Global Asymptotic Stability on Euclidean Spaces

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Abstract

This paper provides sufficient conditions for global asymptotic stability of autonomous dynamical systems on euclidean spaces. For dimension greater than two, the technique combines a version of the argument used by Olech on the bidimensional case and Lyapunov method. A Palais-Smale type condition is used to study the behaviour of unbounded orbits. Global stability for the bidimensional problem is established under hypotheses which do not imply the Markus-Yamabe condition.

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1 Introduction

In this article we study the global asymptotic stability of the autonomous system

$$(AS) \qquad \dot{u}(t) = X(u(t)).$$

where $X : I\!R^m \to I\!R^m$ is a vector field of class C^1 satisfying $X(0) = 0$. We also suppose the origin is a local asymptotic attractor for system (AS).

In our first result, we assume that $m \geq 3$ and write $I\!R^m = I\!R^{2+n} = I\!R^2 \times I\!R^n$ and $X = (F, G) : I\!R^{2+n} \to I\!R^{2+n}$. To establish the global asymptotic stability of system (AS) on this case, we suppose that Markus-Yamabe condition holds on the plane $I\!R^2 = I\!R^2 \times \{0\}$. We also assume the existence of a Lyapunov function on $I\!R^{2+n} \setminus I\!R^2$ satisfying a Palais-Smale type condition with respect to the vector field X. The technique used combines a version of Olech's argument for the planar problem with the well known Lyapunov method.

We recall that a vector field $X : I\!R^m \to I\!R^m$ satisfies the Markus-Yamabe condition [denoted (MY)] if the eigenvalues of $X'(u)$ have negative real part for every $u \in I\!R^m$. By $\mathcal{X}$,

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we denote the space of vector fields of class C^1 from $I\!R^m$ on itself which have the origin as a local attractor for the associated system. The following condition is our basic assumption

(H0) $I\!R^m = I\!R^{2+n} = I\!R^2 \times I\!R^n$, $X = (F, G) : I\!R^{2+n} \to I\!R^{2+n}$ and there exist C^1 maps $L : I\!R^2 \to I\!R^2$, $H : I\!R^{2+n} \to I\!R^2$ satisfying
(i) L satisfies (MY) condition on $I\!R^2$.
(ii) $F(x, y) = L(x) + H(x, y)$, for every $(x, y) \in I\!R^{2+n}$,
(iii) $X(x, 0) = (L(x), 0)$, for every $x \in I\!R^2$.

As observed above, our results are also based on the existence of a Lyapunov function for system (AS). More specifically, we suppose

(H1) There exists a function $V \in C^1(I\!R^{2+n}, [0, \infty))$ satisfying
(i) $\lim_{\|x\| \to \infty} \inf\{V(x, y) \mid \|y\| \geq \delta\} > 0$, for every $\delta > 0$,
(ii) $< \nabla V(x, y), X(x, y) >< 0$, for every $(x, y) \in I\!R^{2+n} \setminus I\!R^2$,

It is worthwhile to mention that condition (H1) does not imply that the origin is a global attractor for (AS) since we may have $V(x, 0) = 0$ for every $x \in I\!R^2$ (See the applications in section 4). Moreover, we emphasize that our Lyapunov conditions do not imply that the solutions of the system are bounded at all. The following conditions allow us to use a variant of Olech's argument [9] for the planar case. Considering $X \in \mathcal{X}$, L given by (H0), and the Lyapunov Function V, given by (H1), we assume

(H2) There exist $c, M, R > 0$ and $\rho \in (0, \infty]$ such that, for every $\|x\| > R$, $\|y\| < \rho$, we have
(i) $| < L(x)^\perp, H(x, y) > | \leq MV(x, y)$,
(ii) $< \nabla V(x, y), X(x, y) >\leq -cV(x, y)$,
and

(H3) There exists $\delta \in [0, 1)$ such that

$$\lim_{\|x\| \to \infty, \|y\| \to 0} \frac{\|H(x, y)\|}{\|L(x)\|} \leq \delta.$$

In (H2), $L^\perp$ represents the vector field orthogonal to L, obtained by a counterclockwise rotation. The folowing definition introduces the notion of Palais-Smale condition [1, 11] with respect to a given vector field X,

Definition 1.1 *Given a vector field $X \in C(I\!R^m, I\!R^m)$, we say that the $V \in C^1(I\!R^m, I\!R)$ satisfies the Palais-Smale condition with respect to X at level $c \in I\!R$ [denoted $(PS)_{(X,c)}$] if every sequence $(u_k) \subset I\!R^m$ such that $V(u_k) \to c$ and $< \nabla V(u_k), X(u_k) >\to 0$, as $k \to \infty$, possesses a bounded subsequence.*

Note that $V \in C^1(I\!R^m, I\!R)$ satisfies $(PS)_c$ condition for $c \in I\!R$, if it satisfes $(PS)_{(\nabla V,c)}$. Now, we are able to state our first result,

Theorem A *Suppose $X \in \mathcal{X}$ satisfies (H0)-(H3), with V satisfying $(PS)_{(X,c)}$ condition for every $c > 0$. Assume further the semi-completivity condition of the solutions of (AS) (i.e. they are defined on $[0,\infty)$). Then, the origin is a global attractor for system (AS).*

The proof of Theorem A is obtained by the verification of two basic steps: First, we use conditions (H0)-(H1) and the fact that V satisfies $(PS)_{(X,c)}$, for $c > 0$,to verify that orbits of (AS) which do not converge to the origin must approach asymptotically the plane $I\!R^2$. Then, we apply a variant of Olech's argument [9] to conclude that the origin is a global attractor for (AS). Concerning the semi-completivity condition assumed above, we observe that in [2] is implied by some geometric hyphoteses which could be useful in our context.

We note that, by Gutierrez [6] (See also [4, 5]) and (H0), $L(x) = X(x,0) : I\!R^2 \to I\!R^2$ is an injective vector field. Consequently, by [9], the origin is a global attractor for the orbits on the plane $I\!R^2$.

In the second part of this article, we present a result of global asymptotic stability for system (AS) on $I\!R^2$ when (MY) condition does not hold. Setting $S_{(-c,c)}(f) = \{u \in I\!R^m \,|\, -c \leq f(u) \leq c\}$, for $f : I\!R^m \to I\!R$ and $c \geq 0$, and denoting by X_i, $i = 1, 2$, the i-coordinate of $X : I\!R^2 \to I\!R^2$, we suppose

(H4) Trace$(X'(u)) < 0$, for every $u \in I\!R^2$,

(H5) There exists $c > 0$ such that

$$\det(X'(u)) \neq 0, \ \forall\, u \in S_{(-c,c)}(X_1),$$

(H6) $\nabla X_1(u) \neq 0$, for every $u \in I\!R^2$.

Recalling that $f \in C^1(I\!R^m, I\!R)$ satisfies (PS) condition when it satisfies $(PS)_c$ for every $c \in I\!R$, we may state

Theorem B *Suppose $X : I\!R^2 \to I\!R^2$ belongs to $\mathcal{X}$ and satisfies (H4)-(H6), with X_1 satisfying (PS). Then, the origin is a global attractor for system (AS).*

If we assume the following version of condition (H0),

$(\hat{H}0)$ $I\!R^m = I\!R^{2+n} = I\!R^2 \times I\!R^n$, $X = (F, G) : I\!R^{2+n} \to I\!R^{2+n}$ and there exist C^1 maps $L : I\!R^2 \to I\!R^2$, $H : I\!R^{2+n} \to I\!R^2$ satisfying (H0)-(ii), (H0)-(iii) and
(iv) L satisfies (H4)-(H6) with L_1 satisfying (PS) condition,

Theorem B and the argument employed in the proof of Theorem A (See Proposition 2.22 and Remark 2.23) provide

Theorem C *Suppose $X \in \mathcal{X}$ satisfies $(\hat{H}0)$, (H1)-(H3) with V satisfying $(PS)_{(X,c)}$ condition for every $c > 0$. Assume further the semi-completivity condition of the the solutions of (AS). Then, the origin is a global attractor for system (AS).*

We should mention that Theorem A was motivated by a recent counter-example of Markus-Yamabe conjecture on $I\!R^3$ [3] which possesses a divergent orbit that approaches asymptotically the plane $I\!R^2 \times \{0\}$. We were also motivated by the observation that a version of the famous Palais-Smale condition, assumed frequently in critical point theory (See [1, 11] and references therein), may be combined with the Lyapunov method to study the behaviour of the orbits of a dynamical system which are not bounded. Finally, we note that Theorem B was inspired by the observation that Olech's result for the bidimensional problem is valid under hypotheses which do not imply (MY) condition.

Based in a former result by Gutierrez and Teixeira [7], we shall state a conjecture that we believe may have a proof similar to our proof of Theorem A. This conjecture is concerned with the behaviour of the orbits of system (AS) on a neighborhood of infinity at the invariant plane $I\!R^2 \times \{0\}$.

We say that a C^1-vector field L on $I\!R^2$ satisfies (GT) condition if:

(i) L has at least one critical point (say 0),

(ii)$Det(L'(u)) > 0$ for every $u \in I\!R^2$,

(iii) there is $\rho > 0$ such that $Trace(L'(u)) < 0$ provided that $\|u\| \geq \rho$,

(iv) $J_L = \int_{I\!R^2} Trace(L'(x,y))dxdy \neq 0$.

The vector field L satisfies the (H00) condition if it satisfies the (GT) condition, (H0)-(ii) and (H0)-(iii). Denoting by $P_\infty = (\infty, 0)$ the point on $I\!R^{2+n}$ representing the ∞ in $I\!R^2 \times 0$, we consider

Conjecture *Assume that $X \in \chi$ satisfies (H00), (H1), (H2), (H3), with V satisfying $(PS)_{(X,c)}$ condition for every $c > 0$. Assume further the semi-completivity condition of the solutions of (AS). Then, P_∞ is a repellor (resp. attractor) for (AS) provided that $J_L < 0$ (resp. $J_L > 0$).*

The article has the following organization: In section 2, we prove Theorem A. There, we also state a version of this theorem when the origin is a global attractor for the bidimensional problem associated to L. In section 3, after some preliminary results, we prove Theorem B. Finally, in section 4, we present applications of Theorems A, B and C.

2 Proof of Theorem A

Arguing by contradiction, we suppose that (AS) possesses a solution $\gamma(t) = \gamma(t, u_0)$, $u_0 = (x_0, y_0) \notin I\!R^2$, satisfying

$$\|\gamma(t, u_0)\| \not\to 0, \text{ as } t \to \infty. \tag{2.1}$$

The proof that such fact is not possible will be achieved by the verification of several steps. First, we observe that we follow the standard notation for Lyapunov functions, i.e.,

$$\begin{cases} V(t) = V(\gamma(t)) \\ \dot{V}(t) = \frac{dV}{dt}(t) = < \nabla V(\gamma(t)), X(\gamma(t)) > . \end{cases}$$

As our first step, we establish that every solution of system (AS) satisfying (2.1) converges asymptotically to the plane $I\!R^2$,

Lemma 2.1 *Suppose $X \in \mathcal{X}$ satisfies (H0), (H1). Assume $\gamma(t) = \gamma(., u_0) : [0, \infty) \to I\!R^{2+n}$ is a solution of (AS) satisfying (2.1). Then, $\|\gamma(t)\| \to \infty$, as $t \to \infty$.*

Proof: Arguing by contradiction, we suppose that the lemma is false. By [6], we must have $\gamma(t) \in I\!R^{2+n} \setminus I\!R^2$, for every $t \in [0,\infty)$. Furthermore, we find $0 < R_1 < R_2 < \infty$ and sequences $0 < t_1 < s_1 < \ldots < t_k < s_k < \ldots$ such that $t_k \to \bar{t} \in I\!R \cup \{\infty\}$, as $k \to \infty$, and, for every $k \in I\!N$,

$$\begin{cases} \|\gamma(t_k)\| = R_1, \\ \|\gamma(s_k)\| = R_2, \\ R_1 \leq \|\gamma(t)\| \leq R_2, \text{ for every } t \in [t_k, s_k]. \end{cases} \tag{2.2}$$

Taking $M_1 = \max\{\|X(x,y)\| \mid R_1 \leq \|(x,y)\| \leq R_2\}$, by (AS), we have

$$R_2 - R_1 \leq \|\gamma(s_k) - \gamma(t_k)\| \leq M_1(s_k - t_k), \ \forall\, k \in I\!N. \tag{2.3}$$

This implies that $\bar{t} = \infty$. Using that V is a Lyapunov function, we get

$$V(t) = V(\gamma(t)) \leq V(0) < \infty, \ \forall\, t \in [0,\infty).$$

Furthermore, since the origin is a local attractor for (AS) and a global attractor for orbits on $I\!R^2$; by condition (H1), and the compactness of $\overline{(B_{R_2}(0) \setminus B_{R_1}(0))}$, we find $d > 0$ such that, for every $k \in I\!N$,

$$V(t) \geq d > 0, \ \forall\, t \in [t_k, s_k].$$

Thus, invoking (H1) one more time, we find $\delta > 0$, independent of $k \in I\!N$, such that

$$\dot{V}(t) \leq -\delta > 0, \ \forall\, t \in [t_k, s_k].$$

This implies, via (2.3), that $V(s_k) \to -\infty$, as $k \to \infty$, contradicting the continuity of $V(x,y)$ and (2.2). The lemma is proved. □

Lemma 2.2 *Suppose $X \in \mathcal{X}$ satisfies (H0) and (H1) with V satisfying $(PS)_{(X,c)}$ for every $c > 0$. Assume $\gamma(t,u_0) = (x(t),y(t)) : [0,\infty) \to I\!R^{2+n}$ is a solution of (AS) satisying (2.1). Then, $\|x(t)\| \to \infty$ and $\|y(t)\| \to 0$, as $t \to \infty$.*

Proof: By Lemma 2.1, it suffices to verify that $\|y(t)\| \to 0$, as $t \to \infty$. First, we claim that there exists a sequence $t_k \to \infty$, as $k \to \infty$, such that

$$\dot{V}(t_k) \to 0, \text{ as } k \to \infty.$$

Effectively, if we assume otherwise, we find $T > 0$ and $K > 0$ such that $\dot{V}(t) \leq -K$, for every $t \geq T$. But this implies $V(t) \to -\infty$, as $t \to \infty$, contradicting (H1). The claim is proved.

Now, we invoke Lemma 2.1, (H1) and we use that V satisfies $(PS)_{(X,c)}$, for every $c > 0$, to conclude that $V(t_k) \to 0$, as $k \to \infty$. Observing that $0 < V(s) \leq V(t)$, for every $s \geq t$, we obtain that $V(t) \to 0$, as $t \to \infty$. Consequently, by (H1), $\|y(t)\| \to 0$, as $t \to \infty$. The lemma is proved. □

Given a continuous curve $\beta : [0,1] \to I\!R^2$, we denote by $l(\beta) = l(\beta([0,1])$ its length. The following basic result will be used to estimate the length of a closed curve which winds around the origin,

Lemma 2.3 *Suppose $\beta : [0,1] \to I\!R^2$ is a closed continuous curve. Assume β satisfies*

(β_1) *The origin belongs to a bounded component of $I\!R^2 \setminus \beta([0,1])$,*

(β_2) *There exist $t_0 \in [0,1]$ and $d > 0$ such that*

$$\|\beta(t_0)\| \geq d > 0.$$

Then, $l(\beta) \geq 2d$.

Proof: Without loss of generality, we may suppose that $t_0 = 0$. By (β_1), there exist $t \in (0,1)$ and $\lambda > 0$ such that $\beta(t) = -\lambda\beta(0)$. Consequently, by (β_2), $l(\beta) = l(\beta([0,t]) + l(\beta([t,1]) \geq 2d$. The lemma is proved. □

Corollary 2.4 *Let $\beta : [0,1] \to I\!R^2 \setminus \{0\}$ be a closed simple curve of class C^1 by parts satisfying (β_2). Suppose $T : I\!R^2 \to I\!R^2$ is a vector field of class C^1 satisfying*

(T_1) *$T(0) = 0$ and $T(x) \neq 0$, for every $x \in I\!R^2 \setminus \{0\}$.*

(T_2) *$< T(\beta(t), (\beta'(t))^{\perp} > \geq 0 (\leq 0)$, for every $t \in [0,1]$ such that $\beta'(t)$ is defined.*

Then, $l(\beta) \geq 2d$.

Remark 2.5 *As observed in the introduction, Gutierrez [6] has proved that $L, L^{\perp} : I\!R^2 \to I\!R^2$ is injective if it satisfies (MY) condition. Hence, under this condition and $L(0) = 0$, (T_1) holds.*

Proof: Consider the autonomous system associated to T,

$$\dot{x}(t) = T(x(t)).$$

Using (T_1), (T_2), $\beta([0,1]) \subset I\!R^2 \setminus \{0\}$, and the fact that β is a closed simple curve, we conclude that the origin must belong to the bounded component of $I\!R^2 \setminus \beta([0,1])$. Hence, by (β_2) and Lemma 2.3, $l(\beta) \geq 2d$. The Corollary is proved. □

In the following step, we apply a version of Olech's argument [9], making use of Green's Theorem in $I\!R^2$, to obtain a contradiction. For that, we fix $\gamma(t) = (x(t), y(t)) = \gamma(t, u_0)$, $u_0 = (x_0, y_0) \notin I\!R^2$ such that γ satisfies (2.1)

Considering $R, \rho > 0$ given by (H2) and taking $R > 0$ larger and $\rho > 0$ smaller if necessary, we invoke the injectivity of L, (H0) and (H3) to find $d > 0$ and $0 \leq \hat{\delta} < 1$ such that

$$\begin{cases} \|L(x)\| \geq d > 0, & \forall \, \|x\| \geq R, \\ \|H(x,y)\| \leq \hat{\delta}\|L(x)\|, & \forall \, \|x\| \geq R, \ \|y\| \leq \rho. \end{cases} \tag{2.4}$$

Now, we use Lemma 2.2 to find $T \geq 0$ such that

$$\begin{cases} \|x(t)\| \geq 3R, & \forall \, t \geq T, \\ \|y(t)\| \leq \rho, & \forall \, t \geq T. \end{cases} \tag{2.5}$$

The following lemma provides an estimate for the flow of L across the projection on $I\!R^2$ of the orbit $\gamma([T, \infty))$,

Lemma 2.6 *There exists $T_1 \geq T$, T given by (2.5), such that*

$$\int_{T_1}^{\infty} |< L(x(s)), F^{\perp}(x(s), y(s)) >| \, ds < \frac{dR}{2}.$$

Proof: By (H0), (H2)(i) and (2.5), for every $S \geq T$, we get

$$\int_{S}^{\infty} |< L(x(s)), F^{\perp}(x(s), y(s)) >| \, ds \leq b \int_{S}^{\infty} V(s) \, ds.$$

On the other hand, by (H2)(ii) and (2.5), we have $V(s) \leq V(S)e^{-c(s-S)}$ for every $s \geq S \geq T$. Hence,

$$\int_{S}^{\infty} V(s) \, ds \leq bV(S) \int_{S}^{\infty} e^{-c(s-S)} \, ds = \frac{bV(S)}{c}.$$

Since $V(S) \to 0$, as $S \to \infty$, we obtain the desired estimate by taking $T_1 = S > 0$ sufficiently large. The lemma is proved. □

Considering $x_{T_1} = (x(T_1), 0) \in \mathbb{R}^2$, we take $\gamma(t, x_{T_1})$, the solution of (AS) with $\gamma(0) = x_{T_1}$. Since $X = (F, G)$ satisfies (H0), we have that $\gamma(t, x_{T_1}) \in \mathbb{R}^2$, for every $t \in \mathbb{R}$, and $\gamma(t, x_{T_1}) \to 0$, as $t \to \infty$.

Remark 2.7 *Using that system (AS) is autonomous, it is not difficult to show that we may suppose $T_1 = T = 0$ and*

$$x((0, \infty)) \cap \gamma((0, \infty), x_0) = \emptyset. \tag{2.6}$$

Now, we study the behaviour of the curve $x : [0, \infty) \to \mathbb{R}^2$.

Lemma 2.8 *The application $x : [0, \infty) \to \mathbb{R}^2$ is locally injective.*

Proof: Arguing by contradiction, we suppose there exist $s_0 \in [0, \infty)$ and sequences $(t_k), (s_k) \subset [0, \infty)$ such that

$$\begin{cases} t_k < s_k, \ \forall \ k \in \mathbb{N}, \\ x(t_k) = x(s_k), \ \forall \ k \in \mathbb{N}, \\ t_k \to s_0, s_k \to s_0, \text{ as } k \to \infty. \end{cases} \tag{2.7}$$

Since $\gamma(s, u_0) = (x(s), y(s))$ solves (AS), we have

$$x(s_k) = x(t_k) + \int_{t_k}^{s_k} \left(L(x(s)) + H(x(s), y(s)) \right) \, ds.$$

Taking the inner product with $L(x(s_0))$ and considering (2.7), we get

$$\|L(x(s_0))\| \leq \left(\max_{t_k \leq s \leq s_k} \{ \|L(x(s)) - L(x(s_0))\| + \|H(x(s), y(s))\| \} \right).$$

Hence, by (2.4) and (2.5), we obtain

$$0 < (1-\hat{\delta})d \leq \max_{t_k \leq s \leq s_k} \|L(x(s)) - L(x(s_0))\|.$$

However, this last relation contradicts (2.7) and the continuity of $L(x(s))$. The Lemma is proved. □

Given $\tau \in I\!R$, we consider $\eta(t,\tau) = \eta(t, \gamma(\tau, x_0))$, the solution in $I\!R^2$ of the system

$$(AS)^{\perp} \qquad \begin{cases} \dot{x}(t) = L^{\perp}(x(t)) \\ x(0) = \gamma(\tau, x_0) \in I\!R^2. \end{cases}$$

Denoting by $(w^-(\tau), w^+(\tau))$ the maximum interval of definition for the solution of $(AS)^{\perp}$, we set

$$\mathcal{O} = \{(s,t,\tau) \in I\!R^3 \,|\, s \in I\!R, \tau \in I\!R, t \in (w^-(\tau), w^+(\tau))\},$$

and we define $\Phi : \mathcal{O} \to I\!R^2$ by $\Phi(s,t,\tau) = x(s) - \eta(t,\tau)$, for $(s,t,\tau) \in \mathcal{O}$. Considering $L = (L_1, L_2)$ and $H = (H_1, H_2)$, when $\Phi(s,t,\tau) = 0$, we get

$$D_{s,t}\Phi(s,t,\tau) = \begin{bmatrix} L_1(x(s)) + H_1(x(s), y(s)) & L_2(x(s)) + H_2(x(s), y(s)) \\ -L_2(x(s)) & L_1(x(s)) \end{bmatrix}.$$

Hence, By (2.4), (2.5), whenever $\Phi(s,t,\tau) = 0$ and $s \geq 0$, we obtain

$$\det[D_{s,t}\Phi(s,t,\tau)] \geq \|L(x(s))\|(\|L(x(s))\| - \|H(x(s), y(s))\|) > 0.$$

The following proposition is a direct consequence of the above inequality and the Implicit Function Theorem.

Proposition 2.9 *Given $(s_0, t_0, \tau_0) \in \mathcal{O}$ such that $\Phi(s_0, t_0, \tau_0) = 0$ and $s_0 \geq 0$, we may find a neighborhood U_{τ_0} of τ_0 and unique functions of class C^1, $\phi_1(\tau), \phi_2(\tau) : U_{\tau_0} \to I\!R$ such that $(\phi_1(\tau_0), \phi_2(\tau_0)) = (s_0, t_0)$ and*

$$\Phi((\phi_1(\tau), \phi_2(\tau), \tau)) = 0, \ \forall \ \tau \in U_{\tau_0}. \tag{2.8}$$

Furthermore, if $s \in \phi_1(U_{\tau_0})$, $t \in \phi_2(U_{\tau_0})$ and $\tau \in U_{\tau_0}$ satisfy $\Phi(s,t,\tau) = 0$, then $(s,t) = (\phi_1(\tau), \phi_2(\tau))$.

Corollary 2.10 *Applying Proposition 2.9 to $(s_0, t_0, \tau_0) = (0,0,0)$, we may suppose that $\phi_1 : U_0 \to I\!R$ is an increasing function.*

Proof: Derivating (2.8) with respect to τ at $\tau_0 = 0$, and taking the inner product with $L(x_0)$, we get

$$(\|L(x_0)\|^2 - < H(x_0, y_0), L(x_0) >)\dot{\phi}_1(o) = \|L(x_0)\|^2.$$

Hence, by (2.4) and (2.5), we have that $\dot{\phi}_1(0) > 0$ and, consequently, we may assume that $\phi_1 : U_0 \to I\!R$ is an increasing function with $\phi_1(\tau) > 0$, for every $\tau \in U_0$, $\tau > 0$. The corollary is proved. □

Remark 2.11 *By (2.6), Corollary 2.10, and the fact that $\gamma(t,x_0)$ is not periodic, we have $\phi_2(\tau) \neq 0$, for every $\tau \in U_0$, $\tau > 0$. Since the proof on the other case uses a similar argument, without loss of generality, we suppose that $\phi_2(\tau) > 0$, for every $\tau \in U_0$, $\tau > 0$.*

Now, we let $M > 0$ and $\bar{\tau} > 0$ be such that

$$\begin{cases} 2R < \|\gamma(r,x_0)\| \leq M, \ \forall\, \tau \in [0,\bar{\tau}] \\ \|\gamma(\bar{\tau},x_0)\| = 2R. \end{cases} \tag{2.9}$$

By Lemma 2.2, there exists $\bar{s} > 0$ such that

$$\|x(s)\| \geq M + 2R, \ \forall\, s \geq \bar{s}. \tag{2.10}$$

Taking $M_1 > 0$ such that

$$\begin{cases} \|L(x(s)) + H(x(s),y(s))\| \leq M_1, \ \forall\, 0 \leq s \leq \bar{s}, \\ \|L(\gamma(\tau,x_0))\| \leq M_1, \ \forall\, 0 \leq \tau \leq \bar{\tau}, \end{cases} \tag{2.11}$$

we set,

$$\begin{cases} \hat{\tau}_1 = \min\{\frac{R}{2M_1}, \bar{\tau}\} \\ \hat{s}_1 = \min\{\frac{R}{2M_1}, \bar{s}\}. \end{cases}$$

¿From (2.11), (AS) and $(AS)^\perp$, we obtain

$$\begin{cases} l(x([0,\hat{s}_1])) < R, \\ l(\gamma([0,\hat{\tau}_1],x_0)) < R. \end{cases} \tag{2.12}$$

Lemma 2.12 *$x : [0,\hat{s}_1] \to I\!R^2$ is injective.*

Proof: Arguing by contradiction, we suppose that there exist $0 \leq s_1^* < s_2^* \leq \hat{s}_1$ such that $x(s_1^*) = x(s_2^*)$. First, we claim that we may assume that $x : [s_1^*, s_2^*] \to I\!R^2$ is a closed simple curve. Effectively, considering

$$s_2 = \sup\{s \geq s_1^* \,|\, x : [s_1^*, s_2] \to I\!R^2 \ \text{is injective}\},$$

by Lemma 2.8, we have that $s_1^* < s_2 \leq s_2^*$. Using the definition of s_2 and Lemma 2.8 one more time, we find $0 < \epsilon < s_2 - s_1^*$, $t_0 \in [s_1, s_2 - \epsilon]$ and sequences $(t_k) \subset [s_1, s_2 - \epsilon]$, $(\tau_k) \subset [s_2, s_2 + \epsilon]$ such that $x : [s_2 - \epsilon, s_2 - \epsilon] \to I\!R^2$ is injective, $x(t_k) = x(\tau_k)$, for every $k \in I\!N$, and $t_k \to t_0$, $\tau_k \to s_2$, as $k \to \infty$. Hence, $x(t_0) = x(s_2)$. Moreover, it is not difficult to verify that $x : [t_0, s_2] \to I\!R^2$ is a closed simple curve. The claim is proved.

¿From (2.5), $x[s_1^*, s_2^*] \to I\!R^2$ satisfies (β_2), with $d = 3R$, and $x([s_1^*, s_2^*]) \subset I\!R^2 \setminus \{0\}$. By (2.4), (2.5) and (H0), $L^\perp$ is transversal to $x([s_1^*, s_2^*])$. Hence, $L^\perp$ satisfies (T_2). Since $L^\perp$ satisfies (T_1), we may invoke Corollary 2.4 to conclude that $l(x([s_1^*, s_2^*])) \geq 6R$. But, this contradicts (2.12). The lemma is proved. □

Now, we consider $A_1 \subset [0, \hat{\tau}_1)$, the set formed by the points $\tau \in [0, \hat{\tau}_1)$ such that there exist $t \in [0, w^+(\tau))$ and $s \in [0, \hat{s}_1)$ satisfying

$$\begin{cases} \eta(t,\tau) = x(s), \\ l(\eta([0,t]),\tau) < R. \end{cases} \tag{2.13}$$

Note that $A_1 \neq \emptyset$ because $0 \in A_1$. Moreover,

Lemma 2.13 *Given $\tau \in A_1$, there exists a unique $t \in [0, w^+(\tau))$ satisfying (2.13).*

Proof: Arguing by contradiction, we suppose that there exists $\tau \in A_1$ and $0 \le t_1 < t_2 < w^+(\tau)$ satisfying (2.13). Let $0 \le s_1, s_2 < \hat{s}_1$ be such that

$$\begin{cases} \eta(t_1,\tau) = x(s_1), \\ \eta(t_2,\tau) = x(s_2). \end{cases} \tag{2.14}$$

We claim that $s_1 \neq s_2$. Assuming otherwise, we have that the curve $\eta(t,\tau)$ is periodic with respect to the variable t. Moreover, by (2.5), (2.13) and (2.14), $\eta(.,\tau) : [t_1, t_1] \to I\!R^2 \setminus \{0\}$ and satisfies (β_2) with $d \ge 3R$. Since L is transversal to $\eta([t_1,t_2],\tau)$ and satisfies (T_1), Corollary 2.4 implies that $l(\eta([t_1,t_2]) \ge 6R$. But, this contradicts (2.13). The claim is proved. Without loss of generality, we suppose that $0 \le s_1 < s_2 < \hat{s}_1$. Using that $L^\perp$ is transversal to the curve $x([0,\infty))$ and taking $t_2 > t_1$ smaller if necessary, we may assume

$$\eta(r,\tau) \notin x([s_1,s_2]), \ \forall\, r \in (t_1,t_2). \tag{2.15}$$

By the argument used in the above claim, we have that $\eta([t_1,t_2],\tau)$ is a simple curve. Using this fact, (2.5), (2.13), (2.15) and Lemma 2.12, we conclude that $\Gamma = \eta([t_1,t_2],\tau) \cup x([s_1,s_2])$ is a closed simple curve satisfying $\Gamma \subset I\!R^2 \setminus \{0\}$. The transversality of $L^\perp$ with respect to $x([s_1,s_2])$ (via (2.4)-(2.5)) and $(AS)^\perp$ imply that $L^\perp$ satisfies condition (T_2) with respect to Γ. Since L satisfies (T_1), by (2.5) and Corollary 2.4, we have that $l(\Gamma) \ge 6R$. On the other hand, by (2.12) and (2.13), $l(\Gamma) \le 2R$. The lemma is proved. □

As a direct consequence of Lemmas 2.12 and 2.13, we have

Corollary 2.14 *Given $\tau \in A_1$, there exists a unique $s \in [0, \hat{s}_1)$ satisfying (2.13).*

Based on Lemma 2.13 and Corollary 2.14, we may define $T_1 : A_1 \to [0,\infty)$, $S_1 : A_1 \to [0,\hat{s}_1)$ by $T_1(\tau) = t$, $S_1(\tau) = s$, where t, s are given by (2.13). Considering $[0,\hat{\tau}_1)$ with the topology induced by the real line, we obtain

Lemma 2.15 *A_1 is an open subset of $[0,\hat{\tau}_1)$.*

Proof: By Remark 2.11, $\phi_2(\tau) > 0$ for every $\tau \in U_0$, $\tau > 0$. Hence, by Proposition 2.9, there exists $\epsilon > 0$ such that $[0,\epsilon) \subset A_1$. Consequently, $\tau = 0$ is an interior point of A_1. Now, given $\tau \in A_1 \setminus \{0\}$, we invoke (2.6) and the the fact that $\gamma(\tau, x_0)$ is not periodic to conclude that $T_1(\tau) > 0$. We claim that $S_1(\tau) > 0$. Indeed, if we suppose otherwise, then $\eta(T_1(\tau),\tau) = x(0) = x_0$. Taking $t^* \in [0, T_1(\tau))$ and $\tau^* \in (0,\tau]$ such that

$$\begin{cases} \eta(t^*,\tau) = \gamma(\tau^*, x_0), \\ \eta(r,\tau) \notin \gamma([0,\tau^*], x_0), \ \forall\, r \in (t^*, T_1(\tau)), \end{cases}$$

we consider $\Gamma = \gamma([0,\tau^*], x_0) \cup \eta((t^*, T_1(\tau)), \tau)$ and argue as in earlier results to derive that $l(\Gamma) \geq 6R$. However, this contradicts (2.12) and (2.13). The claim is proved.

Now, we apply Proposition 2.9 at the point $(\tau, T_1(\tau), S_1(\tau))$ and we use $T_1(\tau) \in (0, w^+(\tau))$, $S_1(\tau) \in (0, \hat{s}_1)$ and (2.13) to conclude that there exists an open neighborhood U_τ of τ such that $U_\tau \subset A_1$. The lemma is proved. □

Taking $V_1 = [0, \tau_1) \subset [0, \hat{\tau}_1)$, the component of A_1 which contains the origin, we have

Lemma 2.16 *$T_1, S_1 : [0, \tau_1) \to I\!R$ are continuous functions satisfying $T_1(0) = 0$, $S_1(0) = 0$, and*

(p_1) $T_1(\tau) > 0$, $0 < S_1(\tau) < \hat{s}_1$, for every $\tau \in (0, \tau_1)$.

(p_2) There exists $M > 0$ such that

$$|T_1(\tau)| \leq M < \infty, \ \forall \ \tau \in [0, \tau_1).$$

(p_3) $S_1(\tau) : [0, \tau_1) \to I\!R$ is an increasing function.

<u>Proof:</u> By definition, $T_1(0) = 0$ and $S_1(0) = 0$. Furthermore, the argument used in the proof of Lemma 2.15 shows that $T_1, S_1 : [0, \tau_1) \to I\!R$ are continuous and satisfy (p_1). Now, from (2.9) and (2.13), we get

$$R \leq \|\eta(t, \tau)\| \leq M + R, \ \forall \ t \in [0, T_1(\tau)], \ \tau \in [0, \tau_1). \tag{2.16}$$

Therefore, by [6] and $(AS)^\perp$, we find $\delta > 0$ such that $\|\dot{\eta}(t, \tau)\| \geq \delta > 0$, for every $t \in [0, T_1(\tau)]$. Consequently, invoking (2.13) one more time, we obtain

$$R > l(\eta([0, T_1(\tau)], \tau) \geq \delta T_1(\tau).$$

Hence, (p_2) holds. Finally, we shall verify (p_3): By Proposition 2.9, Corollary 2.10 and Lemma 2.12, $S_1(\tau) = \phi_1(\tau)$ for every $\tau \in U_0$. Thus, invoking Corollary 2.10 one more time, we have $\dot{s}(0) > 0$. Consequently, S_1 is locally injective and increasing on a neighborhood of $\tau = 0$. Hence, to prove (p_3) it suffices to verify that S_1 is locally injective on $(0, \tau_1)$. Arguing by contradiction, we suppose there exist $\tau_0 \in (0, \tau_1)$ and sequences (τ_k^1), $(\tau_k^2) \subset (0, \tau_1)$ such that

$$\begin{cases} \tau_k^i \to \tau_0, \ \text{as}\, k \to \infty, \ i = 1, 2. \\ \tau_k^1 \neq \tau_k^2, \ \forall \ k \in I\!N, \\ 0 < S_1(\tau_k^1) = S_1(\tau_k^2), \ \forall \ k \in I\!N. \end{cases} \tag{2.17}$$

By the transversality of $L^\perp$ and $\gamma((0, \infty), x_0)$, there exists $\epsilon > 0$ such that if $t \in (-\epsilon, \epsilon)$, $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$ and $\eta(t, \tau) \in \gamma((\tau_0 - \epsilon, \tau_0 + \epsilon), x_0)$, then $t = 0$. By (2.17), $\gamma(\tau_k^2, x_0) = \eta(T_1(\tau_k^1) - T_1(\tau_k^2), \tau_k^1)$, for every $k \in I\!N$. Using the continuity of $T_1 : [0, \tau_1) \to I\!R$ and the first relation in (2.17), we obtain that $\tau_k^2 = \tau_k^1$ for k sufficiently large. But, this contradicts $\tau_k^2 \neq \tau_k^1$, for every $k \in I\!N$. The lemma is proved. □

We now consider a sequence $0 < \tilde{\tau}_1 < \ldots < \tilde{\tau}_k < \ldots < \tau_1 \leq \hat{\tau}_1$ satisfying $\tilde{\tau}_k \to \tau_1$, as $k \to \infty$. We also consider $0 < S_1(\tilde{\tau}_1) < \ldots < S_1(\tilde{\tau}_k) < \ldots < \hat{s}_1$ and $0 <$

$T_1(\tilde{\tau}_1), \ldots, T_1(\tilde{\tau}_k), \ldots < \infty$ the associated sequences. By (p_3), $S_1(\tilde{\tau}_k) \nearrow s_1 \leq \hat{s}_1$. Furthermore, invoking (p_2), we may suppose without loss of generality that

$$T_1(\tilde{\tau}_k) \to t_1, \text{ as } k \to \infty. \tag{2.18}$$

As a direct consequence of (2.13), we have

$$\begin{cases} \eta(t_1, \tau_1) = x(s_1), \\ l(\eta([0, t_1], \tau_1) \leq R. \end{cases} \tag{2.19}$$

The following result shows that we have a strict inequality on the second relation of (2.19)

Lemma 2.17 *Considering t_1 given by (2.18), we have*

$$l(\eta([0, t_1], \tau_1) < R.$$

<u>Proof:</u> First, we claim that the curve

$$\Gamma_\tau = \gamma([0, \tau], x_0) \cup \eta((0, T_1(\tau)], \tau) \cup x((0, S_1(\tau)))$$

is a simple closed curve for every $\tau \in (0, \tau_1)$. Effectively, invoking Corollary 2.4 and using the argument employed earlier, we obtain that $\gamma([0, \tau], x_0)$ and $\eta([0, T_1(\tau)], \tau)$ are simple curves. Lemma 2.12 implies that $x([0, S_1(\tau)])$ is also a simple curve. By Lemma 2.13 and (2.6), we have $\eta([0, T_1(\tau)], \tau) \cap x([0, S_1(\tau))) = \emptyset$ and $\gamma((0, \tau], x_0) \cap x((0, S_1(\tau)]) = \emptyset$, respectively. Hence, to prove the claim, it suffices to verify that $\eta((0, T_1(\tau)], \tau) \cap \gamma([0, \tau], x_0) = \emptyset$. Assuming otherwise, we note that by (2.6) and $\eta(T_1(\tau), \tau) \in x([0, \hat{s}_1))$, we must have $t \in (0, T_1(\tau))$ and $\tau_0 \in [0, \tau]$ such that $\eta(t, \tau) = \gamma(\tau_0, x_0)$. But, on this case $S_1(\tau) = S_1(\tau_0)$. Thus, by (p_3), we must have $\tau = \tau_0$. Consequently, $\eta(., \tau)$ is periodic. Since L satisfies (T_1) and is transversal to $\eta([0, t], \tau)$, by (2.9) and Corollary 2.4, we must have $l(\eta([0, t], \tau)) \geq 4R$. However, this contradicts (2.13) and $t < T_1(\tau)$. The claim is proved.

Taking $\Gamma_{1,k} = \Gamma_{\tilde{\tau}_k}$ and $B_{1,k}$ the bounded component of $I\!R^2 \setminus \Gamma_{1,k}$, by Green's Theorem and (H0), we have

$$\int_{\Gamma_{1,k}} < L, \vec{n}_1 > d\sigma = \int\int_{B_{1,k}} (\text{div} L(x, y))\, dxdy \leq 0,$$

where $\vec{n}_1$ is the normal exterior to $B_{1,k}$ and $d\sigma$ is the arclength. Setting $\Gamma^1_{1,k} = \gamma([0, \tilde{\tau}_k], x_0]$, $\Gamma^2_{1,k} = \eta([0, T_1(\tilde{\tau}_k)], \tilde{\tau}_k)$ and $\Gamma^3_{1,k} = x([0, S_1(\tilde{\tau}_k])$, by (H0), we get

$$\int_{\Gamma^1_{1,k}} < L, \vec{n}_1 > d\sigma = 0,$$

Furthermore, $\vec{n}_1(\eta(r, \tilde{\tau}_k) = L(\eta(r, \tilde{\tau}_k)/\|L(\eta(r, \tilde{\tau}_k)\|$, for every $r \in (0, T_1(\tilde{\tau}_k))$. Using (AS), $(AS)^\perp$, and taking $k \to \infty$, we obtain

$$\int_0^{t_1} \|L(\eta(t, \tau_1))\|^2\, dt - \int_0^{s_1} < L(x(s)), F^\perp(x(s), y(s)) > ds \leq 0. \tag{2.20}$$

Hence, by (2.4), (2.16) and Lemma 2.6, we have

$$dl(\eta([0,t_1],\tau_1)) \le \frac{dR}{2}.$$

This proves the lemma. □

As a direct consequence of (2.5), (2.9), (2.10), (2.19), Proposition 2.9 and Lemmas 2.15 and 2.17, we have that either $\tau_1 = \hat{\tau}_1$ or $s_1 = \hat{s}_1$, and $\tau_1 < \bar{\tau}$, $s_1 < \bar{s}$. Consequently,

$$\tau_1 + s_1 \ge \frac{R}{2M_1}. \tag{2.21}$$

For next step, we follow the same argument. Set

$$\begin{cases} \hat{\tau}_2 = \min\{\tau_1 + \frac{R}{2M_1}, \bar{\tau}\} \\ \hat{s}_2 = \min\{s_1 + \frac{R}{2M_1}, \bar{s}\}. \end{cases}$$

By (2.11), (AS) and $(AS)^\perp$, we also have

$$\begin{cases} l(x([s_1,\hat{s}_2])) < R, \\ l(\gamma([\tau_1,\hat{\tau}_2],x_0)) < R. \end{cases}$$

Moreover,

Lemma 2.18 *$x : [s_1,\hat{s}_2] \to I\!R^2$ is injective.*

We also consider $A_2 \subset [\tau_1,\hat{\tau}_2)$, formed by the points $\tau \in [\tau_1,\hat{\tau}_2)$ such that there exist $t \in [0, w^+(\tau))$ and $s \in [s_1,\hat{s}_2)$ satisfying

$$\begin{cases} \eta(t,\tau) = x(s), \\ l(\eta([0,t]),\tau) < R. \end{cases} \tag{2.22}$$

By (2.19) and Lemma 2.17, $\tau_1 \in A_2$. Furthermore, we may show that for every $\tau \in A_2$ there exist unique $t \in [0, w^+(\tau)$ and $s \in [s_1,\hat{s}_2)$ satisfying (2.22). As before, we define $T_2 : [\tau_1,\hat{\tau}_2) \to [0,\infty)$ and $S_2 : [\tau_1,\hat{\tau}_2) \to [s_1,\hat{s}_2)$ to be such values. The following result holds

Lemma 2.19 *A_2 is an open subset of $[\tau_1,\hat{\tau}_2)$.*

Proof: Applying Proposition 2.9 at the point (s_1,t_1,τ_1), we find a neighborhood U_{τ_1} of τ_1 and functions $\phi_1,\phi_2 : U_{\tau_1} \to I\!R^2$ satisfying $\phi_1(\tau_1) = s_1$, $\phi_2(\tau_1) = t_1$ and (2.8) . Furthermore, by construction, we have $\phi_1(\tilde{\tau}_k) = S_1(\tilde{\tau}_k)$, $\phi_2(\tilde{\tau}_k) = T_1(\tilde{\tau}_k)$, for k sufficiently large. Arguing as in the proof of Lemma 2.12, we may suppose, without loss of generality, that $\phi_1 : U_{\tau_1} \to I\!R^2$ is injective. Since $S_1 : [0,\tau_1) \to [0,\hat{s}_1)$ is an increasing function, we also have that $\phi_1 : U_{\tau_1} \to I\!R$ is increasing and, consequently, $\phi_1(\tau) > s_1$ for every $\tau \in U_{\tau_1}$, $\tau > \tau_1$. This fact, Lemma 2.17, $t_1 > 0$, $s_1 < \hat{s}_2$ and the continuity of ϕ_i, $i = 1,2$, imply that τ_1 is an interior point of A_2 on $[\tau_1,\hat{\tau}_2)$. Now, given $\tau \in A_2 \setminus \{\tau_1\}$, we argue as in the proof of Lemma 2.15 to conclude that $T_2(\tau) > 0$, $S_2(\tau) > s_1$. Then, we use Proposition 2.9 and (2.22) to obtain an open neighborhood U_τ of τ such that $U_\tau \subset A_2$. The lemma is proved. □

Taking $V_2 = [\tau_1,\tau_2) \subset [\tau_1,\hat{\tau}_2)$, the component of A_2 which contains τ_1, we get

Lemma 2.20 *$T_2, S_2 : [0,\tau_2) \to I\!R$ are continuous functions satisfying $T_2(\tau_1) = t_1$, $S_2(\tau_1) = s_1$, and*

($\hat{p}_1$) $T_2(\tau) > 0$, $s_1 < S_2(\tau) < \hat{s}_2$, for every $\tau \in (\tau_1, \tau_2)$.

($\hat{p}_2$) There exists $M > 0$ such that

$$|T_2(\tau)| \leq M < \infty, \ \forall \ \tau \in [\tau_1, \tau_2).$$

($\hat{p}_3$) $S_2 : [\tau_1, \tau_2) \to I\!R$ is an increasing function.

Proof: The proofs of ($\hat{p}_1$) and ($\hat{p}_2$) are similar to the proofs of (p_1) and (p_2), respectively. For that reason, we omit them. For the proof of ($\hat{p}_3$), we first claim that S_2 is injective and increasing on a neighborhood of τ_1. Effectively, considering ϕ_i, $i = 1, 2$, and U_{τ_1} given in the proof of Lemma 2.19, we have that $S_2(\tau) = \phi_2(\tau)$, for $\tau \in U_{\tau_1}$, $\tau > \tau_1$. The claim follows because ϕ_2 is an increasing function on U_{τ_1}. Finally, we note that condition ($\hat{p}_3$) follows by verifying, as in the proof of (p_3), that S_2 is locally injective on (τ_1, τ_2). The lemma is proved. □

Now, we consider a sequence $\tau_1 < \tilde{\tau}_1 < \ldots < \tilde{\tau}_k < \ldots < \tau_2 \leq \hat{\tau}_2$ satisfying $\tilde{\tau}_k \to \tau_2$, as $k \to \infty$. We also have the associated sequences $(S_2(\tilde{\tau}_k)) \subset [s_1, \hat{s}_2)$, $(T_2(\tilde{\tau}_k)) \subset I\!R$. Without loss of generality, we may suppose that $S_2(\tilde{\tau}_k) \nearrow s_2 \leq \hat{s}_2$, $T_2(\tilde{\tau}_k) \to t_2$, as $k \to \infty$. Moreover,

$$\begin{cases} \eta(t_2, \tau_2) = x(s_2), \\ l(\eta([0, t_2], \tau_2) \leq R. \end{cases} \tag{2.23}$$

Lemma 2.21 *Considering t_2 given by (2.23), we have*

$$l(\eta([0, t_2], \tau_2) < R.$$

Proof: Arguing as in the proof of Lemma 2.17, we obtain that

$$\Gamma_\tau = \gamma([\tau_1, \tau], x_0) \cup \eta((0, T_2(\tau)], \tau) \cup x((0, S_2(\tau))) \cup \eta((0, t_1), \tau_1)$$

is a simple closed curve for every $\tau \in (\tau_1, \tau_2)$. Then, we take $\Gamma_{2,k} = \Gamma_{\tilde{\tau}_k}$ and $B_{2,k}$ the bounded component of $I\!R^2 \setminus \Gamma_{2,k}$. By Green's Theorem and (H0), we get

$$\int_{\Gamma_{2,k}} < L, \vec{n}_2 > \, d\sigma = \int\int_{B_{2,k}} (\mathrm{div} L(x,y)) \, dx dy \leq 0,$$

where $\vec{n}_2$ is the normal exterior to $B_{2,k}$ and $d\sigma$ is the arclength. Setting $\Gamma^1_{2,k} = \gamma([\tau_1, \tilde{\tau}_k], x_0)$, $\Gamma^2_{2,k} = \eta([0, T_2(\tilde{\tau}_k], \tilde{\tau}_k)$, $\Gamma^3_{2,k} = x([0, S_2(\tilde{\tau}_k])$, and $\Gamma^4_{2,k} = \eta([0, t_1], \tau_1)$, by (H0), we have

$$\int_{\Gamma^1_{2,k}} < L, \vec{n}_2 > \, d\sigma = 0.$$

Furthermore, $\vec{n}_2(\eta(r,\tilde{\tau}_k)) = L(\eta(r,\tilde{\tau}_k))/\|L(\eta(r,\tilde{\tau}_k))\|$, for every $r \in (0,\tilde{\tau}_k)$, and $\vec{n}_2(\eta(r,\tau_1)) = -L(\eta(r,\tau_1))/\|L(\eta(r,\tau_1))\|$, for every $r \in (0,\tau_1)$. Hence, using (AS), $(AS)^{\perp}$, (2.20), and taking $k \to \infty$, we obtain

$$\int_0^{t_2} \|L(\eta(t,\tau_2))\|^2 \, dt - \int_0^{s_2} < L(x(s)), F^{\perp}(x(s),y(s)) > \, ds \leq 0. \tag{2.24}$$

The lemma is a direct consequence of (2.4), (2.16), (2.24) and Lemma 2.6. □

By (2.5), (2.9), (2.10), (2.23), Proposition 2.9 and Lemmas 2.19 and 2.21, we have that either $\tau_2 = \hat{\tau}_2$ or $s_2 = \hat{s}_2$ and $\tau_2 < \bar{\tau}$, $s_2 < \bar{s}$. Consequently, by (2.21),

$$\tau_2 + s_2 \geq \tau_1 + s_1 + \frac{R}{2M_1} \geq \frac{R}{M_1}.$$

Arguing in a similar way, we obtain sequences $((t_k, s_k, \tau_k)) \subset I\!R^3$, such that $t_k \in (0,\infty)$, $s_k \in (s_{k-1}, \bar{s})$, $\tau_k \in (\tau_{k-1}, \bar{\tau})$, for every $k \in I\!N$, and

$$\tau_k + s_k \geq \frac{kR}{2M_1}, \ \forall\, k \in I\!N.$$

But, this contradicts $\bar{\tau} + \bar{s} < \infty$ and concludes the proof of Theorem A. □

Condition (H0)-(i) has been used only to establish that $I\!R^2$ is on the domain of attraction of the origin and to show that the first equation in (2.4) holds. Thus if, we suppose

$(\tilde{H}0)$ $I\!R^m = I\!R^{2+n} = I\!R^2 \times I\!R^n$, $X(F,G) : I\!R^{2+n} \to I\!R^{2+n}$ and there exist C^1 maps $L : I\!R^2 \to I\!R^2$, $H : I\!R^{2+n} \to I\!R^2$ satisfying (H0)-(ii), (H0)-(iii) and

(iv) The origin is a global attractor for the system associated to L,

(v) $\liminf_{\|x\|\to\infty} \|L(x)\| > 0$,

we obtain

Proposition 2.22 *Suppose $X \in \mathcal{X}$ satisfies $(\tilde{H}0)$, (H1)-(H3) with V satisfying $(PS)_{(X,c)}$ condition for every $c > 0$. Assume further that the solutions of (AS) are defined on $[0,\infty)$. Then, the origin is a global attractor for system (AS).*

Remark 2.23 *It is worthwhile to mention that condition (H2) has been used only to prove Lemma 2.6. Thus, any other condition that provides that lemma implies the global asymptotic stability of system (AS).*

3 Proof of Theorems B and C

In this section we prove Theorems B and C. First, we need to state some preliminary results. The following result is due to Olech. For a question of completeness, we present its proof.

Lemma 3.1 *Suppose $X : I\!R^2 \to I\!R^2$ belongs to $\mathcal{X}$ and satisfies (H4) and*

(H7) $X(u) \neq 0$, for every $u \in I\!R^2 \setminus \{0\}$,

(H8) There exist $\rho, \alpha > 0$ such that

$$\|X(u)\| \geq \alpha, \ \forall \ u \in \mathbb{R}^2, \ \|u\| \geq \rho.$$

Then, the origin is a global attractor for system (AS).

Proof: Using (H4) and Green's Theorem, we obtain that (AS) does not have a periodic solution. Denoting by $w(u)$ the w-limit set of u, from (H4), (H7), (H8) and the argument employed in the proof of Theorem A (See also [9]), we conclude that $A_\infty = \{u \in \mathbb{R}^2 \mid w(u) = \emptyset\}$ is an open set. Since the origin is a local attractor for system (AS), we also have that $A_0 = \{u \in \mathbb{R}^2 \mid w(u) = \{0\}\}$ is an open set. Furthermore, $A_0 \cap A_\infty = \emptyset$ and $A_0 \neq \emptyset$. Hence, to prove Lemma 3.1, it suffices to verify that $\mathbb{R}^2 = A_0 \cup A_\infty$.

Arguing by contradiction, we suppose that there exist $u, v \in \mathbb{R}^2$ such that $v \in w(u) \setminus \{0\}$. Since $w(v) \subset w(u)$ and the intersection of $w(u)$ with a transversal section to X possesses at most a point, we have that $w(v)$ cannot have a regular point of X since, otherwise, $\gamma(t, v)$ would be a periodic solution of (AS). Consequently, by (H7), $w(v) = \{0\}$ or $w(v) = \emptyset$. As A_0 and A_∞ are open sets, by definition of w-limite set, we obtain that $w(u) = \{0\}$ or $w(u) = \emptyset$, respectively. However, this contradicts $v \in w(u) \setminus \{0\}$. The lemma is proved. □

Before stating our next lemma, we need to recall a result proved in [12]: Let E be a real Banach space. Given $f \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$, we denote by $S_c(f)$ and K_c the sets $\{u \in E \mid f(u) = c\}$ and $\{u \in E \mid f(u) = c, \ f'(u) = 0\}$. We say that c is an admissible level of f if either c is a regular value of f, or the components of K_c possesses only a point and c is an isolated critical value of f.

Theorem 3.2 *(The Level Surface Theorem) Suppose $f \in C^1(E, \mathbb{R})$ satisfies (PS). Assume $c \in \mathbb{R}$ is an admissible level of f and that u and v are two distinct points of $S_c(f)$. Then, either*

(i) u and v are in the same component of $S_c(f)$,

or

(ii) f has a critical value $d \neq c$.

Remark 3.3 *Theorem 3.2 is true under a generalized version of (PS) condition as proved in [12]. This implies that Theorem B also holds when L_1 satisfies such conditiom.*

Lemma 3.4 *Suppose $X \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies $X(0) = 0$ and (H5)-(H6), with X_1 satisfying (PS). Then, X satisfies (H7)-(H8).*

Proof: By $X(0) = 0$ and (H5), we have that $\det(X'(0)) \neq 0$. Thus, invoking the Inverse Function Theorem, we obtain two open balls centered at the origin, $B(0, \rho_i) \subset \mathbb{R}^2$, $i = 1, 2$, such that $X : \overline{B(0, \rho_1)} \to \mathbb{R}^2$ is injective and $B(0, \rho_2) \subset X(B(0, \rho_1))$. Thus, to prove Lemma 3.4, it suffices to show that (H8) holds with $\rho = \rho_1$ and $\alpha = \min\{\rho_2, c\}$, with c given by (H5).

Arguing by contradiction, we suppose that there exists $u \in \mathbb{R}^2$ such that $\|X(u)\| < \alpha$ and $\|u\| \geq \rho_1$. By our choice of α, we have that $u \in S_{(-c,c)}(X_1)$ and $X(u) \in B(0, \rho_2)$. Now, let $v \in B(0,\rho_1)$ be such that $X(v) = X(u)$. Since X_1 satisfies (PS), by (H6) and Theorem 3.2, there exists $\gamma : [0,1] \to I\!R^2$ such that $\gamma(0) = v$, $\gamma(1) = u$, and

$$X_1(\gamma(t)) = c_1 = X_1(u), \ \forall\, t \in [0,1]. \tag{3.1}$$

Considering $h : [0,1] \to I\!R^2$ defined by $h(t) = X_2(\gamma(t))$, for $t \in [0,1]$, we have that $h(0) = h(1) = X_2(u)$. Furthermore, from $\alpha \leq c$, (H6), (3.1) and the Implicit Function Theorem, we find $t_0 \in (0,1)$ such that $h'(t_0) =< X_2(\gamma(t_0)), \gamma'(t_0) >= 0$, $< X_1(\gamma(t_0)), \gamma'(t_0) >= 0$, and $\gamma'(t_0) \neq 0$. This implies that $\det(X'(\gamma(t_0))) = 0$. However, this contradicts (H5) since $\gamma(t_0) \in S_{(-c,c)}(X_1)$. The proof of Lemma 3.4 is concluded. □

Theorem B is a direct consequence of Lemmas 3.1 and 3.4. For the proof of Theorem C, we first note that $(\tilde{H}0)$-(v) is verified since, by Lemma 3.4, it satisfies (H8). This fact, Theorem B and Proposition 2.22 imply that the origin is a global attractor for system (AS).

4 Applications

In this section we present applications of the results proved in sections 2 and 3.

1. Consider $X_\lambda = (F, G_\lambda) : I\!R^m \to I\!R^m$, $\lambda \in I\!R$, a vector field of class C^1 satisfying (H_0) with $G_\lambda(x,y) = -Ay + \lambda M(x,y)$, where $A : I\!R^n \to I\!R^n$ is a positive selfadjoint operator. As before, we are considering $I\!R^m = I\!R^{2+n} = I\!R^2 \times I\!R^n$ and $F(x,0) = L$. Suppose X_λ satisfies

   (F1) There exist $p, A, B > 0$ and a function $\varphi \in C(I\!R\, I\!R)$ such that

   $$\begin{cases} \|L(x)\| \leq A\|x\|^p + B, \ \forall\, x \in I\!R^2, \\ < F(x,y), x > \leq A\|x\|^2\varphi(\|y\|) + B, \ \forall\, (x, y \in I\!R^m, \end{cases}$$

   (F2) For $p > 0$, given by (F1), there exist $R, \rho, C > 0$ such that, for every $\|x\| > R$, $\|y\| \leq \rho$, we have

   $$\|H(x,y)\| \leq \frac{C\|y\|^2}{\|x\|^p},$$

   (G1) there exists $D > 0$ such that

   $$\|M(u)\| \leq D, \ \forall\, u \in I\!R^m.$$

   Then, there exists $\lambda_0 > 0$ such that the system

   $$\begin{cases} \dot{x}(t) = L(x(t)) + H(x,y), \\ \dot{y}(t) = -Ay + \lambda M(x,y) \end{cases}$$

   has the origin as a global attractor for every $\lambda \in I\!R$, $|\lambda| < \lambda_0$. Effectively, Consider $V : I\!R^m \to I\!R^m$ defined by

$$V(x,y) = \frac{1}{2} < Ay, y >, \ \forall \ (x,y) \in I\!R^m.$$

Since A is positive definite, V satisfies (H1)-(i). By (G1), there exist $c, \lambda_0 > 0$ such that, for every $|\lambda| < \lambda_0$,

$$< \nabla V(u), X(u) > \leq -cV(u), \ \forall \ u \in I\!R^m.$$

This shows that (H1)-(ii) and (H2)-(ii) also hold. Furthermore, (H2)-(i) is consequence of (F1), (F2) and the definition of V and (H3) is obtained by invoking (F2). Finally, we note that the solutions of the system is defined for every $t \geq 0$, by our choice of V, λ_0 and the second equation on (F1). Theorem A implies that the above system has the origin as the global attractor.

2. Consider the system

$$\begin{cases} \dot{x}_1(t) = g_1(x_1(t), x_2(t)) + h_1(x_1(t), x_2(t), x_3(t))x_3(t), \\ \dot{x}_2(t) = g_2(x_1(t), x_2(t)) + h_2(x_1(t), x_2(t), x_3(t))x_3(t), \\ \dot{x}_3(t) = -x_3(t) \end{cases}$$

where $L = (g_1, g_2)$ and $\hat{H} = (h_1, h_2)$ are of class C^1, $(L + H)(0) = 0$ and L satisfies (MY). Taking $X(u) = (F(u), G(u) = L(x_1, x_2) + \hat{H}(u)x_3$, for every $u = (x_1, x_2, x_3) \in I\!R^3$, we suppose

(X1) There exist $M, R, \rho > 0$ such that, for every $u = (x_1, x_2, x_3) \in I\!R^3$, we have

$$|(-g_1h_2 + g_2h_1)(u)| \leq M, \ \forall \ |(x_1, x_2)| > R, \ |x_3| < \rho.$$

(X2) There exist $A, B > 0$ and $\phi \in C(I\!R, I\!R)$ such that

$$< F(u), u > \leq A\|(x_1, x_2)\|\phi(|x_3|) + B,$$

for every $u = (x_1, x_2, x_3) \in I\!R^3$. Then, the above system is globally asymptotically stable. Consider $V : I\!R^3 \to I\!R$ defined by $V(u) = |x_3|$, for every $u = (x_1, x_2, x_3) \in I\!R^3$. Then, $V \in C^1(I\!R^3 \setminus I\!R^2), I\!R)$. Furthermore, it is not difficult to verify that X satisfies (H0) and (H1) and that V satisfies $(PS)_{(X,c)}$ on $I\!R^3 \setminus I\!R^2$, for every $c > 0$. By (X1), X also stisfies (H2) on $I\!R^3 \setminus I\!R^2$. We also note that condition (X2) implies that the solutions of the system are defined on $[0, \infty)$. Since the proof of Theorem A is the same under these conditions, we obtain that the origin is a global attractor for the above system as claimed.

A particular case is obtained when $X((x_1, x_2, x_3)) = (x_1(x_3 - 1), x_2(x_3 - 1), -x_3)$, for every $(x_1, x_2, x_3) \in I\!R^3$. This simple case provides an example where (MY) condition is not satisfied in $I\!R^3$.

3. Consider $X : I\!R^2 \to I\!R^2$ the vector field of class C^1 defined by

$$X(x_1, x_2) = (-x_1, h(x_1) + \alpha x_2), \ \forall\ (x_1, x_2) \in I\!R^2. \tag{4.1}$$

where $\alpha \in I\!R$ and $h \in C^1(I\!R, I\!R)$ satisfy

(a) $h(s) < 1 - \alpha$, for every $s \in I\!R$,

(b) $h(0) < -\alpha$.

Then, the origin is a global attractor for the associated system. Indeed, since

$$X'(x_1, x_2) = \begin{bmatrix} -1 & 0 \\ h'(x_1)x_2 & h(x_1) + \alpha \end{bmatrix}$$

for every $(x_1, x_2) \in I\!R^2$, (a) implies that X satisfies (H4). Furthermore, it is clear that X_1 satisfies (H6) and (PS). From (b) and the continuity of h, we find $c > 0$ such that

$$\det(X'(x_1, x_2)) = -(h(x_1) + \alpha), \ \forall\ (x_1, x_2) \in I\!R^2, \ |x_1| < c.$$

Consequently, X satisfies (H6). Furthermore, by (H4), $X_1(0) = 0$ and the above relation, we get that $X \in \mathcal{X}$. Invoking Theorem B, we conclude that the origin is a global attractor for the associated system.

4. Suppose $X : I\!R^2 \to I\!R^2$ is a vector field of class C^1 satisfying (4.1), with $\alpha \in I\!R$ and $h \in C^1(I\!R, I\!R)$ satisfying (b) and the following stronger version of (a):

($\hat{a}$) There exists $\hat{\alpha} > \alpha$ such that

$$h(s) \le 1 - \hat{\alpha}, \ \forall\ s \in I\!R.$$

Now assume $Y = (f, g) : I\!R^2 \to I\!R^2$ is a vector a field of class C^1 satisfying $Y(0) = 0$ and

(Y1) There exists $M > 0$ such that

$$\|Y\|_{C^1} = \sup\{\|Y(u)\| + \|Y'(u)\| \mid u \in I\!R^2\} < M < \infty,$$

(Y2) There exist $M_1, C_1 > 0$ such that

$$\lim_{|x_2| \to \infty} \sup_{|x_1| \le C_1} \left| \frac{\partial f}{\partial x_2}(x_1, x_2) x_2 \right| \le M_1 < \infty.$$

Then, there exists $\lambda_0 > 0$ such that the origin is a global attractor for the system

$$\dot{u}(t) = X(u(t)) + \lambda Y(u(t)),$$

for every $\lambda \in I\!R$, $|\lambda| < \lambda_0$. Considering $X_\lambda = X + \lambda Y$, by Theorem B it suffices to verify that X_λ belongs to $\mathcal{X}$ and satisfies (H4)-(H6). First, we note that

$$\text{Trace}(X'_\lambda(x_1,x_2)) = -1 + h(x_1) + \lambda\left(\frac{\partial f}{\partial x_1}(x_1,x_2) + \frac{\partial g}{\partial x_2}(x_1,x_2)\right).$$

Consequently, from $(\hat{a})$ and (Y1), we find $\lambda_1 > 0$ such that X_λ satisfies (H4) for $|\lambda| < \lambda_1$. Now, we use (b), (Y1) and (Y2) to obtain $\lambda_2, c_2 > 0$ such that, for every $|\lambda| < \lambda_2$,

$$\det(X'_\lambda((x_1,x_2)) > 0, \ \forall\ (x_1,x_2) \in I\!R^2, \ |x_1| < c_2. \tag{4.2}$$

Observing that $(X_\lambda)_1(x_1,x_2) = -x_1 + \lambda f(x_1,x_2)$, for $(x_1,x_2) \in I\!R^2$, we find $0 < \lambda_0 < \min\{\lambda_1, \lambda_2\}$ and $c > 0$ such that, for every $|\lambda| < \lambda_0$, we have

$$|x_1| < c_2, \ \forall\ (x_1,x_2) \in S_{(-c,c)}((X_\lambda)_1). \tag{4.3}$$

Using (4.2) and (4.3), we conclude that X_λ satisfies (H5) for every $|\lambda| < \lambda_0$. We also note that $X_\lambda \in \mathcal{X}$ since $X_\lambda(0) = 0$. Furthermore, by taking λ_0 smaller if necessary, we obtain that $(X_\lambda)_1$ satisfies (PS) and (H6). That concludes the verification that the origin is a global attractor for the above system when $|\lambda| < \lambda_0$.

5. Consider the system

$$\begin{cases} \dot{x}_1(t) = -x_1 + x_2 x_3 + x_4, \\ \dot{x}_2(t) = \frac{1}{2}(1 - e^{-x_1^2} + \sin x_1)\arctan x_2 + x_1 x_3 + x_3, \\ \dot{x}_3(t) = -x_3(t) + x_4 \sin x_1, \\ \dot{x}_4(t) = -x_4(t). \end{cases}$$

Then, the origin is a global attractor. First of all, we observe that the associated vector field satisfies $(\hat{H}0)$ with $L : I\!R^2 \to I\!R^2$ and $H : I\!R^4 \to I\!R^2$ given by

$$L(x) = (-x_1, \frac{1}{2}(1 - e^{-x_1^2} + \sin x_1)\arctan x_2), \forall\, x = (x_1,x_2) \in I\!R^2, \tag{4.4}$$

and

$$H(u) = (x_1 x_3 + x_4, x_1 x_3 + x_4), \forall\, u = (x_1,x_2,x_3,x_4) \in I\!R^4. \tag{4.5}$$

Taking $V : I\!R^4 \to I\!R^2$ defined by

$$V(u) = \frac{1}{2}x_3^2 + \frac{1}{2}x_4^2, \forall\, u = (x_1,x_2,x_3,x_4) \in I\!R^4,$$

we obtain that V satisfies (H1) and

$$< \nabla V(u), X(u) > \leq -V(u), \ \forall\ u \in I\!R^4. \tag{4.6}$$

This implies that V satisfies $(PS)_{(X,c)}$ for every $c > 0$. Now, given a solution $\gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ of the system and $a > 0$, we use the two first equations of the system and (4.6) to find $t_0 > 0$ and $A > 0$ such that

$$\frac{d}{dt}(x_1^2(t)+x_2^2(t)) \leq a(x_1^2(t)+x_2^2(t))+A,\ \forall\ t \geq t_0$$

Consequently, there exist $B, C > 0$ such that

$$x_1^2(t)+x_2^2(t) \leq Be^{at}+C,\ \forall\ t \geq 0. \tag{4.7}$$

On the other hand, using the definition of L and H, we obtain $D, E > 0$ such that, for every $t \geq 0$,

$$|< L^{\perp}(\gamma(t)), H(\gamma(t)) >| \leq D(|x_3|+|x_4|)+E(|x_1(t)|^2+|x_1(t)|^2)|x_3(t)|.$$

Invoking (4.6)-(4.7), we conclude that Lemma 2.6 holds. Hence, by Theorem C and Remark 2.23, to show that the origin is a global attractor for the system, it suffices to verify that condition (H3) is satisfied. By Lemma 3.4, L satisfies (H8) since it satisfies (H5)-(H6) and L_1 satisfies (PS). Applying (4.4), (4.5) and (H8), we get

$$\limsup_{\|x\|\to\infty, \|y\|\to 0} \frac{\|H(u)\|}{\|L(x)\|} \leq |x_3| + \frac{\|y\|}{d} = 0.$$

Here, we have considered $u=(x,y)$, $x=(x_1,x_2)$ and $y=(x_3,x_4)$. This concludes the verification that the origin is a global attractor for the system.

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications.* J. F. Anal. **14** (1973), 349-381.

[2] C. Chicone and J. Sotomayor, *On a class of complete polynomial vector fields in the plane.* J. of Diff. Eq. **61** (1986), 398-418.

[3] A. Cima, A. van den Essen, A. Gassull, E. Hubbers and F. Mañosas, *A polynomial counterexample to the Markus-Yamabe Conjecture.* Nijmegen Report 9551 (1995), 1-4.

[4] R. Feßler, *A proof of the two dimensional Markus-Yamabe Stability Conjecture.* Ann. Polonici Math. **62** (1995), 45-75.

[5] A. A. Glutsyuk, The complete solution of the Jacobian problem for vector fields on the plane. Comm. Moscow Math. Soc., Russian Math. Surveys **49** (1994), 185-186.

[6] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture.* Ann. Inst. H. Poincaré Anal. Non Linéare **12** (1995), 627-671.

[7] C. Gutierrez and M. A. Teixeira, *Asymptotic stability at infinity of planar vector fields.* Bol. Soc. Bras. Mat. **26**, N.1, (1995), 57-66.

[8] G. H. Meisters and C. Olech, *Global stability, injectivity and the Jacobian conjecture.* Procc. of the first world congress on nonlinear analysis. Editor. Lakshmikanthan. Walter Gruyter & Co., Tampa, Fl, 1992.

[9] C. Olech, *On the global stability of an autonomous system on the plane.* Cont. to Diff. Eq. **1** (1963), 389-400.

[10] T. Parthasarathy, *On global univalence theorems.* Lecture Notes in Mathematics n^0 977, Springer Verlag, 1983.

[11] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations.* C.B.M.S. Regional Confer. Ser. in Math., n^0 65, Am. Math. Soc., Providence, RI, 1986.

[12] E. A. B. Silva and M. A. Teixeira, to appear in *A version of Rolle's Theorem and applications.*