INFINITELY MANY SOLUTIONS OF NONLINEAR ELLIPTIC SYSTEMS

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1 Introduction

In this paper we study elliptic systems of the form

$$\begin{cases} -\Delta u = H_v(x,u,v) & \text{in } \Omega \\ -\Delta v = H_u(x,u,v) & \text{in } \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain and $H\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function. We shall also consider the case when $\Omega = \mathbb{R}^N$, and in this case the system takes the form

$$\begin{cases} -\Delta u + u = H_v(x,u,v) & \text{in } \mathbb{R}^N \\ -\Delta v + v = H_u(x,u,v) & \text{in } \mathbb{R}^N \end{cases} \tag{1.2}$$

In the bounded case, we look for solutions of (1.1) subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$. In the case when $\Omega = \mathbb{R}^N$ we assume that some symmetry with respect to x holds; for instance, that the x-dependence of H is radial, or that H is invariant with respect to certain subgroups of $O(N)$ acting on R^N. We shall obtain both radial and non-radial solutions in the radial symmetric case, thus observing a symmetry breaking effect.

In order to illustrate the kind of results obtained here, let us state two theorems. We first consider the case when Ω is bounded. In such a case, the following set of hypotheses is assumed. First, the regularity of the Hamiltonian:

(H_1) $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and $H \geq 0$.

Next we assume conditions related to the growth of the right side of (1.1).

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(H$_2$) There exist constants $p, q > 1$ and $c_1 > 0$ with

$$1 > \frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N} \tag{1.3}$$

such that

$$|H_u(x,u,v)| \le c_1(|u|^{p-1} + |v|^{(p-1)q/p} + 1) \tag{1.4}$$

and

$$|H_v(x,u,v)| \le c_1(|v|^{q-1} + |u|^{(q-1)p/q} + 1) \tag{1.5}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

The next condition is a "non-quadraticity" condition at infinity introduced by Costa-Magalhães [5]. It is related to the so-called Ambrosetti-Rabinowitz condition and it is devised to get some sort of Palais-Smale condition for the functionals involved.

(H$_3$) There exist $1 < \alpha < p$ and $1 < \beta < q$ with

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{1.6}$$

and such that

$$\frac{1}{\alpha} H_u(x,u,v)u + \frac{1}{\beta} H_v(x,u,v)v - H(x,u,v) \ge a(|u|^\mu + |v|^\nu - 1) \tag{1.7}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. Here a, μ, ν are positive constants satisfying

$$\mu > \frac{pN}{2} \max\left\{\frac{1}{2} - \frac{1}{p},\ 1 - \frac{1}{p} - \frac{1}{q}\right\}$$

and

$$\nu > \frac{qN}{2} \max\left\{\frac{1}{2} - \frac{1}{q},\ 1 - \frac{1}{p} - \frac{1}{q}\right\}$$

.

Remark 1.1. *Observe that* $1 - \frac{1}{p} - \frac{1}{q}$ *is always positive in view of* (H$_2$). *It follows from* (H$_3$) *that*

$$H(x,u,v) \ge c(|u|^\alpha + |v|^\beta - 1). \tag{1.8}$$

In fact, (1.8) *follows from a condition weaker than* (H$_3$), *namely*

$$\frac{1}{\alpha} H_u(x,u,v)u + \frac{1}{\beta} H_v(x,u,v)v \ge H(x,u,v)$$

for all $x \in \overline{\Omega}$ *and* $|(u,v)| \ge R$; *see Felmer* [10].

Remark 1.2. *Suppose H satisfies the following condition of Ambrosetti-Rabinowitz type: there is $R>0$ and $1<\alpha'<p$ and $1<\beta'<q$ with $\frac{1}{\alpha'}+\frac{1}{\beta'}<1$ and such that*

$$\frac{1}{\alpha'}H_u(x,u,v)u+\frac{1}{\beta'}H_v(x,u,v)v \geq H(x,u,v)$$

for $x\in\overline{\Omega}$ and $|(u,v)|\geq R$. Then condition (H$_3$) *holds. In this case, it follows that H is superquadratic, in the sense that*

$$H(x,u,v)\geq c_1(|u|^{\alpha'}+|v|^{\beta'})-c_2.$$

The next condition provides the symmetry we assume here.

(H$_4$) $H(x,-u,-v)=H(x,u,v)$ for all $(x,u,v)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}$.

Now we are prepared to state the result in the case of Ω bounded. For that matter we introduce a non-increasing sequence of constants δ_n, $n\in\mathbb{N}$, with $\delta_n\to 0$, which will be defined in Section 3, and which depend only on p, q, α and β.

Theorem 1.1. *Suppose that* (H$_1$)-(H$_4$) *hold. Then there is a $k_0\in\mathbb{N}$ such that, if*

$$\liminf_{|(u,v)|\to\infty}\frac{2H(x,u,v)}{|u|^{\alpha}+|v|^{\beta}}>\frac{1}{\delta_K} \tag{1.9}$$

holds for $K\geq k_0$, system (1.1)*, subject to Dirichlet boundary conditions, has $K-k_0+1$ pairs of nontrivial solutions.*
Moreover, if

$$\lim_{|(u,v)|\to\infty}\frac{H(x,u,v)}{|u|^{\alpha}+|v|^{\beta}}=+\infty,$$

(in particular, if H is superquadratic) then system (1.1), subject to Dirichlet boundary conditions has infinitely many solutions.

The solutions obtained in Theorem 1.1 are strong solutions in the sense that $u\in W^{2,p/(p-1)}(\Omega)\cap W_0^{1,p/(p-1)}(\Omega)$ and $v\in W^{2,q/(q-1)}(\Omega)\cap W_0^{1,q/(q-1)}(\Omega)$. The existence of at least one solution for the system (1.1), without symmetry assumptions, has been considered before. See the survey paper [6] for a list of references.

As a corollary of Theorem 1.1 we obtain solutions to two nonlinear eigenvalue problems. We consider first

$$\left\{\begin{array}{rcl} -\Delta u &=& \delta u+\lambda|v|^{\beta-2}v+H_v(x,u,v),\\ \\ -\Delta v &=& \mu|u|^{\alpha-2}u+\delta v+H_u(x,u,v)\end{array}\right. \tag{1.10}$$

in a bounded domain $\Omega\subset\mathbb{R}^N$ subject to Dirichlet boundary conditions $u=v=0$ on $\partial\Omega$. The constants α,β are those from (H_3).

Corollary 1.1. *If H satisfies* (H$_1$)-(H$_4$)*, then for each $k\in\mathbb{N}$, there exists $\Lambda_k>0$, such that (1.10) has k pairs of non-trivial solutions provided λ, $\mu>\Lambda_k$.*

Next we consider the eigenvalue problem

$$\begin{cases} -\Delta u &= \lambda H_v(x,u,v) \\ -\Delta v &= \lambda H_u(x,u,v) \end{cases} \tag{1.11}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$.

Corollary 1.2. *Suppose that H satisfies* (H_1)-(H_4), *and*

$$\liminf_{|(u,v)|\to\infty} \frac{H(x,u,v)}{|u|^\alpha + |v|^\beta} > 0.$$

Then for each $k \in \mathbb{N}$, there exists $\Lambda_k > 0$, such that (1.11) *has k pairs of non-trivial solutions provided $\lambda > \Lambda_k$.*

Let us now state a result for the case when system (1.2) is considered in the whole of $\mathbb{R}^N$. We need a distinct, but similar, set of hypotheses.

(H_1') $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1, $H \geq 0$, $H(x,u,v) > 0$ for $|(u,v)| > 0$ and H is radial in the variable x.

(H_2') There exist positive constants p, q, a, b and c_1 with

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}, \quad 1 < a < p-1, \quad 1 < b < q-1, \tag{1.12}$$

such that

$$|H_u(x,u,v)| \leq c_1(|u|^{p-1} + |v|^{(p-1)q/p} + |u|^a) \tag{1.13}$$

and

$$|H_v(x,u,v)| \leq c_1(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^b) \tag{1.14}$$

for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$.

(H_3') There exist $1 < \alpha < p$ and $1 < \beta < q$ with $\alpha^{-1} + \beta^{-1} < 1$ and such that

$$\frac{1}{\alpha} H_u(x,u,v)u + \frac{1}{\beta} H_v(x,u,v)v \geq H(x,u,v)$$

for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$.

(H_4') There are positive constants c and r such that

$$H(x,u,v) \geq c(|u|^p + |v|^q) \quad \text{for } x \in \mathbb{R}^N \text{ and } |(u,v)| \leq r.$$

(H_5') $H(x,u,v) = H(x,-u,-v)$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$.

Remark 1.3. *It follows from* (H_3') *that there are positive constants c and R such that*

$$H(x,u,v) \geq c(|u|^p + |v|^q) \quad \text{for } |(u,v)| \geq R. \tag{1.15}$$

Then (1.14) *and assumption* (H_4') *imply that*

$$H(x,u,v) \geq c(|u|^p + |v|^q) \quad \text{for all } (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}. \tag{1.16}$$

Theorem 1.2. *Assume that the Hamiltonian H satisfies the hypotheses* (H_1')-(H_5'). *Then system* (1.2) *has infinitely many radial solutions.*

The solutions obtained in Theorem 1.2 are strong solutions in the sense that they satisfy $u \in W^{2,p/(p-1)}_{loc}(\mathbb{R}^N)$ and $v \in W^{2,q/(q-1)}_{loc}(\mathbb{R}^N)$. They also satisfy $u \in H^s(\mathbb{R}^N) \subset L^\gamma(\mathbb{R}^N)$ and $v \in H^t(\mathbb{R}^N) \subset L^\delta(\mathbb{R}^N)$ for some $s, t > 0$ with $s + t = 2$ and $2 < \gamma, 2N/(N-2s)$, $2 < \delta < 2N/(N-2t)$. The existence of at least one solution has been obtained before for special cases of system (1.2) in [9] and recently in [14].

The next result exhibits the breaking of symmetry in certain dimensions. The result extends to the type of systems we have here a result that Bartsch-Willem [3] proved in the scalar case.

Theorem 1.3. *Suppose that* (H_1')-(H_5') *holds. If $N = 4$ or $N \geq 6$ then system* (1.2) *has infinitely many non-radial solutions.*

2 Some Abstract Critical Point Theory

We consider a Hilbert space E and a functional $\Phi \in C^1(E, \mathbb{R})$. Given a sequence $\mathcal{F} = (X_n)$ of finite dimensional subspaces $X_n \subset X_{n+1}$, with $\overline{\bigcup X_n} = E$, we say that Φ satisfies $(PS)^{\mathcal{F}}_c$, at level $c \in \mathbb{R}$, if every sequence z_j, $j \in \mathbb{N}$, with $z_j \in X_{n_j}$, $n_j \to \infty$, such that

$$\Phi(z_j) \to c \quad \text{and} \quad (1 + \|z_j\|)(\Phi|_{X_{n_j}})'(z_j) \to 0 \tag{2.1}$$

has a subsequence which converges to a critical point of Φ. In the case when $X_n = E$ for all $n \in \mathbb{N}$ this form of the Palais-Smale condition is due to Cerami [4]. It is closely related to the standard Palais-Smale condition and to the $(PS)^*$ condition of [1] and [11]. It also yields a deformation lemma. In the present form $(PS)^{\mathcal{F}}_c$ was introduced in Bartsch-Clapp [2].

Remark 2.1. *If Φ has the form*

$$\Phi(z) = \frac{1}{2}\langle Lz, z\rangle - \Psi(z)$$

with $L : E \to E$ a linear Fredholm operator of index zero and $\nabla\Psi : E \to E$ completely continuous, then a bounded $(PS)^{\mathcal{F}}_c$ sequence (z_j) has a convergent subsequence. By a $(PS)^{\mathcal{F}}_c$ sequence we mean a sequence as in (2.1). *Let us prove the above statement. First select a subsequence, denoted again by (z_j) such that $z_j \rightharpoonup z$, weakly in E. Then $\nabla\Psi(z_j) \to \nabla\Psi(z)$, strongly in E. Let $P_n : E \to X_n$ denote the orthogonal projection onto X_n. We have that the sequence*

$$P_{n_j}\nabla\Phi(z_j) = P_{n_j}Lz_j + P_{n_j}\nabla\Psi(z_j)$$

converges to zero in view of (2.1). *So*

$$P_{n_j}Lz_j \to -\nabla\Psi(z) = Lz.$$

Hence $Lz_j \to Lz$. And as a consequence, $z_j \to z$, because $\ker L$ *is finite dimensional.*

Now suppose that E splits as a direct sum $E = E^+ \oplus E^-$. Let $E^\pm_1 \subset E^\pm_2 \subset \ldots$ be a strictly increasing sequence of finite dimensional subspaces of $E^\pm$ such that $\overline{\bigcup_{n=1}^\infty E^\pm_n} = E^\pm$. Setting $E_n = E^+_n \oplus E^-_n$ we can formulate the hypotheses on Φ which are needed for our first abstract theorem.

(Φ_1) $\Phi \in C^1(E,\mathbb{R})$ and satisfies $(PS)_c^{\mathcal{F}}$ for $\mathcal{F} = (E_n)_{n\in\mathbb{N}}$ and $c > 0$.

(Φ_2) For some $k \geq 2$ and some $r > 0$ one has

$$b_k := \inf\{\Phi(z) : z \in E^+, z \perp E_{k-1}, \|z\| = r\} > 0. \tag{2.2}$$

(Φ_3) There exists an isomorphism $T : E \to E$ with $T(E_n) = E_n$, for all $n \in \mathbb{N}$, and there exist $K \geq k$ and $R > 0$ such that

$$\text{for } z = z^+ + z^- \in E_K^+ \oplus E^- \text{ with } \max\{\|z^+\|, \|z^-\|\} = R$$

one has

$$\|Tz\| > r \quad \text{and} \quad \Phi(Tz) \leq 0,$$

where k and r are the constants introduced in (Φ_2).

(Φ_4) $d_K := \sup\{\Phi \circ T(z^+ + z^-) : z^+ \in E_K^+, z^- \in E^-, \|z^+\|, \|z^-\| \leq R\} < \infty$.

(Φ_5) Φ is even, i.e. $\Phi(-z) = \Phi(z)$.

A stronger condition that implies (Φ_4) and holds in our application is:

(Φ_6) Φ maps bounded sets to bounded sets.

Theorem 2.1. *Assume* $(\Phi_1) - (\Phi_5)$. *Then, for every* $b < b_k$, Φ *has at least* $K - k + 1$ *pairs* $\pm z_i$ *of critical points with critical values in* $[b, d_K]$.

Proof: We need to recall the equivariant limit category defined in [2], specialized to our situation. We set $G = \mathbb{Z}/2$ which acts on E via the antipodal map. Given invariant subsets $Z \subset Y \subset X$ of E, we define the $G{-}\mathrm{cat}_X(Y,Z)$ to be the least integer m such that there exists a covering $Y \subset U_0 \cup \ldots \cup U_m$ of Y with invariant open subsets $U_0, \ldots, U_m$ of X with the properties:

(i) $Z \subset U_0$ and there exists a continuous family $h^t : U_0 \to X$, $0 \leq t \leq 1$, of odd maps satisfying $h^0(z) = z$ and $h^1(z) \in Z$ for every $z \in U_0$, and $h^t(z) = z$ for every $z \in Z$ and every $t \in [0,1]$.

(ii) For $i = 1, \ldots, m$ there exists a continuous family $h_i^t : U_i \to X$, $0 \leq t \leq 1$, of odd maps satisfying $h_i^0(z) = z$ for every $z \in U_i$ and such that $h_i^1(U_i) = \{\pm z_i\}$, for some $z_i \in X \setminus \{0\}$.

Now we define the equivariant limit category for G-invariant sets $Z \subset Y \subset E$ by

$$G\text{–cat}_E^{\mathcal{F}}(Y,Z) := \limsup_{n\to\infty} G\text{–cat}_{E_n}(Y \cap E_n, Z \cap E_n)\,.$$

Given $d > b > 0$ Theorem 2.8 of [2] says that Φ has at least $G\text{–cat}_E^{\mathcal{F}}(\Phi^d, \Phi^b)$ pairs of critical points with critical values in $[b,d]$. Therefore it suffices to prove that $G\text{–cat}_E^{\mathcal{F}}(\Phi^{d_K}, \Phi^b) \geq K - k + 1$ for $0 < b < b_k$. This follows from the next lemma.

Lemma 2.1. *Fix* $0 \leq b < b_k$ *and* $n \geq K$. *Then*

$$\gamma := G\text{–cat}_{E_n}(\Phi^{d_K} \cap E_n, \Phi^b \cap E_n) \geq K - k + 1.$$

Proof: For simplicity we set $d := d_K$, and $B := B_R E_K^+ \times B_R E_n^-$ with $R > 0$ from (Φ_3). We also write $S_r E_n$ for the sphere of radius r in E_n. Let

$$\Phi_n^d := \Phi^d \cap E_n \subset U_0 \cup \ldots \cup U_\gamma$$

be a covering as in the definition of G–$\mathrm{cat}_{E_n}(\Phi_n^d, \Phi_n^b)$. There are odd mappings $h^1 : U_0 \to \Phi_n^b$ and $h_i^1 : U_i \to \{\pm z_i\}$. Making U_0 smaller if necessary we may assume that h^1 extends continuously to $\overline{U_0}$. Then we can extend h^1 to an odd mapping $h^1 : E_n \to E_n$ by using Tietze's extension theorem. Now we set

$$\mathcal{O} := \{z \in B : \|h^1(Tz)\| < r\}.$$

For $z \in \partial B$ we have $\|Tz\| > r$ and $\Phi(Tz) \le 0$ by (Φ_3). Thus $Tz \in \Phi_n^0 \subset \Phi_n^b$ and $h^1(Tz) = Tz$, and hence $\|h^1(Tz)\| = \|Tz\| > r$. This implies that $\mathcal{O}$ is an open subset of B with $\overline{\mathcal{O}} \subset \mathrm{int}\, B$. Clearly $\mathcal{O}$ is an invariant neighborhood of 0 in $E_K^+ \oplus E_n^-$.

For $z \in T^{-1}(U_0)$ we have that $h^1(Tz) \in \Phi_n^b \subset E_n \backslash S_r(E_{k-1}^\perp \cap E_n^+)$, in virtue of (Φ_2). For $z \in \partial\mathcal{O}$, we have that $\|h^1(Tz)\| = r$. This implies that

$$h^1\big(\partial\mathcal{O} \cap T^{-1}(U_0)\big) \subset S_r E_n \setminus S_r\big(E_{k-1}^\perp \cap E_n^+\big).$$

The latter space has the sphere $S_r\big(E_{k-1} \oplus E_n^-\big)$ as a strong deformation retract. In particular, there exists an odd mapping

$$S_r E_n \setminus S_r\big(E_{k-1}^\perp \cap E_n^+\big) \longrightarrow S_r\big(E_{k-1} \oplus E_n^-\big).$$

Observe that $S_r\big(E_{k-1} \oplus E_n^-\big) \cong S^{k+n-2}$. Putting these mappings together we obtain an odd mapping

$$g_0 : \partial\mathcal{O} \cap T^{-1}(U_0) \longrightarrow S^{k+n-2} \subset \mathbb{R}^{k+n-1}.$$

The mappings $h_i^1 : U_i \to \{\pm z_i\}$ yield odd maps

$$g_i : \partial\mathcal{O} \cap T^{-1}(U_i) \longrightarrow S^0 = \{\pm 1\}.$$

By (Φ_4) we have that $T(B) \subset \Phi_n^d$. Therefore $T^{-1}(U_0), \ldots, T^{-1}(U_\gamma)$ cover B. Setting $V_i := \partial\mathcal{O} \cap T^{-1}(U_i)$, we obtain an open invariant covering of $\partial\mathcal{O}$. Choose then a partition of the unity $\pi_i : \partial\mathcal{O} \to [0,1]$, $i = 0, \ldots, \gamma$, subordinated to the covering $V_0, \ldots, V_\gamma$ of $\partial\mathcal{O}$. Since the V_i's are invariant we may assume that the π_i's are even. Now we define the mapping

$$g : \partial\mathcal{O} \longrightarrow \mathbb{R}^{k+n-1} \times \mathbb{R}^\gamma, \quad g(z) := (\pi_0(z) g_0(z), \ldots, \pi_\gamma(z) g_\gamma(z)).$$

First, observe that g is well defined. Namely, if $\pi_i(z) \neq 0$, then $z \in V_i$ and so $g_i(z)$ is defined. Obviously, g is odd, since the g_i are odd and the π_i are even. Also, g is continuous. In addition $g(z) \neq 0$ for every $z \in \partial\mathcal{O}$ because there exists $i \in \{0, \ldots, \gamma\}$, with $\pi_i(z) \neq 0$, and hence $z \in V_i$ and $|g_i(z)| = 1$. Thus we have a continuous odd mapping $g : \partial\mathcal{O} \to \mathbb{R}^{k+n-1+\gamma} \setminus \{0\}$, where $\mathcal{O}$ is an invariant bounded open neighborhood of 0 in $E_K^+ \oplus E_n^-$. Now Borsuk's theorem implies that $k + n - 1 + \gamma \ge \dim E_K^+ \oplus E_n^- = K + n$. This shows that $\gamma \ge K - k + 1$ as required. □

As an immediate corollary of Theorem 2.1, we obtain the Fountain Theorem, which we state below. First we introduce the following set of conditions.

(Φ_2') There exists a sequence $r_k > 0$, $k \in \mathbb{N}$, such that $b_k \to +\infty$ as $k \to \infty$. (Here b_k is defined as in (Φ_2) with r_k instead of r.)

(Φ_3') There exists a sequence of isomorphisms $T_k : E \to E$, $k \in \mathbb{N}$, with $T_k(E_n) = E_n$ for all k and n, and there exists a sequence $R_k > 0$, $k \in \mathbb{N}$, such that, for $z = z^+ + z^- \in E_k^+ \oplus E^-$ with $\max\{\|z^+\|, \|z^-\|\} = R_k$, one has

$$\|T_k z\| > r_k \quad \text{and} \quad \Phi(T_k z) < 0$$

where r_k is given in (Φ_2').

(Φ_4') $d_k := \sup\{\Phi\big(T_k(z^+ + z^-)\big) : z^+ \in E_k^+, z^- \in E^-, \|z^+\|, \|z^-\| \le R_k\} < \infty$.

Theorem 2.2. (Fountain Theorem) *Suppose that* (Φ_1), (Φ_2') − (Φ_4'), (Φ_5) *hold. Then* Φ *has an unbounded sequence of critical values.*

Hypothesis (Φ_2') will be checked in the applications later on using the contents of the next remark.

Remark 2.2. *Let* E *be a Hilbert space and* $E_1 \subset E_2 \subset \dots$ *be finite dimensional subspaces such that* $\overline{\bigcup_{n=1}^{\infty} E_n} = E$. *Let* $\Phi \in C^1(E, \mathbb{R})$ *be of the form* $\Phi = P - \Psi$ *such that*

$$P(z) \ge \alpha \|z\|^p \quad \text{for all } z \in E$$

and

$$|\Psi(z)| \le \beta(1 + \|z\|_X^q) \quad \text{for all } z \in E$$

Here X *is a Banach space such that* $E \subset X$ *compactly, and* $q > p$, α, β *are positive constants.*

First we prove that

$$\mu_k := \sup\{\|z\|_X : z \in E, z \perp E_{k-1}, \|z\| = 1\} \to 0 \quad \text{as } k \to \infty.$$

Indeed, suppose by contradiction that this is not the case. So, there is $\varepsilon > 0$ *and a sequence* (z_j) *in* E *with* $z_j \perp E_{k_j - 1}$, $\|z_j\| = 1$, $\|z_j\|_X \ge \varepsilon$ *and* $k_j \to \infty$ *as* $j \to \infty$. *This implies that* $z_j \rightharpoonup 0$, *weakly in* E *hence* $z_j \to 0$, *strongly in* X, *which contradicting* $\|z_j\|_X \ge \varepsilon$.

Next we prove that there are $r_k > 0$, $k \in \mathbb{N}$, *so that*

$$b_k := \inf\{\Phi(z) : z \in E, z \perp E_{k-1}, \|z\| = r_k\} \to \infty.$$

Indeed, for $z \in E$, $z \perp E_{k-1}$, *we have*

$$\begin{aligned} \Phi(z) &= P(z) - \Psi(z) \\ &\ge \alpha\|z\|^p - \beta(1 + \|z\|_X^q) \\ &\ge \alpha\|z\|^p - \beta - \beta\mu_k^q\|z\|^q. \end{aligned} \tag{2.3}$$

Taking $\|z\| = r_k$ *with* $r_k := (p\alpha/q\beta\mu_k^q)^{1/(q-p)}$, *we obtain*

$$\Phi(z) \ge c\mu_k^{p^2/(p-q)} \to +\infty,$$

where c *depends only on* p, q, α, β.

Although the Fountain Theorem is an immediate consequence of Theorem 2.1, we choose to give a direct proof of it which does not employ the equivariant limit category, since this is a result with many applications.

Proof of the Fountain Theorem: By (Φ_2') it suffices to show that Φ has a critical value in $[b_k, d_k]$, for every k with $b_k > 0$. Fix such a k and suppose that $[b_k, d_k]$ contains only regular values. By Proposition 2.6 in [2], for n large, there exists a continuous deformation $h_n^t : \Phi_n^{d_k} \to E_n$, $t \in [0,1]$, such that h_n^t is odd and $h_n^1(\Phi_n^{d_k}) \subset \Phi_n^{b_k - \varepsilon}$, for some $\varepsilon > 0$. Moreover $h_n^t(z) = z$ if $\Phi(z) \le 0$. As above we set $B := B_{R_k}E_k^+ \times B_{R_k}E_n^-$, for $n \ge k$. Consider the set

$$\mathcal{O} = \{z \in B : \|h_1(T_k z)\| < r_k\}.$$

As in the proof of Theorem 2.1 one checks easily that $\mathcal{O}$ is an open invariant neighborhood of 0 in $E_k^+ \oplus E_n^-$, and that $\overline{\mathcal{O}} \subset \operatorname{int} B$. Now we set

$$g := P \circ h_1 \circ T_k : \partial\mathcal{O} \longrightarrow E_{k-1}^+ \oplus E_n^-,\ g(z) := P\big(h_1(T_k z)\big)$$

where $P : E_n \to E_{k-1}^+ \oplus E_n^-$ is the orthogonal projection. Since $\dim(E_k^+ \oplus E_n^-) > \dim(E_{k-1}^+ \oplus E_n^-)$, Borsuk's theorem tells us that g must have a zero. Now $z \in \partial\mathcal{O}$ implies that $\|h_1(Tz)\| = r_k$, and $g(z) = 0$ implies that $h_1(T_k z) \in E_n^+$, $h_1(T_k z) \perp E_{k-1}^+$. It follows from (Φ_2') that $\Phi(h_1(T_k z)) \ge b_k$. This contradicts the fact that $T_k z \in \Phi_n^{d_k}$ by (Φ_3') and $h_1(\Phi_n^{d_k}) \subset \Phi_n^{b_k-\varepsilon}$. □

3 The Variational Setting

3.1 The spaces in the case of a bounded domain in $\mathbb{R}^n$. Let φ_n, $n \in \mathbb{N}$, be an orthonormal basis of $L^2(\Omega)$ made up of eigenfunctions of the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Let λ_n be the corresponding eigenvalues. For all real numbers $s > 0$ we define, for $u = \sum_{j=1}^\infty a_j\varphi_j,\ v = \sum_{j=1}^\infty b_j\varphi_j$:

$$H^s(\Omega) := \left\{u \in L^2(\Omega) : \textstyle\sum_{j=1}^\infty \lambda_j^s |a_j|^2 < \infty\right\}.$$

This is a Hilbert space with respect to the inner product $\langle u, v\rangle_s := \sum_{j=1}^\infty \lambda_j^s a_j b_j$. Clearly, the operator

$$A^s : H^s(\Omega) \longrightarrow L^2(\Omega),\ \ u \longmapsto \sum_{j=1}^\infty \lambda_j^{s/2} a_j \varphi_j$$

is an isometric isomorphism. It is easy to see that

$$\int_\Omega A^s u\, \phi = \int_\Omega u\, A^s \varphi \qquad \text{for all } u, \phi \in H^s(\Omega)$$

which is used to prove the regularity of weak solutions. One has also the Sobolev imbeddings

$$H^s(\Omega) \subset L^p(\Omega)$$

continuously if $1 \le p \le \frac{2N}{N-2s}$ and compactly if $1 \le p < \frac{2N}{N-2s}$.

3.2 The spaces in the case $\Omega = \mathbb{R}^n$. In this case, the space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1+|\xi|^s)^2 \cdot |\widehat{u}(\xi)|^2 d\xi < \infty\right\}$$

where $\widehat{u}$ denotes the Fourier transform of u. This is a Hilbert space with respect to

$$\langle u, v\rangle := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^s)^2 \cdot \widehat{u}(\xi) \cdot \widehat{v}(\xi)\, d\xi.$$

The operator

$$A^s : H^s(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \quad u \longmapsto ((1+|\xi|^s)\widehat{u})^\vee,$$

(where $\omega^\vee$ denotes the inverse Fourier transform of ω) is usually written as $A^s = (1+|D|^s)$. It is readily seen that it is an isometric isomorphism. It is also easy to see that

$$\int_{\mathbb{R}^n} A^s u\, \phi = \int_{\mathbb{R}^n} u\, A^s \phi \qquad \text{for all } u, \phi \in H^s(\mathbb{R}^N).$$

We observe that $A^2 = u - \Delta u$ since

$$A^2 u = \big((1+|\xi|^2)\widehat{u}\big)^\vee \quad \text{for } u \in H^2(\mathbb{R}^N).$$

This explains the form of the system (1.2) in the case of $\Omega = \mathbb{R}^n$.

If G is a subgroup of $O(N)$, then we set

$$L^2_G(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : u(gx) = u(x) \text{ for } g \in G, \text{ and } x \in \mathbb{R}^n \text{ a.e.}\}$$

and $H^s_G := H^s \cap L^2_G$. Clearly we have that

$$A^s(H^s_G(\mathbb{R}^n)) = L^2_G(\mathbb{R}^n)$$

In the case of $\mathbb{R}^n$ there is a loss of the compact imbeddings. However, depending on the group G acting on $\mathbb{R}^n$, we can still recover them. We mention the following result due to P.-L. Lions [12]:

Proposition 3.1. *If*

$$G = O(N_1) \times \ldots \times O(N_k)$$

with $N_i \geq 2$ *and* $\sum_{i=1}^k N_i = N$, *then the imbedding*

$$H^s_G(\mathbb{R}^n) \subset L^\gamma(\mathbb{R}^n) \quad \text{for } 2 < \gamma < \frac{2N}{N-2s}$$

is compact.

The case when $G = O(N)$ was first proved by Strauss [15].

3.3 The "quadratic" forms and the functionals. In the sequel we write E^s to denote both $H^s(\Omega)$ in the case of a bounded domain Ω, and $H^s(\mathbb{R}^N)$. Let us consider the Cartesian product $E := E^s \times E^t$ with $s, t \geq 0$, which is also a Hilbert space endowed with the inner product

$$\langle z, \eta\rangle := \langle u, \phi\rangle_s + \langle v, \psi\rangle_t, \quad \text{for } z = (u,v),\, \eta = (\phi, \psi) \in E.$$

We consider the bilinear form

$$B : E \times E \longrightarrow \mathbb{R}, \quad B[z,\eta] := \int (A^s u\, A^t \psi + A^s \phi\, A^t v),$$

where $\int$ denotes the integral in both cases, over Ω or over $\mathbb{R}^n$. Associated to B, we have the quadratic form

$$Q(z) := \frac{1}{2} B[z,z] = \int A^s u\, A^t v\,.$$

It is easy to see (cf. [7]) that the bounded self-adjoint operator $L : E \to E$ defined by $\langle Lz, \eta\rangle := B[z,\eta]$ has exactly two eigenvalues $+1$ and -1, and that the corresponding eigenspaces E^+ and E^- are given by

$$E^+ = \{(u, A^{-t}A^s u) : u \in E^s\} \quad \text{and} \quad E^- = \{(u, -A^{-t}A^s u) : u \in E^s\}$$

where we are using the notation $A^{-t} = (A^t)^{-1}$.

Now consider the Hamiltonian $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ from Section 1. We consider the functional

$$\Phi(z) := Q(z) - \int H(x,u,v)\, dx \tag{3.1}$$

where $z = (u,v)$. By (H_2) or (H_2'), there exist $s,t > 0$, with $s + t = 2$ and

$$\frac{1}{p} > \frac{1}{2} - \frac{s}{n} \quad \text{and} \quad \frac{1}{q} > \frac{1}{2} - \frac{t}{n}. \tag{3.2}$$

This implies that we have continuous imbeddings $E^s \subset L^p$ and $E^t \subset L^q$. We fix s and t with this property so that Φ is well defined in E by (H_2) or (H_2'). Moreover, (H_2) or (H_2') imply that $\Phi \in C^1(E,\mathbb{R})$ with

$$\langle \Phi'(z), \eta\rangle = B[z,\eta] - \int \Big(H_u(x,u,v)\phi \,+\, H_v(x,u,v)\psi\Big)\, dx$$

for $z, \eta \in E$. From this one deduces that a critical point $z = (u,v)$ of Φ corresponds to a weak solution of (1.1) or (1.2). Namely

$$\int A^s u\, A^t \psi \;=\; \int H_v(x,u,v)\, \psi \qquad \text{for all } \psi \in E^t$$

and

$$\int A^s \phi\, A^t v \;=\; \int H_u(x,u,v)\, \phi \qquad \text{for all } \phi \in E^s.$$

As shown in [7] for the case of Ω bounded these solutions are strong in the sense that

$$u \in W^{2,p/(p-1)}(\Omega) \cap W_0^{1,p/(p-1)}(\Omega) \;\text{ and }\; v \in W^{2,q/(q-1)}(\Omega) \cap W_0^{1,q/(q-1)}(\Omega),$$

and they satisfy (1.1). In the case of $\mathbb{R}^n$ we conclude that $u \in W_{loc}^{2,p/(p-1)}(\mathbb{R}^N)$ and $v \in W_{loc}^{2,q/(q-1)}(\mathbb{R}^N)$, and that they satisfy (1.2).

In order to apply Theorem 2.1 in the next sections, let us introduce the following notations. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of E^s. Clearly the $f_j := A^{-t}A^s e_j$, $j \in \mathbb{N}$, constitute an orthonormal basis of E^t. We set

$$E_n^s := \text{span}\{e_j : j = 1,...,n\} \quad \text{and} \quad E_n^t := \text{span}\{f_j : j = 1,...,n\}.$$

The following result can be readily seen.

Lemma 3.1. *With the above notations, we have*

$$E^{\pm} = \overline{\bigcup_{n=1}^{\infty} E_n^{\pm}}, \quad E = E^+ \oplus E^-, \quad E_n := E_n^+ \oplus E_n^- = E_n^s \times E_n^t$$

□

Next we check that the functional Φ defined in (3.1) satisfies the condition (Φ_2') in both cases, Ω bounded or $\mathbb{R}^N$.

Lemma 3.2. *Assume* (H_1)–(H_2) *or* (H_1')-(H_2')*. Then there exists a sequence of positive real numbers* r_k*,* $k \geq 1$*, such that*

$$b_k := \inf\{\Phi(z) : z \in E^+,\, z \perp E_{k-1}^+,\, \|z\| = r_k\} \to +\infty.$$

Proof: We use Remark 2.2 applied to the Hilbert space E^+ with $P(z) := Q(z)$, and

$$\Psi(z) := \int H(x, u, v)$$

where $z = (u, v)$. We know that $Q(z) = \frac{1}{2}\|z\|^2$ for $z \in E^+$, and from (H_2) we obtain

$$\begin{aligned} \left| \int H(x,u,v) \right| &\leq C \left(\int |u|^p + \int |v|^q + 1 \right) \\ &\leq C' \left(\|z\|_X^r + 1 \right) \end{aligned}$$

where $X = L^p(\Omega) \times L^q(\Omega)$ and $r = \max\{p, q\} > 2$. Also, assuming (H_2') we obtain:

$$\begin{aligned} \left| \int H(x,u,v) \right| &\leq C \left(\int |u|^p + \int |v|^q + \int |u|^{a+1} + \int |v|^{b+1} \right) \\ &\leq C' \|z\|_X^r \end{aligned}$$

for some $r > 2$. □

4 The case Ω bounded.

In this section we prove Theorem 1.1. With the notation from Section 3, we want to apply Theorem 2.1 to the functional

$$\Phi : E \to \mathbb{R}, \quad \Phi(z) = Q(z) - \int H(x, u, v)\, dx \tag{4.1}$$

where $z = (u, v) \in E = E^s \times E^t$. First we show that Φ satisfies a Palais-Smale condition.

Lemma 4.1. *Assume* (H_1), (H_2) *and* (H_3)*. Then* Φ *as defined in* (4.1) *satisfies* $(PS)_c^{\mathcal{F}}$ *for every* $c \in \mathbb{R}$ *and* $\mathcal{F} = (E_n)_{n\in\mathbb{N}}$*.*

Proof: In view of Remark 2.1, it suffices to prove that a $(PS)_c^{\mathcal{F}}$ sequence is bounded in E. This follows as in Section 3 of [8], up to the point where we are to get some bounds for $\|u_n\|_{E^s}$ and $\|v_n\|_{E^t}$. At this point we then use the fact that $\int A^s u_n\, A^t \psi = 0$, for all $\psi \in (E_n^t)^{\perp}$. □

Next we check the other assumptions of Theorem 2.1. It has already been proved that Φ satisfies condition (Φ_2'). In particular, condition (Φ_2) is satisfied.

For each $\lambda > 0$, let us define the isomorphism $T_\lambda : E \to E$ by

$$T_\lambda(u,v) = (\lambda^{\beta-1}u, \lambda^{\alpha-1}v).$$

Clearly $T_\lambda E_n = E_n$ for all $\in \mathbb{N}$. Observe however that $E_n^+ \cap T_\lambda E_n^+ = \{0\}$ and $E_n^- \cap T_\lambda E_n^- = \{0\}$ if $\alpha \neq \beta$.

For each $k \in \mathbb{N}$ consider the finite dimensional subspace E_k^s. Since all norms are equivalent in finite dimensional spaces, we have positive constants σ_k, σ_k', τ_k and τ_k' such that

$$\|u\|_{L^2} \geq \sigma_k \|u\|_{E^s} \quad \text{and} \quad \|u\|_{E^s} \geq \sigma_k' \|u\|_{L^\beta} \quad \text{for all } u \in E_k^s$$

and

$$\|v\|_{L^2} \geq \tau_k \|v\|_{E^t} \quad \text{and} \quad \|v\|_{E^t} \geq \tau_k' \|v\|_{L^\alpha} \quad \text{for all } v \in E_k^t.$$

These constants are going to compose the constants δ_k announced in the introduction.

Lemma 4.2. *Assume that there are constants c_2 and c_3, such that*

$$H(x,u,v) \geq \frac{1}{2}c_2(|u|^\alpha + |v|^\beta) - c_3 \tag{4.2}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. Then, for each $\lambda > 0$ and each $k \in \mathbb{N}$, one has

$$\Phi(T_\lambda z) \leq \lambda^{\alpha+\beta}(1 - c_2\delta_k) + c_3|\Omega|$$

for all $z = z^+ + z^- \in E_k^+ \oplus E^-$, with $\lambda := \|z^+\|$, where

$$\delta_k := \min\left\{ \left(\frac{\sigma_k^2\, \sigma_k'}{2}\right)^\alpha, \ \left(\frac{\tau_k^2\, \tau_k'}{2}\right)^\beta \right\}$$

Proof: For $z = z^+ + z^- \in E_k^+ \oplus E^-$ we write $z^- = z_1^- + z_2^-$ where $z_1^- \in E_k^-$, $z_2^- \perp E_k^-$. We also write $\overline{z} := z^+ + z_1^-$ and $z = (u,v)$, $z^\pm = (u^\pm, v^\pm), \ldots$ Using (4.2) we have

$$\int_\Omega H(x, T_\lambda z) \geq \frac{1}{2}c_2\big(\lambda^{(\beta-1)\alpha}\|u\|_{L^\alpha}^\alpha + \lambda^{(\alpha-1)\beta}\|v\|_{L^\beta}^\beta\big) - c_3|\Omega|. \tag{4.3}$$

Since α and β are conjugate exponents and $u_2^- \perp \overline{u}$ in L^2, we obtain

$$\begin{aligned} \|u\|_{L^\alpha} &\geq |\langle u, \overline{u}\rangle_{L^2}| \cdot \|\overline{u}\|_{L^\beta}^{-1} \\ &= \|\overline{u}\|_{L^2}^2 \|\overline{u}\|_{L^\beta}^{-1} \\ &\geq \sigma_k^2 \sigma_k' \|\overline{u}\|_{E^s} \\ &=: \widetilde{\sigma}_k \|\overline{u}\|_{E^s}. \end{aligned} \tag{4.4}$$

Similarly

$$\|v\|_{L^\beta} \geq \tau_k^2\, \tau_k' \|\overline{v}\|_{E^t} =: \widetilde{\tau}_k \|\overline{v}\|_{E^t}. \tag{4.5}$$

Next observe that $\overline{u} = u^+ + u_1^-$ and $\overline{v} = A^{-t}A^s(u^+ - u_1^-)$. So

$$\|\overline{u}\|_{E^s} = \|u^+ + u_1^-\|_{E^s} \quad \text{and} \quad \|\overline{v}\|_{E^t} = \|u^+ - u_1^-\|_{E^s}. \tag{4.6}$$

Using (4.3)-(4.6) we obtain the following estimate on the Hamiltonian:

$$\int_\Omega H(x, T_\lambda z)$$
$$\geq \frac{1}{2} c_2 \left(\lambda^{(\beta-1)\alpha} \tilde{\sigma}_K^\alpha \|u^+ + u_1^-\|_{E^s}^\alpha \; + \; \lambda^{(\alpha-1)\beta} \tilde{\tau}_K^\beta \|u^+ - u_1^-\|_{E^s}^\beta \right) \; - c_3|\Omega|$$

Since $\|u^+\|_{E^s} = \|z^+\|/2 = \lambda/2$, we have that either $\|u^+ + u_1^-\|_{E^s} \geq \lambda/2$ or $\|u^+ - u_1^-\|_{E^s} \geq \lambda/2$. In either case

$$\int_\Omega H(x, T_\lambda z) \geq \frac{1}{2} c_2 \, \delta_k \, \lambda^{\alpha+\beta} \; - c_3|\Omega|. \tag{4.7}$$

On the other hand we have that

$$Q(T_\lambda z) = \lambda^{\alpha+\beta-2} Q(z) = \frac{1}{2} \lambda^{\alpha+\beta-2} (\|z^+\|^2 - \|z^-\|^2)$$

and so for $\|z^+\| = \lambda$ we obtain

$$Q(T_\lambda z) \leq \frac{1}{2} \lambda^{\alpha+\beta}. \tag{4.8}$$

Thus it follows from (4.7) and (4.8) that

$$\Phi(T_\lambda z) \leq \frac{1}{2}(1 - c_2 \delta_k) \lambda^{\alpha+\beta} \; + c_3|\Omega| \quad \text{ for all } \lambda = \|z^+\| \leq \|z^-\|. \tag{4.9}$$

□

An immediate consequence of Lemma 4.2 is the next result which establishes (Φ_3).

Lemma 4.3. *Suppose that* (4.2) *holds and there is* $K \in \mathbb{N}$ *such that*

$$1 - c_2 \, \delta_K < 0 \tag{4.10}$$

Then, fixing $r > 0$ *there is a* $\lambda_0 > 0$ *such that, for all* $\lambda \geq \lambda_0$ *one has, for* $z = z^+ + z^- \in E_K^+ \oplus E^-$*:*

$$\Phi(T_\lambda z) < 0 \quad \text{if } \|z^+\| = \lambda, \quad \text{and} \quad \|T_\lambda z\| > r \quad \text{if } \max\{\|z^+\|, \|z^-\|\} = \lambda.$$

□

Proof of Theorem 1.1: As mentioned before we apply Theorem 2.1. First we observe that Lemma 4.1 gives condition (Φ_1). Lemma 3.2 implies that there exists a $k_0 \in \mathbb{N}$ such that $b_{k_0} > 0$, which then gives (Φ_2) and (Φ_2'). Now suppose that (1.9) holds for some $K \geq k_0$. Then we can apply Lemmas 4.2 and 4.3 and conclude that (Φ_3) holds. (Φ_4) is implied by the fact that Φ maps bounded sets of E into bounded sets of $\mathbb{R}$. Finally, condition (Φ_5) is a consequence of (H_4). □

5 The case $\Omega = \mathbb{R}^n$

With the notations introduced in Section 3 the weak solutions of (1.2) are the critical points of the functional

$$\Phi(z) = \int_{\mathbb{R}^N} A^s u \, A^t v \; - \; \int_{\mathbb{R}^N} H(x, u, v) \tag{5.1}$$

acting in $E = E^s \times E^t := H^s(\mathbb{R}^N) \times H^t(\mathbb{R}^N)$, where s and t satisfy (3.2). We shall consider the functional Φ restricted to certain subspaces of E where we have compact imbeddings due to symmetry properties. Let us start with the group $G = O(N)$ acting in $\mathbb{R}^n$, and let us look for critical points of Φ in the subspace X of E given by $X = H^s_{O(N)}(\mathbb{R}^n) \times H^t_{O(N)}(\mathbb{R}^n)$. All subspaces introduced in Section 3 are now restricted to spherically symmetric functions. Observe that

$$X := \mathrm{Fix}(G) = \{(u,v) \in E : gu = u,\, gv = v, \text{ for all } g \in O(N)\}$$

where gu means $(gu)(x) = u(gx)$, for all $x \in \mathbb{R}^n$. We see also that Φ is invariant with respect to G, i.e. $\Phi(gu, gv) = \Phi(u,v)$. Hence, it follows from the Palais Principle of Symmetric Criticality, see [13] or [16], that the critical points of Φ restricted to X are critical points of Φ considered in the whole space E.

In order to prove Theorem 2.1, we have to check that $\Phi\big|_X$ satisfies the assumptions of the Fountain Theorem.

For each $\lambda > 0$, let us define the isomorphism $T_\lambda : E \to E$, by

$$T_\lambda(u,v) = (\lambda^\mu u,\, \lambda^\nu v),$$

where $\mu = \frac{m-p}{p}$ and $\nu = \frac{m-q}{q}$, and $m > \max(p,q)$. Clearly $T_\lambda E_n = E_n$.

Lemma 5.1. *Assume conditions* (H_1')-(H_4'). *Then, there is a sequence $\lambda_k > 0$, $k \in \mathbb{N}$, such that* (Φ_3') *holds with $T_k := T_{\lambda_k}$ and $R_k := \lambda_k$.*

Proof: Let us use the notation introduced in the proof of Lemma 4.2. It follows from (1.16) that for any $\lambda > 0$ we have

$$\int_{\mathbb{R}^N} H(x, T_\lambda z) \geq c\left(\lambda^{\mu p}\int_{\mathbb{R}^N}|u|^p \,+\, \lambda^{\nu q}\int_{\mathbb{R}^N}|v|^q\right). \tag{5.2}$$

Using Hölder's inequality, we obtain

$$\|u\|_{L^p}\,\|\overline{u}\|_{L^{p'}} \geq |\langle u, \overline{u}\rangle_{L^2}| = \|\overline{u}\|^2_{L^2} \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1,$$

Next, from the finite dimensionality of E^s_k we have that there is a positive constant γ_k such that

$$\|u\|_{L^p} \geq \gamma_k \|\overline{u}\|_{E^s} \quad \text{for } u \in E^s_k.$$

Similarly there is $\widetilde{\gamma} > 0$ with

$$\|v\|_{L^q} \geq \widetilde{\gamma}_k \|\overline{v}\|_{E^t} \quad \text{for } v \in E^t_k.$$

Thus it follows from (5.2) that

$$\int_{\mathbb{R}^N} H(x, T_\lambda z) \geq c\big(\lambda^{\mu p}\gamma_k^p \|\overline{u}\|^p_{E^s} \,+\, \lambda^{\nu q}\widetilde{\gamma}_k^q \|\overline{v}\|^q_{E^t}\big).$$

As in the proof of Lemma 4.2 we obtain

$$\int_{\mathbb{R}^N} H(x, T_\lambda z) \geq c \cdot \min\left\{\frac{1}{2^p}\lambda^{\mu p}\gamma_k^p \lambda^p,\ \frac{1}{2^q}\lambda^{\nu q}\widetilde{\gamma}_k^q \lambda^q\right\} \geq \sigma_k \lambda^m$$

provided $\|z^+\| = \lambda$.

On the other hand,

$$Q(T_\lambda z) = \lambda^{\mu+\nu}(\|z^+\|^2 - \|z^-\|^2) \leq \lambda^{\mu+\nu+2}$$

for $\|z^+\| = \lambda$. Consequently we have

$$\Phi(T_\lambda z) \leq \lambda^{\mu+\nu+2} - \sigma_k \lambda^m.$$

Since $m > \mu + \nu + 2$, it follows that there is a $\lambda_0(k) > 0$ such that $\Phi(T_{\lambda_k} z) < 0$ if $\lambda_k > \lambda_0(k)$. Also

$$\|T_\lambda z\| \geq \lambda^{\min\{\mu,\nu\}} \|z\|^2,$$

which implies that

$$\|T_{\lambda_k} z\| \geq \lambda_k^{\min\{\mu,\nu\}+2} \qquad \text{for } \max\{\|z^+\|, \|z^-\|\} = \lambda_k.$$

Therefore, we can choose λ_k such that

$$\Phi(T_{\lambda_k} z) < 0 \quad \text{and} \quad \|T_{\lambda_k} z\| \geq r_k$$

for any given r_k. □

Proof of Theorem 1.2: First we observe that hypotheses (H_1') and (H_2') imply that Φ is C^1 in E. And using (H_3') we prove easily that a $(PS)_c^{\mathcal{F}}$ sequence is bounded in X. So it follows from Remark 2.1 and Proposition 3.1 that $\Phi\big|_X$ satisfies the $(PS)_c^{\mathcal{F}}$ condition. Hence (Φ_1) holds. Condition (Φ_2') has already been checked in Lemma 3.2. Condition (Φ_3') is proved in Lemma 5.1. Condition (Φ_4') is trivially verified, and finally (Φ_5) is a consequence of (H_5). So we apply the Fountain Theorem and conclude. □

We omit the **Proof of Theorem 1.3** since it parallels a similar result of Bartsch-Willem [3] for the scalar case. The result in our case follows from an application of the Fountain Theorem, using Proposition 3.1. □

References

[1] A. Bahri and H. Berestycki, Existence of forced oscillations for some nonlinear differential equations, Comm. Pure Appl. Math. 37 (1984), 403-442.

[2] T. Bartsch and M. Clapp, Critical point theory for indefinite functionals with symmetries, J. Fctl. Anal. 138 (1996), 107-136.

[3] T. Bartsch and M. Willem, Infinitely many nonradial solutions of a Euclidean scalar field equation, J. Fctl. Anal. 117 (1993), 447-460.

[4] G. Cerami, Un criterio di esistenza per i punti critici su varietà illimitate, Rec. Ist. Lomb. Sci. Lett. 112 (1978), 332-336.

[5] D.G. Costa and C.A. Magalhães, A unified approach to a class of strongly indefinite functionals, J. Diff. Eq. 122 (1996), 521-547.

[6] D.G. de Figueiredo, Semilinear elliptic systems, Proceedings of the Second School on Nonlinear Functional Analysis and Applications to Differential Equations, ICTP Trieste 1997, World Scientific Publishing Company (to appear).

[7] D.G. de Figueiredo and P. Felmer, On superquadratic elliptic systems, Trans. AMS 343 (1994), 99-116.

[8] D.G. de Figueiredo and C.A. Magalhães, On nonquadratic Hamiltonian elliptic systems, Advances in Diff. Eq. 1 (1996), 881-898.

[9] D.G. de Figueiredo and J. Yang, Decay, Symmetry and existence of solutions of semilinear elliptic systems, to appear in Nonlinear Analysis, TMA.

[10] P. Felmer, Periodic solutions of "superquadratic" Hamiltonian systems, J. Diff. Eq. 102 (1993), 188-207.

[11] S. Li and J.Q. Liu, Some existence theorems on multiple critical points and their applications, Kexue Tongbao 17 (1984) [in chinese].

[12] P.-L. Lions, Symétrie et compacité dans les espaces de Sobolev, J. Funct. Anal. 49 (1982) 315-334.

[13] R.S. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69(1979), 19-30.

[14] B. Sirakov, On the existence of solutions of elliptic systems in $\mathbb{R}^N$, Preprint.

[15] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. (1977), 149-162.

[16] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser 1996.