MINIMAL IMMERSIONS AND HARMONIC MAPS FROM RIEMANN SURFACES TO COMPLEX FLAG MANIFOLDS

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Dedicated to Prof. M. do Carmo on
the occasion of his 70th birthday

Abstract. Some beautiful and important results in minimal and harmonic surfaces on complex flag manifolds are looked back. And several interesting open problems have been proposed. .

§1. Introduction

There is plantiful accomploishment in harmonic surfaces on complex flag manifold, in particular, conformal harmonic surfaces on complex Grassmannians (equivalently, branched minimal surfaces). Consider flag manifolds as reductive homogeneous spaces, they can be distributed two classes. The one

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has height 1, i.e. symmetric complex flag manifolds, i.e. complex Grassmannians. The another has height ≥ 2, i.e. non-symmetric flag manifolds. The propose of the article is reviev some developments following three aspects:

1. Harmonic surfaces on a complex projective space including: Eells-Wood' classification; isotropic condition; Veronese sequence; Keahler angle and curvature, non-isotropic harmonic surfaces; harmonic sequence;

2. The relation between harmonic surfaces on symmetric and non-symmetric flag manifolds;

3. Harmonic surfaces on non-symmetric complex flag manifold.

§2. Minimal and harmonic surfacs

Let (M, g) and (N, h) be two smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map, which we suppose $\operatorname{supp}\phi := \overline{\{x \in M, d\phi(x) \neq 0\}} \subset D$ (a compact doman of M). The energy of ϕ is the number

$$E(\phi) := \frac{1}{2}\int_D < g, \phi^* h > {}^*1$$

A map $\phi : (M, g) \to (N, h)$ is harmonic if and only if it is an extremal of the energy, i.e. for arbitray a family of maps ϕ_t such that $\phi_0 = \phi$ and $\operatorname{supp}\phi_t \subset D$. Then $\frac{d}{dt}|_{t=0}E(\phi_t) = 0$.

Now we consider maps from Riemann surface M. A branched minimal immersion from M to (N, h) is a weakly conformal harmonic map.

In this paper, we restrict N to complex flag manifold, i.e.

$$N := \frac{U(n)}{U(r_1) \times \cdots \times U(r_k)}; \quad r_1 + \cdots + r_k = n$$

In particular, when $k = 2$, N is symmetric, we call N complex Grassmannian, i.e.

$$N := \frac{U(n)}{U(r_1) \times U(n - r_1)} := G(r_1, n)$$

Specially, N is said $(n-1)-$dimensional complex projective space if $r_1 = 1$, i.e.

$$N := \frac{U(n)}{U(1)\times U(n-1)} := G(1,\,n) := \mathbb{CP}^{n-1}$$

§3. $\mathbb{CP}^n$ CASE

Study of harmonic surfaces has a long history which contains many beautiful and interesting results. However one of the most famous results, I think, is Eells-Wood' work. Their result was influced from two aspects.

Suppose that $\phi : S^2 \to S^n \subset \mathbb{R}^{n+1}$ is a minimal immersion. In 1967, E. Calabi[Ca] showed that ϕ associate a holomorphic map $f : S^2 \to \mathbb{CP}^n$ (the directix of ϕ). Furthermore, if ϕ is full, i.e. $\mathrm{Im}\phi \nsubseteq S^{n-1}$, then $n = 2m$, for some $m \in \mathbb{Z}$ and the area of ϕ with respect to induced metric on S^2 is given by

$$A(\phi) = 4\pi d, \qquad \mathbb{Z} \ni d \geq \frac{m(m+1)}{2}$$

13 years later two physicsists, A.M.Din and W.J.Zakrzewski [DZ1,DZ2] given a correspondence from

$$\{\phi : S^2 \xrightarrow[\text{full}]{\text{harmonic}} \mathbb{CP}^n\}$$

onto

$$\left\{(f,\,r)|f : S^2 \xrightarrow[\text{full}]{\text{holomorphic}} \mathbb{CP}^n, \quad r \in \{0,\cdots,n\}\right\}$$

After that, J.Eells and J.C.Wood discussed arbitrary harmonic surfaces M on $\mathbb{CP}^n$[EW]. Their creative work are: starting from full holomorphic map $f : M \to \mathbb{CP}^n$, define

$$\phi = \phi_r =: f_{r-1}^{\perp} \cap f_r \tag{1}$$

for $r \in \{0, 1, \cdots, n\}$, where

$$f_\alpha := [f_u \wedge \frac{\partial f_u}{\partial z} \wedge \cdots \wedge \frac{\partial^\alpha f_u}{\partial z^\alpha}] : M \to G(\alpha+1,\,n+1)$$

is α-th associate curve of f and z is the local complex coordinate on M. They proved that ϕ_r is a full harmonic map from M to $\mathbb{CP}^n$. Furthermore, if $0 < r < n$, they showed that ϕ is not holomorphic and anti-holomorphic. They called ϕ_r isotropic map. And (f, r) is the directix of ϕ_r, i.e.

However, L.Lemaire constructed many non-isotropic harmonic maps from surface into $\mathbb{CP}^1 = S^2$. Hence finding the isotropy conditions for harmonic surfaces on $\mathbb{CP}^n$ is important and interesting.

§4. Isotropy

It is clear that any harmonic topological sphere is isotropic by Din-Zakrzewski (ref. §3). In their article[EW], J.Eells and J.C.Wood showed that any harmonic topological torus with non-zero Brouwer degree is isotropic, too.

For arbitrary closed Riemann surface M with genus g, the isotropic condition due to Jensen and Regoli, in 1989, is

$$r(\partial_0) + r(\bar{\partial}_0) \geq 2(n+1)(g-1)$$

where $r(\partial_0)$ (resp. $r(\bar{\partial}_0)$ is the ∂' (resp. ∂'') second fundamental form of ϕ and $r(\partial_0)$ denote the ramification index of ∂_0.

Given $\mathbb{CP}^n$ standard Fubini-Study metric such that it has constant holomorphic sectional curvature 4. The first author showed that, in 1991, any harmonic map $\phi : M \to \mathbb{CP}^n$ is isotropic whenever $|\deg\phi| > |2(g-1)E(\phi)/\pi|$ where $E(\phi)$ denote the energy of ϕ (ref. §2).

In 1992, using the method of algebra geometry Dong[D] obtained isotropic condition by completely topological invariants :i.e.

$$|\deg\phi| > 2(n-1)(g-1)$$

Up to now, Liao's result maybe is most beautiful in this field[L], i.e.

$$|\deg\phi| > n(g-1)$$

Perhaps the most important isotropic harmonic maps from surfaces into $\mathbb{CP}^n$ are Veronese curve and its associate harmpnic topological spheres. Their history can be returned to Boruvka sphere. 1933, Boruvka[B] constructed minimal spheres in S^{2p} with constant Gauss curvature $\frac{1}{p(p+1)}$. When $p=2$, $S^2 \xrightarrow{\text{min.}} S^4$ is called Veronese surface. It is a very interesting example of minimal immersion in Chern-do Carmo-Kobayashi' pinching theorem[CDK]. Suppose $X : M^m \to S^n$ is a minimal immersion from closed m-dimensional Riemannian manifold into n-dimensional standard Euclidean sphere, and B its second fundamental form. If $\sigma := \|B\|^2 \leq \frac{n}{q}(q := 2 - \frac{1}{n-m})$. then either $\sigma \equiv 0$ ($\Longleftrightarrow X$ is totaqlly geodesic) or $\sigma \equiv \frac{n}{q}$ ($\Longleftrightarrow M$ is Veronese surface or Clifford minimal hypersurface). Thinking S^n as the bouble cover space of real projective space $\mathbb{RP}^n$, and $\mathbb{RP}^n$ as totally geodesic submanifold of $\mathbb{CP}^n$, 1987, S.Bando and Y.Ohnita[BO] constructed Veronese holomorphic curve

$$f(z) = \left[1, \sqrt{\binom{n}{1}}z, \cdots, \sqrt{\binom{n}{r}}z^r, \cdots, z^n\right]$$

where z is the complex coordinate on $S^2 \cong \mathbb{C} \cap \{\infty\}$ and its associated topological harmonic spheres (ref. §1)

$$\phi_1, \phi_2, \cdots, \phi_n$$

In particular, when $n = 2p$, ϕ_p is Brouvka sphere up to a holomorphic isotropy of $\mathbb{CP}^n$. Usually $f = \phi_0, \phi_1, \cdots, \phi_n$ is called Veronese sequence. Furthermore, Bando and Ohnita showed that each ϕ_j $(j = 0, \cdots, n)$ has constant (Gauss) curvature with respect to induced metric. Then, a nice characterization of all full harmonic two-sphere on $\mathbb{CP}^n$ with constant Gauss curvature was obtained by J.Bolton, G.Jensen, G.Regoli and L.M.Woodward[BJRW]. They proved these spheres belong to Veronese sequence. Moreover, each element of Veronese sequence has constant Kähler angle.

§5. Kähler angle

The Kähler angle for minimal immersion from Riemannian surface into Kähler manifold was introduced by S.S.Chern and J.G.Wolfson in 1983[CW1]. Let $\phi : M \to \mathbb{CP}^n$ be a minimal immersion and Ω the Kähler form of $\mathbb{CP}^n$ with respect to its standard Fubini-Study metric, then the Kähler angle, θ, of ϕ is defined by

$$\phi^*\Omega = \cos\theta dA$$

where dA is area element related to metric on M. Here $\cos\theta$ is called Kähler function[EGT]. Kähler angle is very important geometric invariant for minimal immersion into $\mathbb{CP}^n$ because it gives a measure of the failure of ϕ to be holomorphic. Precisely, ϕ is holomorphic map if and only if its Kähler angle θ is 0; ϕ is anti-holomorphic if and only if θ is π; and ϕ is totally real if and only if θ is $\frac{\pi}{2}$, where ϕ is totally real means that

$$J(\phi_* TM) \subset T^\perp M$$

(J is complex structure of $\mathbb{CP}^n$ and $T^\perp M$ is the normal space of ϕ).

There are several interesting open problem on relation between Kähler angle and Gauss curvature. For instance:

(1)(by Bolton-Jensen-Regoli-Woodward) Let $\psi : S^2 \to \mathbb{CP}^n$ be a minimal immersion with constant Kähler angle, and suppose that ϕ is neither holomorphic, antiholomorphic or totally real. Then ϕ has constant Gauss curvature[BJRW].

(2)(by Y. Ohnita) Suppose that $\phi : M \to \mathbb{CP}^n$ is a minimal immersion from Riemannian surface with constant Gauss curvature. Then ϕ has constant Kähler angle[O].

Question (1) was considered firstly by Bolton-Jensen-Rigoli-Woodward in 1988. They gave a positive answer when $n \leq 4$[BJRW].

In 1994, the first author obtained another additional condition for this problem, i.e. $|\cos\theta| \geq \frac{1}{5}$, where $\cos\theta$ is Kähler function of minimal immersion[M2]. Exactly, Bolton et al's conjecture holds if θ is not too close to $\frac{\pi}{2}$.

However, in 1995, Z.Li found a counterexample $\phi : S^2 \to \mathbb{CP}^{10}$ for (1) with $\cos\theta = \frac{1}{7}$[Li]. Hence, now, as far I know (1) become

(1)' If $5 \le n \le 9$ or $|\cos\theta| \in (\frac{1}{7}, \frac{1}{5})$, is conjecture (1) true?

About the question (2). It is easy to see, by combining two results due to Bolton-Jensen-Rigoli-Woodward (ref. §4), Ohnita's conjecture is true for any topological harmonic two sphere. For the Riemannian surface with arbitriry genus, recently, Kenmotsu and Masuda getted a positive answer when $n = 2$[KM]. Together with Eschenburg-Guadalupe-Tribuzy' classification theorem in this case M is one of the following

(1). holomorphic and totally geodesic $\mathbb{CP}^1$;

(2). Veronese curve (ref. §4);

(3). totally real and totally geodesic $\mathbb{RP}^2$;

(4). Clifford flat torus.

Notice that Clifford torus is non-isotropic harmonic surface apart from Examples due to Lemaire[Le]. Its construction originates Kenmotsu's totally real minimal planes in $\mathbb{CP}^n$. Suppose z is the complex coordinate on $\mathbb{R}^2$, defined $\phi : \mathbb{R}^2 \to \mathbb{CP}^n$ by

$$\phi(z) = \left[r_0 e^{\mu_0 z - \overline{\mu_0 z}}, \cdots, r_n e^{\mu_n z - \overline{\mu_n z}} \right] \tag{2}$$

where $r_0, \cdots, r_n$ are non-negative real numbers, and

$$\mu_0, \cdots, \mu_n$$

are complex numbers of unit modulus satisfying $\sum_{j=0}^{n} r_j^2 = 1$. In 1985, Kenmotsu showed that ϕ is an non-isotropic totally real minimal immersion if $c_1 = c_2 = 0$ where $c_k := \sum_{j=0}^{n} r_j^2 \mu_j^k$.

Using the method of moving frame, the first author getted the harmonic equation of examples of (1), i.e.

$$c_1 + \bar{c}_1 c_2 = 2c_1 |c_1|^2 \tag{3}$$

Furthermore, the author proved that all harmonic surfaces which come from (2) and (3) satisfy $\phi^*\Omega \equiv 0$[M3].

Recently, Jensen and Liao reexamined (2)[JL]. They showed that Clifford solution of (2), i.e.

$$\mu_j = e^{\frac{2\pi j}{n+1}\sqrt{-1}} \qquad r_j = \frac{1}{n+1}$$

factors through T^2 if and only if $n = 2, 3, 5$. Moreover, ϕ has isotropy order $\geq r$ when $c_3 = \cdots = c_r = 0$ where ϕ has isotropy order k means that ϕ is orthogonal to $\phi_1, \cdots, \phi_k$ but not to ϕ_{k+1} and $\phi = \phi_0, \phi_1, \cdots$ is the harmonic sequrnce[W].

§6. Harmonic sequence and harmonic covering

For arbitrary harmonic map $\phi : M \to \mathbb{CP}^n$, the image of the second fundamental forms are still harmonic[CW2,BW]. Repeated these process, J.G.Wolfson obtained a beautiful sequence of harmonic maps

$$\phi_0, \phi_1, \phi_2, \cdots$$

called harmonic sequence. Furthermore, Wolfson showed that harmonic map $\phi : M \to \mathbb{CP}^n$ is isotropic if and only if the isotropy order of $\phi = +\infty$[W]. In this case, the second author proved that $\Phi = (\phi_0, \phi_1, \cdots, \phi_n) : M \to F(n+1)$ is a equi-harmonic map, i.e., each isotropic harmonic map can be covered by some equi-harmonic (i.e. harmonic with respect to each left invariant metric) map into full complex flag manifold[N1]. Among all non-isotropic harmonic surfaces on $\mathbb{CP}^n$, perhaps most important ones are superconformal harmonic surfaces. They are defined by the surfaces with orthogonal periodic harmonic sequence.

In 1993, Bolton, Pedit and Woodward established the bridge between superconformal harmonic surfaces and equi-harmonic surfaces on full flag manifolds[BPW]. Their result is: for each superconformal harmonic map $\phi : M \to \mathbb{CP}^n$,

$$\Phi = (\phi_0, \cdots, \phi_n) : M \to F(n+1)$$

is a equi-harmonic map, too. Recently, Burstall considered all finite isotropy order conformal (i.e. isotropy order≥ 2) harmonic surfaces in $\mathbb{CP}^n$. He showed that

$$\Phi := \big(\phi_0, \phi_1, \cdots, \phi_{r-1}, (\phi_0 \oplus \cdots \oplus \phi_{r-1})^{\perp}\big)$$

$: M \to F(\underbrace{1, \cdots, 1}_{r}, n+1-r; n+1)$ is a equi-harmonic map[B,M].

§7. Non-holomorphic harmonic maps into flag manifolds

Equi-harmonic surfaces on complex flag manifolds play a central role because there are many left-invariant metrics on a non-symmetric flag manifold and the relative metric induced by restricting the Killing form, denoted by g, is not well behaved from the point of complex geometry. For instance, the second author showed that g is never Kähler metric[N2]. Recently, authors have proved that g is $(1, 2)$-symplectic if and only if the height of flag manifold $= 2$[MN1].

However, in symmetric flag manifold case, up to a positive number, we have unique left-invariant metric, i.e. standard Fubini-Study metric. Hence equi-harmonicity is equivalent to harmonicity.

The fundamental aspect of equi-harmonic surfaces on complex flag manifolds is to construct non-holomorphic equi-harmonic surfaces. The main reason is:

(1). In the symmetric case, Lichnerowicz[Li] (also see [Ra]) each holomorphic curve on complex Grassmannian is harmonic;

(2). In non-symmetric case, Black asserted that each $f-$holomorphic curve on flag manifold with respect to some horizontal $f-$structure is equi-harmonic.

Many non-holomorphic harmonic surfaces on $\mathbb{CP}^n$ or $G(k,\, n)$ have been constructed by several people, such as Eells-Wood for $\mathbb{CP}^n$ (ref. §2)[EW], Erdem-Wood[ErW] and Ishihara[I]. Recently, authors manufactured a large class of equi-harmonic tori on non-symmetric complex flag manifold which are not $f-$holomorphic with respect to each horizontal $f-$structure [MN2].

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