}, \tag{4.17}$$

where $dist[y,\mathcal{M}]$ denotes the distance between $y \in E$ and the set $\mathcal{M}$. We now provide Nagumo-type conditions for an HJM model to be invariant with respect to a given smooth manifold.

Proposition 4.4. *Let $\mathcal{M}$ be a C^1-submanifold in E, closed as a set and $\mathcal{M} \subset$ Dom (A). Then $\mathcal{M}$ is invariant for the stochastic equation (4.1) if and only if*

$$Ax + \mathcal{S}_H\sigma(t) + \sigma\nu \in T_x\mathcal{M}, \tag{4.18}$$

for each $x \in \mathcal{M}$, $t \in [0,T]$ and $\nu \in U$

Proof. Let $\mathcal{E}$ be the set of U-valued piecewise constant functions. We claim that a closed set K is invariant for equation (4.16) if and only if its mild solution satisfies the following condition: For each $x \in K$ and $v \in \mathcal{E}$ we have $y^{(x,v)}(t) \in K$ for all $t \in [0,T]$. We fix an arbitrary $u \in L^2(0,T;U)$ and let us consider a sequence of step functions u_n converging to u in $L^2(0,T;U)$. Then

$$\|y^{(x,u_n)}(t) - y^{(x,u)}(t)\|_E \leq \sup_{0\leq t\leq T} \| \int_0^t S(t-s)\sigma I_{0+}^{H-1/2}(u_n-u)(s)ds\|_E. \tag{4.19}$$

By Hölder inequality we have

$$\sup_{0\leq t\leq T} \| \int_0^t S(t-s)\sigma I_{0+}^{H-1/2}(u_n-u)(s)ds\|_E \leq C_1\Big(\int_0^T \|\sigma(u_n-u)(r)\|_E^2 dr\Big)^{1/2}, \tag{4.20}$$

where C_1 is a positive constant which depends on T and H. Since σ is bounded we then have inequalities (4.19) and (4.20) imply that a closed set K is invariant for (4.16) if and only if $y^{(x,v)}(t) \in K$, $t \in [0,T]$ for all $x \in K$ and all piecewise constant $U-$valued function v. Thus proving our first claim.

Now let $\mathcal{M} \subset E$ be a closed C^1- submanifold where $\mathcal{M} \subset \mathrm{Dom}(A)$. By Theorem 2 in Jachimiak ([26]) and Theorem 4.1, we know that $\mathcal{M}$ is invariant with respect to equation (4.1) if and only if, for each $x \in \mathcal{M}$, $t \in [0,T]$ and $v \in \mathcal{E}$

$$\liminf_{\alpha \downarrow 0} \frac{1}{\alpha} dist[S(\alpha)(x) + \alpha(\mathcal{S}_H \sigma(t) + I_{0+}^{H-\frac{1}{2}} \sigma v(t)), \mathcal{M}] = 0.$$

By assumption $\mathcal{M}$ is contained in the domain of A and therefore the above condition can be replaced by

$$\liminf_{\alpha \downarrow 0} \frac{1}{\alpha} dist[(x + \alpha(Ax + \mathcal{S}_H \sigma(t) + I_{0+}^{H-\frac{1}{2}} \sigma v(t)), \mathcal{M}] = 0 \tag{4.21}$$

Since U is linear and σ is a bounded linear operator on U, the condition (4.21) is equivalent to

$$\liminf_{\alpha \downarrow 0} \frac{1}{\alpha} dist[(x + \alpha(Ax + \mathcal{S}_H \sigma(t) + \sigma\nu), \mathcal{M}] = 0,$$

for each $x \in \mathcal{M}$, $t \in [0,T]$ and $\nu \in U$. □

We end this section with the characterization of a given finite- dimensional invariant submanifold. In fact, by using Proposition 4.4 the proof of the following results are minor modifications of the arguments given in [30].

Lemma 4.2. *Let $\mathcal{M} \subset E$ be a finite-dimensional C^1-submanifold and closed as a set. If $\mathcal{M}$ is invariant for (4.1) then every $r^{x_0}(t)$ mild solution of equation (4.1) is also a strong solution for every $x_0 \in \mathcal{M}$. In particular, $\mathcal{M} \subset$ Dom (A).*

Proof. Let $a \in \mathrm{Dom}\ A^*$ where A^* denotes the adjoint of A. By using a stochastic Fubini theorem for the fractional Brownian motion (see ([27]) we obtain for $t \in [0,T]$

$$\langle a, r^{x_0}(t)\rangle = \langle a, x_0\rangle + \int_0^t \langle A^* a, r^{x_0}(s)\rangle ds + \int_0^t \langle a, \mathcal{S}_H \sigma(s))\rangle ds + \int_0^t \langle a, \sigma d\tilde{B}(s)\rangle, \quad a.s.$$

Now we may apply the same arguments as in Lemma 2.3 in [30] to show that if $\mathcal{M}$ satisfies the above assumptions and it is invariant for the stochastic equation (4.1) then $\mathcal{M} \subset \mathrm{Dom}\ (A)$ and therefore

$$\mathcal{Q}(r^{x_0}(t) \in \mathrm{Dom}\ (A), \forall t \in [0,T]) = 1, \quad \text{for every } x_0 \in \mathcal{M}.$$

This concludes the proof. □

Now we are in position to prove the main result of this section. In the sequel, we write $\mathrm{Im}\ \sigma := \sigma U$.

Theorem 4.2. *Let $\mathcal{M}$ be a finite-dimensional C^1- submanifold in E (closed as a set). Then $\mathcal{M}$ is invariant for an HJM model given by (4.1) if and only if $\mathcal{M} \subset$ Dom (A), and*

$$Ax \in T_x\mathcal{M}, \tag{4.22}$$

$$\mathcal{S}_H \sigma(t) + Im\ \sigma \subset T_x\mathcal{M}, \tag{4.23}$$

for every $t \in [0,T]$ and $x \in \mathcal{M}$.

Proof. Similar to the proof of Lemma 4.2 one can show that the above conditions imply that every mild solution of the equation (4.16) is also a strong solution which is given by

$$y^{(x,h)}(t) = x + \int_0^t Ay^{(x,h)}(s)ds + \int_0^t \mathcal{S}_H\sigma(s)(s)ds + \int_0^t \sigma I_{0+}^{H-1/2}h(s)ds,$$

for $h \in L^2(0,T;U)$. Therefore differentiating the above expression at $t = 0$, we conclude that $Ax \in T_x\mathcal{M}$ for every $x \in \mathcal{M}$. Proposition 4.4 implies that for each $x \in \mathcal{M}$, we have $\mathcal{S}_H\sigma(t) + \sigma\nu \in T_x\mathcal{M}$ for every $t \in [0,T]$ and $\nu \in U$. Conversely, let $x \in \mathcal{M}$, $v \in U$ and $t \in [0,T]$. By Proposition 4.4 it is sufficient to check (4.18). But this is a straightforward calculation using the parametrizations in $\mathcal{M}$ □

Next we examine concrete short rate models and smooth finite-dimensional manifolds, in particular, Ho-Lee and Hull-White models together with the well-known Nelson-Siegel family [31] are investigated.

5. Nelson-Siegel family

In this section, we are interested in investigating the Nelson-Siegel [31] exponential family $\mathcal{M} = \{F(\cdot,y); y \in \mathcal{Y}\}$ widely used to fit term structure of interest rates. The form of the curve is given by the following expression

$$F(x,y) = y_1 + y_2e^{-y_4x} + y_3xe^{-y_4x}, \quad x \geq 0, \tag{5.1}$$

where we restrict the parameters to the following state space $\mathcal{Y} := \{y = (y_1,\ldots,y_4) \in \mathbb{R}^4 | y_4 \neq 0\}$. Obviously, if $\eta = F(\cdot,y)$ is a generic element of $\mathcal{M}$, then the tangent space of $\mathcal{M}$ at the point η is given by $T_\eta\mathcal{M} = Im\ F_y(\cdot,y)$, where $F_y(\cdot,y)$ is the Frechet derivative at point $y \in \mathcal{Y}$ with respect to y. Therefore, the tangency conditions (4.22) and (4.23) can be written as

$$F_x(\cdot,y) \in Im\ F_y(\cdot,y) \tag{5.2}$$

$$\mathcal{S}_H\sigma(t) + Im\ \sigma \subset Im\ F_y(\cdot,y) \tag{5.3}$$

for every $(t,y) \in [0,T] \times \mathcal{Y}$, where $F_x(\cdot,y)$ is the derivative of $F(\cdot,y)$ with respect to the variable x for each fixed $y \in \mathcal{Y}$. Straightforward computation yields

$$F_x(x,y) = (y_3 - y_2y_4 - y_3y_4x)e^{-y_4x}, \tag{5.4}$$

$$F_y(x,y) = [1, e^{-y_4x}, xe^{-y_4x}, -(y_2 + y_3x)xe^{-y_4x}]. \tag{5.5}$$

As an example, we now study simple interest rate models. By using Theorem 4.2 (in particular relations (4.22) and (4.23)), the calculations are minor modifications from [6] so we just sketch the details.

5.1. **Ho-Lee model.** Let us consider the one-factor model under the Ho-Lee volatility structure given by a constant volatility, i.e., $\sigma_t(x) = \sigma$, for all $(t,x) \in \mathbb{R}^2_+$. In this case, the drift restriction is given by

$$\mathcal{S}_H\sigma(t,x) = \sigma^2\big(\varrho^1_H(t,x) + \varrho^2_H(t)\big), \quad (t,x) \in \mathbb{R}^2_+; \tag{5.6}$$

where $\varrho^1_H(t,x) = 2x\int_0^t \phi_H(t-\theta)d\theta$ and $\varrho^2_H(t) = t\int_0^t \phi_H(t-\theta)d\theta - \int_0^t \theta\phi_H(t-\theta)d\theta$. One should notice that relation (5.2) is satisfied but because of the term $\varrho^1_H(t,x)$ in (5.6), relation (5.3) is not possible and therefore by Theorem 4.2 we conclude that the Ho-Lee model is not consistent with the Nelson-Siegel family. On the other hand, by restricting the state space we may obtain consistency as follows. Let us consider the degenerate Nelson-Siegel family: We take $\mathcal{Y}^0 = \{y = (y_1,\ldots,y_4) \in \mathbb{R}^4 | y_2 = y_4 = 0\}$ and in this case

$$F(x,y) = y_1 + y_3 x, \quad (x,y) \in \mathbb{R}_+ \times \mathcal{Y}^0.$$

Moreover, one can easily check invariance of the Ho-Lee model with general affine manifolds.

5.2. **Hull-White model.** Let us consider the one-factor model under the Hull-White volatility structure given by $\sigma_t(x) = \sigma e^{-\alpha x}$ where α and σ are positive constants. In this case, the drift restriction is given by

(5.7)
$$\mathcal{S}_H\sigma(t,x) = \frac{\sigma^2}{\alpha}\, e^{-\alpha x}\left[\int_0^t \Big(1+e^{-\alpha(t-\theta)}\Big)\phi_H(t-\theta)d\theta\right] - \frac{2\sigma^2}{\alpha}\, e^{-2\alpha x}\int_0^t e^{-\alpha(t-\theta)}\phi_H(t-\theta)d\theta,$$

for each $(t,x) \in \mathbb{R}^2_+$. By considering the full state space $\mathcal{Y}$, clearly the Hull-White model cannot be consistent with the Nelson-Siegel family. By restricting the state space to $\mathcal{Y}^\alpha = \{y = (y_1,\ldots,y_4) | y_4 = \alpha\}$, the curve shape is then given by

$$F(x,y) = y_1 + y_2 e^{-\alpha x} + y_3 x e^{-\alpha x}.$$

Due to the second term in (5.7), the fractional Hull-White model is not consistent with the Nelson-Siegel family on $\mathcal{Y}^\alpha$. So an alternative is to consider the following curve shape

$$F(x,y) = y_1 e^{-\alpha x} + y_2 e^{-2\alpha x}. \tag{5.8}$$

Due to the term $e^{-2\alpha x}$ we now notice that the family given in (5.8) is consistent with the fractional Hull-White model. In fact, the following result is not surprising in view of the previous examples.

Proposition 5.1. *There is no nontrivial fractional interest rate model with deterministic volatility which is consistent with the Nelson-Siegel family.*

Proof. By using Theorem 4.2 the proof is analogous to [6; Proposition 7.1]. □

Using the same ideas, we could also study consistency of concrete multi-factor HJM models by checking relations (5.2) and (5.3). See the works [15, 16, 5]. We conclude this paper by discussing possible extensions of our results. The first extension is a direct consequence of the results of this paper. The other two extensions are less obvious and require further investigation.

5.3. **Mixed noises and multiple scales.** One can consider the case where the HJM equation is driven by several independent fBms with different values of the Hurst parameter $\tilde{H} = (H^1, \ldots, H^d) \in (1/2, 1)^d$:

$$dr(t) = \Big(\frac{d}{dx}r(t) + \mathcal{S}_{\tilde{H}}\sigma(t)\Big)dt + \sum_{i=1}^{d}\sigma^i d\beta^i_{H^i}(t).$$

It can be seen that in this case, there is no conceptual obstruction to the use of the method of the proof of this work. In this case, the proof follows the same lines as before since we are able to perform a Wong-Zakai approximation for the above equation. In fact, the strategy used in this paper works for fairly general continuous Gaussian process.

5.4. **Fractional gaussian noises with jumps.** Alternatively, one can consider the case where the fractional HJM is perturbed by a jump process as follows

$$dr_t = \Big(\frac{d}{dx}r_t + \mathcal{S}_H\sigma_t\Big)dt + \sum_{i=1}^{d}\sigma^i d\beta^i_t + \delta(t,y)m(dt,dy),$$

where $m(dt, dy)$ is a marked point process. In this case, $\log Z$ is no longer a continuous process and therefore the full support no-arbitrage argument given in Lemma 3.1 does not work. Further investigation is needed on the relation between càdlàg (right-continuous with left-limits) price processes and condition (2.18). In the continuous case, the (conditional) full support for $\log Z$ has a prominent role in determining absence of arbitrage. In the càdlàg case, new topological conditions have to be derived to ensure no-arbitrage via Proposition 2.1. See the works [20, 21, 22] for more details in the case of stock markets.

5.5. **Fractional HJM models with multiplicative noise in Skorohod sense.** One can naturally ask if the results of this paper can be extended to multiplicative noise as follows

$$dr_t = \Big(\frac{d}{dx}r_t + \mathcal{S}_H\sigma(r_t)\Big)dt + \sum_{i=1}^{d}\sigma^i_t(r_t)d\beta^i_t.$$

The stochastic integral appearing above is considered in the Skorohod sense. In this case, as already commented in the Introduction, it will require using subsequent work in the stochastic analysis of the fBm. Technical difficulties arise in the Picard iteration of the above equation when the noise enters in a nonlinear way. At our knowledge we only know how to solve such equations when the noise enters linearly which becomes unsuitable for financial applications (see e.g [13, 32]).

Moreover, assuming the existence of solution, more refined probabilistic estimates are required. In fact, by performing Itô formula in (2.13) it will appear additional random terms in the SDE of $\log Z_t(T)$ which involve the Gross-Sobolev derivatives of $\sigma^i(r_\cdot)$ and $\mathcal{S}_H\sigma(r_\cdot)$. Here $\mathcal{S}_H\sigma : E \to E$ must be a suitable Lipschitz mapping chosen in such way that $\mathbb{E}_{\mathbb{Q}}Z_t(T) = P(0,T)$ for every $(t,T) \in \Delta^2$ and $\log Z$ admits full support. In this case, a mild sufficient condition on the volatility structure $(\sigma^i)_{i=1}^{\infty}$ to produce full support for $\log Z$ is not clear in this context. In fact, more refined properties on the support of Skorohod integrals plus bounded variation processes should be developed in this context.

APPENDIX

6. Integration for $C(\mathbb{R}_+;\mathbb{R})-$valued process

In this section we introduce a suitable integral to deal with bond markets driven by fBm. Let $\big(\Omega, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}\big)$ be a stochastic basis where the filtration $(\mathcal{F}_t)_{t\ge o}$ satisfies the usual hypotheses. We denote $\mathcal{M}(\mathbb{R}_+)$ the space of (finite) signed measures on $\mathbb{R}_+$ with the total variation topology $\|\cdot\|_{TV}$. We also write $C_0(\mathbb{R}_+;\mathbb{R})$ the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ converging to zero at infinity. For $m \in \mathcal{M}(\mathbb{R}_+)$ and $l \in C_0(\mathbb{R}_+;\mathbb{R})$ we put

$$lm := \int l(\theta)m(d\theta) \tag{6.1}$$

Let us consider elementary measure - valued processes of the following form

$$\mu_t(\omega,\cdot) := \sum_{i=0}^{N-1} \chi_{F_i}(\omega)\chi_{(t_i,t_{i+1}]}(t)m_i \tag{6.2}$$

where $m_i \in \mathcal{M}(\mathbb{R}_+)$, $0 = t_0 < \ldots < T_N < \infty$ and $F_i \in \mathcal{F}_{t_i}$. We assume that the support of μ is concentrated on $[t,\infty)$ for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$. We denote by $\mathcal{S}_b$ the set of elementary processes of the form (6.2). We endow $\mathcal{S}_b$ with the following norm

$$\|\mu\|^2_{\mathrm{V}} := \mathbb{E} \sup_{0\le t<\infty} \|\mu_t\|^2_{TV} \tag{6.3}$$

The class of integrators will be $C_0(\mathbb{R}_+;\mathbb{R})$- valued stochastic processes satisfying the following hypothesis.

Assumption (A1). Let $\{G(t,T);(t,T)\in\mathbb{R}^2_+\}$ be a jointly continuous real-valued stochastic process such that

$$\mathbb{E} \sup_{(t,T)\in\mathbb{R}^2} |G(t,T)|^2 < \infty.$$

If $\mu \in \mathcal{S}_b$ and G satisfies (A1) then we define

$$\int_0^t \mu_s dG_s := \sum_{i=0}^{N-1} \chi_{F_i}\Big(G_{t_{i+1}\wedge t} - G_{t_i\wedge t}\Big)m_i.$$

By Hölder inequality it follows that

$$\mathbb{E} \sup_{0\le t<\infty} \Big|\int_o^t \mu_s dG_s\Big| \le \|\mu\|_{\mathrm{V}} \mathbb{E}^{1/2} \sup_{0\le s,t<\infty} \|G_s - G_t\|^2_\infty < \infty, \tag{6.4}$$

where $\|\cdot\|_\infty$ denotes the usual (uniform topology) norm on the space $C(\mathbb{R}_+;\mathbb{R})$. Let V be the completion of $\mathcal{S}_b$ with respect to (6.3). By the estimate (6.4) and the definition of V we may easily define $\int_0^\cdot \mu_s dG_s$ for all $\mu \in$ V. Next we present some elementary technical results.

Lemma 6.1. *Fix* $0 < T^* < \infty$ *and consider* $t_i^n := \frac{iT^*}{2^n}$ *for* $i = 0, 1, \ldots, 2^n; \quad n \geq 1$. *Then if* $\mu \in \mathrm{V}$ *and* G *satisfies assumption A1 then*

$$lim_{n\to\infty}\mathbb{E}\Bigg\|\sum_{i=0}^{2^n-1}\mu_{t_i^n}(G_{t_{i+1}^n\wedge\cdot}-G_{t_i^n\wedge\cdot})-\int_0^{\cdot}\mu_s dG_s\Bigg\|_{\infty}=0$$

Proof. Straightforward estimates. □

Next we fix $0 < T^* < \infty$ and consider

$$M_i^{(n)}(T) := \sup_{t_i^n \leq t \leq t_{i+1}^n} G(t,T); \quad T \geq 0,$$

$$m_i^{(n)}(T) := \inf_{t_i^n \leq t \leq t_{i+1}^n} G(t,T); \quad T \geq 0,$$

where $t_i^n := \frac{iT^*}{2^n}$ for $i = 0, 1, \ldots, 2^n; \quad n \geq 1$. With these objects we then define

$$\overline{G}_n(s) := \sum_{i=0}^{2^n-1} \chi_{(t_i^n, t_{i+1}^n]}(s) M_i^{(n)},$$

$$\underline{G}_n(s) := \sum_{i=0}^{2^n-1} \chi_{(t_i^n, t_{i+1}^n]}(s) m_i^{(n)},$$

and

$$\int_0^t \overline{G}_n(s) d\mu_s := \sum_{i=0}^{2^n-1} M_i^{(n)} (\mu_{t_{i+1}^n \wedge t} - \mu_{t_i^n \wedge t}),$$

$$\int_0^t \underline{G}_n(s) d\mu_s := \sum_{i=0}^{2^n-1} m_i^{(n)} (\mu_{t_{i+1}^n \wedge t} - \mu_{t_i^n \wedge t}).$$

We denote $\mathcal{P}_{T^*}$ the set of all partitions of $[0, T^*]$. In the sequel we consider the following assumption:

Assumption (A2).

$$\Pi_{T^*}(\mu) := \sup_{\pi \in \mathcal{P}_{T^*}} \sum_{t_i \in \pi} \|\mu_{t_{i+1}} - \mu_{t_i}\|_{TV} \quad \text{is square integrable.}$$

Lemma 6.2. *Assume that* $\mu \in \mathrm{V}$ *where (A2) holds and consider* G *a stochastic process such that (A1) holds. Then*

(a) $\lim_{n\to\infty} \mathbb{E} \sup_{0 \leq t \leq T^*} \left| \int_0^t \overline{G}_n d\mu - \int_0^t \underline{G}_n d\mu \right| = 0,$

(b) $\lim_{n,m\to\infty} \mathbb{E} \sup_{0 \leq t \leq T^*} \left| \int_0^t \overline{G}_n d\mu - \int_0^t \overline{G}_m d\mu \right| = 0.$

Proof. We notice that

$$\begin{aligned}\left|\int_0^t \overline{G}_n d\mu - \int_0^t \underline{G}_n d\mu\right| &\le \sum_{i=0}^{2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty \|\mu_{t_{i+1}^n\wedge t} - \mu_{t_i^n\wedge t}\|_{TV} \\ &\le \max_{i;i=0,\dots,2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty \Pi(\mu) \quad a.s,\ 0\le t\le T^*.\end{aligned}$$

By continuity $\lim_{n\to\infty} \max_{i;i=0,\dots,2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty = 0 \quad a.s.$ Moreover, there exists a constant C such that

$$\max_{i;i=0,\dots,2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty \le C \sup_{0\le s,T\le\infty} |G(s,T)| \quad a.s\ \forall n\ge 1.$$

By assumptions (A1-A2) and the dominated convergence theorem we conclude part (a). Similarly, $\sup_{0\le s\le T^*, T\ge 0} |\overline{G}_n(s;T) - \overline{G}_m(s;T)|$ goes to zero a.s as $n,m\to\infty$. Moreover, it is bounded by $C\sup_{0\le s,T<\infty}|G(s,T)|$. Again, by assumptions (A1-A2) and dominated convergence theorem we conclude part (b). □

By Lemma 6.2 we shall define

$$\int_0^t G_s d\mu_s := \lim_{n\to\infty}\int_0^t \overline{G}_n(s)d\mu_s = \lim_{n\to\infty}\int_0^t \underline{G}_n(s)d\mu_s.$$

The next result is a straightforward integration by part formula.

Proposition 6.1. *Assume that assumptions (A1) and (A2) hold. Then*

$$\int_0^{T^*} G_s d\mu_s + \int_0^{T^*} \mu_s dG_s = G_{T^*}\mu_{T^*} - G_0\mu_0 \tag{6.5}$$

Proof. By writing a telescoping sum we have

$$\begin{aligned}\sum_{i=0}^{2^n-1} (G_{t_{i+1}^n} - G_{t_i^n})(\mu_{t_{i+1}^n} - \mu_{t_i^n}) &= G_{T^*}\mu_{T^*} - G_0\mu_0 \\ &- \sum_{i=0}^{2^n-1}(G_{t_{i+1}^n} - G_{t_i^n})\mu_{t_i^n} - \sum_{i=0}^{2^n-1}(\mu_{t_{i+1}^n} - \mu_{t_i^n})G_{t_i^n},\end{aligned}$$

a.s for all $n\ge 1$. By Lemma 6.1 and Lemma 6.2 we only need to show that the left-side goes to zero as $n\to\infty$. But this is an immediate consequence of hypotheses (A1) and (A2) together with the continuity of G. □

References

[1] Aida, S., Kusuoka, S. and Stroock, D.(1993). On the Support of Wiener Functionals. *Asymptotic problems in probability theory: Wiener functionals and asymptotics.* Longman Sci. and Tech., Pitman Research Notes in Math. Series **294**, N.Y., 3 - 34.

[2] Alòs. E., Mazet, O. and Nualart, D. (2001). Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29**, 766-801.

[3] Alòs, E., and Nualart, D. (2003). Stochastic integration with respect to the fractional Brownian motion. *Stochastics. Stochastics. Rep.* **75**, 129-152.

[4] Backus, D. and Zin, S. (1993). Long memory inflation uncertainty: evidence of term structure of interest rate. *J. Money, Cred. Bank.* **25**, 687-700.

[5] Björk, T., Di Masi, G., Kabanov, Y Runggaldier, W. (1997). Towards a general theory of bond markets. *Finance Stoch.* **1**, 141-174.

[6] Björk, T. and Christensen, B. (1999). Interest rate dynamics and consistent forward rate curves. *Math. Finance.* **9**, 323-348.
[7] Cajueiro, D. and Tabak, B. (2006). Testing for Long-Range Dependence in the Brazilian Term Structure of Interest Rates. *XXX Brazilian Econometric Metting.*
[8] Da Prato, G., and Zabczyk, J. (1992). *Stochastic Equations in Infinite Dimensions.* Cambridge Univ. Press.
[9] Decreusefond. L., and Ustunel. A.S. (1998). Stochastic analysis of the fractional Brownian motion. *Potential Analysis.* **10**, 177-214.
[10] Delbaen F., and Schachermayer, W. (1994). A General Version of the Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, **300**, 463-520.
[11] De Donno, M. and Pratelli, M (2007). A theory os stochastic integration for bond markets. *Ann. Appl. Probab.* **15**, 2773-2791.
[12] Duncan, T. E., Maslowski, B., Pasik-Duncan, B. (2002). Fractional Brownian motions and stochastic equations in Hilbert spaces. *Stoch. Dyn.* **2**, 225-250.
[13] Duncan, T. E., Maslowski, B., Pasik-Duncan, B. (2005). Stochastic equations in Hilbert spaces with a multiplicative fractional gaussian noise. *Stoch. Proc. Appl.* **115**, 1357-1383.
[14] Filipovic, D. (1999). A note on the Nelson-Siegel family. *Math. Finance* **9**, 349-359.
[15] Filipovic, D. (2001). Consistency problems for Heath-Jarrow-Morton interest rate models. *Lecture Notes in Math.* **1760**, Springer-Verlag.
[16] Filipovic, D.(2000). Consistency Problems for HJM Interest Rate Models. Thesis. ETH.
[17] Filipovic, D. Invariant manifolds for weak solutions to stochastic equations. *Probab. Theory Relat. Fields.* **118**, 323-341.
[18] Filipovic, D., and Teichmann. J. (2003). Existence of invariant manifolds for stochastic equations in infinite dimension. *J. Func. Anal.* **197**, 398-432.
[19] Filipovic, D., and Teichmann. J. (2004). On the geometry of the term structure of interest rates. *Proc. R. Soc. Lond. A.* **460**, 129-167.
[20] Guasoni, P. (2006). No Arbitrage under Transaction Costs with Fractional Brownin Motion and Beyond. *Math. Finance* **16**, 569-582.
[21] Guasoni, P., Rásonyi, M., and Schachermayer, W. (2007). Consistent Price Systems and Face-Lifting Pricing under Transaction Costs. Forthcoming in *Ann. App.Probab.*
[22] Guasoni, P., Rásonyi, M., and Schachermayer, W. (2007). The Fundamental Theorem of Asset Pricing for Continuous Process under Small Transaction Costs. *Preprint.*
[23] Hairer, M., and Ohashi, A (2007). Ergodic theory for SDEs with extrinsic memory. *Ann. Probab.* **35**, 1950-1977.
[24] Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new metodology for contingent claims valuation. *Econometrica.* **60**, 77-105.
[25] Hu, Y. (2005). Integral transformations and anticipative calculus for fractional Brownian motions. *Mem. Amer. Math. Soc.* **175**, no 825.
[26] Jashimiak, W. (1997). A Note on Invariance for Semilinear Differential Equations. *Bull. Polish Acad. Sciences* **45**.
[27] Krvavich, Y.V., and Mishura, Y.S. (2001). Differentiability of fractional integrals whose kernels contain fractional Brownian motions. *Ukranian Math. J.* **53**, 35-47.
[28] Mccarthy, J., DiSario, R., Saraoglu, H. and Li, H. (2004). Tests of long-range dependence of interest rates using wavelets. *Quarterly R. Economics. Finance.* **44**, 180-189.
[29] Musiela, M. (1993). Stochastic PDEs and term structures models. *Journées Internationales de Finance.* IGR-AFFI, La Baule.
[30] Nakayama, T. (2004). Viability Theorem for SPDEs Including HJM Framework. *J. Math. Sci. Univ. Tokyo.* **11**, 313-324.
[31] Nelson, C., and Siegel, A. Parsimonious modeling of yield curves. *J. Business.* **60**, 473-489.
[32] Nourdin, I., and Tudor, C.A (2006). Some linear fractional stochastic equations. *Stochastics* **78**, 51-65.
[33] D. Nualart. (2003). Stochastic calculus with respect to the fractional Brownian motion and applications. *Contemporary Mathematics.* 336, 3-39.
[34] Norros, I., Valkeila, E., and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli.* **5**, 571-587.
[35] Russo, F. and Vallois, P. (2000). Stochastic claculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.* **70**, 1-40.

[36] Samko, S.G., Kilbas, A.A., and Marichev,O.I. (1993). *Fractional Integrals and Derivatives.* Gordon and Breach Science. Yverdon.

[37] Samorodnitsky, G., and Taqqu, M,S. *Stable non-Gaussian Random processes.* Chapman and Hall. New York.

[38] Shiryaev, A. (1999). *Essentials of Stochastic Finance.* World Scientific.

[39] Tsay, W. (2000). Long memory story of the real interest rate. *Economic Letters.* **67**, 325-330.

[40] Zabczyk, J. (2000). Stochastic invariance and consistency of financial models. *Rend. Mat. Acc. Lincei.* **11**, 67-80.

Departamento de Matemática. Universidade Estadual de Campinas - UNICAMP, 13.081-97-Campinas, SP, Brazil

E-mail address: ohashi@ime.unicamp.br