SEMILINEAR ELLIPTIC SYSTEMS

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Introduction

In this series of lectures we survey and discuss results on the existence of solutions for the system

$$-\Delta u = g(x,u,v), \quad -\Delta v = f(x,u,v) \text{ in } \Omega, \tag{0.1}$$

subject to Dirichlet boundary conditions on $\partial\Omega$. Ω is a bounded domain in $\mathbb{R}^N, N \geq 3$. Due to the use of some results on the regularity of solutions of elliptic problems we shall at some points assume implicitely some regularity on $\partial\Omega$. We do not discuss the case $N = 2$, where the imbedding theorems of Trudinger-Moser allow the treatment of nonlinearities which have a growth faster than the polynomial growth required by the Sobolev imbeddings.

Clearly we do not present in full the proofs of all the results discussed. The methodology used here is the following. We state the results, comment the key points in the proofs, explaining the techniques used, compare the results and hint questions that can be object of further study. With these purposes in mind, we do not state the more general results available in the literature. For that matter, some results are valid for more general second order operators instead of the Laplacian. Also other boundary conditions can be considered. A careful guide to the literature is presented all along these lectures.

We say that system(0.1) is *variational* if either one of the following conditions holds:

(I) There is a real-valued differentiable function $F(x,u,v)$ for $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that $\dfrac{\partial F}{\partial u} = g$ and $\dfrac{\partial F}{\partial v} = f$. In this case, the system is said to be *gradient*.

(II) There is a real-valued differentiable function $H(x,u,v)$ for $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that $\dfrac{\partial H}{\partial u} = f$ and $\dfrac{\partial H}{\partial v} = g$. In this case, the system is said to be *Hamiltonean*.

The terminology *variational* comes from the fact that in both cases, the above system is the Euler-Lagrange equations of a functional naturally associated to the system. Indeed, if we work with functions u and v in $H_0^1(\Omega)$, the functional associated to the gradient system is

$$\Phi(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2) - \int_\Omega F(x,u,v). \tag{0.2}$$

while the one associated to a Hamiltonean system is

$$\Phi(u,v) = \int_\Omega \nabla u . \nabla v - \int_\Omega H(x,u,v), \tag{0.3}$$

provided F and H have the appropriate growth in order to get their integrability. We shall come back to this question later.

Therefore, these two types of systems can be treated using Critical Point Theory. However, if the system is not variational we make recourse to topological methods.

We emphasize that we will be mostly interested in superlinear systems. For instance if f and g have growth with respect to u and v faster than linear. For Hamiltonean systems we will have a notion of superlinearity which takes into account that we have a coupled system. It is expected that superlinear systems, as the case of superlinear equations, are much harder to study, no matter which method one uses. Variationally, we are confronted with questions of compactness of the functional, as well as with an intrincated geometry, in most cases. Topologically, the question of a priori bounds comes as a difficult problem. We plan to address all these questions.

In Lecture 1, we discuss a class of non-variational systems, showing how to prove the existence of positive solutions. We start there the discussion of a priori bounds, which is taken in detail in Lecture 2, using the Blow-up Method. This discussion leads naturally to questions which are answered by Theorems of Liouville type. And such theorems are the object of Lecture 3. In Lecture 4 we discuss gradient systems and in Lecture 5 we consider Hamiltonean systems.

1 Lecture 1: Existence of Positive Solutions for a Nonvariational System

In this lecture we discuss the existence of positive solutions for system (0.1). Since there is no variational structure we shall use topological methods through general propositions concerning nonlinear mappings which take a cone in a Banach space into itself. There is a large literature on this subject, cf. the papers by Amann [1], Benjamin [3], Nussbaum [43], the books of Krasnosels'kii [35], Deimling [20], and the survey article [19], where many further references can be found.

We recall the following result, Theorem 3.1 of [19].

Theorem 1.1 *Let $\mathcal{C}$ be a cone in a Banach space X and $T : \mathcal{C} \to \mathcal{C}$ a compact mapping such that $T(0) = 0$. Assume that there are real numbers $0 < r < R$ and $t > 0$ such that*

(i) $x \neq tTx$ for $0 \leq t \leq 1$ and $x \in \mathcal{C}, ||x|| = r$, and

(ii) There exists a compact mapping $H : \overline{B}_R \times [0, \infty) \to \mathcal{C}$ (where $B_\rho = \{x \in \mathcal{C} : ||x|| < \rho\}$) such that

(a) $H(x, 0) = Tx$ for $||x|| = R$,

(b) $H(x, t) \neq x$ for $||x|| = R$ and $t \geq 0$

(c) $H(x, t) = x$ has no solution $x \in \overline{B}_R$ for $t \geq t_0$

Then

$$i_c(T, B_r) = 1, \; i_c(T, B_R) = 0, \; i_c(T, U) = -1,$$

where $U = \{x \in \mathcal{C} : r < ||x|| < R\}$, and i_c denotes the Leray-Schauder index. As a consequence T has a fixed point in U.

Let us assume the following conditions on the nonlinearity:

(A1) $f, g : \overline{\Omega} \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \to \overline{\mathbb{R}_+}$ are locally lipschtzian.

(A2) $f(x, u, v) = o(|u| + |v|), g(x, u, v) = o(|u| + |v|)$ uniformly in x and for $|u| + |v| \to 0$.

We apply Theorem 1.1 in the following context. Consider the space

$$X = \{U = \begin{bmatrix} u \\ v \end{bmatrix} : u, v \in C^0(\overline{\Omega}), u = v = 0 \text{ on } \partial\Omega\}$$

endowed with the norm $||U|| = ||u||_{L^\infty} + ||v||_{L^\infty}$, which makes it a Banach space. Let $S : X \to X$ be the solution operator defined by $S \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$, where u and v are the solution of

$$-\Delta u = \varphi, \qquad -\Delta v = \psi \text{ in } \Omega$$
$$u = v = 0 \text{ on } \partial\Omega.$$

It is well known that S is a linear compact operator. Let

$$\mathcal{C} = \{U = \begin{bmatrix} u \\ v \end{bmatrix} \in X : u(x) \geq 0, v(x) \geq 0, x \in \overline{\Omega}\}.$$

It follows from the Maximum Principle that $S(\mathcal{C}) \subset \mathcal{C}$. Using (A1) we see that the mapping defined next takes $\mathcal{C}$ into $\mathcal{C}$:

$$T \begin{bmatrix} u \\ v \end{bmatrix} = S \begin{bmatrix} g(x, u, v) \\ f(x, u, v) \end{bmatrix}.$$

It follows also that T is compact and $T(0) = 0$, where (A1) and (A2) are used again.

In order to show that (i) holds, assume by contradiction that for all $r > 0$ there is a $t \in [0,1]$ such that $U = tTU$, which can be written as

$$-\Delta u = tg(x,u,v), \quad -\Delta v = tf(x,u,v). \tag{1.1}$$

Take $\varepsilon < \lambda_1/2$ where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Using (A2), we see that there is $r_0 > 0$ such that for all $0 < r \leq r_0$ one has $|f(x,u,v)| \leq \varepsilon(|u| + |v|)$ and a similar estimate for g. Multiplying the equations by φ_1, integrating by parts and using these estimates we get $\lambda_1 \int u\varphi_1 \leq t\varepsilon(\int u\varphi_1 + \int v\varphi_1)$ and a similar one corresponding to the second equation. Using these inequalities we conclude that $\lambda_1 \leq 2\varepsilon$, which is a contradiction.

In order to verify (ii) of Theorem 1.1 we introduce

$$H\left(\begin{bmatrix} u \\ v \end{bmatrix}, t\right) = S\begin{bmatrix} g(x, u+t, v+t) \\ f(x, u+t, v+t) \end{bmatrix}.$$

Clearly (a) holds and (c) follows readily from some sort of "superlinearity". Any of the conditions below gives (c):

(A3i) There are real numbers $\mu > \lambda_1$ and $C > 0$ such that $f(x,u,v) \geq \mu v - C$, uniformly in $x \in \overline{\Omega}, u \in \mathbb{R}^+$.

(A3ii) There are real numbers $\mu > \lambda_1$ and $C > 0$ such that $g(x,u,v) \geq \mu u - C$, uniformly in $x \in \overline{\Omega}, v \in \mathbb{R}^+$

(A3iii) There are positive real numbers μ_1, μ_2 and C such that $\mu_1\mu_2 > \lambda_1^2$ and

$$f(x,u,v) \geq \mu_1 u - C, \text{uniformly in } x \in \overline{\Omega}, v \in \mathbb{R}^+$$

and

$$g(x,u,v) \geq \mu_2 v - C, \text{ uniformly in } x \in \overline{\Omega}, u \in \mathbb{R}^+.$$

Finally, condition (b) is an a priori bound for the parametrized system

$$\begin{cases} -\Delta u = g(x, u+t, v+t), \quad -\Delta v = f(x, u+t, v+t) \\ u = v = 0 \text{ on } \partial\Omega \end{cases} \tag{1.2}$$

In summary we have the following result.

Theorem 1.2 *Assume (A1), (A2) and one of the conditions (A3). Suppose that there exists a constant $C > 0$ such that*

$$||u||_{L^\infty}, ||v||_{L^\infty} \leq C$$

for all eventual solutions of (1.1). Then system (0.1) has a non-negative non trivial solution (u,v).

Remark 1.1 *Using this sort of ideas a similar result was previously proved for the scalar case in [28]. The case of system was studied in [46], where a linear part more general than the one here was considered. Lecture 2 is devoted to get a priori bounds for such systems via the blow-up method. Let us next illustrate other two methods for obtaining a priori bounds, which apply to some special cases: (i) via Hardy-Sobolev, (ii) via Moving Planes.*

So, let us now assume the following growth conditions on f and g

(A4) There exist $q \geq 1$ and $\sigma' \geq 0$ such that

$$|f(x,u,v)| \leq C(|u|^q + |v|^{q\sigma'} + 1)$$

uniformly in $x \in \overline{\Omega}$.

(A5) There exist $p \geq 1$ and $\sigma \geq 0$ such that

$$|g(x,u,v)| \leq C(|v|^p + |u|^{p\sigma} + 1)$$

uniformly in $x \in \overline{\Omega}$.

Theorem 1.3 *Let $N \geq 4$. Assume conditions (A1), (A3iii), (A4) and (A5) with p, q, σ and σ' satisfying*

$$\frac{1}{p+1} + \frac{N-1}{N+1}\,\frac{1}{q+1} > \frac{N-1}{N+1} \tag{1.3}$$

$$\frac{1}{p+1}\,\frac{N-1}{N+1} + \frac{1}{q+1} > \frac{N-1}{N+1} \tag{1.4}$$

and

$$\sigma = \frac{L}{\max(L,K)}, \qquad \sigma' = \frac{K}{\max(L,K)}$$

where

$$K = \frac{p}{p+1} - \frac{2}{N} > 0 \text{ and } L = \frac{q}{q+1} - \frac{2}{N} > 0.$$

Let (u,v) be a positive solution of (0.1). Then there exists a constant $C > 0$ such that $||u||_{L^\infty} \leq C$ and $||v||_{L^\infty} \leq C$.

Remark 1.2 *The above result is Theorem 2.1 in [15]. How about $N = 3$?. The method of proof is the use of Hardy type of inequalities in interpolated forms with Sobolev imbedding inequalities. For instance, one has the following result which was proved by Brézis-Turner [10] in the case $q = 2$ and used to get a priori bounds in the scalar case. The more general case is due to Kavian [34]. A discussion of Hardy's inequalities can be seen in [15].*

Theorem 1.4 *Let $u \in W_0^{1,q}(\Omega), q < N$. Then for any $\tau \in [0,1]$ one has*

$$\left|\left|\frac{u}{\varphi^\tau}\right|\right|_{L^r} \leq C\,||\nabla u||_{L^q} \tag{1.5}$$

where

$$\frac{1}{r} = \frac{1}{q} - \frac{1-\tau}{N}.$$

The first step in the proof of Theorem 1.3 is to use (A3iii) and prove that there is a constant $c > 0$ such that $\int u\varphi_1 \leq c$ and $\int v\varphi_1 \leq c$ for all eventual positive solutions of (0.1). This implies that u and v are uniformly bounded in L^1_{loc}. So the whole problem is to control them near boundary. This is precisely the role of (1.5). Observe that if $p = q$ then (1.3)-(1.4) reduces to $p < (N+1)/(N-1)$ which is exponent in the work in [10] for the scalar case. As in the scalar case, p and q restricted by (1.3)-(1.4) is not the best growth admissible. See the next theorem.

Theorem 1.5 *Assume (A1) and (A3iii) for a function f which depends only on u and a g which depends only on v. Assume that there are numbers $\alpha, \beta \in]0, \infty[$ and $1 \leq p, q < \infty$ such that*

$$\lim_{u\to\infty} \frac{f(u)}{u^q} = \alpha, \quad \lim_{v\to\infty} \frac{g(v)}{v^p} = \beta \tag{1.6}$$

and

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}\ , \ N \geq 3. \tag{1.7}$$

Then there is a constant $C > 0$ such that $||u||_{L^\infty} \leq C$ and $||v||_{L^\infty} \leq C$ for all positive solutions of (0.1).

Remark 1.3 *This is Theorem 2.1 in [12]. Condition (1.7) says that p, q are below the critical hyperbola, see more on that in Lecture 3. The idea in the proof of Theorem 1.5 is, once you have that u, v are bounded in L^1_{loc}, to proceed to estimate them near the boundary. For that matter, one proceeds as in [28] and use the Method of Moving Planes. For this method see, for instance, [9].*

2 Lecture 2: A Priori Bounds for Positive Solutions of a Non-Variational System

In order to appreciate the symmetry of our hypotheses, we shall use in this lecture the notation u_1 and u_2 for the unknown functions in system (0.1), in place of u and v. In f and g we separate the leading part and consider a system of the form

$$\begin{cases} -\Delta u_1 = a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + h_1(x, u_1, u_2) \\ -\Delta u_2 = c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + h_2(x, u_1, u_2) \end{cases} \tag{2.1}$$

subject to Dirichlet boundary conditions on $\partial\Omega$, where Ω is some smooth bounded domain in $\mathbb{R}^N$. We assume at the outset the hypotheses below. Further conditions will come later. In this lecture we discuss positive solutions of system (2.1). And in fact this is already reflected in the way we wrote the system; observe that the exponents α_{ij} are not necessarily integers.

(A1) $a, b, c, d : \overline{\Omega} \to [0, \infty)$ are continuous functions.

(A2) $\alpha_{ij} \geq 0 \quad i, j = 1, 2$

(A3) There exist positive constants c_1 and c_2 such that

$$|h_1(x,u_1,u_2)| \le c_1(1+|u_1|^{\beta_{11}}+|u_2|^{\beta_{12}})$$
$$|h_2(x,u_1,u_2)| \le c_2(1+|u_1|^{\beta_{21}}+|u_2|^{\beta_{22}})$$

where

$$0 \le \beta_{ij} < \alpha_{ij} \quad i,j=1,2$$

We remark that one can consider more general elliptic operators in system (2.1) in place of the Laplacian, as well as non-homogeneous boundary conditions. We mention the work of Souto [46] [47], Montenegro [40], [41], [42], Birindelli-Mitidieri [8]. In [40] systems with $m > 2$ equations are studied, and in Clément-Manásevich-Mitidieri [17] systems involving p-Laplacians are considered.

The blow-up method to obtain a priori bounds.

This method was first used by Gidas-Spruck [30] in the context of a single equation. Later applications to systems can be seen in [46], [17], [8], and the most general results in [40], whose approach we follow next. This method explores distinct homogeneities of the several terms in the equations.

Our aim is to prove that under further conditions which will be introduced soon, the non- negative solutions of system (2.1) are bounded in L^∞, i.e., there is a constant $C > 0$ such that $||u_1||_{L^\infty} \le C$ and $||u_2||_{L^\infty} \le C$. In this section we are considering classical solutions, i.e., $u_i \in C^2(\Omega)\cap C^0(\overline{\Omega})$. We also assume that Ω satisfies the exterior cone condition.

Suppose, by contradiction, that there is no such a priori bound. That is, there is a sequence of positive solutions $(u_{1,n}, u_{2,n})$ such that

$$\max(||u_{1,n}||_{L^\infty}, ||u_{2,n}||_{L^\infty}) \to \infty.$$

Let β_1 and β_2 be two positive real numbers to be chosen later. We may assume without loss of generality that $||u_{1,n}||_{L^\infty} \to +\infty$ and

$$||u_{1,n}||_{L^\infty}^{\frac{1}{\beta_1}} \ge ||u_{2,n}||_{L^\infty}^{\frac{1}{\beta_2}} .$$

Let $x_n \in \Omega$ be the point where $u_{1,n}(x)$ assumes its maximum. Let $\lambda_n > 0$ be such that $\lambda_n^{\beta_1}||u_{1,n}||_{L^\infty} = 1$. Clearly $\lambda_n \to 0$ as $n \to \infty$. Let $\Omega_n = \{y \in \mathbb{R}^N : \lambda_n y + x_n \in \Omega\}$ and define the new unknowns $v_{i,n} : \Omega_n \to \mathbb{R}$ by

$$v_{i,n}(y) = \lambda_n^{\beta_i} u_{i,n}(\lambda_n y + x_n).$$

We observe that $v_{1,n}(0) = 1$ and that $0 \le v_{i,n}(y) \le 1$ for all $y \in \Omega_n$ and $i = 1,2$.

Due to compactness of $\overline{\Omega}$ we may assume that $x_n \to x_0 \in \overline{\Omega}$. So there are two cases to consider, $x_0 \in \Omega$ and $x_0 \in \partial\Omega$.

(i)*First case*, $x_0 \in \Omega$. Then given $R > 0$ there is an $n_0 \in \mathbb{N}$ such that $\Omega_n \supset \overline{B_R(0)}$. Since

$$\Delta_y v_{i,n} = \lambda_n^{\beta_i}\Delta_y u_{i,n}(\lambda_n y + x_n) = \lambda_n^{\beta_i+2}\Delta_x u_{i,n}(\lambda_n y + x_n),$$

we see that

$$-\Delta v_{1,n} = \lambda_n^{\beta_1+2}\{a(*)u_{1,n}^{\alpha_{11}}(*) + b(*)u_{2,n}^{\alpha_{12}}(*) + h_1(*, u_{1,n}(*), u_{2,n}(*))\},$$

where the $*$ stands for $\lambda_n y + x_n$, and a similar equation for $v_{2,n}$. In this way, we obtain the following system for $v_{i,n}$:

$$\begin{cases} -\Delta v_{1,n} = \lambda_n^{\beta_1+2-\beta_1\alpha_{11}} a(*)v_{1,n}^{\alpha_{11}} + \lambda_n^{\beta_1+2-\beta_2\alpha_{12}} b(*)v_{2,n}^{\alpha_{12}} + \\ \quad \lambda_n^{\beta_1+2} h_1(*, \lambda_n^{-\beta_1} v_{1,n}, \lambda_n^{-\beta_2} v_{2,n}) \\ \\ -\Delta v_{2,n} = \lambda_n^{\beta_2+2-\beta_1\alpha_{21}} c(*)v_{1,n}^{\alpha_{21}} + \lambda_n^{\beta_2+2-\beta_2\alpha_{22}} d(*)v_{2,n}^{\alpha_{22}} + \\ \quad \lambda_n^{\beta_2+2} h_2(*, \lambda_n^{-\beta_1} v_{1,n}, \lambda_n^{-\beta_2} v_{2,n}) \end{cases} \tag{2.2}$$

Now we fix $R > 0$ such that $\overline{B_{2R}} = \overline{B_{2R}(0)} \subset \Omega_n$, for all n sufficiently large. Clearly the sequences $(v_{1,n})$ and $(v_{2,n})$ are uniformly bounded in $L^\infty(B_{2R})$, namely by 1. Viewing bounds in $C^{1,\alpha}$ we use L^p-regularity ([32] Theorem 9.11) to get

$$||v_{i,n}||_{W^{2,p}(B_R)} \leq C(||v_{i,n}||_{L^p(B_{2R})} + ||(RHS)_i||_{L^p(B_{2R})}),$$

where $(RHS)_i$ stands for the right side of equation $i = 1, 2$ in system (2.2). Since we can take $p > N$, we conclude that (v_{in}) is uniformly bounded in $C^{1,\alpha'}(\overline{B}_R)$. Passing to a subsequence we may assume that $v_{i,n} \to v_i$ in $C^{1,\alpha}(\overline{B}_R)$ for $0 < \alpha < \alpha' < 1$ and using L^p-regularity again we may also assume that this convergence is in $W^{2,p}(B_R)$. Now the idea is to pass to the limit in system (2.2). So we should first know what happens with the coefficients of the leading terms. Clearly we want that all exponents of λ_n should be ≥ 0. If one of them is positive, then the corresponding term disappears at the limit. If one of them is equal to zero, then the corresponding term has a definite limit, which we shall see soon. At this point, we introduce two classes of systems: (i) weakly coupled and (ii) strongly coupled.

Definition 1. System (2.1) is *weakly coupled* if there are positive numbers β_1, β_2 such that

$$\begin{aligned} \beta_1 + 2 - \beta_1\alpha_{11} = 0 \quad , \quad \beta_1 + 2 - \beta_2\alpha_{12} > 0 \\ \beta_2 + 2 - \beta_1\alpha_{21} > 0 \quad , \quad \beta_2 + 2 - \beta_2\alpha_{22} = 0 \end{aligned} \tag{2.3}$$

Definition 2. System (2.1) is *strongly coupled* if there are positive numbers β_1, β_2 such that

$$\begin{aligned} \beta_1 + 2 - \beta_1\alpha_{11} > 0 \quad , \quad \beta_1 + 2 - \beta_2\alpha_{12} = 0 \\ \beta_2 + 2 - \beta_1\alpha_{21} = 0 \quad , \quad \beta_2 + 2 - \beta_2\alpha_{22} > 0 \end{aligned} \tag{2.4}$$

Remark 2.1 *It follows that system (2.1) is weakly coupled if* $\alpha_{11} > 1, \alpha_{22} > 1$ *and*

$$\alpha_{12} < \frac{\alpha_{22} - 1}{\alpha_{11} - 1}\alpha_{11} \quad \text{and} \quad \alpha_{21} < \frac{\alpha_{11} - 1}{\alpha_{22} - 1}\alpha_{22}, \tag{2.5}$$

and, in this case, we choose

$$\beta_1 = \frac{2}{\alpha_{11} - 1} \quad \text{and} \quad \beta_2 = \frac{2}{\alpha_{22} - 1}. \tag{2.6}$$

Remark 2.2 *System (2.1) is strongly coupled if* $\alpha_{12}\alpha_{21} > 1$ *and*

$$\alpha_{11} < \frac{\alpha_{21}+1}{\alpha_{12}+1}\alpha_{12} \quad \text{and} \quad \alpha_{22} < \frac{\alpha_{12}+1}{\alpha_{21}+1}\alpha_{21}, \tag{2.7}$$

and, in this case, we choose

$$\beta_1 = \frac{2(\alpha_{12}+1)}{\alpha_{12}\alpha_{21}-1} \quad \text{and} \quad \beta_2 = \frac{2(\alpha_{21}+1)}{\alpha_{12}\alpha_{21}-1}. \tag{2.8}$$

Weakly Coupled Systems.

Let us now analyse what happens with the lower order terms, firstly in the case of a weakly coupled system. Using (A3) we can estimate

$$|\lambda_n^{\beta_1+2} h_1(*, \lambda_n^{-\beta_1} v_{1,n}, \lambda_n^{-\beta_2} v_{2,n})| \leq C\lambda_n^{\beta_1+2}(1 + |\lambda_n^{-\beta_1} v_{1,n}|^{\beta_{11}} + |\lambda_n^{-\beta_2} v_{2,n}|^{\beta_{12}}).$$

and use (2.3) to see that

$$\beta_1 + 2 - \beta_1\beta_{11} > 0 \quad \text{and} \quad \beta_1 + 2 - \beta_2\beta_{12} > 0$$

and then conclude that

$$\lim_{n\to\infty} \lambda_n^{\beta_1+2} h_1(*, \lambda_n^{-\beta_1} v_{1,n}, \lambda_n^{-\beta_2} v_{2,n}) = 0.$$

A similar conclusion we have for the term h_2 is the second equation. Thus, in the limit we have

$$\begin{cases} -\Delta v_1(y) = a(x_0) v_1^{\alpha_{11}}(y) \\ -\Delta v_2(y) = d(x_0) v_2^{\alpha_{22}}(y) \end{cases} \tag{2.9}$$

for all $y \in B_R(0)$. By a standard diagonal process we have that (2.9) holds for all $y \in \mathbb{R}^N$. Assuming that $a(x_0)$ and $d(x_0)$ are positive, it follows by scaling that there are functions w_1 and w_2 of class C^2 defined in the whole of $\mathbb{R}^N$ with $w_1 \geq 0$, $w_1 \not\equiv 0$,$w_2 \geq 0$ satisfying

$$\begin{array}{l} -\Delta w_1 = w_1^{\alpha_{11}} \\ -\Delta w_2 = w_2^{\alpha_{22}} \end{array} \quad , \quad \text{in} \quad \mathbb{R}^N.$$

In this way we come to a contradiction if $0 < \alpha_{11}, \alpha_{22} < (N+2)/(N-2)$, in view of Theorem 3.1.

(ii) *Second Case*: $x_0 \in \partial\Omega$.

It still remains to discuss the case when $x_0 \in \partial\Omega$. In this case, we can change the independent variables in such a way that the boundary $\partial\Omega$ in the neighborhood of x_0 is a piece of the hyperplane $x_N = 0$, and $x_0 = 0$. The effect of this is that the Laplacean is replaced by a general second order strongly elliptic operator. However this causes no problem, since in the limit we come to an elliptic operator with constant coefficients and then an orthogonal transformation takes it to the Laplacean. As we have remarked before, we could have been working with a general second order elliptic operator instead of Δ. Let

$d_n = dist(x_n, \partial\Omega)$ and as before $\Omega_n = \{y \in \mathbb{R}^N : \lambda_n y + x_n \in \Omega\}$. Given $R > 0$, there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\Omega_n \supset B_{2R}(0) \cap \{y \in \mathbb{R}^N : y_N \geq -\frac{d_n}{\lambda_n}\}.$$

If $d_n/\lambda_n \to +\infty$, we have that, fixed $R > 0, B_{2R}(0) \subset \Omega_n$, and we are as in the case $x_0 \in \Omega$. Otherwise, we claim that there is a $C > 0$ such that $d_n/\lambda_n \geq C$ for all n. Indeed, we may assume that $d_n/\lambda_n < M$ for some positive constant M. So, given $R > 0$, for n sufficiently large the point $(0', -\frac{d_n}{\lambda_n})$ belongs to $B_R(0) \cap \{y : y_N > -M\}$. Now let $\widehat{\Omega}_n = \Omega_n \cap B_R(0) \cap \{y : y_N > -M\}$. By results of De Giorgi-Nash type ([32], Corollary 9.29, p.252) applied to the equations of system (2.2) we conclude that there is a constant $C > 0$, independent of n, such that $||v_{i,n}||_{C^\alpha(\widehat{\Omega}_n)} \leq C$. Hence we get

$$\left| v_{1,n}(0) - v_{1,n}\left(0', -\frac{d_n}{\lambda_n}\right) \right| \leq C \left(\frac{d_n}{\lambda_n}\right)^\alpha$$

and since $v_{1,n}(0) = 1$ and $v_{1,n}(0', \frac{d_n}{\lambda_n}) = 0$ we have the claim proved. So, we may assume that $d_n/\lambda_n \to s > 0$. Arguing as in the first case we conclude that

$$\begin{aligned} &-\Delta v_1 = a(x_0) v_1^{\alpha_{11}} \\ &-\Delta v_2 = d(x_0) v_2^{\alpha_{22}} \quad \text{in} \quad \{y : y_N > -s\} \\ &v_1 = v_2 = 0 \quad \text{on} \quad \{y : y_N = -s\} \end{aligned}$$

and by scaling, one obtains a pair of non-negative functions w_1 and w_2 with $w_1 \not\equiv 0$ such that

$$-\Delta w_1 = w_1^{\alpha_{11}}, \quad -\Delta w_2 = w_2^{\alpha_{22}} \quad \text{in} \quad \{y : y_N > -s\}$$

So a contradiction arrives if we have the same assumptions that were made in the case that $x_0 \in \Omega$. Cf. Theorem 3.2.

In summary, we have proved the following result.

Theorem 2.1 *Let (2.1) be a weakly coupled system satisfying conditions (A1), (A2), (A3) and such that $a(x), d(x) \geq c_0 > 0$ for $x \in \overline{\Omega}$. Assume also that $0 < \alpha_{11}, \alpha_{22} < (N+2)/(N-2)$. Then there is a constant $C > 0$ such that*

$$||u_1||_{L^\infty}, ||u_2||_{L^\infty} \leq C$$

for all positive solutions $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of system (2.1).

Strongly Coupled System. As in the case of a weakly coupled system, we see that the terms of (2.2) involving h_i go to 0 as $n \to \infty$. The remaining part of the argument is similar to this previous case. So the contradiction assumption of nonexistence of a priori bounds leads to: (i) the existence of a nontrivial non-negative C^2 solution (ω_1, ω_2) of the system

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \quad \text{in} \quad \mathbb{R}^N \tag{2.10}$$

or (ii) the existence of a nontrivial non-negative C^2 solution (ω_1, ω_2) of

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } (\mathbb{R}^N)^+ \tag{2.11}$$

with

$$\omega_1(x', 0) = \omega_2(x', 0) = 0$$

So a contradiction comes if we set conditions on the exponents such that (2.10) and (2.11) have only the trivial solution $\omega_1 = \omega_2 \equiv 0$. In summary, we have proved the following result, where we use the notation $\mathbb{R}^N_+ = \{y = (y', y_N) \in \mathbb{R}^N : y_N > 0\}$.

Theorem 2.2 *Let (2.1) be a strongly coupled system satisfying (A1), (A2) and (A3), and such that $b(x), c(x) \geq c_0 > 0$ for $x \in \overline{\Omega}$. Assume that the following conditions hold:*

(L1) The exponents α_{12} and α_{21} are such that the only non- negative solution of

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } \mathbb{R}^N$$

is $w_1 = \omega_2 \equiv 0$.

(L2) The only non-negative solution of

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } \mathbb{R}^N_+$$

with $\omega_1(x', 0) = \omega_2(x', 0) = 0$ is $\omega_1 = \omega_2 \equiv 0$. Then there is a constant $C > 0$ such that

$$||u_1||_{L^\infty}, ||u_2||_{L^\infty} \leq C$$

for all non-negative solutions (u_1, u_2) of system (2.1).

Remark 2.3 *Which conditions should be imposed on the exponents α_{12} and α_{21} in such a way that (L1) and (L2) holds? In the next lecture we address this question and state some sufficient conditions that insure the validity of (L1) and (L2).*

3 Lecture 3: Theorems of Liouville Type

The classical Liouville Theorem from Function Theory says that every bounded entire function is constant. In terms of a differential equation one has: if $(\partial/\partial\overline{z})f(z) = 0$ and $|f(z)| \leq C$ for all $z \in \mathbb{C}$ then $f(z)$ =const. Hence results with a similar contents are nowadays called Liouville theorems. For instance, a superharmonic function defined in the whole plane $\mathbb{R}^2$, which is bounded below, is constant. Also, all results discussed in this section have this nature. For completeness, we survey also results on a single equation, namely

$$-\Delta u = u^p \tag{3.1}$$

Remark 3.1 *If the equation is considered in* $\mathbb{R}^2$*, then a non-negative solution of (3.1) is necessarily null. So the interesting case is* $\mathbb{R}^N, N \geq 3$*, which we discuss next.*

Theorem 3.1 *Let* u *be a non-negative* C^2 *function defined in the whole of* $\mathbb{R}^N$*, such that (3.1) holds in* $\mathbb{R}^N$*. If* $0 < p < (N+2)/(N-2)$*, then* $u \equiv 0$*.*

Remark 3.2 *This result was proved by Gidas-Spruck [31] in the case* $1 < p < (N+2)/(N-2)$*. A simpler proof using the method of moving parallel planes was given by Chen-Li [16], and it is valid in the whole range of* p*. A very elementary proof valid for* $p \in (0, \frac{N}{N-2})$ *was given by Souto [47]. In fact, his proof is valid for the case of* u *being a non-negative supersolution, i.e.*

$$-\Delta u \geq u^p \quad \text{in } \mathbb{R}^N, \tag{3.2}$$

with p *in the same restricted range.*

Theorem 3.2 *Let* $u \in C^2(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ *be a non- negative function such that*

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^N_+ \\ u(x',0) = 0 \end{cases} \tag{3.3}$$

If $p \leq (N+2)/(N-2)$ *then* $u \equiv 0$*.*

Remark 3.3 *This is Theorem 1.3 of [30], plus Remark 2 on page 895 of the same paper. It is remarkable that in the case of the half-space the exponent* $(N+2)/(N-2)$ *is not the right one for theorems of Liouville type. Indeed, Dancer [18] has proved the following result.*

Theorem 3.3 *Let* $u \in C^2(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ *be a non- negative bounded solution of (3.3). If* $1 < p < (N+1)/(N-3)$ *for* $N \geq 4$ *and* $1 < p$ *for* $N = 3$*, then* $u \equiv 0$*.*

Remark 3.4 *If* $p = (N+2)/(N-2), N \geq 3$*, then (3.1) has a two-parameter family of bounded positive solutions:*

$$U_{\varepsilon,x_0}(x) = \left[\frac{\varepsilon\sqrt{N(N-2)}}{\varepsilon^2 + |x-x_0|^2}\right]^{\frac{N-2}{2}},$$

which are called instantons.

Next we state some results on supersolutions still in the scalar case.

Theorem 3.4 *Let* $u \in C^2(\mathbb{R}^N)$ *be a non-negative supersolution of (3.2). If* $1 \leq p \leq \frac{N}{N-2}$*, then* $u \equiv 0$*.*

Remark 3.5 *This result is proved in Gidas [29] for $1 < p \leq N/(N-2)$. The case $p = 1$ is included in Souto [47]. The proof in [47] can be slightly changed to give a simple proof of the next result for $p \in [1, \frac{N}{N-2}]$.*

Theorem 3.5 *Let $u \in C^2(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ be a non-negative supersolution of (3.3). If $1 \leq p \leq \dfrac{N}{N-2}$ then $u \equiv 0$.*

Liouvile for systems.

Now we come to systems of the form

$$-\Delta u = v^p, \quad -\Delta v = u^q. \tag{3.4}$$

Here the dividing line between existence and non-existence of positive solutions (u, v) defined in the whole of $\mathbb{R}^N$ should be the so-called *critical hyperbola* introduced independently in the work of Clément-deFigueiredo - Mitidieri [12] and Peletier-van der Vorst [45]. Such hyperbola is defined by

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}, \quad p, q > 0 \tag{3.5}$$

In analogy with the scalar case one may conjecture that (3.4) has no bounded positive solutions defined in the whole of R^N if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \quad p, q > 0. \tag{3.6}$$

This conjecture has not been setlled in full so far. Why such a conjecture? In answering, let us have some history. The critical hyperbola appeared in the study of existence of positive solutions for superlinear elliptic systems of the form

$$-\Delta u = g(v), \quad -\Delta v = f(u) \tag{3.7}$$

subject to Dirichlet boundary conditions in a bounded domain Ω of $\mathbb{R}^N$. If $g(v) \sim v^p$ and $f(u) \sim u^q$ as $u, v \to \infty$, then system (3.7) is said to be sub-critical if p, q satisfy (3.6). For such systems [in analogy with sub-critical scalar equations, $-\Delta u = f(u)$, $f(u) \sim u^p$ and $1 < p < (N+2)/(N-2)$] one can establish a priori bounds of positive solutions, prove a Palais-Smale condition and put through an existence theory by a topological or a variational method. This sort of work initiated in [12] and [45] has been continued. We shall survey all this in Section 5. Recall the critical scalar case, where the problem: $-\Delta u = |u|^{2^*-2}u$ in Ω, $u = 0$ on $\partial\Omega$, Ω a starshaped bounded domain in $\mathbb{R}^N, N \geq 3$, has no solution $u \neq 0$; here $2^* = 2N/(N-2)$]. In analogy, in the case of sytems, the critical hyperbola appears in the statement: if Ω is a bounded starshaped domain in $\mathbb{R}^N, N \geq 3$, the Dirichlet problem for the system below has no non-trivial solution:

$$-\Delta u = |v|^{p-1}v, \quad -\Delta v = |u|^{q-1}u$$

if, p, q satisfy (3.5). This follows from an identity of Pohozaev-type, see Mitidieri [36]; also Pucci-Serrin [44] for general forms of Pohozaev-type identities.

Next we describe several Liouville-type theorem for systems.

Theorem 3.6 *Let $p, q > 0$ satisfying (3.6). Then system (3.4) has no non-trivial radial positive solutions of class $C^2(\mathbb{R}^N)$.*

Remark 3.6 *This result settles the conjecture in the class of radial functions. It was proved in [36] for $p, q > 1$, and for p, q in the full range by Serrin-Zou [48]. The proof explores the fact that eventual positive radial solutions of (3.4) have a definite decay at ∞; this follows from an interesting observation (cf. Lemma 3.1 in [36]): If $u \in C^2(R^N)$ is a positive radial superharmonic function, then*

$$ru'(r) + (N-2)u(r) \geq 0, \text{for all } r > 0.$$

Theorem 3.6 is sharp as far as the critical hyperbola is concerned. Indeed, there is the following existence result of Serrin-Zou [50].

Theorem 3.7 *Suppose that $p, q > 0$ and that*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq 1 - \frac{2}{N} \tag{3.8}$$

Then there exist infinitely many values $\xi = (\xi_1, \xi_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that system (3.4) admits a positive radial solution (u, v) with central values $u(0) = \xi_1, v(0) = \xi_2$. Moreover $u, v \to 0$ as $|x| \to \infty$, so that the solution is in fact a ground state for (3.4).

Let us now mention some results on the nonexistence of positive solutions of (3.4), without the assumption of being radial.

Theorem 3.8 *Let $u, v \in C^2(\mathbb{R}^N)$ be non-negative solutions of*

$$-\Delta u \geq v^p, \quad -\Delta v \geq u^q \quad \text{in } \mathbb{R}^N, \tag{3.9}$$

where $p, q > 0$ and

$$\frac{1}{p+1} + \frac{1}{q+1} \geq \frac{N-2}{N-1}. \tag{3.10}$$

Then $u = v \equiv 0$.

This result is due to Souto [46], [47]. The idea of his proof is to reduce the problem to a question concerning a scalar equation. Suppose, by contradiction, that u and v are positive solutions of (3.9) in $\mathbb{R}^N$. Introduce a function $\omega = uv$. So

$$\Delta\omega \leq 2\nabla u \nabla v - u^{q+1} - v^{p+1}. \tag{3.11}$$

Using the inequality

$$a \cdot b \le \frac{1}{4}|a+b|^2 \quad a, b \in \mathbb{R}^N$$

we get that

$$2\nabla u \nabla v \le \frac{1}{2}\omega^{-1}|\nabla\omega|^2.$$

On the other hand, choose $r > 0$ such that $\frac{1}{r} = \frac{1}{p+1} + \frac{1}{q+1}$. Then by Young's inequality

$$\omega^r = u^r v^r \le \frac{r}{q+1}u^{q+1} + \frac{r}{p+1}v^{p+1} \le u^{q+1} + u^{p+1}.$$

So

$$\Delta\omega \le \frac{1}{2}\omega^{-1}|\nabla\omega|^2 - \omega^r. \tag{3.12}$$

Replacing ω by f^2 in (3.12) one obtains

$$-\Delta f \ge \frac{1}{2}f^{2r-1} \text{ in } \mathbb{R}^N,$$

with $f > 0$ in $\mathbb{R}^N$. Using Theorem 3.4, we see that this is a contradiction, since $2r - 1 \le N/(N-2)$.

Theorem 3.9 *Suppose that $p, q > 1$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} \ge 1 - \frac{2}{N-2}\max\left(\frac{1}{p+1}, \frac{1}{q+1}\right). \tag{3.13}$$

Then system (3.9) has no nontrivial solution of class $C^2(\mathbb{R}^N)$.

The above result is Corollary 2.1 in [37], let us comment the proof in [37], which uses spherical means; see the definition in ([22]p.302): let $v \in C(\mathbb{R}^N)$, then the *spherical mean* of v at x of radius ρ is

$$M(v; x, \rho) = \frac{1}{\text{meas}[\partial B_\rho(x)]}\int_{\partial B_\rho(x)} v(y)d\sigma(y).$$

Changing coordinates we see that

$$M(v; x, \rho) = \frac{1}{\omega_N}\int_{|\nu|=1} v(x + \rho\nu)d\omega \tag{3.14}$$

where ω_N denotes the surface area of the unit sphere of $\mathbb{R}^N$ and ν ranges over this unit sphere. Then, one has Darboux formula

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{N-1}{\rho}\frac{\partial}{\partial\rho}\right)M(v; x, \rho) = \Delta_x M(v; x, \rho). \tag{3.15}$$

Now let us use these ideas for the functions u and v in system (3.9):

$$\Delta_x M(u;x,\rho) = \frac{1}{\omega_N}\int_{|\nu|=1}\Delta_x u(x+\rho\nu)d\omega \leq -\frac{1}{\omega_N}\int_{|\nu|=1}[v(x+\rho\nu)]^p d\omega$$

and using Jensen's inequality we obtain

$$\Delta_x M(u;x,\rho) \leq -[M(v;x,\rho)]^p. \tag{3.16}$$

Denoting

$$M(u(x);x,\rho) = u^{\#}(\rho)\ ,\ M(v(x);x,\rho) = v^{\#}(\rho)$$

and using (3.15) and (3.16) we obtain

$$-\Delta_\rho u^{\#} \geq (v^{\#})^\rho \quad -\Delta_\rho v^{\#} \geq (u^{\#})^q \tag{3.17}$$

where $\Delta_\rho = \left(\frac{\partial^2}{\partial\rho^2} + \frac{N-1}{\rho}\frac{\partial}{\partial\rho}\right)$.

To proceed we need the following result, which follows readily from (3.14), and the second part is proved using the divergence theorem.

Theorem 3.10 *If $v \in C^2(\mathbb{R}^N)$, then $M(v;x,\rho)$ is also $C^2(\mathbb{R}^N)$ in the variable x and $C^2([0,\infty))$ in the variable ρ. Moreover,*

$$\left(\frac{d}{d\rho}v^{\#}\right)(0) = 0, \quad \text{and} \left(\frac{d}{d\rho}v^{\#}\right)(\rho) \leq 0$$

. So $v^{\#}(\rho)$ is non-increasing.

We recall Lemma 3.1 in [36]:

Lemma 3.1 *If $u \in C^2(\mathbb{R}^N)$ is a positive radial superharmonic function, then*

$$ru'(r) + (N-2)u(r) \geq 0 \quad \text{for} \quad r > 0. \tag{3.18}$$

It follows readily from this lemma that $u(r) \geq Cr^{2-N}$ for large $r > 0$. Now the question reduces to the following one-dimensional result:

Theorem 3.11 *Let $u(\rho),\ v(\rho)$ be two C^2 functions defined and non-increasing in $[0,\infty)$, such that $u'(0) = v'(0) = 0$ and*

$$-\Delta_\rho u \geq v^p, \quad -\Delta_\rho v \geq u^q. \tag{3.19}$$

Suppose that $p,q > 1$ and that (3.13) holds. Then $u = v \equiv 0$.

So to complete the proof of Theorem 3.9 it remains to prove Theorem 3.11, which we do next. The first equation in (3.19) can be written as

$$-[\rho^{N-1}u'(\rho)]' \geq \rho^{N-1}[v(\rho)]^p$$

Integrating from 0 to r and observing that v is non-increasing we get

$$-u'(r) \geq \frac{1}{N}v(r)^p$$

and using (3.18) we obtain

$$v(r)^p \leq \frac{N(N-2)}{r}u(r). \tag{3.20}$$

Similarly, working with the second equation in (3.19) we get

$$u(r)^q \leq \frac{N(N-2)}{r}v(r) \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$u(r) \leq [N(N-2)]^{\frac{1+p}{pq-1}} r^{-\frac{2(1+p)}{pq-1}} \tag{3.22}$$

and

$$v(r) \leq [N(N-2)]^{\frac{1+q}{pq-1}} r^{-\frac{2(1+q)}{pq-1}} \tag{3.23}$$

Using Lemma 3.1, we conclude that the assumption of u and v positive is impossible if

$$N-2 < \frac{2(1+q)}{pq-1} \quad \text{or} \quad N-2 < \frac{2(1+\rho)}{pq-1}$$

which is exactly (3.13).

Theorem 3.12 *A) If $p > 0$ and $q > 0$ are such that $p, q \leq (N+2)/(N-2)$, but not both equal to $(N+2)/(N-2)$, then the only non-negative solution of (3.4) is $u = v = 0$.*

B) If $\alpha = \beta = (N+2)/(N-2)$, then u and v are radially symmetric with respect to some point of $\mathbb{R}^N$.

This theorem is due to deFigueiredo-Felmer [24]. The proof uses the method of Moving Planes. A good basic reference of this method is [9]. The idea in the proof of the above theorem is to use Kelvin's transform in the solutions u, v of (3.4), which a priori have no known (or prescribed) behavior at infinite. By means of Kelvin u and v are transformed in new unknowns ω and z, which now have a definite decay at ∞. Consequently the method of moving planes can start.

Theorem 3.13 *Let $p > 0$ and $q > 0$ satisfying (3.6) then there are no positive solutions of (3.4) satisfying*

$$u(x) = o(|x|^{-\frac{N}{q+1}}), \quad v(x) = o(|x|^{-\frac{N}{p+1}}), \text{ as } |x| \to \infty. \tag{3.24}$$

The above result is due to Serrin-Zou [48], where the next result is also proved.

Theorem 3.14 *Let $N = 3$, and $p, q > 0$ satisfying (3.6). Then there are no positive solutions of (3.4) for which either u or v has at most algebraic growth at infinity.*

Remark 3.7 *Observe that Theorem 3.13 extends Theorem 3.6, since radial positive solutions have a decay at infinity given in (3.22) and (3.23). Observe that (3.6) implies that*

$$\frac{2(1+p)}{pq-1} > \frac{N}{q+1} \quad \text{and} \quad \frac{2(1+q)}{pq-1} > \frac{N}{p+1}.$$

The proof of Theorem 3.13 is based on an interesting L^2 estimate of the gradient of a superharmonic function, namely,

Lemma 3.2 *Let $\omega \in C^2(\mathbb{R}^N)$ be positive, superharmonic (i.e. $-\Delta\omega \geq 0$ in $\mathbb{R}^N$) and*

$$\omega(x) = o(|x|^{-\gamma}) \quad \text{as} \quad |x| \to \infty. \tag{3.25}$$

Then

$$\int_{B_{2R}\setminus B_R} |\nabla\omega|^2 = o(R^{N-2-2\gamma}) \quad \text{as} \quad R \to \infty, \tag{3.26}$$

where B_R is the ball of radius R in $\mathbb{R}^N$ centered at the origin.

Another basic ingredient in the proof of Theorem 3.13 is an identity of Pohožaev-type, a special case of a general identity [44], namely,

Lemma 3.3 *Let (u, v) be a positive solution of (3.4) and let a_1 and a_2 be constants such that $a_1 + a_2 = N - 2$. Then*

$$\begin{aligned} \int_{B_\rho} \left\{ \left(\tfrac{N}{p+1} - a_1\right) v^{p+1} + \left(\tfrac{N}{q+1} - a_2\right) u^{q+1} \right\} &= \int_{\partial B} \left\{ \tfrac{v^{p+1}}{p+1} + \tfrac{u^{q+1}}{q+1} \right\} \\ + \int_{\partial B} \left(2\tfrac{\partial u}{\partial r}\,\tfrac{\partial v}{\partial r} - \nabla u \cdot \nabla v\right) &+ \int_{\partial B} \left(a_1 \tfrac{\partial u}{\partial r} v + a_2 u \tfrac{\partial v}{\partial r}\right). \end{aligned} \tag{3.27}$$

Proof of Theorem 3.13. using these two lemmas. Choose a_1 and a_2 in such a way that

$$\frac{N}{p+1} - a_1 = \frac{N}{q+1} - a_2 = \delta, \; a_1 + a_2 = N - 2.$$

Next, dividing (3.27) by ρ and integrating with respect to ρ between some R and $2R$ and estimating we get

$$\delta \ln 2 \int_{B_R} (u^{q+1} + v^{p+1}) \leq \int_{B_{2R} \setminus B_R} \left(\frac{u^{q+1}}{q+1} + \frac{v^{p+1}}{p+1} \right)$$
$$\int_{B_{2R} \setminus B_R} |\nabla u . \nabla v| + R^{-1} \int_{B_{2R} \setminus B_R} (v|\nabla u| + u|\nabla v|). \tag{3.28}$$

Now using the hypothesis (3.24), we see that the first integral in the right side of (3.28) is $o(1)$. Next one uses Lemma 3.2 with $\omega = u$, $\gamma = \dfrac{N}{q+1}$ and $\omega = v, \gamma = \dfrac{N}{p+1}$. With that we can estimate the second and third integrals using Cauchy -Schwarz and get that they are $o(R^{N-2-\frac{N}{p+1}-\frac{N}{q+1}})$ which is $o(1)$. This contradicts (3.28) as $R \to \infty$.

Theorem 3.15 *Let $p, q > 1$ satisfying*

$$\frac{N-1}{2}(pq-1) \leq \max(p+1, q+1). \tag{3.29}$$

Then the system of inequalities

$$-\Delta u \geq v^p, \quad -\Delta v \geq u^q \ \text{ in } \ \mathbb{R}_+^N \tag{3.30}$$

has no non-negative nontrivial solution.

Remark 3.8 *This result is due to Birindelli-Mitidieri [8], where, instead of a half- space, more general cones are considered. Observe that (3.29) can be written as*

$$\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{N-1} \max\left(\frac{1}{p+1}, \frac{1}{q+1} \right)$$

and compare it with (3.13) in Theorem 3.9.

Final remarks on Liouville Theorem For Systems.

(i) The conjecture on the validity of a Liouville theorem in the whole of $\mathbb{R}^N$ for all p and q below the critical hyperbola seems to be unsetlled at this moment. In dimension $N = 3$ and for bounded functions, the conjecture has been proved in [48], see Theorem 3.14 above.

(ii) Liouville theorems for systems of inequalities in the whole of $\mathbb{R}^N$ are Theorems 3.8 and 3.9. Is inequality in (3.13) sharp? Observe that if $p = q$, (3.13) yields $p \leq N/(N-2)$, which is the value obtained in Theorem 3.4.

(iii) Observe that a Liouville theorem for a system of inequalities in $\mathbb{R}_+^N$ is given by Theorem 3.15. Compare with the following result of [46]

Theorem 3.16 *Let $u, v \in C^2(\mathbb{R}_+^N) \cap C^0(\overline{\mathbb{R}_+^N})$ be non-negative solutions of (3.30) with $u = v = 0$ on ∂R_+^N. If $1 \leq p, q \leq \dfrac{N+2}{N-2}$ then $u = v \equiv 0$.*

(iv) It is an open question to know if Theorem 3.16 can be improved. That is, if there is some analogue to Dancer result, Theorem 3.3.

(v) Liouville-type theorems for systems of p-Laplacians have been studied recently by Mitidieri-Pohozaev [38].

(vi) Liouville theorems for equations with a weight have been considered in Berestycki, Capuzzo Dolcetta- Nirenberg [5].

4 Lecture 4: Gradient Systems

The theory of gradient systems is sort of similar to that of scalar equations. We shall discuss it in the more general context of p-Laplacians,

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1$$

We consider the system of equations

$$-\Delta_p u = F_u(x,u,v)\ , \quad -\Delta_q v = F_v(x,u,v) \tag{4.1}$$

subject to Dirichlet boundary condition. The idea is to look for the solutions of (4.1) as critical points of the functional

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \int_\Omega F(x,u,v), \tag{4.2}$$

whose Euler-Lagrange equations are the weak form of equations (4.1). The functional (4.2) is to be defined in the Cartesian product $E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. For that matter, due to Sobolev imbeddings, we require

(F1) $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and

$$|F(x,u,v)| \leq C(1 + |u|^{p^*} + |v|^{q^*})$$

where $p^* = \frac{Np}{N-p}$, which comes from the continuous imbedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Similarly for q^*. Here we are assuming that $1 < p, q < N$. This means that, in the case of the Laplacian, we are with $N \geq 3$. Condition (F1) implies that Φ is well defined in E. However, in order to have it in the C^1 class, we have to require stronger assumptions. Namely, F has to be C^1 and the following bounds on the partial derivatives are assumed

$(F2)$ $|F_u(x,u,v)| \leq C(1 + |u|^{p^*-1} + |v|^{\frac{q^*(p^*-1)}{p^*}})$

$|F_v(x,u,v)| \leq C(1 + |v|^{q^*-1} + |u|^{\frac{p^*(q^*-1)}{q^*}})$.

Next we discuss a work of Boccardo-deFigueiredo [6], which unifies part of some previous work by Boccardo-Fleckinger-de Thélin [7] and de Thélin-Vélin [54]. Let us be more precise about the growth of F at ∞: let $0 < r \leq p^*$ and $0 < s \leq q^*$, and assume

$(F3)$ $|F(x,u,v)| \leq C(1 + |u|^r + |v|^s)$.

We discuss three non-critical cases:

(I) $r < p$ and $s < q$, ("sublinear-like"),

(II) $r > p$, $s > q$, and $r < p^*$, $s < q^*$, ("superlinear-like"),
(III) $r = p$ and $s = q$, ("resonant type").

Theorem 4.1 *(The coercive case). Assume (F2) and (F3) with r and s as in (I). Then Φ achieves a global minimum at $(u_0, v_0) \in E$, which is then a weak solution of (4.1).*

Remark 4.1 *This is an imediate consequence of the theorem on the minimization of coercive weakly lower semicontinuous functionals. Next, if we assume*

(F4) $\quad F(x,0,0) = F_u(x,0,0) = F_v(x,0,0) = 0, \forall x \in \overline{\Omega},$

then $u = v = 0$ is a solution of (4.1). The next result gives conditions for the existence of non-trivial solutions.

Theorem 4.2 *(The coercive case, non-trivial solutions). Assume (F2), (F4) and (F3) with r and s as in (I). Then Φ achieves a global minimum at a point $(u_0, v_0) \neq (0,0)$, provided that there are positive constants R and $\Theta < 1$, and a continuous function $K : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

(F5) $\quad F(x, t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) \geq t^{\Theta} K(x,u,v),$

for $x \in \overline{\Omega}$, $|u|, |v| \leq R$ and small $t > 0$.

Remark 4.2 *As in Theorem 4.1, Φ achieves its infimum. All we have to do is to show that there is a point $(u_1, v_1) \in E$ where $\Phi(u_1, v_1) < 0$. Let φ be a first eigenfunction of the p-Laplacian*

$$-\Delta_p \varphi = \lambda_1(p) |\varphi|^{p-2} \varphi \quad \text{in} \quad \Omega, \quad \varphi = 0 \quad \text{on} \quad \partial\Omega.$$

The function φ can be taken > 0 in Ω, and we know that $\varphi \in C^{1,\alpha}(\overline{\Omega})$, see [21], [52]. So we can use $u_1 = t^{\frac{1}{p}} \varphi$ and $v_1 = t^{\frac{1}{q}} \psi$, where $\psi > 0$ is a first eigenfunction of the q-Laplacian, and $t > 0$ is small.

Now let us go to the "superlinear cases". Viewing the need of a Palais-Smale condition we assume a sort of Ambrosetti-Rabinowitz condition

(F6) $0 \leq F(x,u,v) \leq \theta_p u F_u(x,u,v) + \theta_q v F_v(x,u,v)$,

for all $x \in \overline{\Omega}$ and $|u|, |v| \geq R$, where R is some positive number and

$$\frac{1}{p^*} < \theta_p < \frac{1}{p}, \ \frac{1}{q^*} < \theta_q < \frac{1}{q}$$

Theorem 4.3 *Assume (F2), (F4), (F6) and (F3) with r and s as in (II). Assume also that there are positive constants C and ε, and numbers $\overline{r} > p$ and $\overline{s} > q$, such that*

(F7) $\quad |F(x,u,v)| \leq C(|u|^{\overline{r}} + |v|^{\overline{s}}),$

for $|u|, |v| \leq \varepsilon, x \in \overline{\Omega}$. Then Φ has a non- trivial critical point.

Remark 4.3 *The proof goes by an application of the Mountain-Pass Theorem [2]. Can condition (F7) be weakened? Condition (F6) is used to prove that Φ satisfies (PS).*

The analysis of the resonant case requires the study of an eigenvalue problem for a system involving p-Laplacians. Let $G : \mathbb{R}^2 \to [0,\infty)$ be a C^1 even function such that

(G) $\qquad G(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) = tG(u,v), \quad G(u,v) \leq k(|u|^p + |v|^q).$

Examples of such functions: $G(u,v) = c_1|u|^p + c_2|v|^q,\ G(u,v) = c|u|^\beta|v|^\gamma$ with $\dfrac{\beta}{p} + \dfrac{\gamma}{q} = 1$.

Theorem 4.4 *Given $a \in L^\infty(\Omega)$, there are a real number $\lambda_1(a)$ and a pair of functions $(u_0, v_0) \in E$, with $u_0, v_0 > 0$ in Ω such that*

$$\begin{cases} -\Delta_p u_0 - aG_u(u_0,v_0) = \lambda_1(a)u_0|u_0|^{p-2} \\ -\Delta_q v_0 - aG_v(u_0,v_0) = \lambda_1(a)v_0|v_0|^{q-2} \end{cases}$$

and

$$\frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q - \int aG(u,v) \geq \lambda_1(a)\left[\frac{1}{p}\int |u|^p + \frac{1}{q}\int |v|^q\right]$$

for all $(u,v) \in E$, with equality for (u_0,v_0).

Remark 4.4 *The lemma is proved by a minimization argument. The eigenfunction (u_0,v_0) is $C^1(\overline{\Omega})$ by the regularity results in [52] and their positivity follows from Vasquez maximum principle [53] for the p-Laplacian.*

Theorem 4.5 *Assume (F2) and (F3) with r and s as in (III). Suppose that there is a function G satisfying condition (G) above and such that*

(F8) $\lambda_1(a) > 0$, where $\limsup_{|u|,|v|\to\infty} \dfrac{F(x,u,v)}{G(u,v)} \leq a(x) \in L^\infty(\Omega)$,

and $\lambda_1(a)$ is the one defined in Theorem 4.4. Then Φ is bounded below and its infimum is achieved.

Theorem 4.6 *Assume (F2), (F4) and (F3) with r and s as in (III). Suppose that there exist positive numbers C, R, μ and ν such that*

(F9) $\qquad \dfrac{1}{p}uF_u + \dfrac{1}{q}vF_v - F \geq C(|u|^\mu + |v|^\nu)$ *for $|u|,|v| \geq R$.*

Assume also that there are positive numbers $\overline{R}$ and ε, and L^∞ functions $b(x)$ and $c(x)$ such that

(F10) $\qquad \lambda_1(b) < 0,\ F(x,u,v) \geq b(x)G(u,v), |u|,|v| \geq \overline{R}$

(F11) $\qquad \lambda_1(c) > 0,\ F(x,u,v) \leq c(x)\tilde{G}(u,v), |u|,|v| \leq \varepsilon$

where G and $\tilde{G}$ are functions satisfying condition (G). Then the functional Φ has a non trivial critical point.

Remark 4.5 *The proof here uses the Mountain Pass Theorem. Condition (F11) at the origin assures that $(0,0)$ is a local minimum. (F10) gives the behaviour at infinity, a sort of "superlinearity". Compare (F10) with (F8), which is a "sublinearity". (F9) is a type of condition introduced by Costa-Magalhães [13], [14]. It implies a compactness condition of the (PS) type, namely Cerami* *condition [11], for short (Ce) condition. We say that $\Phi : E \to R$ satisfies (Ce) condition if all $(u_n, v_n) \in E$ such that*

$$|\Phi(u_n, v_n)| \leq c, \quad (1 + ||u_n||_{W^{1,p}} + ||v_n||_{W^{1,p}})\Phi'(u_n, v_n) \to 0,$$

contains a convergent subsequence in the norm of E.

Final remark. It would be interesting to study the critical case $r = p^*, s = q^*$, as well as mixed cases involving, for instance, (i) $r < p, s > q$, (ii) $r = p, s > q$, (iii) $r = p^*, s < q$. etc. The geometry of the functional Φ in each case could be intrincated and we may eventually need other propositions on critical points.

5 Lecture 5: Hamiltonean Systems

In this lecture we discuss the existence of solutions of the Dirichlet problem for systems of the form

$$-\Delta u = H_v(x, u, v), \quad -\Delta v = H_u(x, u, v) \tag{5.1}$$

where H is a C^1 real-valued function defined in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. Here H_u denotes the partial derivative $\frac{\partial H}{\partial u}$, and Ω is a bounded domain $\mathbb{R}^N$, $N \geq 3$. We are interested in superlinear problems, a notion that we will define later and which takes into account the type of coupling established in these Hamiltonean systems. We have already met Hamiltonean systems in Section 3, where we studied them in the whole of $\mathbb{R}^N$. The so-called critical hyperbola introduced there also plays an important role here. Indeed, a special case of (5.1) is the system

$$-\Delta u = g(u), \quad -\Delta v = f(u) \tag{5.2}$$

where f and g are as in (1.6). In this case $H(u, v) = F(u) + G(v)$, where $F(s) = \int_0^s f(t)dt$ and $G(s) = \int_0^s g(t)dt$. We have already explained how to treat (5.2) by topological methods. It is not at all clear how to deal with the general case of (5.1) by similar methods. Variational methods instead have been used with success. The following assumptions have been found adequate for obtaining a functional, whose Euler-Lagrange equation give the weak solution of (5.1):

(H1) $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function.

(H2) There are parameters $p, q > 0$ and a positive constant C such that

$$|H(u, v)| \leq C(|u|^{p+1} + |v|^{q+1} + 1), \quad \forall (x, u, v).$$

(H3) There is a positive constant C, such that

$$|H_u(x, u, v)| \leq C(|u|^p + |v|^{\frac{p(q+1)}{p+1}}), \quad \forall (x, u, v)$$

$$|H_v(x, u, v)| \leq C(|v|^q + |u|^{\frac{q(p+1)}{q+1}}), \quad \forall (x, u, v).$$

We have observed before that if we want to work with the functional

$$\Phi(u,v) = \int \nabla u \nabla v - \int H(x,u,v), \tag{5.3}$$

this will naturally lead to the fact that we have to take $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. On its turn, however, this would imply that p and q should be less or equal to $(N+2)/(N-2)$ in order to have $H(x,u(x),v(x))$ in $L^1(\Omega)$. The following example shows that this is not the best choice. Indeed, consider the system

$$-\Delta u = v, \quad -\Delta v = |u|^{p-1}u \tag{5.4}$$

subject to Dirichlet boundary conditions. Such a system is equivalent to the single equation

$$\Delta^2 u = |u|^{p-1}u \tag{5.5}$$

subject to Navier boundary conditions, namely $u = \Delta u = 0$ on $\partial\Omega$. Since (5.5) is a fourth order elliptic equation, weak solutions are to be found in $W^{2,2}(\Omega)$. Hence we could consider powers p up to $(N+4)/(N-4)$, which is larger than $(N+2)/(N-2)$. This indicates that one should be able to treat cases when, for instance, the growth of H_u with respect to u is larger than $(N+2)/(N-2)$. As we shall see, this is possible provided the growth of H_v with respect to v is smaller than $(N+2)/(N-2)$. This is the point where the notion of the critical hyperbola enters.

System (5.1) is said to be *subcritical* if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \tag{5.6}$$

and it is said to be *superlinear* if

$$1 > \frac{1}{p+1} + \frac{1}{q+1}. \tag{5.7}$$

The possibility of using powers larger than $(N+2)/(N-2)$ calls for the use of fractional Sobolev spaces. They will be defined using Fourier expansions on the eigenfunctions of $(-\Delta, H_0^1(\Omega))$; it is well-known that the eigenvalue problem

$$-\Delta u = \lambda u \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega, \tag{5.8}$$

has an increasing sequence of eigenvalues (φ_n), $\varphi_n \in H_0^1(\Omega), \int |\varphi_n|^2 = 1$, with the properties

(i) λ_1 is a positive and simple eigenvalue, and $\varphi_1(x) > 0$ for $x \in \Omega$.

(ii)$\lambda_n \to +\infty$.

(iii)$\int \varphi_i \varphi_j = \int \nabla\varphi_i \nabla\varphi_j = 0$, for $i \neq j$.

So (φ_n) is an orthonormal system in $L^2(\Omega)$ and an orthogonal system in $H_0^1(\Omega)$, and it is known that these are complete systems.

Definition. For $s \geq 0$, we define

$$E^s = \{u = \sum a_n \varphi_n \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^s a_n^2 < \infty\}. \tag{5.9}$$

Here $a_n = \int_\Omega u\varphi_n$. E^s is a Hilbert space with the inner-product given by

$$< u, v >_{E^s} = \sum_{n=1}^{\infty} \lambda_n^s a_n b_n, \quad \text{where} \quad v = \sum_{n=1}^{\infty} b_n \varphi_n. \tag{5.10}$$

Let us define the following maps

$$\begin{array}{rcl} A^s : E^s & \longrightarrow & L^2 \\ u = \sum_{n=1}^{\infty} a_n \varphi_n & \longmapsto & A^s u = \sum_{n=1}^{\infty} \lambda_n^{s/2} a_n \varphi_n \end{array} \tag{5.11}$$

Clearly A^s is an isometric isomorphism, that is

$$\int A^s u A^s v = < u, v >_{E^s} . \tag{5.12}$$

Observe that for $s = 1$, one has

$$\int_\Omega A^1 u A^1 v = \int_\Omega \nabla u \nabla v. \tag{5.13}$$

The Sobolev imbedding theorem says that
"$E^s \subset L^p$ continuously if $\frac{1}{p} \geq \frac{1}{2} - \frac{s}{N}$, and compactly if the previous inequality is strict".

As observed above (5.11) is not the right quadratic part of the functional. What would be right one? Assume that p, q satisfy (5.6) and (5.7). Choose $s, t > 0$, such that $s + t = 2$ and

$$\frac{1}{p} > \frac{1}{2} - \frac{s}{N}, \frac{1}{q} > \frac{1}{2} - \frac{t}{N}.$$

Thus $E^s \subset L^p(\Omega)$, and $E^t \subset L^q(\Omega)$, with compact immersions.

Let now $E = E^s \times E^t$. If $z = (u, v) \in E$, then $H(x, u, v) \in L^1$. So the functional below

$$\Phi(z) = \int_\Omega A^s u A^t v - \int_\Omega H(x, u, v) \tag{5.14}$$

is well defined for $z = (u, v) \in E$ and it is of class C^1. Its derivative is given by the following expression

$$< \Phi'(z), \eta > = \int_\Omega A^s u A^t \psi + A^s \phi A^t v - \int_\Omega H_u \phi + H_v \psi,$$

where $\eta = (\phi, \psi)$. So the critical points of the functional Φ given by (5.12) are the weak solutions $(u, v) \in E^s \times E^t$ of the system

$$\int_\Omega A^s \phi A^t v = \int_\Omega H_u \phi, \forall \phi \in E^s \tag{5.15}$$

$$\int_\Omega A^s u A^t \psi = \int_\Omega H_v \psi, \forall \psi \in E^t. \tag{5.16}$$

Remark 5.1 *The following regularity theorem was proved in [25]: "these weak solutions* (u,v) *are indeed* $u \in W_0^{1,\frac{p+1}{p}}(\Omega) \cap W^{2,\frac{p+1}{p}}$ *and* $v \in W_0^{1,\frac{q+1}{q}}(\Omega) \cap W^{2,\frac{q+1}{q}}$*, which we call strong solutions of (5.1)".*

The following result was proved in [25]:

Theorem 5.1 *Assume (H1), (H2) and (H3) with* $p, q > 0$ *satisfying (5.6) and (5.7). In addition, assume*

(H4) There exists $R > 0$ *such that*

$$\frac{1}{p+1} H_u(x,u,v)u + \frac{1}{q+1} H_v(x,u,v)v \geq H(x,u,v) > 0$$

for all $x \in \overline{\Omega}$ *and* $|(u,v)| \geq R$.

(H5) There exist $r > 0$ *and* $c > 0$ *such that*

$$|H(x,u,v)| \leq c(|u|^{p+1} + |v|^{q+1}),$$

for all $x \in \overline{\Omega}$ *and* $|(u,v)| \leq r$.
Then, system (5.1) has a strong solution.

Remarks on the proof of Theorem 5.1. The proof consists in obtaining a critical point of the functional (5.14). First we observe that Φ is strongly indefinite. In fact, the space E decomposes into $E = E^+ \oplus E^-$, where $E^\pm$ are infinite dimensional subspaces and the quadratic part

$$Q(z) = \int_\Omega A^s u A^t v, \quad \text{for} \quad z = (u,v)$$

is positive definite in E^+ and negative definite in E^-. This fact and (H5) induce a geometry on the functional Φ that calls for the use of a linking theorem of Benci-Rabinowitz [4] in a version due to Felmer [23]. Conditions (H2), (H3) and (H4) are used to prove a Palais-Smale condition.

Remark 5.2 *Condition (H5) in the previous theorem excludes cases when* H_u *and* H_v *have linear terms. Indeed, on one hand the superlinearity condition (5.7) is equivalent to* $pq > 1$*. And on the other hand, linear terms would imply that (H5) should hold with* $p = q = 1$*, which then is not possible. Let us now treat this case.*

Suppose now that H has a quadratic part, namely $\frac{1}{2}cu^2 + \frac{1}{2}bv^2 + auv$. In this case the system becomes

$$-\Delta u = au + bv + H_v \ , \qquad -\Delta v = cu + av + H_u \ , \tag{5.17}$$

where H satisfies the assumption of the previous theorem. This situation has been studied in special cases by Hulshof-van der Vorst [33]and deFigueiredo-Magalhães [26]. The result we present below is the more general result in this line and it is due to deFigueiredo-Ramos [27].

Theorem 5.2 *Let a, b, c be real constants. For p, q as in (5.6) and (5.7), suppose that H satisfies (H1)- (H5). Then system (5.17) admits a nonzero strong solution.*

Remark 5.3 *In both [25] and [27] one allows more general Hamiltoneans H. In fact, the growth at ∞ can be different from the its behavior at zero.*

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