HORIZONTAL f-STRUCTURES, ϵ-MATRICES AND EQUI-HARMONIC MOVING FLAGS

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Dedicated to Prof. M. do Carmo on the occasion of his 70th birthday

Abstract. In this paper, we establish a one-one correspondence between invariant $f-$structures on non-symmetric complex flag manifolds and skew-symmetric matrices valued in $\{-1, 0, 1\}$ and study the geometry of invariant $f-$structures and $f-$holomorphic curves on flag manifolds. In particular, we construct equi-harmonic tori on full complex flag manifolds which are not $f-$holomorphic with respect to any horizontal $f-$structure and characterize some important horizontal f-structures such as almost complex and primitive horizontal $f-$structures.

§0. Introduction

Complex flag manifolds are the most typical and important reductive homogeneous spaces. They include symmetric (height$=1$) and non-symmetric (height≥ 2) cases. Many beautiful results in harmonic surfaces on the symmetric flag manifolds (i.e. Grassmannians, in particular, projective spaces) have been obtained in the recent years. In this case since the difference of partial energies is a smooth homotopy invariant[Li], the holomorphicity of maps implies their harmonicity. It follows that the fundamental aspect of harmonic

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surfaces on Grassmannians is to construct non-holomorphic harmonic maps. For instance, start from holomorphic maps, Eells-Wood manufactured non-holomorphic harmonic maps[EW]. They called them (complex) isotropic maps.

However, in the non-symmetric case, a complex flag manifold has many left-invariant metrics. Furthermore the relative metric induced by restricting the Killing form is not well behaved from the point of view of complex geometry. Hence the equi-harmonic surfaces on non-symmetric flag manifolds play a central role, where equi-harmonicity means harmonicity with respect to all left-invariant metrics. Black's theorem tells us, if map $\phi : M \to F = F(r_1, \cdots, r_n; N)$ from a Riemann surface to the complex flag manifold with height $n-1$ is $f-$holomorphic with respect to some horizontal f-structure on F, then ϕ is equi-harmonic; and for an equi-weakly conformal surface the converse is still true.

A natural question is: is there an equi-harmonic surface which is not $f-$holo -morphic with respect to any horizontal $f-$structure on F?

In this article, we will give a positive answer for this question. In fact, a stronger result will be obtained. We will manufacture equi-harmonic tori on full complex flag manifolds which are not $f-$holomorphic with respect to any invariant $f-$structure on full flag manifolds.

Our main approach is to study the geometry of invariant $f-$structures and $f-$holomorphic curves on non-symmetric flag manifolds by encoding invariant $f-$structures into $\epsilon-$matrices and some ideas of Uhlenbeck as in [Uh] for example. The $f-$structures on a Riemannian manifold, introduced by Yano in 1963, extend almost complex (resp. contact) structures on a even (resp.odd)-dimensional manifold, and they are applied widely in study of geometry of harmonic maps[Bu1,Lo,R]. A nice way to understand invariant almost complex structures on a flag manifold is tournament. Using technique of tournoment and Birkhoff-Grothendieck decomposition theorem, Burstall-Salamon obtained their factorisation theorem with strictly decreasing length[BS]. In fact, there is a $1:1$ correspondence between invariant almost complex structures on a complex flag manifold with height $n-1$ and n-tournaments. Inspired by [BS], in this paper we establish one-one correspondence between invariant $f-$structures and skew-symmetric matrices valued in $\{-1, 0, 1\}$. We call these matrices ϵ-matrices. Using the theory of ϵ-matrices, we characterize some of the main horizontal $f-$structures, i.e. almost complex structure and the $f-$structures associate to primitive maps.

The invariant $f-$structure related to primitive surfaces on a non-symmetric complex flag manifold F is naturally induced from the $k(\geq 3)-$symmetric structure of F. The importance of primative surfaces are not only their equiharmonicity but also their covering property for each conformal harmonic surface on a symmetric complex flag manifold. For instance, "Eells-Wood" surfaces[N2] covering all complex isotropic harmonic surfaces are primitive surfaces (ref. corollary 4.4 and (5.1)). Bolton, Pedit and Woodward studied non-isotropic harmonic surfaces with orthogonal harmonic sequences in $\mathbb{CP}^n$. They established a one-one correspondence between these surfaces (called them superconformal ones) and $\tau-$primitive surfaces on a full complex flag manifold. And the latter are closely related to affine Toda fields [BPW]. Burstall considered all finite isotropy order conformal harmonic surfaces in $\mathbb{CP}^n$. He showed that these surfaces can be covered by the primitive surfaces on flag bundle over $\mathbb{CP}^n$ [Bu2](which can be identified to surfaces on not-necessarily-full flag manifolds[M]). Recently, this covering property has been extended to surfaces on Grassmannian by Udagawa[Ud].

In this paper, we characterize the invariant $f-$structures associated to perimative surfaces on non-symmetric flag manifolds $F(\underbrace{1, \cdots, 1}_{n-1}, N-n+1; N)$. More precisely, we show that these $f-$structures have exactly maximal rank among all the horizontal ones.

The horizontality of the standard complex structure J_1 and the associate parabolic almost complex structure J_2 have been discussed in [MN], they proved that J_1 is never horizontal, and J_2 is not horizontal if the complex flag manifold has height≥ 3. We will prove that a horizontal $f-$structure on a non-symmetric complex manifold is almost complex if and only if it is the standard parabolic almost complex structure J_2 on $2-$height flag manifold.

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§1. f-STRUCTURES ON COMPLEX FLAG MANIFOLDS

Consider the complex flag manifold

$$F(r_1, \cdots, r_n; N) = \frac{U(N)}{U(r_1) \times \cdots \times U(r_n)}$$

where $r_1 + \cdots + r_n = N$. $F(r_1, \cdots, r_n; N)$ is a completely reductive homogeneous space with reductive splitting

$$u(N) = [u(r_1) + \cdots + u(r_n)] \oplus [\oplus_{i<j} m_{ij}] \tag{1.1}$$

where

$$m_{ij} = \{A = (A_{kl}) \in u(N)|,\, A_{kl} = 0 \text{ if } (k,l) \neq (i,j) \text{ and } (j,i)\}$$

$A_{kl} \in gl(r_k \times r_l; C)$ is the isotropy representation. It will be necessary to consider the complexified version of (1.1). We have

$$\begin{aligned} m_{ij}^{\mathbb{C}} &= \{A = (A_{kl}) \in gl(N;\mathbb{C})|\, A_{kl} = 0 \text{ if } (k,l) \neq (i,j) \text{ and } (j,i)\} \\ &= E_{ij} \oplus E_{ji} \end{aligned}$$

and

$$E_{ij} := \{A = (A_{kl}) \in gl(N,\mathbb{C})|\, A_{kl} = 0 \text{ if } (k,l) \neq (i,j)\} \tag{1.2}$$

is $U(r_1) \times \cdots \times U(r_n)$ invariant and irreducible. For arbitrary $A = (A_{kl}) \in E_{ij}$ we have

$$A = (A_{kl}^1) + \sqrt{-1}(A_{kl}^2)$$

where

$$A_{kl}^1 = \begin{cases} \frac{A_{ij}}{2} & (k,l) = (i,j) \\ -\frac{\bar{A}_{ij}^t}{2} & (k,l) = (j,i) \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_{kl}^2 == \begin{cases} \frac{A_{ij}}{2\sqrt{-1}} & (k,l) = (i,j) \\ -\frac{\bar{A}_{ij}^t}{2\sqrt{-1}} & (k,l) = (j,i) \\ 0 & \text{otherwise} \end{cases}$$

so it's easy to see that

$$\bar{A} = (A_{kl}^1) - \sqrt{-1}(A_{kl}^2) \in E_{ji}$$

and vice versa, we get

$$\overline{E}_{ij} = E_{ji}$$

$$[\oplus_{i<j} m_{ij}]^{\mathbb{C}} = \oplus_{i \neq j} E_{ij} \tag{1.3}$$

Definition 1.1[Bl]. An f-structure on

$$F = F(r_1, \cdots, r_n;\, N)$$

is a section $\mathcal{F}$ of $End(TF(r_1, \cdots, r_n;\, N))$ such that $\mathcal{F}^3 + \mathcal{F} = 0$.

An $U(N)$ invariant f-structure on $F(r_1, \cdots, r_n;\, N)$ may be identified with an H equivariant endomorphism,$\mathcal{F}$, of $\oplus_{i<j}$ such that $\mathcal{F}^3 + \mathcal{F} = 0$ where $H = U(r_1) \times \cdots \times U(r_n)$. Using Schur's Lemma, all the $U(N)$-invariant f-structures may be constructed as following: Put

$$\epsilon = (\epsilon_{ij})$$

a $n \times n$ skew-symmetric matrix with values in the set $\{1,\, 0,\, -1\}$. Define

$$\begin{aligned} \sqrt{-1} \quad &\text{eigenspace of} \quad \mathcal{F} = \oplus_{\epsilon_{ij}=1} E_{ij} \\ -\sqrt{-1} \quad &\text{eigenspace of} \quad \mathcal{F} = \oplus_{\epsilon_{ij}=1} \overline{E}_{ij} = \oplus_{\epsilon_{ij}=-1} E_{ij} \\ 0 \quad &\text{eigenspace of} \quad \mathcal{F} = \oplus_{\epsilon_{ij}=0} E_{ij} \end{aligned} \tag{1.4}$$

Determining the eigenspaces in this way defines an H equivariant endomorphism $\mathcal{F}$ of $\oplus_{i \neq j} E_{ij}$, which is seen to be the C-linear extension of an H equivariant endomorphism of $\oplus_{i<j} m_{ij}$ since $\oplus_{\epsilon_{ij}=1} E_{ij}$ and $\oplus_{\epsilon_{ij}=-1} E_{ij}$ are conjugate and

$$[\oplus_{\epsilon_{ij}=1} E_{ij}] \cap [\oplus_{\epsilon_{ij}=-1} E_{ij}] = \{0\}$$

.

Definition 1.2[Bu1]. The complex dimension of $\sqrt{-1}$ eigenspace of $\mathcal{F}$ is the rank of $\mathcal{F}$.

Suppose that f-structure $\mathcal{F}$ is defined by $\epsilon(\mathcal{F}) = (\mathcal{F}_{ij})$ then

$$\begin{aligned} \text{rank}\mathcal{F} :&= \dim_{\mathbb{C}}[\sqrt{-1} \quad \text{eigenspace of} \quad \mathcal{F}] \\ &= \dim_{\mathbb{C}} \oplus_{\mathcal{F}_{ij}=1} E_{ij} \\ &= \sum_{\mathcal{F}_{ij}=1} \dim_{\mathbb{C}} E_{ij} = \sum_{\mathcal{F}_{ij}=1} r_i r_j \end{aligned}$$

In particular, if $2\text{rank}\mathcal{F} = \dim F(r_1, \cdots, r_n;\, N)$, i,e

$$2 \sum_{\mathcal{F}_{ij}=1} r_i r_j = N^2 - r_1^2 - \cdots - r_n^2 = \sum_{i \neq j} r_i r_j$$

it follows that 0 eigenspace $= \{0\}$, so $\mathcal{F}$ is an almost complex structure. And vice versa. We have shown that

Theorem 1.3. *There is a 1 : 1 correspondence between $U(N)$ invariant f-structure $\mathcal{F}$ on $F(r_1, \cdots, r_n; N)$ and $n \times n$ skew-symmetric matrices $\epsilon(\mathcal{F}) = (\mathcal{F}_{ij})$ with values in the set $\{1, 0, -1\}$ such that*

$$\operatorname{rank}\mathcal{F} = \sum_{\mathcal{F}_{ij}=1} r_i r_j = \sum_{\mathcal{F}_{ij}=-1} r_i r_j = \frac{1}{2} \sum_{\mathcal{F}_{ij}\neq 0} r_i r_j \tag{1.5}$$

and $\mathcal{F}$ is almost complex structure if and only if $\mathcal{F}_{ij} \neq 0$ for any $i \neq j$.

Corollary 1.4. *There are $3^{\binom{n}{2}} U(N)$-invariant f-structures on a complex flag manifold $F(r_1, \cdots, r_n; N)$, obtained by choosing, for each $(i < j)$, ϵ_{ij}.*

§2. Horizontal f-structures

Among all f-structures on a complex flag manifold, an important class is so-called horizontal f-structures, because any f-holomorphic map with respect to a horizontal f-structure is harmonic for all invariant metrics and its harmonicity is preserved under homogeneous projection.

Definition 2.1. An invariant f-structure $\mathcal{F}$ on a complex flag manifold with the property that $[\mathcal{F}_+, \mathcal{F}_-] \subset h$ will be called a horizontal f-structure, where

$$\mathcal{F}_+ = +\sqrt{-1} \quad \text{eigenspace}$$

$$\mathcal{F}_- = -\sqrt{-1} \quad \text{eigenspace}$$

$$h = u(r_1) + \cdots + u(r_n)$$

For an arbitrary $A \in E_{ij}$ put

$$A = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (0, \cdots, 0, A^i, 0, \cdots, 0)$$

where $A_j = (0, \cdots, 0, A_{ij}, 0, \cdots, 0)$ and $A^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{ij} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Similarly for $B \in E_{kl}$.

Hence

$$\begin{aligned}
[A,\, B] &= AB - BA \\
&= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} (0, \cdots, 0, B^k, 0, \cdots, 0) \\
&- \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B_l \\ 0 \\ \vdots \\ 0 \end{pmatrix} (0, \cdots, 0, A^i, 0, \cdots, 0) \\
&= \begin{pmatrix} & & \\ & A_j B^k & \\ & & \end{pmatrix} - \begin{pmatrix} & & \\ & B_l A^i & \\ & & \end{pmatrix} \qquad (2.1)
\end{aligned}$$

It follows that

$$[E_{ij},\, E_{kl}] = \begin{cases} 0 & \text{if} \quad i,\, j,\, k,\, l \quad \text{are distinct or} \quad j \neq l \\ E_{il} & \text{if} \quad j = k,\, i \neq l \\ E_{ii} - E_{jj} & \text{if} \quad j = k,\, i = l \end{cases} \qquad (2.2)$$

The following theorem characterize the horizontal f-structures on a complex flag manifold in terms of ϵ-matrices.

Theorem 2.2. *Let $\mathcal{F}$ be an invariant f-structure on F. Then $\mathcal{F}$ is horizontal if and only if there is an n-permutation σ such that*
i) σ and σ^2 have no fixed points;
ii) $\{(i, j)|\mathcal{F}_{ij} = 1\} \subset \{(k, \sigma(k))|k = 1, 2, \cdots, n\}$

Proof. An equivalent condition of horizontality is $[\mathcal{F}_+, \bar{\mathcal{F}_+}] \subset h^{\mathbb{C}}$[Bl, p41]. Notice that $\mathcal{F}_+ = \sqrt{-1}$ eigenspace of $\mathcal{F} = \oplus_{\mathcal{F}_{ij}=1} E_{ij}$. So $[\mathcal{F}_+, \bar{\mathcal{F}_+}] \subset h^{\mathbb{C}}$ if and only if $\mathcal{F}_{ij} = \mathcal{F}_{kl} = 1, (i, j) \neq (k, l)$ implies that $i \neq k$, and $j \neq l$. It now follows, from the skew-symmetricity of $\epsilon(\mathcal{F})$, that $\mathcal{F}$ is horizontal if and only if there exists an n-order permutation σ such that i) and ii) hold.

Proposition 2.3. *Let $F = F(r_1, \cdots, r_n; N)$ be a complex flag manifold with height > 1 and J an almost complex structure on F. Then J is horizontal if and only if $n = 3$ and J is non-integrable.*

Proof. Sufficient is clear. Suppose that J is horizontal. From theorem 1.2 and theorem 2.2 one gets that $\epsilon(J)$ is a 3×3 matrix. Furthermore up to a sign

$$\epsilon(J) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Thus J is non-integrable[MN].

§3. n-SYMMETRIC SPACES AND THEIR INDUCED f-STRUCTURES

Let $(E_1, \cdots, E_n)$ denote the legs of $F(r_1, \cdots, r_n; N)$ at the identity coset. It is easy to see that

$$\begin{aligned} u(N)^{\mathbb{C}} &\cong \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N) \\ &\cong \overline{\mathbb{C}}^N \otimes \mathbb{C}^N \\ &\cong (\bar{E}_1 \oplus \cdots \oplus \bar{E}_n) \otimes (E_1 \oplus \cdots \oplus E_n) \\ &\cong \oplus_{j,k}(\bar{E}_j \otimes E_k) \cong \oplus_{j,k}\mathrm{Hom}(E_j, E_k) \end{aligned}$$

Set

$$E_{j+mn} = E_j \qquad \text{for} \quad m \in Z \tag{3.1}$$

Define $Q \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)$ by

$$Q = \zeta^j \quad \text{on} \quad E_j \quad \text{for} \quad j \in Z$$

where $\zeta = \exp(2\pi\sqrt{-1}/n)$. Clearly Q is unitary and conjugation by Q gives a periodic automorphism of $U(N)$ with order n. Thus $F(r_1, \cdots, r_n; N)$ becomes an n-symmetric space as in [K].

This automorphism induces one on $u(N)^{\mathbb{C}}$ as following: $\tau : \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N) \to \mathrm{Hom}(C^N, C^N)$

$$\tau(\xi) = Q \circ \xi \circ Q^{-1}$$

Then we have

Proposition 3.1. *The ζ^k-eigenspace of τ, denoted by g_k, is*

$$\oplus_{j \in T} Hom(E_j, E_{j+k})$$

where $T = \{1, 2, \cdots, n\}$.

Proof. For any $\xi \in \mathrm{Hom}(E_j, E_{j+k})$, $\quad V \in E_j$ we have

$$\begin{aligned}[\tau(\xi)](QV) &= Q \circ \xi(V) \\ &= \zeta^{j+k}\xi(V) \\ &= \zeta^k \xi(\zeta^j V) = \zeta^k \xi(QV)\end{aligned}$$

It follows that $\tau(\xi) = \zeta^k \xi$, so $\xi \in g_k$. Conversely, for any $\xi \in g_k \subset g^{\mathbb{C}} := \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)$, we may write

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n$$

where $\xi_j \in \oplus_{i \in T} \mathrm{Hom}(E_i, E_{i+j})$. Together with $\xi \in g_k$ we get that

$$\begin{aligned}\zeta^k \xi =& \tau(\xi) \\ =& \tau(\Sigma \xi_j) \\ =& \Sigma \tau(\xi_j) = \Sigma_{j=1}^n \zeta^j \xi_j\end{aligned}$$

This implies that $\xi = \xi_k \in \oplus_{j \in T} \mathrm{Hom}(E_j, E_{j+k})$.

Corollary 3.2. $g_k \cap g_{-k} = \{0\}$ *if and only if* $k \neq \frac{n}{2}, 0$

Proof. In fact (see §1)

$$\mathrm{Hom}(E_i, E_j) \cong E_{ij} \tag{3.2}$$

Now our conclusion can be obtained from (3.1) (3.2) and proposition 3.1 immediately.

Corollary 3.3. *When $k \neq \frac{n}{2}$, 0, g_k determines an unique invariant horizontal f-structure $\mathcal{F}_k$ on $F(r_1, \cdots, r_n; N)$ such that $g_k = \sqrt{-1}-$eigenspace of $\mathcal{F}_k$.*

Proof. From (3.2) each $\mathrm{Hom}(E_i, E_j)$ is Ad_H invariant. Combine with proposition 3.1 we get that each g_k is Ad_H invariant. Since g_k is ζ^k- eigenspace of τ one has

$$\bar{g_j} = g_{-j} \tag{3.3}$$

Furthermore because τ is the derivative of the automorphism of order n,

$$[g_i, g_j] \subset g_{i+j} \tag{3.4}$$

Assume that $k \neq 0, \frac{n}{2}$. Define

$g_k =$ the eigenspace associated to $\sqrt{-1}$

$g_{-k} =$ the eigenspace associated to $-\sqrt{-1}$

$\oplus_{j \in T\setminus\{k, n-k, n\}} g_j =$ the eigenspace associated to 0

From (3.3) and proposition 3.1 we obtain an invariant f-structure $\mathcal{F}_k$ on $F(r_1, \cdots, r_n; N)$ (see §1). Using (3.3) and (3.4) we see that

$$\begin{aligned} [g_k, \bar{g_k}] =& [g_k, g_{-k}] \\ =& g_0 = \mathrm{Lie} H \end{aligned}$$

which implies that $\mathcal{F}_k$ is horizontal.

§4. ϵ-MATRICES AND f-HOLOMORPHIC CURVES

Let M be a Riemannian surface with local complex coordinate z and

$$\phi : M \to F(r_1, \cdots, r_n; N)$$

a map into a flag manifold with its moving flag $\{E_i\}$ (see [BS]). Set

$$A'_{ij} = \pi_j \circ \frac{\partial}{\partial z} \circ \pi_i \tag{4.1}$$

where π_i denotes the orthogonal projection onto E_i. When $i \neq j$, A'_{ij} is called the second fundamental form of ϕ.

Remark. The notation of the second fundamental forms is the same as in [BW].

Definition 4.1. A map $\phi : M \to F(r_1, \cdots, r_n; N)$ is said to be subordinate to an ϵ-matrix (ϵ_{ij}) if $A'_{ij} = 0$ whenever $\epsilon_{ij} \neq 1, i \neq j$. We recall that $\epsilon_{ij} \in \{-1, 0, 1\}$ for any i, j.

Proposition 4.2. *A map $\phi : M \to F(r_1, \cdots, r_n; N)$ is f-holomorphic relative to an invariant f-structure $\mathcal{F}$ on F if and only if it is subordinate to $\epsilon(\mathcal{F})$.*

Proof. A map $\phi : M \to F(r_1, \cdots, r_n; N)$ is f-holomorphic if and only if $d\phi$ interwines the f-structures, i,e,

$$d\phi \circ J = F \circ d\phi \tag{4.2}$$

where J is the standard complex structure on Riemann surface. It is easy to see that (4.2) holds if and only if $d\phi(\frac{\partial}{\partial z}) \in \sqrt{-1}-$eigenspace of $\mathcal{F}$[R,p.90]. Notice that the Maurer-Cartan form gives the familiar isomorphism

$$\begin{aligned} \phi^{-1}TF(r_1, \cdots, r_n; N)^C &= \oplus_{i \neq j} \bar{E}_i E_j \\ &= \oplus_{i \neq j} \operatorname{Hom}(E_i,\, E_j) \end{aligned}$$

Furthermore under this isomorphism the component of $d\phi(\frac{\partial}{\partial z})$ in $\operatorname{Hom}(E_i,\, E_j)$ is A'_{ij}[BS][U]. By the conjugation it is clear to see that the subspace E_{ij} (see §1) corresponds to $\bar{E}_i E_j = \operatorname{Hom}(E_i,\, E_j)$, which combine with (1.4) we have

$$\sqrt{-1} \quad \text{eigenspace of} \quad \mathcal{F} = \oplus_{\mathcal{F}_{ij}=1} \operatorname{Hom}(E_i,\, E_j)$$

It follows that ϕ is $f-$ holomorphic related to $\mathcal{F}$ if and only if $\mathcal{F}_{ij} \neq 1, i \neq j$ implies that $A'_{ij} = 0$

Combine with Black's result[Bl][Bu2], we have

Corollary 4.3. *Suppose that $\phi : M^2 \to F(r_1, \cdots, r_n; N)$ is subordinate to an horizontal $\epsilon-$matrix (i.e. it is associated to a horizontal f-structure). Then $\phi = (\phi_1, \cdots, \phi_n)$ is an equi-harmonic map and each $\phi_j : M^2 \to G_{r_j,N}$ is harmonic for $j = 1, 2, \cdots, n$.*

Corollary 4.4[N2]. *The Eells-Wood maps: $\Phi : M^2 \to F(n)$ are equi-harmonic.*

Proof. Let $\phi : M^2 \to CP^{n-1}$ be a full isotropic harmonic map. Then its diagram is [BW]:

$$\underset{\phi_0}{\cdot} \longrightarrow \underset{\phi_1}{\cdot} \longrightarrow \cdots \longrightarrow \underset{\phi_{n-1}}{\cdot} \longrightarrow 0$$

Hence $\Phi = (\phi_0, \phi_1, \cdots, \phi_{n-1}) : M^2 \to F(n)$ is subordinate to horizontal $\epsilon-$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & & & \cdots & 0 \\ -1 & 0 & 1 & 0 & & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & & 1 \\ 0 & & \cdots & 0 & -1 & & 0 \end{pmatrix}$$

where

$$\phi_0, \phi_1, \cdots, \phi_{n-1}$$

is the harmonic sequence of ϕ[W]. It follows that Φ is equiharmonic from corollary 4.3.

§5. Algebraic characterization of horizontal f-structures associated to primitive maps

Using the ζ^k- eigenspace decomposition (see §3) and conjugation we have the decomposition of trivial bundle over a complex flag manifold $\underline{g} := u(N)^C \times F(r_1, \cdots, r_n; N)$ i.e.

$$\underline{g} = \underline{g}_0 \oplus \underline{g}_1 \oplus \cdots \oplus \underline{g}_{n-1}$$

where $\underline{g}_j$ has its fibre $\mathrm{Ad}_b g_j$ at $x = bH$. Because each g_j is Ad_H invariant, so is $\mathrm{Ad}_b g_j$.

On the other hand, differentiations of the orbit maps on complex flag manifold induce a bundle homomorphism α. Its kernel is exactly equal to $\underline{g}_0$. Restricting α on $\underline{g}_1 \oplus \cdots \oplus \underline{g}_{n-1}$ we have a bundle isomorphism. Its inverse map is the so called Maurer-Cartan form β on $F(r_1, \cdots, r_n; N)$. β is a natural extension of the (left) Maurer-Cartan form on $U(N)$.

Definition 5.1. Let $\psi : M \to F(r_1, \cdots, r_n; N)$ be a map of a Riemann surface to a complex manifold. ψ is primative if $\psi^*\beta^{(1,0)}$ takes values in $\underline{g}_1$ where $\beta^{(1,0)}$ denotes the (1, 0)-component of the Maurer-Cartan form on $F(r_1, \cdots, r_n; N)$.

Remark 5.2. By the definition 5.1, the primative map ψ means that $\psi_*(\frac{\partial}{\partial z}) \in \sqrt{-1}-$ eigenspace of the horizonal f-structure associated to $\underline{g}_1$ where z is a local complex coordinate on M. Hence ψ is an f-holomorphic map. In particular, in the case $n = 3$, a primitive map is just a holomorphic map with respect to the unique (up to a sign) non-integrable almost complex structure (ref. prop. 2.3).

Now we are in the position of characterizing the horizontal f-structures associate to primitive maps in terms of their ranks. We restrict ourselves to $F(\underbrace{1, \cdots, 1}_{n-1}, N-n+1; N)$.

Theorem 5.3. *The maximal rank of all horizontal invariant f-structures on $F(1, \cdots, 1, N-n+1; N)$ is $2N-n$. Furthermore, if $\mathcal{F}$ is a horizontal invariant f-structure such that*

$$\operatorname{rank}\mathcal{F} = 2N - n$$

Then $\mathcal{F}$ is the one associated to $\zeta-$ eigenspace $\underline{g}_1$ up to a permutation of order in the row index of $\mathcal{F}$.

Proof. From (1.5) and theorem 2.2 a horizontal f-structure $\mathcal{F}$ attaches maximal rank if and only if its ϵ-matrix satisfies that

$$\{(i,\, j)|\mathcal{F}_{ij} = 1\} = \{(j,\, \sigma(j))|j \in T\}$$

where σ is a permutation and $T = \{1, \cdots, n\}$. Combine with (1.5) it is easy to see that

$$\max \operatorname{rank}\mathcal{F} = 2N - n$$

for all invariant horizontal $f-$structures on $F(1, \cdots, 1, N-n+1; N)$. On the other hand, the horizontal $f-$structure associated to $\underline{g}_1$ satisfies that

$$\{(i,\, j)|\mathcal{F}_{ij} = 1\} = \{(1,\, 2),\ (2,\, 3),\ \cdots,\ (n-1,\, n),\ (n,\, 1)\} \tag{5.1}$$

Hence each invariant horizontal f-structure with maximal rank is identitical to the one associated to $\underline{g}_1$ up to a permutation of order in the row index of $\mathcal{F}$.

§6. Closed surfaces on full complex flag manifolds

From this section, we restrict ourselves to full complex flag manifolds i.e.

$$F(n) := F(\underbrace{1, \cdots, 1}_{n}; n)$$

Let $\tilde{\phi} : M \to U(n)$ be the lift map of $\phi : M \to F(n)$, i.e.

$$\phi = \pi \circ \tilde{\phi}$$

where $\pi : U(n) \to F(n)$ is the natural projection. Let $e_1, \cdots, e_n$ be standard basis in $\mathbb{C}^n$, i.e.

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We denote π_j the matrix of the orthogonal projection onto E_j with respect to $e_1, \cdots, e_n$ (ref. §4). Then

$$\pi_j : M \to gl(n, C)$$

satisfies that

$$A'_{ji}(e_1, \cdots, e_n) = (e_1, \cdots, e_n) A_z^{ij} \tag{6.1}$$

where $A_z^{ij} := \pi_i \frac{\partial \pi_j}{\partial z}$. For $V \in \Gamma\left(\phi^* TF(n)\right)$, we set

$$q = \phi^* \beta(V)$$

where $\phi^*\beta : \phi^* F(n) \to M \times u(n)$ is the pull-back of Maurer-Cartan form. Define the variation of ϕ by

$$\phi_t(x) := \pi\left(exp(-tq)\tilde{\phi}\right) \tag{6.2}$$

Denote associate objects by $\pi_j(t)$, $A_z^{ij}(t)$ etc. Then we have

Lemma 6.1.

$$1). \frac{\partial}{\partial t}|_{t=0} \pi_j(t) = [\pi_j, q] \tag{6.3}$$

$$2). \frac{\partial}{\partial z}[\pi_j, q] = [\frac{\partial \pi_j}{\partial z}, q] + [\pi_j, \frac{\partial q}{\partial z}] \tag{6.4}$$

$$3). \frac{\partial}{\partial t}|_{t=0} A_z^{ij}(t) = [A_z^{ij}, q] - \pi_i \frac{\partial q}{\partial z} \pi_j \tag{6.5}$$

Proof. 1). From (6.1) we have

$$\pi_j = \tilde{\phi} E_j \tilde{\phi}^*$$

where (from now to the end) E_j will denote the matrix which has 1 in the (j,j)-position and zero elsewhere. Together with (6.2) one gets

$$\pi_j(t) = e^{-tq}\pi_j e^{tq}$$

Hence

$$\frac{\partial}{\partial t}|_{t=0}\pi_j(t) = -q\pi_j + \pi_j q = [\pi_j, q]$$

2). It is obvious.

3). Using 1) and 2) we have

$$\begin{aligned}
\frac{\partial}{\partial t}|_{t=0}A_z^{ij}(t) &= \frac{\partial}{\partial t}|_{t=0}[\pi_i(t)\frac{\partial \pi_j(t)}{\partial z}] \\
&= \left(\frac{\partial}{\partial t}|_{t=0}\pi_i(t)\right)\frac{\partial \pi_j}{\partial z} + \pi_i\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}|_{t=0}\pi_j(t)\right) \\
&= [\pi_i,\, q]\frac{\partial \pi_j}{\partial z} + \pi_i\frac{\partial}{\partial z}[\pi_j,\, q] \\
&= [\pi_i,\, q]\frac{\partial \pi_j}{\partial z} + \pi_i\left([\frac{\partial \pi_j}{\partial z},\, q] + [\pi_j,\, \frac{\partial q}{\partial z}]\right) \\
&= [A_z^{ij},\, q] - \pi_i\frac{\partial q}{\partial z}\pi_j
\end{aligned}$$

where notice that $\pi_i\pi_j = 0$ whenever $i \neq j$.

The inner product on $gl(n,\, C)$ is defined by

$$< A,\, B >:= tr(AB^*) \qquad \forall A, B \in gl(n,\, C) \tag{6.6}$$

It is easy to check that

$$< A,\, B >= \overline{< B,\, A >} \tag{6.7}$$

$$< A,\, [B,\, C] >=< [B^*,\, A],\, C > \tag{6.8}$$

In patricular we have

$$< A,\, B > + < B,\, A >= 2Re < A,\, B > \tag{6.9}$$

Furthermore, the inner products are preserved under correspondence (6.1). Let

$$ds^2_\Lambda := \sum \lambda_{ij}\omega_{i\bar{j}}\omega_{\bar{i}j} \tag{6.10}$$

be a left-invariant metric on $F(n)$, where

$$\omega = (\omega_{i\bar{j}})$$

is the Maurer-Cartan form on $U(n)$ and

$$\lambda_{ij} = \lambda_{ji} \begin{cases} > 0 & \text{if} \quad i \neq j \\ = 0 & \text{if} \quad i = j \end{cases} \tag{6.11}$$

Let (M, g) be a closed Riemann surface. Then with respect to ds^2_Λ, the energy of ϕ_t is defined by

$$E(\phi_t) := \int_M \sum \lambda_{ij} |A_z^{ij}(t)|^2 v_g \tag{6.12}$$

where

$$v_g = \sqrt{-1} dz \wedge d\bar{z}$$

From (6.5)(6.9) and (6.12) we have

$$\begin{aligned} \frac{d}{dt}|_{t=0} E(\phi_t) &= \int_M \sum \lambda_{ij} \frac{\partial}{\partial t}|_{t=0} |A_z^{ij}(t)|^2 v_g \\ &= 2Re \int_M \sum \lambda_{ij} < A_z^{ij}, \, \frac{\partial}{\partial t}|_{t=0} A_z^{ij}(t) > v_g \\ &= 2Re \int_M \sum \lambda_{ij} < A_z^{ij}, \, [A_z^{ij}, \, q] - \pi_i \frac{\partial q}{\partial z} \pi_j > v_g \end{aligned} \tag{6.13}$$

so we get

$$\frac{1}{2} \frac{d}{dt}|_{t=0} E(\phi_t) = I + II \tag{6.14}$$

where

$$I = Re \int_M \sum \lambda_{ij} < A_z^{ij}, \, [A_z^{ij}, \, q] > v_g \tag{6.15}$$

$$II = -Re \int_M \sum \lambda_{ij} < A_z^{ij}, \, \pi_i \frac{\partial q}{\partial z} \pi_j > v_g \tag{6.16}$$

Lemma 6.2.

$$1).Re < [A_{\bar{z}}^{ji},\, A_z^{ij}],\, q >= 0 \tag{6.17}$$

$$2).< A_z^{ij},\, \pi_i B \pi_j >=< A_z^{ij},\, B >, \qquad \forall B \in gl(n, C) \tag{6.18}$$

where $A_{\bar{z}}^{ji} := \pi_j \circ \frac{\partial \pi_i}{\partial \bar{z}}$.

Proof. 1).It is easy to see that

$$(A_{\bar{z}}^{ji})^* = A_z^{ij} \tag{6.19}$$

so we have

$$[A_{\bar{z}}^{ji},\, A_z^{ij}]^* = [A_{\bar{z}}^{ji},\, A_z^{ij}] \tag{6.20}$$

By using (6.6),(6.9) and (6.20) one gets

$$\begin{aligned} 2Re < [A_{\bar{z}}^{ji},\, A_z^{ij}],\, q > &=< [A_{\bar{z}}^{ji},\, A_z^{ij}],\, q > + < q,\, [A_{\bar{z}}^{ji},\, A_z^{ij}] > \\ &= tr([A_{\bar{z}}^{ji},\, A_z^{ij}]q^*) + tr(q[A_{\bar{z}}^{ji},\, A_z^{ij}]^*) \\ &= -tr([A_{\bar{z}}^{ji},\, A_z^{ij}]q) + tr(q[A_{\bar{z}}^{ji},\, A_z^{ij}]) = 0 \end{aligned}$$

2). Notice that $\pi_i \pi_j = 0$, $i \neq j$ and $\pi_i^2 = \pi_i$ we have

$$\begin{aligned} < A_z^{ij},\, \pi_i B \pi_j > &= tr(A_z^{ij} \pi_j^* B^* \pi_i^*) \\ &= tr(\pi_i \frac{\partial \pi_j}{\partial z} \pi_j B^* \pi_i) \\ &= tr(\pi_i \frac{\partial \pi_j}{\partial z} \pi_j B^*) \\ &= -tr(\frac{\partial \pi_i}{\partial z} \pi_j B^*) \\ &= tr(\pi_i \frac{\partial \pi_j}{\partial z} B^*) =< A_z^{ij},\, B > \end{aligned}$$

It is clear to see that, from (6.8)(6.15)(6.17) and (6.19)

$$I = -2Re \int_M \sum \lambda_{ij} < [A_{\bar{z}}^{ji},\, A_z^{ij}],\, q > v_g = 0$$

For II, we use (6.18) and the Stokes' theorem and yield

$$\begin{aligned} II &= -Re\int_M \sum \lambda_{ij} < A_z^{ij},\, \frac{\partial q}{\partial z} > v_g \\ &= Re\int_M \sum \lambda_{ij} < \frac{\partial A_z^{ij}}{\partial \bar{z}},\, q > v_g - Re\int_M \sum \lambda_{ij} \frac{\partial}{\partial \bar{z}} < A_z,\, q > v_g \\ &= Re\int_M < \frac{\partial A_z^{\Lambda}}{\partial \bar{z}},\, q > v_g \end{aligned} \tag{6.21}$$

where

$$A_z = \sum_{i,j} \lambda_{ij} A_z^{ij} \tag{6.22}$$

and

$$\frac{\partial A_z}{\partial \bar{z}} : M \to u(n) \tag{6.23}$$

(easily checked!). We have

Proposition 6.3. *$\phi : (M,\, g \to (F(n),\, dS_\Lambda^2)$ is harmonic if and only if $\frac{\partial A_z^\Lambda}{\partial \bar{z}} = 0$ if and only if*

$$\frac{\partial A_x^\Lambda}{\partial x} + \frac{\partial A_y^\Lambda}{\partial y} = 0 \tag{6.24}$$

where

$$A_x^\Lambda := \sum \lambda_{ij}\pi_i \frac{\partial \pi_j}{\partial x}, \qquad A_y^\Lambda = \sum \lambda_{ij}\pi_i \frac{\partial \pi_j}{\partial y} \tag{6.25}$$

Proof. In fact

$$\begin{aligned} 4\frac{\partial A_z}{\partial \bar{z}} &= \sum_{i,j} \lambda_{ij} \left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y} \right) (A_x^{ij} - \sqrt{-1} A_y^{ij}) \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + (*) \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\sqrt{-1}}(*) &= \sum_{i,j} \lambda_{ij} \left(\frac{\partial A_x^{ij}}{\partial y} - \frac{\partial A_y^{ij}}{\partial x} \right) \\ &= \sum_{i,j} \lambda_{ij} \left[\frac{\partial}{\partial y}(\pi_i \frac{\partial \pi_j}{\partial x}) - \frac{\partial}{\partial x}(\pi_i \frac{\partial \pi_j}{\partial y}) \right] = 0 \end{aligned}$$

because $\lambda_{ij} = \lambda_{ji}$.

§7 Non$-f-$holomorphic equi-harmonic tori

Suppose $\phi : R^2 \to F(n)$ is defined by

$$\phi = \pi \circ \tilde{\phi} \tag{7.1}$$

where

$$\tilde{\phi}(x,y) = e^{Ax+By} \tag{7.2}$$

and $A, B \in u(N), \quad [A,B]=0$. Then

$$\tilde{\phi}(x,y) = e^{By}e^{Ax} \tag{7.3}$$

$$\frac{\partial \tilde{\phi}}{\partial x} = \tilde{\phi}A \tag{7.4}$$

$$\frac{\partial \tilde{\phi}^*}{\partial x} = \left(\frac{\partial \tilde{\phi}}{\partial x}\right)^* = -A\tilde{\phi}^* \tag{7.5}$$

Combine with the proof of lemma 6.1, we have

$$\frac{\partial \pi_i}{\partial x} = \frac{\partial}{\partial x}(\tilde{\phi}E_i\tilde{\phi}^*) = \tilde{\phi}[A,\, E_i]\tilde{\phi}^* \tag{7.6}$$

So

$$A_x^{ji} = \pi_j\frac{\partial \pi_i}{\partial x} = \tilde{\phi}E_j[A,\, E_i]\tilde{\phi}^* = \tilde{\phi}E_jAE_i\tilde{\phi}^* \tag{7.7}$$

Similarly we have

$$A_y^{ji} = \tilde{\phi}E_jBE_i\tilde{\phi}^* \tag{7.8}$$

Hence the second fundamental forms of ϕ satisfy that

$$A_z^{ji} = \tilde{\phi}E_j\chi E_i\tilde{\phi}^* \tag{7.9}$$

where

$$\chi = \frac{1}{2}(A - \sqrt{-1}B)$$

Now let $\mathcal{F}$ be a invariant f-structure with associated ϵ-matrix $(\mathcal{F}_{ij})$. Then we have

Proposition 7.1. *ϕ is $f-$holomorphic with respect to $\mathcal{F}$ if and only if $b_{ij} = \sqrt{-1}a_{ij}$ whenever $\mathcal{F}_{ij} \neq 1, i \neq j$.*

Proof. Put

$$A = (a_{ij}) \qquad B = (b_{ij}) \qquad \chi = (x_{ij})$$

Then ϕ is $f-$ holomorphic with respect to $\mathcal{F}$

$$\overset{Prop.4.2}{\Longleftrightarrow} i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad A'_{ij} = 0$$

$$\overset{(6.1)}{\Longleftrightarrow} i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad A^{ji}_{z} = 0$$

$$\overset{(7.7)}{\Longleftrightarrow} i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad E_j \chi E_i = 0$$

$$\Longleftrightarrow i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad x_{ji} = 0$$

$$\Longleftrightarrow i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad a_{ji} = \sqrt{-1}b_{ji}$$

$$\Longleftrightarrow i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad b_{ij} = \sqrt{-1}a_{ij}$$

In fact, we have stronger conditions as following:

Proposition 7.2. *The equivalent condition of ϕ to be $f-$holomorphic with respect to f-structure $\mathcal{F}$ is*

1) $a_{ij} = b_{ij} = 0$ *if* $\mathcal{F}_{ij} = 0, \quad i \neq j$
2) $b_{ij} = -\sqrt{-1}a_{ij}$ *if* $\mathcal{F}_{ij} = 1$
3) $b_{ij} = \sqrt{-1}a_{ij}$ *if* $\mathcal{F}_{ij} = -1$

Proof. From proposition 7.1 it is enough to show the necessite. If $\mathcal{F}_{ij} = 0$ and $i \neq j$, then proposition 7.1 implies that

$$b_{ij} = \sqrt{-1}a_{ij}, \qquad b_{ji} = \sqrt{-1}a_{ji}$$

it follows that $a_{ij} = b_{ij} = 0$ since $A, B \in u(n)$. If $\mathcal{F}_{ij} = 1$ then $\mathcal{F}_{ji} = -1$, so we get

$$b_{ji} = \sqrt{-1}a_{ji}$$

Take conjugation we have

$$b_{ij} = -\sqrt{-1}a_{ij}$$

Now we are in the position to invastigate the harmonicity of ϕ (defined in (7.1) and (7.2)) with double periods. From (7.4)(7.5) and (7.7) it is easy to see that

$$\frac{\partial A_x^{ij}}{\partial x} = \frac{\partial}{\partial x}\left(\tilde{\phi} E_i A E_j \tilde{\phi}^*\right) = \tilde{\phi}[A,\, E_i A E_j]\tilde{\phi}^* \tag{7.10}$$

Similarly we have

$$\frac{\partial A_y^{ij}}{\partial y} = \tilde{\phi}[B,\, E_i B E_j]\tilde{\phi}^* \tag{7.11}$$

Substitute (7.10) and (7.11) into (6.22), we have

Proposition 7.3. *Suppose that $\phi : R^2 \to F(n)$ defined in (7.1) and (7.2) has double periods. Then ϕ is harmonic with respect to ds_Λ^2 if and only if*

$$[A,\, \sum \lambda_{ij} E_i A E_j] + [B,\, \sum \lambda_{ij} E_i B E_j] = 0 \tag{7.12}$$

Now we construct two classes of non$-f-$holomorphic equi-harmonic tori into full complex flag manifolds.

Theorem 7.4. *Let $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k \in \mathbb{Q}\backslash\{0\}$ (where $\mathbb{Q}$ denotes the set of rational numbers) and*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_j = \begin{pmatrix} \alpha_j X & 0 \\ 0 & \beta_j X \end{pmatrix} \tag{7.13}$$

$$B_j = \begin{pmatrix} \beta_j X & 0 \\ 0 & \alpha_j X \end{pmatrix} \quad j = 1, \cdots, k \le \frac{n}{4} \tag{7.14}$$

$$A = \sqrt{-1}\begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \tag{7.15}$$

$$B = \sqrt{-1}\begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \tag{7.16}$$

Then

1).$\phi(x,\, y) = \pi(e^{Ax+By})$ *has double periods;*

2).$\phi : T^2 \to F(n)$ *is equi-harmonic;*

3).ϕ *is not* $f-$*holomorphic with respect to any invariant* $f-$*structure on* $F(n)$.

Proof.

1). For $l \in \{1, 2, \cdots\}$

$$A^l = (\sqrt{-1})^l \begin{pmatrix} \alpha_1^l X^l & & & & & & & \\ & \beta_1^l X^l & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_k^l X^l & & & & \\ & & & & \beta_k^l X^l & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

where

$$X^l = \begin{cases} X & \text{if} \quad l = \text{odd} \\ I_2 & \text{if} \quad l = \text{even} \end{cases}$$

and

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$
\begin{aligned}
e^{Ax} &= I + Ax + \frac{A^2x^2}{2!} + \cdots \\
&= \begin{pmatrix}
\cos\alpha_1 x I_2 & & & & & & & \\
& \cos\beta_1 x I_2 & & & & & & \\
& & \ddots & & & & & \\
& & & \cos\alpha_k x I_2 & & & & \\
& & & & \cos\beta_k x I_2 & & & \\
& & & & & 0 & & \\
& & & & & & \ddots & \\
& & & & & & & 0
\end{pmatrix} \\
&+ \sqrt{-1} \begin{pmatrix}
\sin\alpha_1 x X & & & & & & & \\
& \sin\beta_1 x X & & & & & & \\
& & \ddots & & & & & \\
& & & \sin\alpha_k x X & & & & \\
& & & & \sin\beta_k x X & & & \\
& & & & & 0 & & \\
& & & & & & \ddots & \\
& & & & & & & 0
\end{pmatrix}
\end{aligned}
$$

Combine with $[A,\, B] = 0$, there exists a $\nu \in \mathbb{Z}\backslash\{0\}$, such that

$$\phi(x + 2\pi n\nu,\, y + 2\pi m\nu) = \phi(x,\, y)$$

Hence we have

$$\phi : T^2 = \frac{\mathbb{R}^2}{2\pi\nu(\mathbb{Z} \oplus \mathbb{Z})} \to F(n)$$

2). For any left-invariant ds^2_Λ on $F(n)$, from (7.13) (7.14) and (7.15) we get $\sum \lambda_{ij} E_i A E_j$

$$
= \sqrt{-1} \begin{pmatrix}
\alpha_1\lambda_{12} X & & & & & & & \\
& \beta_1\lambda_{34} X & & & & & & \\
& & \ddots & & & & & \\
& & & \alpha_k\lambda_{4k-3} X & & & & \\
& & & & \beta_k\lambda_{4k-2} X & & & \\
& & & & & 0 & & \\
& & & & & & \ddots & \\
& & & & & & & 0
\end{pmatrix}
$$

so
$A\cdot\sum\lambda_{ij}E_iAE_j$

$$= -\begin{pmatrix} \alpha_1^2\lambda_{12}X^2 & & & & & & & \\ & \beta_1^2\lambda_{34}X^2 & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_k^2\lambda_{4k-3}X^2 & & & & \\ & & & & \beta_k^2\lambda_{4k-2}X^2 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

$= (\sum\lambda_{ij}E_iAE_j)\cdot A$
It follows that

$$[A, \sum\lambda_{ij}E_iAE_j] = 0 \tag{7.17}$$

Similarly we have

$$[B, \sum\lambda_{ij}E_iBE_j] = 0 \tag{7.18}$$

Subutitute (7.17) and (7.18) into (7.12), we see that ϕ is equi-harmonic.

3). Suppose that ϕ is $f-$holomorphic with respect to the invariant $f-$structure $\mathcal{F}$. And $(\mathcal{F}_{ij})$ is the ϵ-matrix of $\mathcal{F}$. From proposition 7.2 one of following is true:
$i).\sqrt{-1}\alpha_1 = \sqrt{-1}\beta_1 = 0$
$ii).\sqrt{-1}\beta_1 = \alpha_1$
$iii).\sqrt{-1}\beta_1 = -\alpha_1$
However this is impossible because $\alpha_1,\ \beta_1 \in \mathbb{Q}\backslash\{0\}$.

Similarly, we can show

Theorem 7.5. *Let* $\alpha_1,\cdots,\alpha_k \in \mathbb{Q}\backslash\{0\}, \qquad 2k\le n$
$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
and

$$A = \sqrt{-1}\begin{pmatrix} \alpha_1X & & & & & & \\ & \alpha_2X & & & & & \\ & & \ddots & & & & \\ & & & \alpha_kX & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

Then $\phi : T^2 \to F(n)$ *defined by*

$$(x,\, y) \to \pi(e^{A(x+y)})$$

is an equi-harmonic map but not $f-$*holomorphic with respect to any invariant* $f-$*structure on* $F(n)$.

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