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Dessislava H. Kochloukova

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ESTATÍSTICA E COMPUTAÇÃO CIENTÍFICA

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Modules of type FP_2 over the integral group algebra of a metabelian group

Dessislava H. Kochloukova*
UNICAMP-IMECC,Cx. P. 6065,
13083-970 Campinas, SP, Brazil
desi@ime.unicamp.br

Abstract

We demonstrate a sufficient condition for some modules M over the group algebra $\mathbb{Z}[G]$ to be of homological type FP_2, where G is a finitely generated split extension of abelian groups. This generalises a result of Bieri-Strebel when M is the trivial module $\mathbb{Z}$ [7] and is a special case of a conjecture suggested in [13, Conj. 7].

1 Introduction

In this paper we consider a result about the homological type of some special (left) modules M over the group algebra $\mathbb{Z}[G]$, where $G = N \rtimes Q$ is a finitely generated group, a split extension of abelian groups N and Q. We assume that N acts trivially on M. Our main result is the existence of a sufficient condition (in terms of the Bieri-Strebel invariant Σ introduced in [7]) for M to be of homological type FP_2 over $\mathbb{Z}[G]$.

Theorem 1 *Let Q be a finitely generated abelian group, M, N be finitely generated (left) $\mathbb{Z}[Q]$-modules. Let $G = N \rtimes Q$ be the split extension of N by Q, where the action of Q on N via (left) conjugation is the original action of Q on N. Suppose that G is finitely presented and*

$$0 \notin \mathbb{R}_{>0}\Sigma^c_M(Q) + conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q)).$$

Then M is of type FP_2 over $\mathbb{Z}[G]$, where G acts on M via the canonical projection $G \to Q$.

Theorem 1 is a particular case of a more general conjecture classifying modules M of type FP_m [13, Conj. 7]. This conjecture is a generalization of the FP_m- Conjecture for metabelian groups [2].

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Conjecture [13, Conj. 7] *Let Q be a finitely generated abelian group, M, N be finitely generated (left) $\mathbb{Z}[Q]$-modules and $G = N \rtimes Q$ be a split extension of N by Q, where the action of Q on N via (left) conjugation is the original action of Q on N. Then M is of type FP_m over $\mathbb{Z}[G]$, where G acts on M via the canonical projection $G \to Q$ if and only if $0 \notin \mathbb{R}_{>0}\Sigma^c_M(Q) + conv_{\leq m}(\mathbb{R}_{>0}\Sigma^c_N(Q))$ and $0 \notin conv_{\leq m}(\mathbb{R}_{>0}\Sigma^c_N(Q))$.*

In Theorem 1 and in the above Conjecture $\mathbb{R}_{>0}\Sigma^c_M(Q)$ and $\mathbb{R}_{>0}\Sigma^c_N(Q)$ are subsets of $\mathbb{R}^n$ invariant under multiplication with elements from $\mathbb{R}_{>0}$, where n is the torsion-free rank of Q and for a subset T of $\mathbb{R}^n$, $conv_{\leq m}T$ stands for the union of all convex hulls of subsets of T of at most m elements. There is a homological conjecture, the FP_m-Conjecture, that suggests a classification for the finitely generated metabelian groups of type FP_m. The FP_m-Conjecture states that if G is finitely generated and an extension of abelian groups N by Q then G is of type FP_m if and only if $0 \notin conv_{\leq m}(\mathbb{R}_{>0}\Sigma^c_N(Q))$. The FP_m-Conjecture holds for $m = 2$ [7] and in this case a bit more is true, a finitely generated metabelian group G is of type FP_2 if and only if G is finitely presented.

In the case when M and N have finite prime exponent p as abelian groups the sufficient condition from Theorem 1 is necessary [13, Prop. 29]. We strongly believe that in general the sufficient condition from Theorem 1 is necessary.

Corollary 1 *Let Q be a finitely generated abelian group, M, N be finitely generated (left) $\mathbb{Z}[Q]$-modules of prime exponent p and $G = N \rtimes Q$ be the split extension of N by Q, where the action of Q on N via (left) conjugation is the original action of Q on N. Then M has homological type FP_2 as a $\mathbb{Z}[G]$-module, where G acts on M via the canonical projection $G \to Q$ if and only if $0 \notin \mathbb{R}_{>0}\Sigma^c_M(Q) + conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q))$ and $0 \notin conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q))$.*

We note that there is a monoid version, the Σ^m-Conjecture, of the FP_m-Conjecture that suggests a formula for the calculation of the higher geometric invariants of metabelian groups. The Σ^m-Conjecture for finitely generated metabelian groups G has been proved for $m = 2$ [10], [9] and gives that in the split extension case i.e. $G = N \rtimes Q$ where N and Q are abelian groups

$$conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q)) = \mathbb{R}_{>0}(\Sigma^2(N \rtimes Q, \mathbb{Z})^c \mid_Q),$$

where $\Sigma^m(G, \mathbb{Z})^c = S(G) \backslash \Sigma^m(G, \mathbb{Z})$ is the complement of the homological invariant $\Sigma^m(G, \mathbb{Z})$ in $S(G)$. We note as well that by the main result of [12] we have that in general for a group G with a normal locally polycyclic subgroup H for every real character χ of G such that $\chi(H) \neq 0$ we have $[\chi] \in \cap_{m \geq 1}\Sigma^m(G, \mathbb{Z})$. Then Theorem 1 can be reformulated as follows.

Corollary 2 *Let Q be a finitely generated abelian group, M, N be finitely generated (left) $\mathbb{Z}[Q]$-modules, $G = N \rtimes Q$ be the split extension of N by Q, where the action of Q on N via (left) conjugation is the original action of Q on N and G be finitely presented. Suppose $0 \notin \mathbb{R}_{>0}\Sigma^1(M \rtimes Q, \mathbb{Z})^c + \mathbb{R}_{>0}\Sigma^2(N \rtimes Q, \mathbb{Z})^c$. Then M is of type FP_2 over $\mathbb{Z}[G]$, where G acts on M via the canonical projection $G \to Q$.*

Proof. Note that $\Sigma^1(M \rtimes Q, \mathbb{Z})^c$ has only elements $[\chi]$ such that $\chi \in Hom(M \rtimes Q, \mathbb{R}) \backslash \{0\}$ and $\chi(M) = 0$. For $\chi \in Hom(M \rtimes Q, \mathbb{R}) \setminus \{0\}$ with $\chi(M) = 0$ we have $[\chi] \in \Sigma^1(M \rtimes Q, \mathbb{Z})^c$ if and only if $[\chi \mid_Q] \in \Sigma^c_M(Q)$. Thus Corollary 2 is equivalent to Corollary 1. □

The proof of Theorem 1 is quite long and technical. It generalises some new ideas introduced in [4], where the FP_3-Conjecture for metabelian groups in the split extension case is proved. We have tried to explain the similarities and differences between the methods used in this paper and in [4] in section 5.1. Section 5.1 is self contained and is included to help the reader who has already read [4]. It is interesting to note that the modules M we study are in general non-trivial and the Bieri-Harlander's result is about the homological property FP_3 of the trivial module $\mathbb{Z}$ over a metabelian group. The improvement from trivial to non-trivial module is done at the expense of going one dimension down i.e. we study the property FP_2 of (in general) non-trivial module M instead of the FP_3 property of the trivial module $\mathbb{Z}$.

2 Preliminaries on the homological invariant Σ^m

In this section we revise known results about the homological invariant Σ^m and at the end we prove a technical proposition that will be needed later in the paper.

Let M be a finitely generated left $\mathbb{Z}[Q]$-module, where Q is a finitely generated abelian group. By definition

$$S(Q) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in \mathrm{Hom}\,(Q, \mathbb{R}) \setminus \{0\}\},$$

$$Q_\chi = \{q \in Q \mid \chi(q) \geq 0\}$$

and

$$\Sigma_M(Q) = \{[\chi] \in S(Q) \mid M \text{ is finitely generated over } \mathbb{Z}[Q_\chi]\}.$$

The complement of $\Sigma_M(Q)$ in $S(Q)$ is denoted by $\Sigma_M^c(Q)$. By definition for an element λ of the group algebra $\mathbb{Z}[Q]$ the support $supp(\lambda)$ is the finite set of elements of Q that appear in λ.

Lemma 1 *([5]) Suppose M is a finitely generated (left) $\mathbb{Z}[Q]$-module. Then there exists a finite subset Λ_M of the centraliser of M in $\mathbb{Z}[Q]$ and some positive real number ν_M such that for every $[\mu] \in \Sigma_M(Q)$ there is an element λ in Λ_M with*

$$min\{\mu(q) \mid q \in supp\ \lambda\} > \nu_M.$$

Note that $\mathrm{Hom}\,(Q, \mathbb{R})$ and $Q \otimes_{\mathbb{Z}} \mathbb{R}$ can be identified with $\mathbb{R}^n$, where n is the torsion-free rank of Q. We write the product of elements in Q inside the group algebra $\mathbb{Z}[Q]$ multiplicatively, but when we work only with elements in Q we write the operation additively. Thus we identify $Q/tor(Q)$ with the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$.

Lemma 2 *([7, Lemma 1.1]) Let Λ_M be the set and ν_M be the positive real number given by Lemma 1. Then there is a positive integer $\rho_1(M)$ such that for $x \in \mathbb{R}^n$ with $\mid x \mid \geq \rho_1(M)$, $\frac{x}{|x|} \in -\Sigma_M(Q)$ there is $\lambda \in \Lambda_M$ such that $x + supp\ \lambda$ is a subset of the open ball with centre the origin and radius $\mid x \mid - \frac{\nu_M}{2}$.*

The invariant $\Sigma_M(Q)$ is a particular case of a more general homological invariant defined in [6]. Let H be a finitely generated group and V be a finitely generated (left) $\mathbb{Z}[H]$-module. The invariant $\Sigma^m(H, V)$ is defined as

$$\{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in \mathrm{Hom}\,(H, \mathbb{R}) \setminus \{0\}, V \text{ has homological type } FP_m \text{ over } \mathbb{Z}[H_\chi]\},$$

where $H_\chi = \{h \in H \mid \chi(h) \geq 0\}$. Note that $\Sigma_M(Q) = \Sigma^0(Q, M)$.

Following [6] we consider

$$\mathcal{F} : \ldots \to F_i \to F_{i-1} \to \ldots \to F_0 \to V \to 0$$

a free resolution of V over $\mathbb{Z}[H]$. Let X_i be a basis of F_i as $\mathbb{Z}[H]$-module and we set $X = \cup_{i\geq 0} X_i$. We further assume that $\mathcal{F}$ is admissible in the sense of [6] i.e. for every $x \in X$ we have $\partial(x) \neq 0$. Define the function support with respect to X (denoted by $supp_X$) for elements of $\mathcal{F}$ as in [6, p. 472] i.e. it is function which associates to every element of $\cup_i F_i$ a finite subset of the group H with the following properties:

$$supp_X(0) = \emptyset;$$

$$\text{for } \lambda = \sum_{z_{h,x}\in\mathbb{Z}\setminus\{0\}, h\in H, x\in X_i} z_{h,x}hx \text{ by definition } supp_X(\lambda) = \cup_{z_{h,x}\in\mathbb{Z}\setminus\{0\}} supp_X(hx);$$

$$\text{for } i \geq 1, x \in X_i \text{ and } h \in H \text{ by definition } supp_X(hx) = supp_X(\partial(hx));$$

$$\text{for } x \in X_0 \text{ and } h \in H \text{ by definition } supp_X(hx) = \{h\}.$$

The way the support function $supp_X$ is defined it depends on X. But it is easy to note that for two bases X and $\tilde{X}$ such that

$$X_0 = \tilde{X}_0, HX_i = H\tilde{X}_i \text{ for every } i \geq 1$$

we have

$$supp_X = supp_{\tilde{X}}.$$

For every $\chi \in \mathrm{Hom}\,(H, \mathbb{R}) \setminus \{0\}$ there is a valuation v_χ [6, p. 470, section 2.2] extending χ and depending on the choice of X

$$v_\chi : \cup_{i\geq 0} F_i \to \mathbb{R}_\infty$$

with the following properties:

1. $v_\chi(hf) = \chi(h) + v_\chi(f)$ for every $h \in H, f \in \cup_{i\geq 0} F_i$;
2. $v_\chi(x) = v_\chi(\partial(x))$ for $x \in \cup_{i\geq 1} X_i$ and $v_\chi(x) = 0$ for $x \in X_0$;
3. $v_\chi(\sum_{z_{h,x}\in\mathbb{Z}\setminus\{0\}, h\in H, x\in X_i} z_{h,x}hx) = min_{z_{h,x}\mathbb{Z}\setminus\{0\}}\{v_\chi(hx)\}$ for every $i \geq 0$;
4. $v_\chi(f_1 + f_2) \geq min\{v_\chi(f_1), v_\chi(f_2)\}$ for $i \geq 0$ and $f_1, f_2 \in F_i$;
5. $v_\chi(\partial f) \geq v_\chi(f)$ for every $f \in \cup_{i\geq 0} F_i$;

6. $v_\chi(f) = \infty$ if and only if $f = 0$.

Theorem 2 *[6, Thm 4.7] Let*

$$\mathcal{F} : \ldots \to F_i \to F_{i-1} \to \ldots \to F_0$$

be an admissible free deleted resolution of V *over* $\mathbb{Z}[H]$ *with finitely generated* $m -$ *skeleton i.e.* F_i *has a basis* X_i *such that* $0 \notin \partial(X_i)$ *for every* $i \geq 0$ *and for* $i \leq m$ *the set* X_i *is finite. Then the following three conditions are equivalent for a compact subset* Γ *of*

$$S(H) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in Hom(H, \mathbb{R}) \setminus \{0\}\} :$$

1. $\Gamma \subseteq \Sigma^m(H, V)$*;*

2. there is a finite set ψ *of chain monomorphisms* $\varphi : \mathcal{F} \to \mathcal{F}$*, lifting the identity map* id_V *of* V*, with the property that for each point* $[\chi] \in \Gamma$ *there is some* $\varphi \in \psi$ *with*

$$v_\chi(\varphi(x)) > v_\chi(x) \text{ for every } x \in X^{(m)} = \cup_{i \leq m} X_i,$$

where v_χ *is the valuation extending* χ *defined for the fixed basis* $X = \cup_{i \geq 0} X_i$*;*

3. after replacing $\mathcal{F}$ *by a suitable admissible free resolution, obtained by performing on* $\mathcal{F}$ *a finite sequence of elementary expansions, we can find a set* ψ *as in 2. and for each* $\varphi \in \psi$ *there is a chain homotopy* $\sigma_\varphi : \varphi \simeq Id_{\mathcal{F}}$ *with* $\sigma_\varphi(X_i) \subseteq X_{i+1} \cup \{0\}$ *for every* $0 \leq i \leq m$*.*

From now on we assume that H is of the form $H_1 \rtimes Q$, where Q is a free abelian group of rank n (we will apply later this construction for $H_1 = N$ and $H_1 = 1$). We view Q as the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Let $\pi : H \to Q$ be the canonical map. Define $| h | = | \pi(h) |$, where $| \ |$ is the standard Euclidean norm in $\mathbb{R}^n$ and denote by $(\ , \ )$ the standard inner product in $\mathbb{R}^n$.

Now let Γ be a compact subset of $\Sigma^m(H, V)$ with the additional property that for every $[\mu] \in \Gamma$ we have that $\operatorname{Ker} \pi = H_1 \subseteq \operatorname{Ker} \mu$. An element $u \in \Gamma$ is in fact $u = [\mu]$ for some $\mu \in \operatorname{Hom}(H, \mathbb{R}) \setminus \{0\}$ and we write $u \mid_Q$ for $[\mu \mid_Q] \in S(Q)$. By definition for a subset Γ of $\{[\mu] \in S(H) \mid \mu(H_1) = 0\}$

$$\Gamma_Q = \{[\mu \mid_Q] \mid [\mu] \in \Gamma\}.$$

We fix some admissible free resolution $\mathcal{F}$ with finitely generated $m -$ skeleton of the left $\mathbb{Z}[H]$-module V and a finite set ψ given by part 3 of Theorem 2. Following [6, section 5.5] define for $u \in \Gamma$

$$\rho(u) = max_{\varphi \in \psi} min_{x \in X^{(m)}} (v_u(\varphi(x)) - v_u(x)),$$

where in order to define v_u we think of u as an element of $Hom(H, \mathbb{R}) \setminus \{0\}$ such that $u(H_1) = 0$ and the image of u in $Hom(Q, \mathbb{R}) \simeq \mathbb{R}^n$ lies on S^{n-1}. This defines a continuous and positive real function $\rho : \Gamma \to \mathbb{R}$ that attains its infimum since Γ is compact

$$r = inf\{\rho(u) \mid u \in \Gamma\} > 0.$$

Following [6, section 5.5] we put

$$s = max\{| \sigma_\varphi(y) |, \text{ for } y \in HX^{(m)}, 1_Q \in \pi(supp_X(y)), \varphi \in \psi\} > 0, \tag{1}$$

i.e. the maximum is over the finite set of chain homotopies σ_φ given by part 3 of Theorem 2 and the finite set of elements $\{y \mid y \in HX^{(m)}, 1_Q \in \pi(supp_X(y))\}$. The remark after [6, (5.11)] shows that s is well defined.

Proposition 1 *Let V be a (left) $\mathbb{Z}[H]$-module of type FP_m, Γ a non-empty compact subset of $\Sigma^m(H,V)$ with the additional property that for $[\mu] \in \Gamma$ we have that $H_1 = Ker \pi \subseteq Ker \mu$ and $\mathcal{F}$ be an admissible free resolution of V as in part 3 of Theorem 2. Then for every closed ball B in $\mathbb{R}^n$ with radius $\geq s^2/2r$ and for every $c \in F_m$ satisfying the following two conditions :*

1. $\pi(supp_X(c)) \subseteq B$, $\pi(supp_X(\partial c)) \subseteq B \setminus \partial B$.

2. *for v the centre of B and for every $h \in supp_X(c) \cap \pi^{-1}(\partial B \cap Q)$*

$$\frac{\pi(h) - v}{\mid \pi(h) - v \mid} \in -\Gamma \mid_Q \subseteq -\Gamma,$$

there exists $c_1 \in F_m$ such that $\pi(supp_X(c_1)) \subseteq B \setminus \partial B$ and $\partial c_1 = \partial c$.

Remarks 1. We view $\pi(h) - v$ as an element of $\mathbb{R}^n \simeq Hom(Q, \mathbb{R})$ where the isomorphism sends $r \in \mathbb{R}^n$ to the real character χ_r given by $\chi_r(q) = (r, q)$. Thus $\frac{\pi(h)-v}{|\pi(h)-v|} \in S^{n-1} \simeq S(Q)$.

2. Furthermore we view an element of the unit sphere $S^{n-1} \simeq S(Q)$ in $\mathbb{R}^n$ as element of the unit sphere in $\mathbb{R} \otimes_{\mathbb{Z}} (H/H')$ via the embedding of Q in the abelianization H/H' of H. This gives the embedding of $\Gamma \mid_Q$ in Γ.

Proof. The proof is a modification of the proof of [6, Lemma 5.3]. We induct on the number of elements of the finite set $\pi(supp_X(c)) \cap \partial B$. If this number is 0 then we define $c_1 = c$. If not we show that there exists $\tilde{c} \in F_m$ such that $\pi(supp_X(\tilde{c})) \subseteq B$, $\partial(c) = \partial(\tilde{c})$ and the set $\pi(supp_X \tilde{c}) \cap \partial B$ has less elements than the set $\pi(supp_X c) \cap \partial B$. Then the proof of the proposition is completed by induction.

We fix one element $g \in supp_X(c) \cap \pi^{-1}(\partial B \cap Q)$ and decompose

$$c = c' + c'' \text{ where } c = \sum_{y \in HX_m, n_y \in \mathbb{Z} \setminus \{0\}} n_y y,$$

c' collects all terms $n_y y$ such that $supp_X(y) \cap g\mathrm{Ker}\,\pi \neq \emptyset$.

By assumption

$$-\frac{\pi(g) - v}{\mid \pi(g) - v \mid} \in \Gamma \mid_Q \subseteq \Gamma \subseteq \Sigma^m(H, V).$$

From now on we write u for $-\frac{\pi(g)-v}{|\pi(g)-v|}$ as an element of $\Gamma \subseteq \Sigma^m(H,V) \subseteq S(H)$ and $u \mid_Q$ for $-\frac{\pi(g)-v}{|\pi(g)-v|}$ as an element of $\Gamma_Q \subseteq S(Q)$. Then there is a finite set ψ, an element $\varphi \in \psi$ and $\sigma_{\cdot\varphi}$ as in part 3 of Theorem 2. Furthermore we want φ to be an element of ψ such that

$$r \leq \rho(u) = min_{x \in X^{(m)}}(v_u(\varphi(x)) - v_u(x)). \tag{2}$$

From now on we write σ for $\sigma_{\cdot\varphi}$ and define

$$\tilde{c} = c + \partial\sigma(c') = \varphi(c') - \sigma(\partial c') + c''.$$

hence

$$\partial(\tilde{c}) = \partial(c) + \partial^2\sigma(c') = \partial(c).$$

By [6, Lemma 4.3] for every $f \in \cup_{i \leq m} F_i$ we have

$$v_u(\sigma(f)) \geq v_u(f)$$

and for an element y of HX_i such that y occurs in $\sigma(f)$ but not in f

$$v_u(y) \geq v_u(f) + \rho(u) \geq v_u(f) + r.$$

Note that

$$v_u(\varphi(c')) \geq v_u(c') + r \geq (\pi(g), u \mid_Q) + r,$$

where the first inequality is a corollary of (2) together with [6, Lemma 2.1], the second inequality follows by the fact that $\pi(supp(c')) \subseteq \pi(supp(c)) \subseteq B$. Furthermore

$$v_u(\sigma(\partial c')) \geq v_u(\partial c') = v_u(\partial c - \partial c'') \geq min\{v_u(\partial c), v_u(\partial c'')\} \geq min\{v_u(\partial c), v_u(c'')\},$$

where the first inequality follows from [6, Lemma 4.3]. Note that

$$\pi(supp_X(\partial c)) \subseteq B \setminus \partial B \text{ and } \pi(supp_X(c'')) \subseteq B \setminus \{\pi(g)\},$$

hence

$$min\{v_u(\partial c), v_u(c'')\} > (\pi(g), u \mid_Q).$$

Then

$$v_u(\tilde{c}) = v_u(\varphi(c') - \sigma(\partial c') + c'') \geq min\{v_u(\varphi(c')), v_u(\sigma(\partial c')), v_u(c'')\} >$$
$$min\{v_u(\varphi(c')), v_u(\partial c), v_u(c'')\} > (\pi(g), u \mid_Q)$$

and hence

$$\pi(g) \notin \pi(supp_X(\tilde{c})).$$

Furthermore

$$supp_X(\tilde{c}) \subseteq supp_X(c) \cup supp_X(\partial\sigma(c')),$$
$$supp_X(\partial\sigma(c')) \subseteq supp_X\sigma(c') \subseteq \pi^{-1}(\{t \in \mathbb{R}^n \mid (t, u \mid_Q) \geq (\pi(g), u \mid_Q) + r\} \cap Q) \cup supp_X(c')$$
$$\subseteq \pi^{-1}(\{t \in \mathbb{R}^n \mid (t, u \mid_Q) \geq (\pi(g), u \mid_Q) + r\} \cap Q) \cup supp_X(c),$$

where the last but one inclusion follows from the already discussed [6, Lemma 4.3]. Then

$$\pi(supp_X(\tilde{c}) \setminus supp_X(c)) \subseteq \pi(supp_X(\partial\sigma(c')) \setminus supp_X(c)) \subseteq$$
$$\{t \in \mathbb{R}^n \mid (t, u \mid_Q) \geq (\pi(g), u \mid_Q) + r\} \cap Q.$$

Note that by (1) we have that the length of the elements of $\pi(g^{-1}supp_X\sigma(c'))$ is at most s. Since $\pi(g^{-1}supp_X(\partial\sigma(c'))) \subseteq \pi(g^{-1}supp_X\sigma(c'))$ we deduce that the length of the elements of $\pi(g^{-1}supp_X(\partial\sigma(c')))$ is at most s. Then $\pi(supp_X(\partial\sigma(c')) \setminus supp_X(c))$ is contained in the range exhibited in the picture on [6, p. 489], hence $\pi(supp_X(\partial\sigma(c')) \setminus supp_X(c)) \subseteq B \setminus \partial B$. Then

$$\pi(supp_X(\tilde{c}) \setminus supp_X(c)) \subseteq \pi(supp_X(\partial\sigma(c')) \setminus supp_X(c)) \subseteq B \setminus \partial B.$$

Thus

$$\pi(supp_X(\bar{c})) \subseteq (\pi(supp_X(c)) \cup (B \setminus \partial B)) \setminus \{\pi(g)\}$$

and hence

$$\pi(supp_X(\bar{c})) \cap \partial B \subseteq (\pi(supp_X(c)) \cap \partial B) \setminus \{\pi(g)\}$$

i.e. $\pi(supp_X(\bar{c})) \cap \partial B$ has less elements than $\pi(supp_X(c)) \cap \partial B$. This completes the inductive step of the proof. □

3 A reduction of the main problem

From now on we work under the assumptions of Theorem 1. We start this section justifying why we can assume that Q is a free abelian group. In general Q has a subgroup of finite index that is free abelian. We note that finiteness properties of modules do not change if we consider restriction to subgroups of finite index (the case of the trivial module $\mathbb{Z}$ is proved in [8, VIII, Prop. 5.1] but the proof is quite general and works for any module). This together with the fact that the invariant $\Sigma_M(Q)$ depends only on the torsion-free part of Q shows we can assume that Q is free abelian.

Let $m_1, \ldots, m_k$ be a generating set of M over $\mathbb{Z}[Q]$, $\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i$ be a free $\mathbb{Z}[Q]$-module with basis $D_0 = \{d_1, \ldots, d_k\}$ and

$$\nu : \oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i \to M$$

be a homomorphism of $\mathbb{Z}[Q]$–modules that sends d_i to m_i. Let $\oplus_{1 \leq i \leq k} \mathbb{Z}[G] d_i$ be a free $\mathbb{Z}[G]$-module with basis $D_0 = \{d_1, \ldots, d_k\}$ and

$$\theta : \oplus_{1 \leq i \leq k} \mathbb{Z}[G] d_i \to M$$

be the homomorphism of $\mathbb{Z}[G]$-modules that sends d_i to m_i. We identify $\oplus_{1 \leq i \leq k} \mathbb{Z}[G] d_i$ with $\mathbb{Z}[N] \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)$ via the multiplication map $\mathbb{Z}[N] \times \mathbb{Z}[Q] \to \mathbb{Z}[G]$. Then the kernel of θ is generated as an abelian group by

$$\text{the image } \overline{Aug(\mathbb{Z}[N]) \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)} \text{ of } Aug(\mathbb{Z}[N]) \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)$$

and

$$\text{the image } \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker} \nu} \text{ of } \mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker} \nu$$

where both images are in $\mathbb{Z}[N] \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)$. From now on we write Ω for the augmentation ideal $Aug(\mathbb{Z}[N])$. As $\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i$ and the group ring $\mathbb{Z}[N]$ are free $\mathbb{Z}$-modules the canonical maps from $\Omega \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)$ and $\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker} \nu$ to $\mathbb{Z}[N] \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)$ are inclusions i.e.

$$\overline{\Omega \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)} \simeq \Omega \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i) \text{ and } \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker} \nu} \simeq \mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker} \nu.$$

Note that

$$\operatorname{Ker} \theta / \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker} \nu} \simeq (\Omega \otimes_{\mathbb{Z}} (\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i))/W \simeq \Omega \otimes_{\mathbb{Z}} ((\oplus_{1 \leq i \leq k} \mathbb{Z}[Q] d_i)/\operatorname{Ker} \nu) \simeq \Omega \otimes_{\mathbb{Z}} M,$$

where W is the subset of $\Omega \otimes_{\mathbb{Z}} (\oplus_{1\leq i\leq k}\mathbb{Z}[Q]d_i)$ whose image in $\mathbb{Z}[N] \otimes_{\mathbb{Z}} (\oplus_{1\leq i\leq k}\mathbb{Z}[Q]d_i)$ is $\overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} Ker \nu} \cap \overline{\Omega \otimes_{\mathbb{Z}} (\oplus_{1\leq i\leq k}\mathbb{Z}[Q]d_i)}$.

By the dimension shifting argument [1, Prop. 1.4] M is FP_2 over $\mathbb{Z}[G]$ if and only if $\operatorname{Ker}\theta$ is of type FP_1 (finitely presented) over $\mathbb{Z}[G]$. Now as M is finitely generated over $\mathbb{Z}[Q]$ and $\mathbb{Z}[Q]$ is Noetherian we deduce that $\operatorname{Ker}\nu$ is of type FP_∞ over $\mathbb{Z}[Q]$. Then the induced module

$$\uparrow_{\mathbb{Z}[Q]}^{\mathbb{Z}[G]} \operatorname{Ker}\nu \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}[Q]} Ker\ \nu \simeq (\mathbb{Z}[N] \otimes_{\mathbb{Z}} \mathbb{Z}[Q]) \otimes_{\mathbb{Z}[Q]} Ker\ \nu$$

$$\simeq \mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker}\nu \simeq \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker}\nu}$$

is of type FP_∞ over $\mathbb{Z}[G]$. Again by the dimension shifting argument for the short exact sequence

$$0 \to \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker}\nu} \to \operatorname{Ker}\theta \to \operatorname{Ker}\theta / \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker}\nu} \to 0$$

we see that $\operatorname{Ker}\theta$ is of type FP_1 (finitely presented) over $\mathbb{Z}[G]$ if and only if

$$\operatorname{Ker}\theta / \overline{\mathbb{Z}[N] \otimes_{\mathbb{Z}} \operatorname{Ker}\nu} \simeq \Omega \otimes_{\mathbb{Z}} M$$

is of type FP_1 over $\mathbb{Z}[G]$. We summarise

Lemma 3 *The $\mathbb{Z}[G]$-module M is of type FP_2 if and only if $\Omega \otimes_{\mathbb{Z}} M$ is of type FP_1 over $\mathbb{Z}[G]$, where $\Omega = Aug(\mathbb{Z}[N])$. The action of G on $\Omega \otimes_{\mathbb{Z}} M$ is as follows : N acts via left multiplication in Ω and Q acts diagonally on the tensor product via its (left) action on N and M.*

4 Constructing one infinite presentation of $\Omega \otimes_{\mathbb{Z}} M$

Let

$$\mathcal{R} : \mathbb{Z}B \xrightarrow{d_2} \mathbb{Z}A \xrightarrow{d_1} N \longrightarrow 0$$

be an exact sequence of (left) $\mathbb{Z}[Q]$-modules with B and A disjoint unions of finitely many free Q-orbits. We note that such a sequence exists as N is finitely generated over the Noetherian ring $\mathbb{Z}[Q]$, hence is of type FP_∞ over $\mathbb{Z}[Q]$. In particular N is finitely presented over $\mathbb{Z}[Q]$. We equip A with a linear order invariant under the Q-action.

First we want to "lift" $\mathcal{R}$ to a resolution of $\Omega = Aug(\mathbb{Z}[N])$ over $\mathbb{Z}[G]$, where $G = N \rtimes Q$. To do so we will construct first an exact sequence of (left) $\mathbb{Z}[G]$-modules

$$\mathcal{F} : F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 = \mathbb{Z}[N] \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

where ∂_0 is the augmentation map,

$$F_1 = \oplus_{a\in A}\mathbb{Z}[N]e_a \text{ is a free } \mathbb{Z}[N] - \text{module with basis } \{e_a\}_{a\in A},$$

$$F_2 = (\oplus_{b\in B}\mathbb{Z}[N]e_b) \oplus (\oplus_{a_1<a_2, a_i\in A}\mathbb{Z}[N]e_{a_1}e_{a_2})$$

is a free $\mathbb{Z}[N]$ – module with basis the disjoint union $\{e_{a_1}e_{a_2}\}_{a_1,a_2\in A, a_1<a_2}, \cup \{e_b\}_{b\in B}$.

The action of Q on $\mathcal{F}$ is defined as follows :

$$q(ne_a) = ({}^q n)e_{qa}, q(ne_{a_1}e_{a_2}) = ({}^q n)e_{qa_1}e_{qa_2} \text{ and } q(ne_b) = ({}^q n)e_{qb}.$$

To explain the differential ∂_2 we consider the following infinite presentation of the group N

$$\langle a \mid a_1 a_2 a_1^{-1} a_2^{-1}, \alpha(b)\rangle_{a_1,a_2,a\in A, a_1<a_2; b\in B}$$

where for $d_2(b) = \sum_{1\leq i\leq m} z_i a_i$ with $a_1 < a_2 < \ldots < a_m$ and $z_i \in \mathbb{Z}$ the element $\alpha(b)$ is defined as $a_1^{z_1} a_2^{z_2} \ldots a_m^{z_m}$. Then by [8, Prop. 5.4] there is an exact sequence

$$0 \to R_{ab} = R/[R,R] \xrightarrow{\theta_2} \oplus_{s\in S} \mathbb{Z}[N]e_s \xrightarrow{\theta_1} \mathbb{Z}[N] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where

$$S = A \text{ and } R = \mathrm{Ker}\,(F(S) \xrightarrow{\pi} N)$$

with $F(S)$ the free group with basis S, π the homomorphism of groups defined by $\pi(s) = d_1(s)$ for $s \in S$, ϵ the augmentation map and

$$\theta_1(e_s) = \pi(s) - 1 = d_1(s) - 1.$$

By [8, Pr.3(d),p. 45] θ_2 is given explicitly by Fox "free derivatives" i.e. θ_2 is induced from the map $R \to \oplus_{s\in S}\mathbb{Z}[N]e_s$ sending r to $\sum_{s\in S}(\overline{\partial r/\partial s})e_s$, where overlining stands for the image of elements of $\mathbb{Z}[F(S)]$ in $\mathbb{Z}[N]$ via π. In our previous notations we can define $\partial_2(e_b)$ as the image of $\alpha(b)$ in the above formula i.e as $\sum_{s\in S}(\overline{\partial\alpha(b)/\partial s})e_s$. Thus for $\alpha(b) = a_1^{z_1} \ldots a_m^{z_m}$ we can define

$$\partial_2(e_b) = \sum_{i\leq m}(\overline{\partial(a_1^{z_1} \ldots a_m^{z_m})/\partial a_i})e_{a_i} = \sum_{i\leq m}(\overline{(a_1^{z_1} \ldots a_{i-1}^{z_{i-1}})(\partial a_i^{z_i}/\partial a_i)})e_{a_i}.$$

Note that

$$\partial a_i^{z_i}/\partial a_i = 1 + a_i + \ldots + a_i^{z_i - 1} \text{ for } z_i > 0$$

and

$$\partial a_i^{z_i}/\partial a_i = -a_i^{z_i}(1 + a_i + \ldots + a_i^{-z_i-1}) \text{ for } z_i < 0$$

Hence in both cases

$$\partial a_i^{z_i} \equiv z_i \bmod \Omega = Aug(\mathbb{Z}[N]).$$

Using the Fox free derivatives again we define

$$\partial_2(e_{a_1}e_{a_2}) = \sum_{s\in S}(\overline{\partial a_1 a_2 a_1^{-1} a_2^{-1}/\partial s})e_s = (d_1(a_1) - 1)e_{a_2} - (d_1(a_2) - 1)e_{a_1}$$

and finally we set $\partial_1 = \theta_1$ i.e.

$$\partial_1(e_a) = d_1(a) - 1.$$

The following lemma summarizes some of the properties of the differential ∂_2 of $\mathcal{F}$.

Lemma 4 *The differential $\partial_2 : F_2 \to F_1$ satisfies the following two conditions:*

1. $\partial_2(e_b) \equiv d_2(e_b)$ *modulo* $\oplus_{a \in A} \Omega e_a$, *where* $\Omega = Aug(\mathbb{Z}[N])$.

2. $\partial_2(e_b) = \sum_i \lambda_i e_{a_i}$, *where* $\lambda_i = \overline{a_1^{z_1} \dots a_{i-1}^{z_{i-1}} (\partial(a_i^{z_i})/\partial a_i)} \in \mathbb{Z}[N]$. *Furthermore if for some* i *we have* $z_i = \pm 1$ *then* $\lambda_i \in z_i N$.

Now we consider an exact sequence of $\mathbb{Z}[Q]$-modules

$$\mathcal{T} : \mathbb{Z}F \xrightarrow{\nu_2} \mathbb{Z}D \xrightarrow{\nu_1 = \nu} M \longrightarrow 0$$

such that F and D are disjoint unions of finitely many free Q-orbits, the orbits in D are generated by the elements $d_1, \dots, d_k$ of the set D_0 defined at the beginning of section 3 and $\nu_1 = \nu$. Note that the complex $\mathcal{T}$ exists since M is a finitely generated module over the Noetherian ring $\mathbb{Z}[Q]$, hence is Noetherian and of homological type FP_∞ over $\mathbb{Z}[Q]$.

Let $\mathcal{P}$ be the complex of (left) $\mathbb{Z}[G]$-modules induced from $\mathcal{F}$

$$\mathcal{P} : F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \Omega = Aug(\mathbb{Z}[N]) \longrightarrow 0.$$

We consider the deleted complexes

$$\mathcal{P}^{del} : F_2 \xrightarrow{\partial_2} F_1 \longrightarrow 0$$

and

$$\mathcal{T}^{del} : \mathbb{Z}F \xrightarrow{\nu_2} \mathbb{Z}D \longrightarrow 0$$

and their tensor product after omitting the most left term $F_2 \otimes_{\mathbb{Z}} \mathbb{Z}F$

$$\mathcal{P}^{del} \otimes_{\mathbb{Z}} \mathcal{T}^{del} : (F_1 \otimes_{\mathbb{Z}} \mathbb{Z}F) \oplus (F_2 \otimes_{\mathbb{Z}} \mathbb{Z}D) \longrightarrow F_1 \otimes_{\mathbb{Z}} \mathbb{Z}D \longrightarrow 0$$

that we view as the deleted complex $\widetilde{\mathcal{P}}^{del}$ of the complex

$$\widetilde{\mathcal{P}} : (F_1 \otimes_{\mathbb{Z}} \mathbb{Z}F) \oplus (F_2 \otimes_{\mathbb{Z}} \mathbb{Z}D) \longrightarrow F_1 \otimes_{\mathbb{Z}} \mathbb{Z}D \xrightarrow{\partial_1 \otimes \nu_1} \Omega \otimes_{\mathbb{Z}} M \longrightarrow 0.$$

Note that by the Künneth formula $\widetilde{\mathcal{P}}$ is an exact complex of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[Q] \simeq \mathbb{Z}[G \times Q]$-modules, in particular of $\mathbb{Z}[G]$-modules where N acts via (left) multiplication on the first component of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[Q]$ and Q acts diagonally via the embedding of $Q \times Q$ in $G \times Q$.

From now on we write $\mathbb{Z}D$ as $\oplus_{d \in D} \mathbb{Z} e_d$ and $\mathbb{Z}F$ as $\oplus_{f \in F} \mathbb{Z} e_f$. We write . (or just omit it) for the tensor product in $\widetilde{P}$ and write $e_\beta e_\alpha$ for $(-1)^{deg(\alpha).deg(\beta)} e_\alpha e_\beta$, where $e_\alpha e_\beta$ is a well-defined element of $\widetilde{\mathcal{P}}$. Here $deg(e_a) = 1 = deg(e_d), deg(e_b) = 2 = deg(e_f)$. Then we can rewrite $\widetilde{P}$ in the following form

$$\widetilde{\mathcal{P}} : (\oplus_{a \in A, f \in F} \mathbb{Z}[N] e_a e_f) \oplus (\oplus_{a_1, a_2 \in A, a_1 < a_2, d \in D} \mathbb{Z}[N] e_{a_1} e_{a_2} e_d) \oplus (\oplus_{b \in B, d \in D} \mathbb{Z}[N] e_b e_d) \rightarrow$$

$$\oplus_{a \in A, d \in D} \mathbb{Z}[N] e_a e_d \xrightarrow{\partial_1 \otimes \nu_1} \Omega \otimes M \rightarrow 0 \qquad (3)$$

From now on when we refer to $\widetilde{P}$ we will mean (3). The differential of $\widetilde{\mathcal{P}}^{del}$ is the standard differential of tensor product of complexes and hence the differential ∂ of $\widetilde{\mathcal{P}}$ is the following one

$$\partial(e_a e_f) = -e_a \nu_2(e_f), \ \partial(e_b e_d) = \partial_2(e_b) e_d,$$

$$\partial(e_{a_1}e_{a_2}e_d) = \partial_2(e_{a_1}e_{a_2})e_d, \ \partial(e_a e_d) = \partial_1(e_a)\nu_1(e_d).$$

In order to have more symmetric notations we will write ∂ for all the differentials in the complexes $\mathcal{P}, \widetilde{\mathcal{P}}$ and $\mathcal{T}$ without low index to denote the degree of the differential. It will be clear from the context both the complex and degree we work with. Note that under this agreement the above formulae become:

$$\partial(e_a e_f) = -e_a\partial(e_f), \ \partial(e_b e_d) = \partial(e_b)e_d,$$

$$\partial(e_{a_1}e_{a_2}e_d) = \partial(e_{a_1}e_{a_2})e_d, \ \partial(e_a e_d) = \partial(e_a)\partial(e_d). \tag{4}$$

5 A "cell" structure of the chain complex $\widetilde{\mathcal{P}}^{del}$ and support of "cells"

5.1 Motivation

We start this subsection with a discussion of the construction of 3-dimensional CW-complex X in [4, section 3.2]. Let M_0 be a finitely generated $\mathbb{Z}[Q]$-module (in [4] M_0 is denoted by M). Consider an exact sequence of $\mathbb{Z}[Q]$-modules

$$\mathbb{Z}B_0 \xrightarrow{\partial} \mathbb{Z}A_0 \to M_0. \to 0$$

with B_0 and A_0 free Q-sets with finitely many orbits (in [4] A_0 and B_0 are denoted by A and B respectively). It is convenient to put an ordering on A_0 and B_0 that is compatible with the Q-action. The space constructed in [4, section 3.2] has a single 0-cell, trivial second homotopy group and fundamental group isomorphic to M_0 as Q-group. The complex X is obtained from the 3-skeleton of a $K(\mathbb{Z}A_0, 1)$-complex Z by gluing 2-dimensional cells e_b^2 indexed by $b \in B_0$, 3-dimensional prisms $e^3_{(b,a)}$ indexed by $(b, a) \in B_0 \times A_0$ and finitely many free Q-orbits of 3-dimensional cells e_c^3. The edges of X coincide with the edges of Z are indexed by the elements of A_0 and we denote them by e_a^1 for $a \in A_0$. From these edges arise in Z squares $e^2_{(a_1,a_2)}$ for elements $a_1 < a_2$ of A_0 and cubes $e^3_{(a_1,a_2,a_3)}$ for elements $a_1 < a_2 < a_3$ of A_0.

Let $\tilde{X}$ be the universal covering space of X. As $\pi_1(X) \simeq M_0$ is an isomorphism (of $\mathbb{Z}[Q]$-modules) the group M_0 acts freely on $\tilde{X}$. Furthermore the action of Q on X lifts to Q-action on $\tilde{X}$. This gives rise to action of $G_0 = M_0 \rtimes Q$ on $\tilde{X}$ that is free on cells of dimension at least one but not free on the vertices. The vertices form just one free M_0-orbit. Then the singular chain complex of $\tilde{X}$ is

$$\ldots \to (\oplus_{a\in A_0, b\in B_0}\mathbb{Z}[M_0]e^3_{(b,a)}) \oplus (\oplus_{a_1,a_2,a_3\in A_0, a_1<a_2<a_3}\mathbb{Z}[M_0]e^3_{(a_1,a_2,a_3)}) \oplus (\oplus_{c\in C}\mathbb{Z}[M_0]e^3_c) \to$$

$$(\oplus_{b\in B_0}\mathbb{Z}[M_0]e_b^2) \oplus (\oplus_{a_1,a_2\in A_0, a_1<a_2}\mathbb{Z}[M_0]e^2_{(a_1,a_2)}) \to \oplus_{a\in A_0}\mathbb{Z}[M_0]e^1_a \to \mathbb{Z}[M_0] \to \mathbb{Z} \to 0.$$

As the prism $e^3_{(b,a)}$ is the product of the 2-cell e_b^2 with the edge e_a^1 and squares and cubes are products of two (respectively three) of their edges we write $e_a^1 e_b^2$ for $e^3_{(b,a)}$, $e^1_{a_1}e^1_{a_2}$ for $e^2_{(a_1,a_2)}$

and $e^1_{a_1}e^1_{a_2}e^1_{a_3}$ for $e^3_{(a_1,a_2,a_3)}$. After deleting the upper index corresponding to the dimension of a cell the above complex gives the following exact sequence

$$\mathcal{F}(M_0) : (\oplus_{a\in A_0, b\in B_0}\mathbb{Z}[M_0]e_a e_b) \oplus (\oplus_{a_1,a_2,a_3\in A_0, a_1<a_2<a_3}\mathbb{Z}[M_0]e_{a_1}e_{a_2}e_{a_3}) \oplus (\oplus_{c\in C}\mathbb{Z}[M_0]e_c) \to$$

$$(\oplus_{b\in B_0}\mathbb{Z}[M_0]e_b) \oplus (\oplus_{a_1,a_2\in A_0, a_1<a_2}\mathbb{Z}[M_0]e_{a_1}e_{a_2}) \to \oplus_{a\in A_0}\mathbb{Z}[M_0]e_a \to Aug(\mathbb{Z}[M_0]) \to 0.$$

Now we consider $\mathcal{F}(M_0)$ for the special choice of $A_0 = A, B_0 = B, M_0 = N$

$$\mathcal{F}(N) : (\oplus_{a\in A, b\in B}\mathbb{Z}[N]e_a e_b) \oplus (\oplus_{a_1,a_2,a_3\in A, a_1<a_2<a_3}\mathbb{Z}[N]e_{a_1}e_{a_2}e_{a_3}) \oplus (\oplus_{c\in C}\mathbb{Z}[N]e_c) \to$$

$$(\oplus_{b\in B}\mathbb{Z}[N]e_b) \oplus (\oplus_{a_1,a_2\in A, a_1<a_2}\mathbb{Z}[N]e_{a_1}e_{a_2}) \to \oplus_{a\in A}\mathbb{Z}[N]e_a \to Aug(\mathbb{Z}[N]) \to 0$$

and want to draw the reader's attention to the similarities and differences between $\mathcal{F}(N)$ and $\tilde{\mathcal{P}}$.

1. Both complexes $\mathcal{F}(N)$ and $\tilde{\mathcal{P}}$ have product structure. In fact in the particular case when $M = N$, $F = B, D = A$ both complexes contain free $\mathbb{Z}[N]$-modules generated by the elements $e_a e_b$, $e_{a_1}e_{a_2}e_{a_3}$ and $e_{a_1}e_{a_2}$.

2. The lengths of the complexes $\mathcal{F}(N)$ and $\tilde{\mathcal{P}}$ are different but this is due to the fact that in Theorem 1 we study the FP_2-property of some modules and in [4] the FP_3 property of the trivial modules is discussed.

3. The differentials of the complexes $\mathcal{F}(N)$ and $\tilde{\mathcal{P}}$ are slightly different. This comes from the fact that to obtain $\tilde{\mathcal{P}}$ we first do some dimension shifting and then tensor two complexes. The differentials of $\mathcal{F}(N)$ are standard ones in the sense that we have a product structure and the differential behaves nicely with respect to the product i.e. $\partial(\alpha\beta) = \partial(\alpha)\beta + (-1)^{deg(\alpha)}\alpha\partial(\beta)$.

In [4] the space X is realised as increasing nested union $\cup_{\rho>0}X_\rho$ of Q-finite subcomplexes X_ρ of X and using some homotopy operations it is shown that if

$$0 \notin conv_{\leq 3}(\mathbb{R}_{>0}\Sigma^c_{M_0}(Q))$$

for sufficiently large ρ the subcomplex X_ρ is a retract of X. Then the universal cover of X_ρ has singular chain complex with the property that in dimension 1,2 and 3 all modules are free and finitely generated over $\mathbb{Z}[M_0 \rtimes Q]$. This shows that $Aug(\mathbb{Z}[M_0])$ is of type FP_2 over $\mathbb{Z}[M_0 \rtimes Q]$, hence by dimension shifting argument $\mathbb{Z}$ is of type FP_3 over $\mathbb{Z}[M_0 \rtimes Q]$. i.e. the group $M_0 \rtimes Q$ is of type FP_3.

The philosophy of our proof of Theorem 1 is similar but there are technical differencies due to the many differences between the chain complexes $\tilde{\mathcal{P}}$ and $\mathcal{F}(N)$ listed above . We have tried to keep the notations of the present paper as similar as possible to the notations from [4]. We will show in Section 7 that our complex $\tilde{\mathcal{P}}$ is an union of increasing nested set of G-finite subcomplexes $\tilde{\mathcal{P}}^{(\rho)}$ and that for sufficiently large ρ the subcomplex $\tilde{\mathcal{P}}^{(\rho)}$ is homotopy equivalent to $\tilde{\mathcal{P}}$, hence for sufficiently large ρ the subcomplex $\tilde{\mathcal{P}}^{(\rho)}$ is exact. This will show that $\Omega \otimes M$ is finitely presented over $\mathbb{Z}[N \rtimes Q]$ and will finish off the proof of Theorem 1.

5.2 "Cell" structure and support of "cells"

In this subsection we define a "cell" structure of the beginning $\widetilde{\mathcal{P}}^{del}$ of a deleted resolution of $\Omega \otimes M$. We warn the reader that we do not build a space but just use the label "cell" for some special elements of $\widetilde{\mathcal{P}}^{del}$ in order to keep analogy with the proofs in [4].

We call cells the elements of $X_2 \cup X_3$, where

$$X_2 = \{ne_a e_d\}_{n\in N, a\in A, d\in D} \text{ is a } \mathbb{Z}\text{-basis of } \oplus_{a\in A, d\in D} \mathbb{Z}[N]e_a e_d \subset \widetilde{\mathcal{P}}$$

and

$$X_3 = \{ne_a e_f, ne_b e_d, ne_{a_1} e_{a_2} e_d\}_{n\in N, a_1<a_2, a\in A, b\in B, d\in D, f\in F} \text{ is a } \mathbb{Z}\text{-basis of}$$
$$(\oplus_{a\in A, f\in F}\mathbb{Z}[N]e_a e_f) \oplus (\oplus_{a_1, a_2\in A, a_1<a_2, d\in D}\mathbb{Z}[N]e_{a_1} e_{a_2} e_d) \oplus (\oplus_{b\in B, d\in D}\mathbb{Z}[N]e_b e_d) \subset \widetilde{\mathcal{P}}.$$

The elements of X_2 are called cells of dimension two or squares, the elements of X_3 are called cells of dimension three. Furthermore we call the elements $ne_a e_f$ and $ne_b e_d$ prisms and the elements $ne_{a_1} e_{a_2} e_d$ cubes.

In section 2 we defined support of elements of a free admissible resolution. Here we discuss in detail the support of the resolutions $\mathcal{T}$ and $\mathcal{P}$.

1. We view $\mathcal{T}$ as beginning of a free resolution of M over $\mathbb{Z}[Q]$. Without loss of generality we can assume that the resolution is admissible and consider $\{e_d\}_{d\in D_0}$ as a basis of $\oplus_{d\in D}\mathbb{Z}e_d$ as a free $\mathbb{Z}[Q]$-module and $\{e_f\}_{f\in F}$ as Q-invariant $\mathbb{Z}$-basis of $\oplus_{f\in F}\mathbb{Z}e_f$. Then we can consider support function $supp_Y$ of the elements of $\mathcal{T}$ associated with $Y = \cup_{i\geq 0} Y_i$ where $Y_0 = \{e_d\}_{d\in D_0}$ and $QY_1 = \{e_f\}_{f\in F}$. For $\lambda \in (\oplus_{f\in F}\mathbb{Z}e_f) \cup (\oplus_{d\in D}\mathbb{Z}e_d)$ we write $supp(\lambda)$ for $supp_Y(\lambda)$ i.e.

$$supp(e_d) = \{q\} \text{ where } q^{-1}d \in D_0.$$

2. We view $\mathcal{P}$ as the beginning of a free resolution of $\Omega = Aug(\mathbb{Z}[N])$ over $\mathbb{Z}[N \rtimes Q]$ and represent A as a disjoint union $\cup_{a\in A_0} Qa$ of free Q-orbits. Then $\{e_a\}_{a\in A_0}$ is a basis of $F_1 = \oplus_{a\in A}\mathbb{Z}[N]e_a$ over $\mathbb{Z}[N \rtimes Q]$, $\{ne_{a_1} e_{a_2}\}_{n\in N, a_1<a_2, a_i\in A} \cup \{ne_b\}_{n\in N, b\in B}$ is a $(N \rtimes Q)$-invariant $\mathbb{Z}$-basis of F_2 and as in section 2 we can define a support function $supp_Z$ on $F_1 \cup F_2$ where $Z_0 = \{e_a\}_{a\in A_0}$ and $GZ_1 = \{ne_{a_1} e_{a_2}\}_{n\in N, a_1<a_2, a_i\in A} \cup \{ne_b\}_{n\in N, b\in B}$, $G = N \rtimes Q$. This way we will finish up with support function $supp_Z$ whose images are finite subsets of G. As we prefer working with elements of Q we define a new support function

$$supp = \pi \circ supp_Z : F_1 \cup F_2 \to Set(Q)$$

where $\pi : G \to Q$ is the canonical projection i.e.

$$supp(ne_{qa}) = \{q\}, \text{ where } a \in A_0, q \in Q.$$

3. Finally we define support of the elements of $\tilde{\mathcal{P}}$. By definition the support of a non-zero element of $\widetilde{\mathcal{P}}^{del}$ is a subset of Q and satisfies the following properties:

$$supp(0) = \emptyset \text{ and } supp(\sum_{x_i\in X_j, z_i\in\mathbb{Z}\setminus\{0\}} z_i x_i) = \cup_i supp(x_i) \text{ for fixed } j \in \{2, 3\}$$

$$supp(x) = supp(\partial(x)) \text{ for } x \in X_3,$$

$$supp(ne_a e_d) = supp(ne_a) \cup supp(e_d) \in supp(\mathcal{P}) \cup supp(\mathcal{T})$$

hence

$$supp(ne_a e_d) = \{q_1, q_2\}, \text{ where } q_1^{-1}a \in A_0, q_2^{-1}d \in D_0..$$

Lemma 5 *Let $a \in A, d \in D$, $n \in N$, $\lambda \in \sum_{f\in F}\mathbb{Z}e_f$ and $\mu \in \sum_{b\in B}\mathbb{Z}[N]e_b$. Then*

$$supp(ne_a\lambda) \subseteq supp(e_a) \cup supp(\lambda) \text{ and } supp(\mu e_d) \subseteq supp(\mu) \cup supp(e_d)$$

with equalities for $\lambda \neq 0$ and $\mu \neq 0$, where supp denotes the support functions of $\mathcal{T}$, $\mathcal{P}$ and $\tilde{\mathcal{P}}$ defined above.

Proof. As $supp(0) = \emptyset$ we consider the case $\lambda \neq 0$. Note that for $\lambda = \sum_{z_i\in\mathbb{Z}\setminus\{0\}, f_i\in F} z_i e_{f_i}$

$$supp(ne_a(\sum_i z_i e_{f_i})) = \cup_i supp(ne_a e_{f_i}) = \cup_i supp(\partial(ne_a e_{f_i})) = \cup_i supp(-ne_a\partial(e_{f_i})).$$

Let $\partial(e_{f_i}) = \sum_j \alpha_j d_{i,j}$ where $\alpha_j \in \mathbb{Z}\setminus\{0\}, d_{i,j} \in D$. Then

$$\cup_i supp(-ne_a\partial(e_{f_i})) = \cup_i supp(\sum_j -\alpha_j ne_a e_{d_{i,j}}) = \cup_{i,j} supp(e_a e_{d_{i,j}}) =$$

$$\cup_{i,j}(supp(e_a) \cup supp(e_{d_{i,j}})) = supp(e_a) \cup (\cup_{i,j} supp(e_{d_{i,j}})) = supp(e_a) \cup (\cup_i supp(\partial(e_{f_i}))) =$$

$$supp(e_a) \cup (\cup_i supp(e_{f_i})) = supp(e_a) \cup supp(\lambda).$$

The second part of the lemma is proved similarly. □

The following proposition is corollary of Lemma 5.

Proposition 2 *Let λ_1 be an element of $\mathcal{P}^{del}$ and λ_2 be an element of $\mathcal{T}^{del}$ such that $deg(\lambda_1) + deg(\lambda_2) = 3$ i.e. $\lambda_1\lambda_2 \in \tilde{\mathcal{P}}^{del}$. Then*

$$supp(\lambda_1\lambda_2) \subseteq supp(\lambda_1) \cup supp(\lambda_2).$$

We finish this section with the definition of diameter of a non-zero element of the modules in $\tilde{\mathcal{P}}^{(del)}$, $\mathcal{T}^{(del)}$ and $\mathcal{P}^{(del)}$.

Definition *We define the diameter of a finite subset of Q as the diameter of the smallest closed ball in $\mathbb{R}^n$ that contains this subset. The diameter of a non-zero element of the modules in $\tilde{\mathcal{P}}^{(del)}$, $\mathcal{T}^{del}$ and $\mathcal{P}^{(del)}$ is defined as the diameter of the support of this element. Note that the set of diameters of cells of $\tilde{\mathcal{P}}^{del}$ is a discrete subset of $[0, \infty)$. For a non-negative real number k we define $\tilde{\mathcal{P}}^{(k)}$ and $\mathcal{P}^{(k)}$ as the subcomplexes of $\tilde{\mathcal{P}}$ and $\mathcal{P}$ respectively spanned by $\{$ all cells of diameter not bigger than k $\}$ plus the most right module in the complex in consideration i.e. $\Omega \otimes_{\mathbb{Z}} M$ and Ω respectively.*

6 Some restrictions on the cells of the complex $\tilde{\mathcal{P}}^{del}$

6.1 The restrictions

By the main result of [3] $\Sigma^c_N(Q)$ and $\Sigma^c_M(Q)$ have polyhedral structure. Furthermore by assumption

$$0 \notin \mathbb{R}_{>0}\Sigma^c_M(Q) + conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q)).$$

By the main result of [11] for a finitely presented group $G = N \rtimes Q$

$$[conv_{\leq 2}(\mathbb{R}_{>0}\Sigma_N^c(Q))] = S(Q) \setminus \Sigma^2(N \rtimes Q, \mathbb{Z}) \mid_Q,$$

where [ ] stands for the projection of $Hom(Q, \mathbb{R}) \setminus \{0\}$ to $S(Q)$ and

$$\Sigma^m(N \rtimes Q, \mathbb{Z}) \mid_Q = \{[\chi \mid_Q] \in S(Q) \mid [\chi] \in \Sigma^m(N \rtimes Q, \mathbb{Z}) \text{ and } \chi(N) = 0\}.$$

Let Γ_M and Γ_N be compact subsets of $S(Q)$ such that

$$\Gamma_M \subseteq \Sigma_M(Q) = S(Q) \setminus \Sigma_M^c(Q),$$

$$\Gamma_N \subseteq S(Q) \setminus [conv_{\leq 2}(\mathbb{R}_{>0}\Sigma_N^c(Q))] = \Sigma^2(N \rtimes Q, \mathbb{Z}) \mid_Q \subseteq \Sigma^1(N \rtimes Q, \mathbb{Z}) \mid_Q = \Sigma_N(Q).$$

Then $S(Q) \setminus \Gamma_M$ is an open subset of $S(Q)$ that contains $\Sigma_M^c(Q)$ and $S(Q) \setminus \Gamma_N$ is an open subset of $S(Q)$ that contains $\Sigma_N^c(Q)$. Then using the polyhedral structure of $\Sigma_N^c(Q)$ and $\Sigma_M^c(Q)$ [3] together with the condition $0 \notin \mathbb{R}_{>0}\Sigma_M^c(Q) + conv_{\leq 2}(\mathbb{R}_{>0}\Sigma_N^c(Q))$ we can choose the sets Γ_M and Γ_N such that

$$0 \notin \mathbb{R}_{>0}(S(Q) \setminus \Gamma_N) + \mathbb{R}_{>0}(S(Q) \setminus \Gamma_M). \tag{5}$$

Furthermore there is $\epsilon > 0$ such that for any $u_0 \in S(Q) \setminus \Gamma_N$ and any $v_0 \in S(Q) \setminus \Gamma_M$ the angle between u_0 and v_0 is smaller than $\pi - \epsilon$.

The above compact condition implies the following two lemmas. Note that in the following two lemmas we identify $Q \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ with $Hom(Q, \mathbb{R})$ and the unit sphere S^{n-1} in $\mathbb{R}^n$ with $S(Q)$ via the isomorphism sending $r \in \mathbb{R}^n$ to $\chi_r \in Hom(Q, \mathbb{R})$ given by $\chi_r(q) = (r, q)$, where (,) is the standard inner product in $\mathbb{R}^n$.

Lemma 6 *There is a positive real number m_1 with the following property.*

Let B_0 be the smallest closed ball in $\mathbb{R}^n$ that contains the support of a prism $e_a e_f$, v be the centre of B_0 and r be the diameter of B_0. If $r \geq m_1$ then either

1. $\frac{q_1 - v}{|q_1 - v|} \in -\Gamma_N$, *where* $\{q_1\} = supp(e_a)$

or

2. *for every q from the support of e_f we have* $\frac{q-v}{|q-v|} \in -\Gamma_M$.

Proof. Assume that $u = \frac{q_1 - v}{|q_1 - v|} \notin -\Gamma_N$ and for some $q \in supp(e_f) = supp(\partial e_f)$ we have $t = \frac{q-v}{|q-v|} \notin -\Gamma_M$. By the definition of B_0 as the smallest closed ball that contains $supp(e_a e_f) = supp(e_a) \cup supp(e_f)$ for sufficiently large r the angle between u and t can be very close to π, in particular bigger than $\pi - \epsilon$, where ϵ is defined just after (5). In particular $-u \in S(Q) \setminus \Gamma_N$ and $-t \in S(Q) \setminus \Gamma_M$ and the angle between $-u$ and $-t$ is bigger than $\pi - \epsilon$, a contradiction with the choice of ϵ.

□

Lemma 7 *There is a positive real number m_2 with the following property.*

Let B_0 be the smallest closed ball in $\mathbb{R}^n$ that contains the support of a prism $e_b e_d$, v be the centre of B_0 and r be the diameter of B_0. If $r \geq m_2$ then either

1. $\frac{q_1 - v}{|q_1 - v|} \in -\Gamma_M$ *for* $supp(e_d) = \{q_1\}$

or

2. *for every q from the support of e_b we have* $\frac{q-v}{|q-v|} \in -\Gamma_N$.

Proof. The proof is similar to the proof of the previous lemma. ☐

For T a subset of $S(N \rtimes Q)$ we write $T\mid_Q$ for $\{[\chi\mid_Q] \in S(Q) \mid [\chi] \in T, \chi(N) = 0\}$.

Lemma 8 *For any* $m \geq 0$

$$\Sigma^{m+1}(N \rtimes Q, \mathbb{Z})\mid_Q = \Sigma^m(N \rtimes Q, \Omega)\mid_Q,$$

where Ω *is the augmentation ideal of* $\mathbb{Z}[N]$. *In particular*

$$\Gamma_N \subseteq \Sigma^2(N \rtimes Q, \mathbb{Z})\mid_Q = \Sigma^1(N \rtimes Q, \Omega)\mid_Q$$

Proof. Let $\chi \in Hom(N \rtimes Q, \mathbb{Z}) \setminus \{0\}$ and $\chi(N) = 0$. Note that $[\chi\mid_Q] \in \Sigma^{m+1}(N \rtimes Q, \mathbb{Z})\mid_Q$ is equivalent to the trivial module $\mathbb{Z}$ being of homological type FP_{m+1} over $\mathbb{Z}[(N \rtimes Q)_\chi] = \mathbb{Z}[N \rtimes Q_\chi]$. Consider the short exact sequence of left $\mathbb{Z}[N \rtimes Q_\chi]$-modules

$$0 \to Aug(\mathbb{Z}[N \rtimes Q_\chi]) \to \mathbb{Z}[N \rtimes Q_\chi] \to \mathbb{Z} \to 0,$$

where $Aug(\mathbb{Z}[N \rtimes Q_\chi]) \to \mathbb{Z}[N \rtimes Q_\chi]$ is the inclusion map. Note that the module in the middle is free (of rank 1) over $\mathbb{Z}[N \rtimes Q_\chi]$, hence of type FP_∞ over $\mathbb{Z}[N \rtimes Q_\chi]$. By the dimension shifting argument [1, Prop. 1.4] the trivial module $\mathbb{Z}$ has homological type FP_{m+1} over $\mathbb{Z}[(N \rtimes Q)_\chi] = \mathbb{Z}[N \rtimes Q_\chi]$ if and only if $Aug(\mathbb{Z}[N \rtimes Q_\chi])$ is FP_m over $\mathbb{Z}[N \rtimes Q_\chi]$.

Now by [6, Thm. C] applied for the chain endomorphism φ that is multiplication with an element of $Q_\chi \setminus Ker\ \chi$ we have that $[\chi\mid_Q] \in \Sigma^m(Q, \mathbb{Z})$ for every $m \geq 0$ (in fact the same argument shows that $\Sigma^m(Q, \mathbb{Z}) = S(Q)$), hence the trivial module $\mathbb{Z}$ is always of type FP_∞ over $\mathbb{Z}[Q_\chi]$. Hence the dimension shifting argument for the short exact sequence of $\mathbb{Z}[Q_\chi]$-modules

$$0 \to Aug(\mathbb{Z}[Q_\chi]) \to \mathbb{Z}[Q_\chi] \to \mathbb{Z} \to 0$$

where $Aug(\mathbb{Z}[Q_\chi]) \to \mathbb{Z}[Q_\chi]$ is the inclusion map, shows that $Aug(\mathbb{Z}[Q_\chi])$ is FP_∞ over $\mathbb{Z}[Q_\chi]$. Then $\uparrow_{\mathbb{Z}[Q_\chi]}^{\mathbb{Z}[N \rtimes Q_\chi]} Aug(\mathbb{Z}[Q_\chi])$ is of type FP_∞ over $\uparrow_{\mathbb{Z}[Q_\chi]}^{\mathbb{Z}[N \rtimes Q_\chi]} \mathbb{Z}[Q_\chi] \simeq \mathbb{Z}[N \rtimes Q_\chi]$. Note that the induced module $\uparrow_{\mathbb{Z}[Q_\chi]}^{\mathbb{Z}[N \rtimes Q_\chi]} Aug(\mathbb{Z}[Q_\chi])$ and $\mathbb{Z}[N \rtimes Q_\chi]Aug(\mathbb{Z}[Q_\chi])$ are isomorphic as $\mathbb{Z}[N \rtimes Q_\chi]$-modules. In particular $\mathbb{Z}[N \rtimes Q_\chi]Aug(\mathbb{Z}[Q_\chi])$ is FP_∞ over $\mathbb{Z}[N \rtimes Q_\chi]$.

Now consider the short exact sequence of $\mathbb{Z}[N \rtimes Q_\chi]$-modules

$$0 \to \mathbb{Z}[N \rtimes Q_\chi]Aug(\mathbb{Z}[Q_\chi]) \to Aug(\mathbb{Z}[N \rtimes Q_\chi]) \to Aug(\mathbb{Z}[N]) \to 0,$$

induced by the inclusion of $\mathbb{Z}[N \rtimes Q_\chi]Aug(\mathbb{Z}[Q_\chi])$ in the augmentation ideal $Aug(\mathbb{Z}[N \rtimes Q_\chi])$. As $\mathbb{Z}[N \rtimes Q_\chi]Aug(\mathbb{Z}[Q_\chi])$ is of type FP_∞ over $\mathbb{Z}[N \rtimes Q_\chi]$ by the dimension shifting argument [1, Prop. 1.4] $Aug(\mathbb{Z}[N \rtimes Q_\chi])$ is of type FP_m over $\mathbb{Z}[N \rtimes Q_\chi]$ if and only if $Aug(\mathbb{Z}[N])$ is of type FP_m over $\mathbb{Z}[N \rtimes Q_\chi]$.

Thus the trivial module $\mathbb{Z}$ is of homological type FP_{m+1} over $\mathbb{Z}[(N \rtimes Q)_\chi] = \mathbb{Z}[N \rtimes Q_\chi]$ if and only if $Aug(\mathbb{Z}[N])$ is of type FP_m over $\mathbb{Z}[N \rtimes Q_\chi]$. This is equivalent to the lemma via the definition of the Σ-invariants. ☐

Let Λ_M and Λ_N be the finite subsets of the centralizers of M and N respectively given by Lemma 1. By enlarging the sets B and F with finitely many new free Q-orbits if necessary we can assume that :

1. In the resolution $\mathcal{R}$ defined at the beginning of Section 4 for every $\lambda \in \Lambda_N \subset \mathbb{Z}[Q], a_0 \in A_0$ (remember A is the disjoint union $\cup_{a_0 \in A_0} Qa_0$ and A_0 is a finite set) there is $b_{\lambda,a_0} \in B$ such that

$$d_2(b_{\lambda,a_0}) = (1 - \lambda)a_0. \tag{6}$$

2. In the resolution $\mathcal{T}$ defined after the statement of Lemma 4 for every $\lambda \in \Lambda_M \subset \mathbb{Z}[Q], d_0 \in D_0$ (remember D is the disjoint union $\cup_{d_0 \in D_0} Qd_0$ and D_0 is a finite set) there is $f_{\lambda,d_0} \in F$ such that

$$\partial(e_{f_{\lambda,d_0}}) = (1 - \lambda)e_{d_0}. \tag{7}$$

Remember that we write ∂ for the differentials of $\mathcal{P}$, $\widetilde{\mathcal{P}}$ and $\mathcal{T}$. In the above case ∂ stands for the differential of $\mathcal{T}$ and in the old notations the above differential is

$$\nu_2(f_{\lambda,d_0}) = (1 - \lambda)d_0.$$

The above two types of restrictions together with Lemma 2 imply the following Lemma 9. We remind the reader that the function ρ_1 was defined in Lemma 2 and we write multiplicatively the group operation in Q when viewed as a subset of $\mathbb{Z}[Q]$ and additively when viewed as the subset $\mathbb{Z}^n$ of $\mathbb{R}^n$.

Lemma 9 *Suppose* B_0 *is a closed ball in* $\mathbb{R}^n$ *with radius* $\geq max\{\rho_1(M), \rho_1(N)\}$ *and centre* v *and* q *is an element of* $Q \cap \partial B_0$. *Then*

1. if $\frac{q-v}{|q-v|} \in -\Sigma_M(Q)$ *and* $supp(e_d) = \{q\}$ *then there is* $e_f \in \mathcal{T}$ *with*

$$supp(\partial(e_f) - e_d) \subseteq B_0 \setminus \partial B_0;$$

2. if $\frac{q-v}{|q-v|} \in -\Sigma_N(Q)$ *and* $supp(e_a) = \{q\}$ *then there is* $e_b \in \mathcal{F}$ *and an element* $h \in N$ *such that*

$$supp(h\partial(e_b) - e_a) \subseteq B_0 \setminus \partial B_0.$$

Proof. To prove the first part of the lemma consider $d_0 = q^{-1}d \in D_0$ and $\lambda \in \Lambda_M$ such that $q + supp(\lambda) \subseteq B_0 \setminus \partial B_0$. Then by (7) we can define $f = qf_{\lambda,d_0}$.

To prove the second part consider $a_0 = q^{-1}a \in A_0$ and $\lambda \in \Lambda_N$ such that $q + supp(\lambda) \subseteq B_0 \setminus \partial B_0$. By (6) we have $d_2(b_{\lambda,a_0}) = (1 - \lambda)a_0$. Then by Lemma 4(2) for $b = qb_{\lambda,a_0}$ there is $h \in N$ such that $\partial_2(e_b) = h^{-1}e_a +$ cells with support in $B_0 \setminus \partial B_0$. □

6.2 Some corollaries of the restrictions

We start this section reminding the reader that for the complex

$$\mathcal{P} : F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \Omega = Aug(\mathbb{Z}[N]) \to 0$$

$\mathcal{P}^{(\rho)}$ is the subcomplex of $\mathcal{P}$ spanned by Ω and all cells of diameter $\leq \rho$. We view $\mathcal{P}$ as the beginning of a free resolution of Ω over $\mathbb{Z}[G]$, where $G = N \rtimes Q$, whose components are not all finitely generated i.e. F_2 is not.

Lemma 10 *There exists a positive number* m_3 *such that*

$$\mathcal{P}^{(\rho)} : F_2^{(\rho)} \to F_1^{(\rho)} = F_1 \to \Omega \to 0$$

is exact for every $\rho \geq m_3$.

Proof. We observe first that the lemma is equivalent with Ω being finitely presented over $\mathbb{Z}[G]$. Let $0 \to Aug(\mathbb{Z}[Q]) \to \mathbb{Z}[Q] \to \mathbb{Z} \to 0$ be the short exact sequence of (left) $\mathbb{Z}[Q]$-modules given by the inclusion map $Aug(\mathbb{Z}[Q]) \to \mathbb{Z}[Q]$. As $\mathbb{Z}[Q]$ is a Noetherian ring the trivial module $\mathbb{Z}$ over $\mathbb{Z}[Q]$ is of type FP_∞. Then by dimension shifting argument $Aug(\mathbb{Z}[Q])$ is of type FP_∞ over $\mathbb{Z}[Q]$.

Consider the short exact sequences of left $\mathbb{Z}[G]$-modules induced by the inclusion of $Aug(\mathbb{Z}[G])$ in $\mathbb{Z}[G]$ and the inclusion of $\mathbb{Z}[G]Aug(\mathbb{Z}[Q])$ in $Aug(\mathbb{Z}[G])$

$$0 \to Aug(\mathbb{Z}[G]) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

and

$$0 \to \mathbb{Z}[G]Aug(\mathbb{Z}[Q]) \to Aug(\mathbb{Z}[G]) \to \Omega = Aug(\mathbb{Z}[N]) \to 0.$$

Observe that $Aug(\mathbb{Z}[Q])$ is FP_∞ over $\mathbb{Z}[Q]$, hence the induced module $\mathbb{Z}[G]Aug(\mathbb{Z}[Q]) \simeq \uparrow_{\mathbb{Z}[Q]}^{\mathbb{Z}[G]} Aug(\mathbb{Z}[Q])$ is FP_∞ over $\mathbb{Z}[G]$. Then by applying dimension shifting argument for both exact sequences we have that Ω is FP_m over $\mathbb{Z}[G]$ if and only if $Aug(\mathbb{Z}[G])$ is FP_m over $\mathbb{Z}[G]$ and $Aug(\mathbb{Z}[G])$ is FP_m over $\mathbb{Z}[G]$ if and only if the trivial module $\mathbb{Z}$ is FP_{m+1} over $\mathbb{Z}[G]$. In particular Ω is FP_1 (finitely presented) over $\mathbb{Z}[G]$ if and only if the trivial module $\mathbb{Z}$ is FP_2 over $\mathbb{Z}[G]$. By the main result of [7] $G = N \rtimes Q$ is finitely presented if and only if it is of type FP_2. Thus Ω is finitely presented over $\mathbb{Z}[G]$ and the lemma holds.

□

Now we discuss some corollaries of Proposition 1. In section 2 two numbers r and s were defined and used in Proposition 1. Now we fix r_1 and s_1 to be the values of these r and s for $V = \Omega$, $H = N \rtimes Q$, $m = 1$ and the free resolution $\mathcal{P}^{(m_3)}$ given by Lemma 10. Furthermore we fix r_2 and s_2 to be the values of r and s for $V = M$, $H = Q$, $m = 1$ and the free resolution $\mathcal{T}$ of M. Observe that we can assume that the resolution $\mathcal{P}^{(m_3)}$ and $\mathcal{T}$ satisfy part 3 of Theorem 2, this can be achieved by enlarging B and F with finitely many new free Q-orbits if necessary. The following two lemmas are obvious consequences of Proposition 1.

Lemma 11 *Let* $\beta \in \oplus_{b' \in B}\mathbb{Z}[N]e_{b'} \subset \mathcal{P}^{m_3}$ *such that*

$$supp(\beta) \subseteq B_0 \text{ and } supp(\partial(\beta)) \subseteq B_0 \setminus \partial B_0,$$

where B_0 *is a closed ball with radius* $\geq s_1^2/r_1$ *and centre* v. *Furthermore assume that for every* $q \in supp(\beta) \cap \partial B_0$ *we have*

$$\frac{q - v}{\mid q - v \mid} \in -\Gamma_N \subseteq -\Sigma^1(N \rtimes Q, \Omega) \mid_Q = -\Sigma^2(N \rtimes Q, \mathbb{Z}) \mid_Q .$$

Then there is

$$t \in F_2^{(m_3)} = (\oplus_{b \in B}\mathbb{Z}[N]e_b) \oplus (\oplus_{a_1 < a_2, diam(e_{a_1} e_{a_2}) \leq m_3}\mathbb{Z}[N]e_{a_1}e_{a_2}) \subset F_2$$

such that

$$supp(t) \subseteq B_0 \setminus \partial B_0 \text{ and } \partial t = \partial(\beta).$$

Note that as Q is abelian we have $\Sigma^0(Q,M) = \Sigma^m(Q,M)$ for every m and hence $\Gamma_M \subset \Sigma_M(Q) = \Sigma^0(Q,M) = \Sigma^1(Q,M)$.

Lemma 12 *Let $\beta \in \oplus_{f\in F}\mathbb{Z}e_f \subseteq T$ such that*

$$supp(\beta) \subseteq B_0 \text{ and } supp(\partial(\beta)) \subseteq B_0 \setminus \partial B_0,$$

where B_0 is a closed ball with radius $\geq s_2^2/r_2$ and centre v. Assume further that for every $q \in supp(\beta) \cap \partial B_0$ we have

$$\frac{q-v}{\mid q-v \mid} \in -\Gamma_M \subseteq -\Sigma^1(Q,M).$$

Then there is $t \in \oplus_{f\in F}\mathbb{Z}e_f$ such that

$$supp(t) \subseteq B_0 \setminus \partial B_0 \text{ and } \partial t = \partial(\beta).$$

We finish this section with an obvious lemma that will be used many times in the following sections.

Lemma 13 *Let α be an element of a module in $\tilde{\mathcal{P}}^{(del)}$ and B_0 be a closed ball in $\mathbb{R}^n$ such that*

$$supp(\alpha) \subseteq (B_0 \setminus \partial B_0) \cup T \subseteq B_0,$$

where T is a finite subset of B_0 such that B_0 is not the smallest closed ball that contains T. Then B_0 is not the smallest closed ball containing $supp(\alpha)$. In particular α has diameter smaller than the diameter of B_0.

7 The existence of a chain map μ

We remind the reader that $\tilde{\mathcal{P}}^{(\rho)}$ by definition is the subcomplex of $\tilde{\mathcal{P}}$ spanned by $\Omega \otimes_{\mathbb{Z}} M$ and the "cells" of diameter $\leq \rho$. Our aim in this section is to define a chain map of $\mathbb{Z}[G]$-complexes

$$\mu : \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$$

lifting the identity of $\Omega \otimes_{\mathbb{Z}} M$ with the properties that for

$$r_0 = 2max\{s_1^2/r_1, s_2^2/r_2, \rho_1(M), \rho_1(N), m_1/2, m_2/2, m_3/2,$$

the diameters of the support of all the cells e_a, e_b, e_d and e_f for $a \in A, b \in B, d \in D, f \in F\}$

1. the restriction of μ on $\tilde{\mathcal{P}}^{(r_0)}$ is identity;
2. for $r > r_0$ we have $\mu(\tilde{\mathcal{P}}^{(r)}) \subseteq \cup_{s<r}\tilde{\mathcal{P}}^{(s)}$.

Note that since A, B, D e F form only finitely many free Q-orbits and so r_0 is well defined.

As the set of diameters of cells in $\tilde{\mathcal{P}}$ is a discrete subset of $[0,\infty)$ the existence of μ will imply immediately that the subcomplex $\tilde{\mathcal{P}}^{(r_0)}$ is exact and in fact it gives a finite presentation of $\Omega \otimes M$ over $\mathbb{Z}[G]$. Then Lemma 3 will complete the proof of Theorem 1.

The construction of μ on a cell of diameter $r > r_0$ will be done inductively on r (remember the set of diameters of cells in $\widetilde{\mathcal{P}}$ is a discrete subset of $\mathbb{R}_{\geq 0}$). Assume we have constructed μ on $\cup_{s<r}\widetilde{\mathcal{P}}^{(s)}$. To extend μ to $\widetilde{\mathcal{P}}^{(r)}$ we will construct a chain map

$$\mu_r : \widetilde{\mathcal{P}}^{(r)} \to \cup_{s<r}\widetilde{\mathcal{P}}^{(s)}$$

whose restriction to $\cup_{s<r}\widetilde{\mathcal{P}}^{(s)}$ is identity and define the extension of μ to $\widetilde{\mathcal{P}}^{(r)}$ as the composition

$$\widetilde{\mathcal{P}}^{(r)} \xrightarrow{\mu_r} \cup_{s<r} \widetilde{\mathcal{P}}^{(s)} \xrightarrow{\mu} \cup_{s<r} \widetilde{\mathcal{P}}^{(s)}.$$

7.1 The definition of μ_r on the squares

We start by defining the map μ_r on the squares $e_a e_d$ of diameter r, remember r is a fixed real number such that $r > r_0$. We choose representatives of the Q-orbits of these squares, define μ_r on them and then define μ_r to be $(N \rtimes Q)$-invariant on the squares $\{n e_a e_d\}_{a\in A, d\in D, n\in N}$ of diameter r.

Let $e_a e_d$ be one representative of a Q-orbit of $\{e_a e_d\}_{a\in A, d\in D}$ of diameter r. As the support of a square has at most two points the minimal ball that contains the support $\{q_1, q_2\}$ of $e_a e_d$ is the ball B_0 with centre $v = (q_1 + q_2)/2$, were $supp(e_a) = \{q_1\}$ and $supp(e_d) = \{q_2\}$. By (5) either

$$\frac{q_1 - v}{\mid q_1 - v \mid} \in -\Gamma_N \subseteq -\Sigma^2(N \rtimes Q, \mathbb{Z}) \mid_Q \subseteq -\Sigma^1(N \rtimes Q, \mathbb{Z}) \mid_Q = -\Sigma_N(Q)$$

or

$$\frac{q_2 - v}{\mid q_2 - v \mid} \in -\Gamma_M \subseteq -\Sigma_M(Q).$$

1. If $\frac{q_2 - v}{|q_2 - v|} \in -\Gamma_M$ by Lemma 9(1) there exists element e_f such that the support of $\partial(e_f) - e_d$ is in $B_0 \setminus \partial(B_0)$. By Lemma 5

$$supp(e_a(e_d - \partial(e_f))) \subseteq supp(e_a) \cup supp(e_d - \partial(e_f)) \subseteq supp(e_a) \cup (B_0 \setminus \partial B_0) \subseteq B_0$$

and then by Lemma 13 the diameter of $e_a e_d - e_a\partial(e_f)$ is smaller than r. We observe that

$$\partial(e_a\partial(e_f)) = \partial(e_a)\partial^2(e_f) = \partial(e_a)0 = 0$$

and define

$$\mu_r(e_a e_d) = \mu_r(e_a e_d - e_a\partial(e_f)) = e_a e_d - e_a\partial(e_f).$$

Note that

$$\partial\mu_r(e_a e_d) = \partial(e_a e_d - e_a\partial(e_f)) = \partial(e_a e_d) = \mu_r\partial(e_a e_d)$$

where the last equality comes from the fact that $\partial(e_a e_d) \in \Omega \otimes M$ and $\mu_r \mid_{\Omega\otimes M} = id \mid_{\Omega\otimes M}$. Thus μ_r commutes with ∂ and hence is a chain map.

2. If $\frac{q_2 - v}{|q_2 - v|} \notin -\Gamma_M$ then $\frac{q_1 - v}{|q_1 - v|} \in -\Gamma_N$. In this case by Lemma 9(2) there exists a cell e_b and an element $h \in N$ such that the support of $e_a - \partial(he_b)$ is in the interior of the ball B_0. By Lemma 5

$$supp((e_a - \partial(he_b))e_d) \subseteq supp(e_a - \partial(he_b)) \cup supp(e_d) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_d) \subseteq B_0$$

and by Lemma 13 the diameter of $e_ae_d - \partial(he_b)e_d$ is smaller than r. We observe that

$$\partial(\partial(he_b)e_d) = \partial^2(he_b)\partial(e_d) = 0\partial(e_d) = 0$$

and define

$$\mu_r(e_ae_d) = \mu_r(e_ae_d - \partial(he_b)e_d) = e_ae_d - \partial(he_b)e_d.$$

Note that μ_r is a chain map i.e. commutes with ∂

$$\partial\mu_r(e_ae_d) = \partial(e_ae_d - \partial(he_b)e_d) = \partial(e_ae_d) = \mu_r\partial(e_ae_d).$$

7.2 Defining μ_r on the special prisms

We remind the reader that by definition the prisms are the elements of type ne_ae_f and ne_be_d.

Definition. *One prism e_ae_f is called special if there exists $z \in \mathbb{Z}\setminus\{0\}$ and $d \in D$ such that*

$$supp(\partial(e_f) - ze_d) \subseteq B_0 \setminus \partial B_0,$$

where B_0 is the smallest closed ball that contains $supp(e_ae_f) = supp(e_a) \cup supp(e_f) = supp(e_a) \cup supp(\partial e_f)$. Note that in this case $supp(e_ae_f) \subseteq supp(e_a) \cup supp(e_d) \cup (B_0 \setminus \partial B_0)$, hence B_0 is the smallest closed ball containing $supp(e_ae_d) = supp(e_a) \cup supp(e_d)$.

One prism e_be_d is called special if there exists $\lambda \in \mathbb{Z}[N]\setminus\{0\}$ and $a \in A$ such that

$$supp(\partial(e_b) - \lambda e_a) \subseteq B_0 \setminus \partial B_0,$$

where B_0 is the smallest closed ball that contains $supp(e_be_d) = supp(e_d) \cup supp(e_b) = supp(e_d) \cup supp(\partial e_b)$. Note that in this case $supp(e_be_d) \subseteq supp(e_a) \cup supp(e_d) \cup (B_0 \setminus \partial B_0)$, hence B_0 is the smallest closed ball containing $supp(e_ae_d) = supp(e_a) \cup supp(e_d)$.

Note that the definition of a special prism carries out to the Q-orbit it generates i.e. in one Q-orbit of "cells" of type ne_ae_f or ne_be_d either every element is special or there is no special one.

As before we choose representatives of the Q-orbits of $\{e_ae_f\}_{a\in A, f\in F}$ and $\{e_be_d\}_{b\in B, d\in D}$ and work only with them. We assume now that e_be_d and e_ae_f are two representatives of diameter r that are special prisms.

I. First we define μ_r on e_ae_f. Let B_0 be the smallest closed ball that contains $supp(e_ae_f)$, so B_0 has diameter r, and v be the center of B_0. Since e_ae_f is a special prism there is $z \in \mathbb{Z}\setminus\{0\}$ such that

$$supp(\partial(e_f) - ze_d) \subseteq B_0 \setminus \partial B_0.$$

By Lemma 5 $supp(e_a(\partial(e_f) - ze_d)) \subseteq supp(e_a) \cup supp(\partial(e_f) - ze_d) \subseteq supp(e_a) \cup (B_0 \setminus \partial B_0) \subseteq B_0$ and hence by Lemma 13 $e_a(\partial(e_f) - ze_d)$ has diameter smaller than r. There are two cases to consider :

1. $\mu_r(e_ae_d) = e_ae_d - e_a\partial(e_{f_1})$ for some $f_1 \in F$ such that $supp(\partial(e_{f_1}) - e_d) \subseteq B_0 \setminus \partial B_0$;
2. $\mu_r(e_ae_d) = e_ae_d - \partial(he_{b_1})e_d$ for some $b_1 \in B, h \in N$ such that $supp(e_a - \partial(he_{b_1}) \subseteq B_0 \setminus \partial B_0$.

If 1. holds we have $supp(\partial(e_{f_1}) - e_d) \subseteq B_0 \setminus \partial B_0$ and $supp(\partial(e_f) - ze_d) \subseteq B_0 \setminus \partial B_0$, hence $supp(\partial(e_f) - z\partial(e_{f_1})) \subseteq B_0 \setminus \partial B_0$. Note that

$$\mu_r(\partial(e_a e_f)) = \mu_r(-e_a\partial(e_f)) = \mu_r(-e_a(\partial(e_f) - ze_d)) - z\mu_r(e_a e_d)$$

$$= -e_a(\partial(e_f) - ze_d) - z(e_a e_d - e_a\partial(e_{f_1})) = -e_a(\partial(e_f - ze_{f_1}))$$

and

$$supp(e_a(\partial(e_f - ze_{f_1}))) \subseteq supp(e_a) \cup supp(\partial(e_f - ze_{f_1})) \subseteq supp(e_a) \cup (B_0 \setminus \partial B_0) \subseteq B_0.$$

Then by Lemma 13 the diameter of $e_a(\partial(e_f - ze_{f_1}))$ is smaller than r. Note as well that

$$supp(e_f - ze_{f_1}) = supp(e_f) \cup supp(e_{f_1}) \subseteq supp(\partial e_f) \cup supp(\partial e_{f_1}) =$$

$$supp(\partial(e_f) - ze_d + ze_d) \cup supp(\partial(e_{f_1}) - e_d + e_d) \subseteq$$

$$supp(\partial(e_f) - ze_d) \cup supp(ze_d) \cup supp(\partial(e_{f_1}) - e_d) \cup supp(e_d) \subseteq supp(e_d) \cup (B_0 \setminus \partial B_0)$$

and by the construction of μ_r on the squares

$$\frac{q_2 - v}{\mid q_2 - v \mid} \in -\Gamma_M.$$

where $supp(e_d) = \{q_2\}$. Then by Lemma 12 there is t in $\sum_{f \in F} \mathbb{Z} e_f$ with support in the interior of B_0 and such that $\partial(t) = \partial(e_f - ze_{f_1})$. Then

$$\mu_r(\partial(e_a e_f)) = -e_a(\partial(e_f - ze_{f_1})) = -e_a(\partial t) = \partial(e_a t)$$

and we define

$$\mu_r(e_a e_f) = e_a t.$$

Observe that

$$supp(e_a t) \subseteq supp(e_a) \cup supp(t) \subseteq supp(e_a) \cup (B_0 \setminus \partial B_0) \subseteq B_0$$

and by Lemma 13 the diameter of $e_a t$ is smaller than r. Note that μ_r is a chain map as $\partial\mu_r(e_a e_f) = \partial(e_a t) = \mu_r\partial(e_a e_f)$.

If 2. holds by the construction of he_{b_1} and e_d we have that

$$supp(\partial(he_{b_1}) - e_a) \cup supp(\partial(e_f) - ze_d) \subseteq B_0 \setminus \partial B_0$$

Let $\lambda_1 = \partial(he_{b_1}) - e_a$ and $\lambda_2 = \partial(e_f) - ze_d$. Then

$$\partial(he_{b_1})\partial(e_f) - ze_a e_d = (\lambda_1 + e_a)(\lambda_2 + ze_d) - ze_a e_d = \lambda_1\lambda_2 + z\lambda_1 e_d + e_a\lambda_2.$$

Using Lemma 5 we have

$$supp(\lambda_1 e_d) \subseteq supp(\lambda_1) \cup supp(e_d) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_d) \quad .$$

and by Lemma 13 the diameter of $\lambda_1 e_d$ is smaller than r. Similarly the diameter of $e_a\lambda_2$ is smaller than r. Note that $\lambda_1\lambda_2$ is a $\mathbb{Z}$-linear combination of cells $\lambda_1 e_{d_1}$ for some $supp(e_{d_1}) \subseteq$

B_0, and $\lambda_1 e_{d_1}$ has diameter smaller than r. hence $\lambda_1\lambda_2$ has diameter smaller than r. Thus the diameter of $\partial(he_{b_1})\partial(e_f) - ze_ae_d$ is smaller than r. and

$$\mu_r(\partial(he_{b_1})\partial(e_f) - ze_ae_d) = \partial(he_{b_1})\partial(e_f) - ze_ae_d.$$

Then

$$\mu_r(\partial(he_{b_1})\partial(e_f)) = \mu_r(\partial(he_{b_1})\partial(e_f) - ze_ae_d) + \mu_r(ze_ae_d) = \partial(he_{b_1})\partial(e_f) - ze_ae_d + z\mu_r(e_ae_d) =$$

$$\partial(he_{b_1})\partial(e_f) - ze_ae_d + z(e_ae_d - \partial(he_{b_1})e_d) = \partial(he_{b_1})(\partial(e_f) - ze_d).$$

Note that by Lemma 5

$$supp((e_a - \partial(he_{b_1}))e_f) \subseteq supp(e_a - \partial(he_{b_1})) \cup supp(e_f) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_f) \subseteq B_0$$

and then by Lemma 13 the diameter of $(e_a - \partial(he_{b_1}))e_f$ is smaller than r. As for any element λ in $\tilde{\mathcal{P}}^{del}$ we have $supp(\partial(\lambda)) \subseteq supp(\lambda)$ we deduce that the diameter of $\partial((e_a - \partial(he_{b_1}))e_f)$ is smaller than r. In particular

$$\mu_r(\partial((e_a - \partial(he_{b_1}))e_f)) = \partial((e_a - \partial(he_{b_1}))e_f).$$

Note that

$$\partial(e_ae_f) = \partial((e_a - \partial(he_{b_1}))e_f) + \partial(\partial(he_{b_1})e_f) = \partial((e_a - \partial(he_{b_1}))e_f) - \partial(he_{b_1})\partial(e_f).$$

and hence

$$\mu_r(\partial(e_ae_f)) = \mu_r(\partial((e_a - \partial(he_{b_1}))e_f) - \partial(he_{b_1})\partial(e_f)) = \partial((e_a - \partial(he_{b_1}))e_f) - \mu_r(\partial(he_{b_1})\partial(e_f))$$

$$= \partial((e_a - \partial(he_{b_1}))e_f) - \partial(he_{b_1})(\partial(e_f) - ze_d) = \partial((e_a - \partial(he_{b_1}))e_f - he_{b_1}(\partial(e_f) - ze_d)).$$

We define

$$\mu_r(e_ae_f) = (e_a - \partial(he_{b_1}))e_f - he_{b_1}(\partial(e_f) - ze_d).$$

Note that by Lemma 5

$$supp(he_{b_1}(\partial(e_f) - ze_d)) \subseteq supp(he_{b_1}) \cup supp(\partial(e_f) - ze_d) \subseteq supp(e_{b_1}) \cup (B_0 \setminus \partial B_0) \subseteq B_0$$

and thus by Lemma 13 $he_{b_1}(\partial(e_f) - ze_d)$ has diameter smaller than r.

Then both $(e_a - \partial(he_{b_1}))e_f$ and $he_{b_1}(\partial(e_f) - ze_d)$ have diameter smaller than r, hence $\mu_r(e_ae_f)$ has diameter smaller than r. Finally μ_r is a chain map as

$$\partial\mu_r(e_ae_f) = \partial((e_a - \partial(he_{b_1}))e_f - he_{b_1}(\partial(e_f) - ze_d)) = \mu_r\partial(e_ae_f).$$

This completes the definition of μ_r on e_ae_f.

II. Now we define μ_r on e_be_d. Let B_0 be the smallest closed ball in $\mathbb{R}^n$ containing the support of e_be_d and let v be the center of B_0. Let $\lambda \in \mathbb{Z}[N]$ and $e_a \in \mathcal{A}$ be such that

$$supp(\partial(e_b) - \lambda e_a) \subseteq B_0 \setminus \partial B_0.$$

By Lemma 5

$$supp((\partial(e_b) - \lambda e_a)e_d) \subseteq supp(\partial(e_b) - \lambda e_a) \cup supp(e_d) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_d) \subseteq B_0$$

and by Lemma 13 $(\partial(e_b) - \lambda e_a)e_d$ has diameter smaller than r. There are two cases to consider :

1. $\mu_r(e_a e_d) = e_a e_d - e_a\partial(e_{f_1})$ for some $f_1 \in F$ such that $supp(e_d - \partial(e_{f_1})) \subseteq B_0 \setminus \partial B_0$.
2. $\mu_r(e_a e_d) = e_a e_d - \partial(he_{b_1})e_d$ for some $h \in N$, $b_1 \in B$ such that $supp(e_a - \partial(he_{b_1})) \subseteq B_0 \setminus \partial B_0$;

If 1. holds by Lemma 5

$$supp(e_b(e_d - \partial(e_{f_1}))) \subseteq supp(e_b) \cup supp(e_d - \partial(e_{f_1})) \subseteq supp(e_b) \cup (B_0 \setminus \partial B_0) \subseteq B_0$$

and by Lemma 13 $e_b(e_d - \partial(e_{f_1}))$ has diameter smaller than r. In particular

$$\mu_r(e_b(e_d - \partial(e_{f_1}))) = e_b(e_d - \partial(e_{f_1})).$$

We define

$$\mu_r(e_b e_d) = \mu_r(-\partial(e_b)e_{f_1} + e_b(e_d - \partial(e_{f_1}))) = -\mu_r(\partial(e_b)e_{f_1}) + e_b(e_d - \partial(e_{f_1})),$$

then

$$\mu_r(e_b e_d + \partial(e_b)e_{f_1}) = e_b(e_d - \partial(e_{f_1})),$$
$$\partial\mu_r(e_b e_d + \partial(e_b)e_{f_1})) = \partial(e_b(e_d - \partial(e_{f_1}))) = \partial(e_b e_d - e_b\partial(e_{f_1})) =$$
$$\partial(e_b e_d) - \partial(e_b)\partial(e_{f_1}) = \partial(e_b e_d + \partial(e_b)e_{f_1}) = \mu_r(\partial(e_b e_d + \partial(e_b)e_{f_1})).$$

The last equality comes from the fact that

$$supp(\partial(e_b e_d + \partial(e_b)e_{f_1})) = supp(\partial(e_b e_d - e_b\partial(e_{f_1}))) \subseteq supp(e_b(e_d - \partial(e_{f_1}))$$

has diameter smaller than r.

Note that by the definition of e_{f_1} we have that $supp(\partial(e_{f_1}) - e_d) \subseteq B_0 \setminus \partial B_0$ and hence $\partial(e_b)e_{f_1}$ is a sum of prisms of diameter smaller than r and special prisms of diameter r considered in case I, hence μ_r is already defined on $\partial(e_b)e_{f_1}$ and the restrictions of $\partial\mu_r$ and $\mu_r\partial$ on $\partial(e_b)e_{f_1}$ are already proved equal i.e. $\partial\mu_r(\partial(e_b)e_{f_1}) = \mu_r\partial(\partial(e_b)e_{f_1})$. Then

$$\partial\mu_r(e_b e_d) = \partial\mu_r(e_b e_d + \partial(e_b)e_{f_1})) - \partial\mu_r(\partial(e_b)e_{f_1}) =$$

$$\mu_r\partial(e_b e_d + \partial(e_b)e_{f_1}) - \mu_r\partial(\partial(e_b)e_{f_1}) = \mu_r\partial(e_b e_d)$$

i.e. μ_r is a chain map as required.

If 2. holds

$$\mu_r(\partial(e_b e_d)) = \mu_r(\partial(e_b)e_d) = \mu_r(\partial(e_b)e_d - \lambda e_a e_d) + \mu_r(\lambda e_a e_d) = \partial(e_b)e_d - \lambda e_a e_d + \mu_r(\lambda e_a e_d)$$

$$= \partial(e_b)e_d - \lambda e_a e_d + \lambda(e_a e_d - \partial(he_{b_1})e_d) = (\partial(e_b) - \lambda\partial(he_{b_1}))e_d = \partial(e_b - \lambda he_{b_1})e_d.$$

Note that

$$supp(\partial(e_b - \lambda he_{b_1})) = supp(\partial(e_b) - \lambda e_a + \lambda(e_a - \partial(he_{b_1}))) \subseteq$$

$$supp(\partial(e_b) - \lambda e_a) \cup supp(\lambda(e_a - \partial(he_{b_1}))) \subseteq B_0 \setminus \partial B_0$$

and

$$supp(e_b - \lambda he_{b_1}) \subseteq supp(e_b) \cup supp(e_{b_1}) = supp(\partial(e_b)) \cup supp(\partial(e_{b_1})) =$$
$$supp(\partial(e_b) - \lambda e_a + \lambda e_a) \cup supp(h\partial(e_{b_1}) - e_a + e_a) \subseteq supp(\partial(e_b) - \lambda e_a)$$
$$\cup supp(\lambda e_a) \cup supp(h\partial(e_{b_1}) - e_a) \cup supp(e_a) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_a).$$

By the definition of μ_r on the squares we have for $supp(e_a) = \{q\}$

$$\frac{q - v}{\mid q - v \mid} \in -\Gamma_N.$$

Then by Lemma 11 there is

$$t \in (\oplus_{b' \in B} \mathbb{Z}[N] e_{b'}) \oplus (\oplus_{a_1 < a_2} \mathbb{Z}[N] e_{a_1} e_{a_2})$$

such that

$$\partial(t) = \partial(e_b - \lambda he_{b_1}) \text{ and } supp(t) \subseteq B_0 \setminus \partial B_0.$$

Note that by Lemma 5

$$supp(te_d) \subseteq supp(t) \cup supp(e_d) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_d) \subseteq B_0$$

and by Lemma 13 te_d has diameter smaller than r. In particular as $supp(\partial(te_d)) \subseteq supp(te_d)$ we deduce that $\partial(te_d)$ has diameter smaller than r and hence

$$\mu_r \partial(te_d) = \partial(te_d) = \partial(t)e_d = \partial(e_b - \lambda he_{b_1})e_d = \mu_r \partial(e_b e_d).$$

We define

$$\mu_r(e_b e_d) = te_d,$$

and μ_r is a chain map as $\partial \mu_r(e_b e_d) = \partial(te_d) = \mu_r \partial(e_b e_d)$.

7.3 Defining μ_r on the non-special prisms

Consider representatives $e_a e_f$ and $e_b e_d$ of the Q-orbits of all prisms $\{e_a e_f, e_b e_d\}_{a \in A, f \in F, b \in B, d \in D}$ of diameter r. In this section we define μ_r on the representatives that are not special prisms and extend the definition of μ_r on the $(N \rtimes Q)$-orbits in the unique way such that μ_r commutes with the $(N \rtimes Q)$-action.

I. Let B_0 be the smallest closed ball in $\mathbb{R}^n$ that contains the support of $e_a e_f$ and v be its centre. Then as $r \geq m_1$ by Lemma 6 either

1. $\frac{q_1 - v}{|q_1 - v|} \in -\Gamma_N$, where $\{q_1\} = supp(e_a)$

or

2. for every q from the support of e_f we have $\frac{q-v}{|q-v|} \in -\Gamma_M$.

If 1. holds by Lemma 9(2) there exists e_{b_1} and $h \in N$ such that

$$supp(e_a - \partial(he_{b_1})) \subseteq B_0 \setminus \partial B_0$$

Hence by Proposition 2

$$supp(\partial((e_a - \partial(he_{b_1}))e_f)) \subseteq supp((e_a - \partial(he_{b_1}))e_f) \subseteq$$

$$supp(e_a - \partial(he_{b_1})) \cup supp(e_f) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_f) \subseteq B_0$$

and by Lemma 13 the diameters of $(e_a - \partial(he_{b_1}))e_f$ and $\partial((e_a - \partial(he_{b_1}))e_f)$ are smaller than r. In particular

$$\mu_r((e_a - \partial(he_{b_1}))e_f) = (e_a - \partial(he_{b_1}))e_f \text{ and}$$

$$\mu_r(\partial((e_a - \partial(he_{b_1}))e_f)) = \partial((e_a - \partial(he_{b_1}))e_f)$$

Note that

$$\partial(e_a e_f + he_{b_1}\partial(e_f) - (e_a - \partial(he_{b_1}))e_f) = \partial(he_{b_1}\partial(e_f) + \partial(he_{b_1})e_f) =$$

$$\partial(he_{b_1})\partial(e_f) - \partial(he_{b_1})\partial(e_f) = 0.$$

Then we define

$$\mu_r(e_a e_f) = -\mu_r(he_{b_1}\partial(e_f) - (e_a - \partial(he_{b_1}))e_f) = -\mu_r(he_{b_1}\partial(e_f)) + (e_a - \partial(he_{b_1}))e_f.$$

By the choice of e_{b_1} we have that $he_{b_1}\partial(e_f)$ is a sum of special prisms of diameter r and general prisms of diameter smaller than r. In particular μ_r is already defined on these elements and proved to commute with ∂. Then

$$\partial\mu_r(e_a e_f) = -\partial\mu_r(he_{b_1}\partial(e_f)) + \partial(e_a - \partial(he_{b_1}))e_f) = -\mu_r\partial(he_{b_1}\partial(e_f)) +$$

$$\mu_r\partial(e_a - \partial(he_{b_1}))e_f) = -\mu_r(\partial(he_{b_1}\partial(e_f) - (e_a - \partial(he_{b_1}))e_f)) = \mu_r\partial(e_a e_f).$$

If 2. holds by Lemma 9(1) applied several times for the elements e_d in the support of $\partial(e_f)$ that are on the boundary ∂B_0 there exists an element

$$\alpha \in \oplus_{f' \in F}\mathbb{Z}e_{f'}$$

such that

$$supp(\partial(\alpha + e_f)) \subseteq B_0 \setminus \partial B_0, supp(\alpha) \subseteq supp(e_f) \cup (B_0 \setminus \partial B_0)$$

and for every e_{f_0} such that there is a summand of α in $\mathbb{Z}e_{f_0}$ we have that either $supp(e_{f_0}) \subseteq B_0 \setminus \partial B_0$ and hence $e_a e_{f_0}$ has a diameter less than r, or $supp(e_{f_0}) = supp(\partial(e_{f_0})) \subseteq (B_0 \setminus \partial B_0) \cup \{$ just one point of $\partial B_0\}$, hence $e_a e_{f_0}$ is a special prism of diameter r. Then

$$\mu_r(e_a e_f) = \mu_r(e_a(e_f + \alpha)) - \mu_r(e_a \alpha).$$

By the above assumptions on α the product $e_a\alpha$ is a sum of special prisms of diameter r and general prisms of smaller diameter, hence μ_r is already defined on $e_a\alpha$ and commutes with ∂. Now we consider the element $e_f + \alpha$. Note that for $q \in supp(e_f + \alpha) \cap \partial B_0$ we have $q \in supp(e_f)$, hence

$$\frac{q - v}{\mid q - v \mid} \in -\Gamma_M.$$

Then by Lemma 12 there exists an element

$$t \in \oplus_{f' \in F} \mathbb{Z} e_{f'}$$

such that

$$supp(t) \subseteq B_0 \setminus \partial B_0 \text{ and } \partial t = \partial(e_f + \alpha).$$

Note that by Lemma 5

$$supp(e_a t) \subseteq supp(e_a) \cup supp(t) \subseteq supp(e_a) \cup (B_0 \setminus \partial B_0) \subseteq B_0$$

and by Lemma 13 $e_a t$ has diameter smaller than r. Then

$$\partial(e_a(e_f + \alpha)) = -e_a \partial(e_f + \alpha) = -e_a \partial t = \partial(e_a t) \text{ has diameter smaller than } r,$$

$$\mu_r(\partial(e_a(e_f + \alpha))) = \mu_r(\partial(e_a t)) = \partial(e_a t)$$

and we define

$$\mu_r(e_a(e_f + \alpha)) = e_a t.$$

Note that

$$\partial \mu_r(e_a(e_f + \alpha)) = \partial(e_a t) = \mu_r \partial(e_a(e_f + \alpha)),$$

hence μ_r is a chain map i.e.commutes with ∂.

II. Now we define μ_r on the representatives of Q-orbits of prisms of the type $e_b e_d$ that are not special. Fix one such representative. Let B_0 be the smallest closed ball in $\mathbb{R}^n$ that contains the support of $e_b e_d$ and v be its centre. Then as $r \geq m_2$ by Lemma 7 either

1. $\frac{q_1 - v}{|q_1 - v|} \in -\Gamma_M$ for $supp(e_d) = \{q_1\}$

or

2. for every q from the support of e_b we have $\frac{q - v}{|q - v|} \in -\Gamma_N$.

Now we can repeat the argument of case I. For completeness of the proof we give the details. If 1. holds by Lemma 9(1) there exists e_{f_1} such that $e_d - \partial(e_{f_1})$ has support in the interior of B_0. Note that

$$\partial(e_b e_d + \partial(e_b) e_{f_1} - e_b(e_d - \partial(e_{f_1}))) = \partial(\partial(e_b) e_{f_1} + e_b \partial(e_{f_1})) = -\partial(e_b)\partial(e_{f_1}) + \partial(e_b)\partial(e_{f_1}) = 0$$

and we define

$$\mu_r(e_b e_d) = -\mu_r(\partial(e_b) e_{f_1} - e_b(e_d - \partial(e_{f_1}))) = -\mu_r(\partial(e_b) e_{f_1}) + \mu_r(e_b(e_d - \partial(e_{f_1}))).$$

Note that by Proposition 2 $supp(e_b(e_d - \partial(e_{f_1}))) \subseteq supp(e_b) \cup supp(e_d - \partial(e_{f_1})) \subseteq supp(e_b) \cup (B_0 \setminus \partial B_0) \subseteq B_0$ and by Lemma 13 $e_b(e_d - \partial(e_{f_1}))$ has support of diameter smaller than r. Hence

$$\mu_r(e_b(e_d - \partial(e_{f_1}))) = e_b(e_d - \partial(e_{f_1})).$$

By the choice of e_{f_1} we have that $\partial(e_b) e_{f_1}$ is a sum of special prisms of diameter r and general prisms of diameter smaller than r. In particular μ_r is already defined on these elements and commutes with ∂. Hence

$$\partial \mu_r(e_b e_d) = -\partial \mu_r(\partial(e_b) e_{f_1}) + \partial \mu_r(e_b(e_d - \partial(e_{f_1}))) =$$

$$-\mu_r\partial(\partial(e_b)e_{f_1}) + \mu_r\partial(e_b(e_d - \partial(e_{f_1}))) = \mu_r\partial(e_be_d).$$

If 2. holds by Lemma 9(2) applied several times for the elements of $supp(\partial e_b)$ that are on the boundary ∂B_0 of B_0 there exists an element

$$\alpha \in \oplus_{b' \in B}\mathbb{Z}[N]e_{b'}$$

such that

$$supp(\partial(\alpha + e_b)) \subseteq B_0 \setminus \partial B_0, supp(\alpha) \subseteq supp(e_b) \cup (B_0 \setminus \partial B_0)$$

and for every e_{b_0} such that there is a summand of α inside $\mathbb{Z}[N]e_{b_0}$ we have that either $supp(e_{b_0}) \subseteq B_0 \setminus \partial B_0$ and hence $e_{b_0}e_d$ has a diameter less than r or $supp(e_{b_0}) \subseteq (B_0 \setminus \partial B_0) \cup \{$ one point of $\partial B_0\}$, hence $e_{b_0}e_d$ is a special prism of diameter r. Then we define

$$\mu_r(e_be_d) = \mu_r((e_b + \alpha)e_d) - \mu_r(\alpha e_d).$$

By the above assumptions on α the product αe_d is a sum of special prisms of diameter r and general prisms of diameter smaller than r, hence μ_r is already defined on αe_d and commutes with ∂.

Now we consider the element $e_b + \alpha$. Note that for $q \in supp(e_b + \alpha) \cap \partial B_0$ we have $q \in supp(e_b)$, hence $\frac{q-v}{|q-v|} \in -\Gamma_N$. Then by Lemma 11 there exists an element

$$t \in (\oplus_{b' \in B}\mathbb{Z}[N]e_{b'}) \oplus (\oplus_{a_1 < a_2}\mathbb{Z}[N]e_{a_1}e_{a_2}) \text{ such that } supp(t) \subseteq B_0 \setminus \partial B_0 \text{ and } \partial t = \partial(e_b + \alpha).$$

Then

$$supp(te_d) \subseteq supp(t) \cup supp(e_d) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_d) \subseteq B_0$$

and by Lemma 13 the diameter of te_d is smaller than r. Note that

$$\partial((e_b + \alpha)e_d) = (\partial(e_b + \alpha))e_d = (\partial t)e_d = \partial(te_d),$$

$$\mu_r\partial((e_b + \alpha)e_d) = \mu_r(\partial(te_d)) = \partial(te_d)$$

where the last equality comes from the fact that $supp(\partial(te_d)) \subseteq supp(te_d)$ has diameter smaller than r. Finally we define

$$\mu_r((e_b + \alpha)e_d) = te_d$$

and verify that μ_r is a chain map

$$\mu_r\partial((e_b + \alpha)e_d) = \partial(te_d) = \partial\mu_r((e_b + \alpha)e_d).$$

7.4 Defining μ_r on the cubes

To define μ_r on cubes it is sufficient to choose representatives of the Q-orbits of cubes $e_{a_1}e_{a_2}e_d$, where $a_1 < a_2$ are elements of A, $d \in D$. Fix one such representative $e_{a_1}e_{a_2}e_d$ and let B_0 be the smallest closed ball in $\mathbb{R}^n$ that contains the support of $e_{a_1}e_{a_2}e_d$. Let v be the centre of B_0. Note that the support of $e_{a_1}e_{a_2}e_d$ is $supp(e_{a_1}) \cup supp(e_{a_2}) \cup supp(e_d)$ and at least two elements of this support lie on the boundary of B_0. The case when precisely two elements of the support lie on the boundary of B_0 can occur when these 2 elements are antipodal. We construct μ_r on $e_{a_1}e_{a_2}e_d$ by induction on the number $j = j(e_{a_1}e_{a_2}e_d)$ of elements in the set $supp(e_{a_1}e_{a_2}e_d) \cap \partial B_0$. Assume we have constructed μ_r for cubes of diameter r with number of support elements on the boundary of the minimal closed ball that contains the support strictly smaller than j. We aim to construct a chain map

$$\mu_{r,j} : \tilde{\mathcal{P}}^{(r,j)} \to \tilde{\mathcal{P}}^{(r,j-1)},$$

where $\tilde{\mathcal{P}}^{(r,i)}$ is the subcomplex of $\tilde{\mathcal{P}}$ spanned by all cells of diameter smaller than r, all prisms and squares of diameter r, all cubes c of diameter r such that $j(c) \leq i$ and $\mu_{r,j}$ is identity on $\tilde{\mathcal{P}}^{(r,j-1)}$. Once μ_r is defined on $\tilde{\mathcal{P}}^{(r,j-1)}$ we extend it to $\tilde{\mathcal{P}}^{(r,j)}$ as the composition

$$\tilde{\mathcal{P}}^{(r,j)} \xrightarrow{\mu_{r,j}} \tilde{\mathcal{P}}^{(r,j-1)} \xrightarrow{\mu_r} \tilde{\mathcal{P}}^{(r,j-1)}.$$

From now on we work on the construction of the map $\mu_{r,j}$.

Note that the definition of B_0 as the smallest closed ball that contains the support of $e_{a_1}e_{a_2}e_d$ implies that the center v of B_0 is inside or on the boundary of the triangle spanned by the support of $e_{a_1}e_{a_2}e_d$. As by assumption $0 \notin conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q)) + (\mathbb{R}_{>0}\Sigma^c_M(Q))$ and $0 \notin conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^c_N(Q))$ one of the following holds:

1. $supp(e_{a_1}) = \{q_1\}$,$q_1 \in \partial B_0$ and $\frac{q_1 - v}{|q_1 - v|} \in -\Sigma_N(Q)$,
2. $supp(e_{a_2}) = \{q_2\}$,$q_2 \in \partial B_0$ and $\frac{q_2 - v}{|q_2 - v|} \in -\Sigma_N(Q)$,
3. $supp(e_d) = \{q_3\}$,$q_3 \in \partial B_0$ and $\frac{q_3 - v}{|q_3 - v|} \in -\Sigma_M(Q)$.

As a_1, a_2 have an antisymmetric role we can assume that 1 or 3 holds. The case 2 is the same as case 1.

If 1. holds by Lemma 9(2) there exists a cell e_{b_1} and $h \in N$ such that

$$supp(\partial(he_{b_1}) - e_{a_1}) \subseteq B_0 \setminus \partial B_0 \tag{8}$$

We claim that

$$\partial(e_{a_1}e_{a_2}e_d + (d_1(a_2) - 1)he_{b_1}e_d - (e_{a_1} - \partial(he_{b_1}))e_{a_2}e_d) = 0, \tag{9}$$

hence

$$\mu_{r,j}\partial(e_{a_1}e_{a_2}e_d + (d_1(a_2) - 1)he_{b_1}e_d - (e_{a_1} - \partial(he_{b_1}))e_{a_2}e_d) = 0.$$

Note that $e_{b_1}e_d$ is a prism, hence

$$(d_1(a_2) - 1)he_{b_1}e_d \in \tilde{\mathcal{P}}^{(r,j-1)}.$$

By (8) $(e_{a_1} - \partial(he_{b_1}))e_{a_2}e_d$ is a $\mathbb{Z}$-linear combination of cubes of type $ne_ae_{a_2}e_d$, where $n \in N$ and $supp(e_a) \in B_0 \setminus \partial B_0$. In particular $supp(ne_ae_{a_2}e_d) = supp(e_a) \cup supp(e_{a_2}) \cup supp(e_d) \subseteq (B_0 \setminus \partial B_0) \cup supp(e_{a_2}) \cup supp(e_d)$ and hence $ne_ae_{a_2}e_d \in \tilde{\mathcal{P}}^{(r,j-1)}$. Thus

$$(e_{a_1} - \partial(he_{b_1}))e_{a_2}e_d \in \tilde{\mathcal{P}}^{(r,j-1)}.$$

We define $\mu_{r,j}(e_{a_1}e_{a_2}e_d)$ to be

$$-\mu_{r,j}((d_1(a_2) - 1)he_{b_1}e_d - (e_{a_1} - \partial(he_{b_1}))e_{a_2}e_d) = -(d_1(a_2) - 1)he_{b_1}e_d + (e_{a_1} - \partial(he_{b_1}))e_{a_2}e_d.$$

Then by (9)

$$\partial\mu_{r,j}(e_{a_1}e_{a_2}e_d) = \partial(-(d_1(a_2)-1)he_{b_1}e_d+(e_{a_1}-\partial(he_{b_1}))e_{a_2}e_d) = \partial(e_{a_1}e_{a_2}e_d) = \mu_{r,j}\partial(e_{a_1}e_{a_2}e_d),$$

where the last equality follows from the fact that $\partial(e_{a_1}e_{a_2}e_d) \in \widetilde{\mathcal{P}}^{r,j-1}$. Now we prove the claim

$$\partial(e_{a_1}e_{a_2}e_d+(d_1(a_2)-1)he_{b_1}e_d-(e_{a_1}-\partial(he_{b_1}))e_{a_2}e_d) = \partial((d_1(a_2)-1)he_{b_1}e_d+\partial(he_{b_1})e_{a_2}e_d) =$$

$$(d_1(a_2)-1)\partial(he_{b_1})e_d + \partial(\partial(he_{b_1})e_{a_2})e_d = ((d_1(a_2)-1)\partial(he_{b_1}) + \partial(\partial(he_{b_1})e_{a_2}))e_d = 0e_d = 0$$

If 3. holds by Lemma 9(1) there exists e_f such that $e_d - \partial(e_f)$ has support in the interior of B_0. As in the previous case it is sufficient to show that

$$\partial(e_{a_1}e_{a_2}e_d + \partial(e_{a_1}e_{a_2})e_f - (e_d - \partial(e_f))e_{a_1}e_{a_2}) = 0$$

and then as $\partial(e_{a_1}e_{a_2})e_f - (e_d - \partial(e_f))e_{a_1}e_{a_2} \in \mathcal{P}^{(r,j-1)}$ we can define

$$\mu_{r,j}(e_{a_1}e_{a_2}e_d) = -\mu_{r,j}(\partial(e_{a_1}e_{a_2})e_f - (e_d - \partial(e_f))e_{a_1}e_{a_2}) = -\partial(e_{a_1}e_{a_2})e_f + (e_d - \partial(e_f))e_{a_1}e_{a_2}.$$

Note that

$$\partial\mu_{r,j}(e_{a_1}e_{a_2}e_d) = \partial(-\partial(e_{a_1}e_{a_2})e_f + (e_d - \partial(e_f))e_{a_1}e_{a_2}) = \partial(e_{a_1}e_{a_2}e_d) = \mu_{r,j}\partial(e_{a_1}e_{a_2}e_d),$$

where the last equality follows from the fact that $\partial(e_{a_1}e_{a_2}e_d) \in \widetilde{\mathcal{P}}^{(r,j-1)}$. Finally to justify the claim note that $\partial(e_{a_1}e_{a_2}e_d + \partial(e_{a_1}e_{a_2})e_f - (e_d - \partial(e_f))e_{a_1}e_{a_2})$ is equal to

$$\partial(\partial(e_{a_1}e_{a_2})e_f + e_{a_1}e_{a_2}\partial(e_f)) = -\partial(e_{a_1}e_{a_2})\partial(e_f) + \partial(e_{a_1}e_{a_2})\partial(e_f) = 0.$$

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IMECC-UNICAMP
Universidade Estadual de Campinas
CP 6065
13083-970, Campinas-SP
Brasil