For the volume in recognition of Jack Hale's seventieth birthday

Reversible Vector Fields with 1:1 Resonance

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In honour of Jack Hale's seventieth birthday

Abstract

We consider a class of reversible vector fields with (1:1) resonance at an equilibrium. By means of an efficient normal form we study the local behavior of the systems, showing the existence of invariant varieties and reversible periodic solutions. Moreover, we obtain an analogue of the Lyapunov-Devaney theorem in the 1:1 reversible setting, which can be treated as a limit case where two pairs of non-resonant purely imaginary eigenvalues tend to accumulate.

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1 Introduction

Consider a one-parameter family of vector fields having 1:1 resonance, i.e., systems having a form

$$\dot{p} = f(\mu, p), \quad p \in \mathbb{R}^4, , \quad \mu \in \mathbb{R}, \tag{1.1}$$

where f is a smooth function such that $f(0,\mu) = 0$. The eigenvalues of the matrix $L(0)$ in the linearized system $\dot{p} = L(\mu)p$ has two pairs of purely imaginary eigenvalues $\pm\omega i$, with a two dimensional Jordan block. Moreover, we assume that the system concerned is time reversible. That is, there exists an involution $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ ($\varphi^2 = id.$) such that the relation $f(\mu, \varphi p) = -\varphi'(x) \cdot f(\mu, p)$ holds.

With the above assumption, we can write the above matrix $L(0)$ in the form

$$L(0) = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}$$

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and its generic deformation (see [1])

$$L(\mu)=\begin{pmatrix} 0 & \omega+\alpha\mu & 1 & 0\\ -\omega-\alpha\mu & 0 & 0 & 1\\ \beta\mu & 0 & 0 & \omega\\ 0 & \beta\mu & -\omega & 0\end{pmatrix},$$

where α and β are constant. Since the eigenvalues of $L(\mu)$ are $\lambda_{\pm}=\pm i(\omega+\alpha\mu)\pm\sqrt{\beta\mu}$, it follows that if $\beta\neq 0$ then the eigenvalues of $L(\mu)$ are either two pairs of purely imaginary numbers or two pairs of complex numbers. Conventionally, $\beta\neq 0$ is called transversality condition.

We denote by S_φ the fixed point set of the involution φ, i.e., $S_\varphi = Fix\varphi$. Then S_φ is a manifold whose dimension depends on the type of φ (see [11]). Throughout the paper we assume that S_φ is of dimension 2 and that φ takes the form $\varphi(x_1,x_2,y_1,y_2)=(x_2,x_1,-y_2,-y_1)$. Observe that this is not a restriction since by the Montgomery-Bochner theorem [9] any involution φ with $dim\ Fix\varphi=2$ can be brought to this form by a smooth change of coordinates.

By the Poincare-Dulac theorem, one can formally put any φ-time reversible system in resonant normal form. In fact, a reversible system, based on its linear part, can be formally put in normal form in such a way that the normalization remains in the centralizer of the involution. (see [10]. Vector field (1.1) is formally conjugated to the following form in $\mathbb{R}^4$.

$$\begin{cases} \dot{x}_1=\omega x_2+y_1+x_2F(\mu,u,v)\\ \dot{x}_2=-\omega x_1+y_2-x_1F(\mu,u,v)\\ \dot{y}_1=\omega y_2+y_2F(\mu,u,v)+x_1G(\mu,u,v)\\ \dot{y}_2=-\omega y_1-y_1F(\mu,u,v)+x_2G(\mu,u,v),\end{cases}\tag{1.2}$$

where F and G are real series of u,v, $u=x_1^2+x_2^2$, $v=x_1y_2-x_2y_1$. This means that given a smooth (C^∞) vector field one can normalize it up to a given order $O(|u|+|v|)^N$ with arbitrary N.

In the paper we shall first classify all such 4-dimensional reversible systems from their linear parts and then focus the study on systems with $(1:1)$ resonance. We normalize the normal form (1.2) and show that, with repsect to its principle nonlinear part, the normalization can be taken from the centralizer of the involution. As to the dynamics of the system around the equilibrium, by means of the normal form the, we analyze the existence of the symmetric periodic orbits. Remind that a periodic orbit of a reversible system is called symmetric if it encounters the fixed point set of the involution. We point out that although the normalizing changes of coordinates in general are divergent and thus one can argue that the methods here is essentially the same as the method of 'truncation" of the system, the normal form given here does convince one that generically the knowledge of the 3-jet of the system is essential (see Theorem A). This corresponds to the non-degenerate assumptions which are traditionally imposed on the 3-jet of the system (see [11]).

Vector fields with 1:1 resonance have been studied by several authors. In [14] there is an exposition about 1:1 Hamiltonian systems and in [11] a comprehensive and clear study about reversible systems. In [7], the existence and persistence of homoclinic orbits of 1:1 reversible systems are studied. The relation concerning the orbital equivalence between 1:1 Hamiltonian and 1:1 reversible systems is established in [10]. In this paper we lay emphasis on the detailed account of geometric invariants and symmetric properties of the system due to the reversibility. As the result, the local behavior of any smooth 1:1 reversible vector field which is C^0 equivalent to our normal form is completely understood. Moreover, all the descriptions given here are in an explicit way so that possible applications in the related subjects can be performed ([13]).

Concerning the existence of the symmetric periodic orbits, our results can be illustrated by the following example. This concise example holds the essential part of the discussion of [7] on the existence of the periodic porbits.

Example 1 Consider vector field

$$\begin{cases} \dot{x}_1 = x_2 + y_1 + \alpha\mu x_2 \\ \dot{x}_2 = -x_1 + y_2 - \alpha\mu x_1 \\ \dot{y}_1 = y_2 + \alpha\mu y_2 + x_1(\beta\mu + cr^2) \\ \dot{y}_2 = -y_1 - \alpha\mu y_1 + x_2(\beta\mu + cr^2) \end{cases} \tag{1.3}$$

where $r^2 = x_1^2 + x_2^2$.

Let $c = 1$ and $\beta\mu < 0$ in (1.3). The system has two first integrals: $I_1 = x_1y_2 - x_2y_1$ and $I_2 = y_1^2 + y_2^2 - \frac{1}{2}r^4 - \beta\mu r^2$. In (I_1, I_2) plane, we consider the parameterized equations $I_1^2 = -\tau^2(\tau + \beta\mu)$, $I_2 = -\frac{3}{2}\tau^2 - \beta\mu\tau$, $0 \leq \tau \leq -\beta\mu$. This gives a closed curve C which has two "cusp" points P and P' (when $\tau = -\frac{2}{3}\beta\mu$, see Fig. 1). One sees that for any given pair (I_1, I_2) if it falls on C then the algebraic system $I_1 = |r\rho|$, $I_2 = \rho^2 - \frac{1}{2}r^4 - \beta\mu r^2$ has one root of multiplicity two and one single root (r, ρ). If it falls inside C then there are three different roots, whereas outside C there is only one real root. The two points P and P' correspond to a root of multiplicity 3.

Notice that those and only those orbits with starting points on C are symmetric periodic orbits. In fact, the system has the following two 2-parameter families of symmetric periodic orbits around the equilibrium 0:

$$(r\cos(\sigma t + \theta_0), -r\sin(\sigma t + \theta_0), \mp\rho\sin(\sigma t + \theta_0), \mp\rho\cos(\sigma t + \theta_0)), \tag{1.4}$$

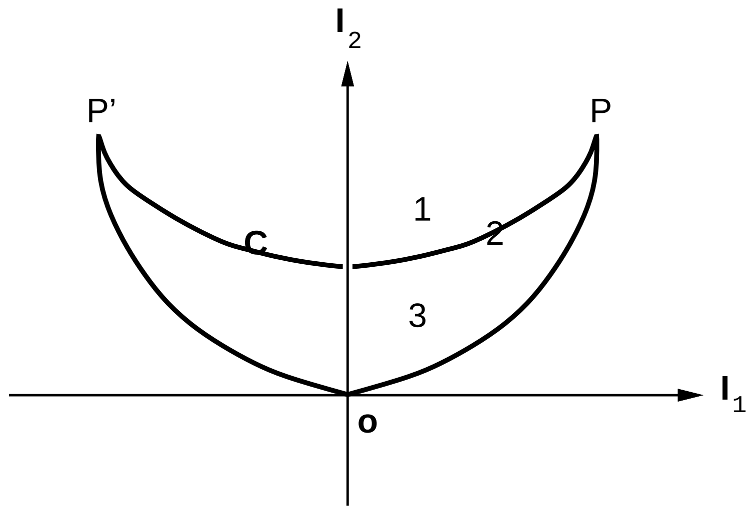

Figure 1: The curve of double roots of the equations of Example 1

where $\sigma = 1 + \alpha\mu \pm \sqrt{-(r^2 + \beta\mu)}$, $\rho = r\sqrt{-(r^2 + \beta\mu)}$, and r and θ_0 are parameters, $0 \leq r \leq \sqrt{-\beta\mu}$. The first family of orbits meet S_φ, the fixed point set of the involution φ, at two points $P_1(-\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}\rho, \frac{\sqrt{2}}{2}\rho)$ and $P_2(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}\rho, -\frac{\sqrt{2}}{2}\rho)$, and the second family at the points $P_1'(-\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}\rho, -\frac{\sqrt{2}}{2}\rho)$ and $P_2'(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}\rho, \frac{\sqrt{2}}{2}\rho)$.

The periods of the both families of solutions go to 2π as the initial points approach the origin.

If $c = -1$ in (1.3), then the system has no symmetric periodic orbits around the origin, but outside a neighborhood of the origin there are two 2-parameter families of symmetric periodic orbits (1.4), where $\sigma = 1 + \alpha\mu \pm \sqrt{r^2 - \beta\mu}$, and $\rho = r\sqrt{(r^2 - \beta\mu)}$.

In the following theorems we first give the normal form of a smooth φ-time reversible vector field with 1:1 resonance and then we extend and generalize the results of the above example. For the methods of normalization of vector fields, see [3, 12]. In [15] also there is a short exposition on the normal forms of vector fields.

Theorem A *Let X be a smooth φ-reversible vector field having 1:1 resonance.*

Then in generic case, X is formally conjugated to the following reversible normal form

$$\begin{cases} \dot{x}_1 = \omega x_2 + y_1 + x_2 f(r^2, v) \\ \dot{x}_2 = -\omega x_1 + y_2 - x_1 f(r^2, v) \\ \dot{y}_1 = \omega y_2 + y_2 f(r^2, v) + x_1(cr^2 + dv) \\ \dot{y}_2 = -\omega y_1 - y_1 f(r^2, v) + x_2(cr^2 + dv), \end{cases} \tag{1.5}$$

where $r^2 = x_1^2 + x_2^2$, $v = x_1 y_2 - x_2 y_1$ and $f(r^2, v) = \sum_{k=1}^{\infty}(a_k r^2 + b_k v)v^{k-1}$.

The genericity conditions, in terms of (1.5), are as follows:

$$d \neq 0, \quad a_1 d - b_1 c \neq 0. \tag{1.6}$$

We shall throughout the paper consider the following deformed normal form.

$$\begin{cases} \dot{x}_1 = \omega x_2 + y_1 + x_2 f(\mu, r^2, v) \\ \dot{x}_2 = -\omega x_1 + y_2 - x_1 f(\mu, r^2, v) \\ \dot{y}_1 = \omega y_2 + y_2 f(\mu, r^2, v) + x_1(\beta\mu + cr^2 + dv) \\ \dot{y}_2 = -\omega y_1 - y_1 f(\mu, r^2, v) + x_2(\beta\mu + cr^2 + dv), \end{cases} \tag{1.7}$$

where f is a function of its arguments, $f(0,0,0) = 0$, $\beta \neq 0$, $c \neq 0$.

We shall prove the following theorem.

Theorem B *Let X be given by (1.7). If $\beta\mu < 0$ then X always has two 2-parameter families of symmetric periodic solutions which shrink to the equilibrium as the initial conditions tend to 0. The periods of both families tend to $2\pi/\omega$ as the initial conditions tend to 0.*

If $c < 0$ and $\beta\mu > 0$ then outside a small neighborhood of the origin X also has two 2-parameter families of symmetric periodic orbits.

If $c > 0$ and $\beta\mu > 0$ then there is no symmetric periodic orbits.

Theorem B in fact is an extension of the Lyapunov-Devaney theorem to the (1:1) resonance case (see [4]. By the Devaney-Lyapunov center theorem, if X has eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, ($\omega_2 > \omega_1 > 0$), then in the neighborhood of the equilibrium X has a family of short periodic solutions whose periods tend to $2\pi/\omega_2$, moreover, if $\omega_2 \neq k\omega_1$, (k an integer), then there is a family of long periodic solutions whose periods tend to $2\pi/\omega_1$. Therefore it seems reasonable to explain the existence of the two families of symmetric periodic solutions in (1:1) case as follows. Treat the eigenvalues ($\pm i\omega$, $\pm i\omega$) as a limit case of $\pm i\omega_1$ and $\pm i\omega_2$ ($\omega_2 > \omega_1 > 0$). Since $\omega \neq 0$, take ω_1 and ω_2 so close to ω that ω_2/ω_1 is not an integer. Then applying the Devaney-Lyapunov theorem, we get two families of periodic solutions, one is of short periods and another long periods. Both families of periods tend to, as $\omega_i \to \omega$, $i = 1, 2$, the same value $2\pi/\omega$.

We notice the similarity of Theorem B in the Hamiltonian setting. The corresponding results for Hamiltonian systems has been well known for decades. Recall that in [10] it is proved that any reversible vector field with non-semisimple 1:1 resonance can be decomposed into Hamiltonian part and non-Hamiltonian part and the original system is orbitally equivalent to its Hamiltonian part. Therefore Theorem B incidentally verifies the validity of [10].

The paper is organized as follows. In section 2 we give a brief classification of 4-dimensional reversible vector fields in order to locate our objects (see [8, 5]). Section 3 is devoted to the normalization of the 1:1 reversible vector fields. In section 4 we present a detailed discussion of geometric invariant of such systems.

2 Classification of reversible vector fields from the linear approximation

It is easy to see that the eigenvalues of reversible vector fields always occur in pairs. In fact, Let X be a reversible vector field on $\mathbb{R}^4$ with linear part $\dot{x} = Ax$. Then the eigenvalues of A always satisfy the following relation

$$\lambda^4 + a\lambda^2 + b = 0, \tag{2.8}$$

where a and b are constant. Thus the linearization at the origin contains all possible qualitatively distinct combinations of eigenvalues when a and b run over all possibilities. This fact makes us possible to give a classification of reversible systems from their linear parts (see Fig. 2):

At the linear level, we have the following classification:

- **Generic case**: for generic a, b, the four eigenvalues are mutually different. In Fig. 2 this means that a, b falls in Regions 1-4. More exactly, in Regions 3 and 4, the linearized vector fields are hyperbolic, the eigenvalues have the forms $(\pm\alpha_1, \pm\alpha_2)$ and $\pm(\alpha \pm \beta i)$, $(\alpha\beta \neq 0)$, respectively. In Regions 1 and 2, the system is non-hyperbolic since the eigenvalues have the forms $(\pm\beta_1 i, \pm\beta_2 i)$ and $(\pm\alpha, \pm\beta i)$, respectively.

  Systems corresponding to different regions are qualitatively distinct. They are bounded by four curves $C_1 - C_4$.

- **Codimension 1 case**: If a and b satisfy an extra algebraic condition then the corresponding system is of codimension 1. In (a, b) plane this

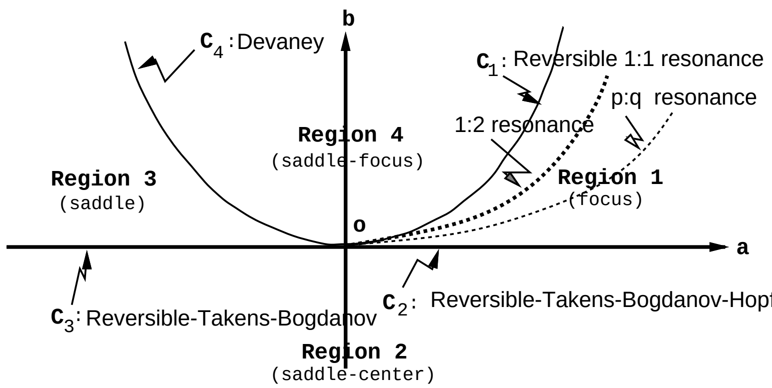

Figure 2: Characteristic of the eigenvalues

extra condition generally gives a rise to a curve. For example, the four boundaries separating the regions are typical codimension-1 curves.

It is worthy pointing out that in Region 1 there are several most interesting codimension-1 curves. Besides C_1 which is traditionally called (1:1) resonance curve, there are other important curves, among them are $(1:N)$ resonances, $N = 2, 3, \ldots$, and subharmonic resonances $(p:q)$. One can see that if $b = \frac{p^2q^2}{(p^2+q^2)^2}a^2$ then $(p:q)$ resonance occurs. Notice also that the curves C_2 and C_3 correspond to vector fields of codimension 1.

- **Codimension 2 case**: This means that two constrains are imposed on vector fields. For example, vector fields having 0 as quadruple eigenvalues and having a 4-dimensional Jordan block are of codimension 2. Vector fields having (1:1) resonant but with vanishing nilpotent part is of at least codimension 2.

3 Normal Forms

The normal form of Theorem A is obtained by the standard Lie algebraic methods. Given a φ reversible vector field X, the first step of normalization is to put X in the resonant normal form by the Poincare-Dulac theorem:

$$\begin{cases} \dot{x}_1 = i\omega x_1 + y_1 + x_1 f_1 + y_1 g_1 \\ \dot{x}_2 = -i\omega x_2 + y_2 + x_2 f_2 + y_2 g_2 \\ \dot{y}_1 = i\omega y_1 + y_1 f_3 + x_1 g_3 \\ \dot{y}_2 = -i\omega y_2 + y_2 f_4 + x_2 g_4 \end{cases} \tag{3.9}$$

where f_j and g_j are functions of $(x_1x_2, x_1y_2, x_2y_1, y_1y_2)$, $f_j(0) = g_j(0) = 0$, $j = 1, \dots, 4$.

The reversibility of X implies that the following relations hold.

$$x_2 f_2 + y_2 g_2 = -x_2 f_1 + y_2 g_1, \quad y_2 f_4 + x_2 g_4 = -y_2 f_3 + x_2 g_1. \tag{3.10}$$

On the other hand, due to the existence of nilpotent part in the linear part of X, the Belitskii theorem is applicable (see [2]), that is, more terms in the functions f_j and g_j can be removed. More precisely, we have

$$\begin{aligned} x_1(\varphi_j)_{y_1} + x_2(\varphi_j)_{y_2} &= 0,\ j = 1, 2 \\ x_1(\varphi_j)_{y_1} + x_2(\varphi_j)_{y_2} &= \varphi_{j-2},\ j = 3, 4, \end{aligned} \tag{3.11}$$

where $\varphi_j = x_j f_j + y_j g_j$ for $j = 1, 2$, and $\varphi_j = y_{j-2} f_j + x_{j-2} g_j$ for $j = 3, 4$.

To draw the general forms of f_j and g_j satisfying (3.10) and (3.11), we notice that $u = x_1x_2$, $v = x_1y_2 - x_2y_1$ satisfy (3.11_1). Therefore we assume that φ_i has the general form $\varphi_i = x_i F_i(u, v) + y_i G_i(u, v)$, $i = 1, 2$. Because of (3.10_1) we have $G_i = 0$. As a result, $\varphi_i = x_i F_i(u, v)$. Again due to the relation (3.10_1), we have $F_1 = -F_2$.

From (3.11_2) we have $\varphi_3 = y_1 F_1(u, v) + x_1 G_3(u, v)$ and $\varphi_4 = -y_2 F_1(u, v) + x_2 G_4(u, v)$. The relation (3.10_2), however, implies $G_4 = G_3$.

Collecting the above facts, we arrive at the normal form

$$\begin{cases} \dot{x}_1 = i\omega x_1 + y_1 + x_1 f(x_1x_2, x_1y_2 - x_2y_1) \\ \dot{x}_2 = -i\omega x_2 + y_2 - x_2 f(x_1x_2, x_1y_2 - x_2y_1) \\ \dot{y}_1 = i\omega y_1 + y_1 f(x_1x_2, x_1y_2 - x_2y_1) + x_1 g(x_1x_2, x_1y_2 - x_2y_1) \\ \dot{y}_2 = -i\omega y_2 - y_2 f(x_1x_2, x_1y_2 - x_2y_1) + x_2 g(x_1x_2, x_1y_2 - x_2y_1). \end{cases} \tag{3.12}$$

The next step of normalization is to work on the nonlinear part of X. We shall prove that in (3.12) all the following monomials are removable:

$$
\begin{array}{lllll}
zu^k\frac{\partial}{\partial z}, & \ldots, & zu^2v^{k-2}\frac{\partial}{\partial z} & & \\
x_iu^k\frac{\partial}{\partial y_i}, & \ldots, & x_iu^2v^{k-2}\frac{\partial}{\partial y_i}, & x_iuv^{k-1}\frac{\partial}{\partial y_i}, & x_iv^k\frac{\partial}{\partial y_i},
\end{array}
\tag{3.13}
$$

where z runs over x_1, x_2, y_1, y_2, $i = 1, 2$, and $k = 2, 3, \ldots$, $u = x_1x_2$ and $v = x_1y_2 - x_2y_1$.

We are seeking for the existence of a series of homogeneous transformations $\Xi: \ x \to x + \xi(x)$, where $\xi(x)$ contains homogeneous resonant monomials. Moreover, we request that Ξ respects the reversibility of X, in other words, we take Ξ from the centralizer of R: $\{\xi : \xi \circ R = R\,\xi\}$.

To eliminate the terms of degree $(2k+1)$ specified in (3.13), take the transformation $\Xi: \ x \to x + \xi(x)$, where

$$
\xi = \sum_{m+n=k-1} \begin{pmatrix} \alpha_{m,n}x_1 \\ \alpha_{m,n}x_2 \\ \alpha_{m,n}y_1 + \beta_{m,n}x_1 \\ \alpha_{m,n}y_2 - \beta_{m,n}x_2 \end{pmatrix} u^m v^n, \tag{3.14}
$$

and consider the Lie bracket $[j^3X, \xi]$. Then the elimination of terms in (3.13) is equivalent to the solvability, in term of α and β, of the homologic system

$$
\begin{aligned}
[j^3X, \xi] &= \textstyle\sum_{m+n=k, m\geq 2, n\geq 0} \mu_{m,n}(x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2})u^m v^n \\
&\quad + \textstyle\sum_{m+n=k, m,n\geq 0} \nu_{m,n}(x_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial y_2})u^m v^n
\end{aligned}
\tag{3.15}
$$

for any numbers $(\mu_{k,0}, \ldots, \mu_{2,k-2})$ and $(\nu_{k,0}, \ldots, \nu_{0,k})$.

Denote in (3.12) $j^1f(u,v) = a_1u + b_1v$, $j^1g(u,v) = cu + dv$. Doing a little calculation, one sees that the solvability of the equation (3.15) is equivalent to the solvability of the linear system

$$
\begin{cases}
b_1(\alpha_{i,k-1-i} - \beta_{i-1,k-i}) + a_1\alpha_{i-1,k-i} = \mu_{i,k-i} & \text{for } i = 2, \ldots k \\
d(\alpha_{i,k-1-i} - \beta_{i-1,k-i}) + c\alpha_{i-1,k-i} = \nu_{i,k-i} & \text{for } i = 0, 1, 2, \ldots k.
\end{cases}
\tag{3.16}
$$

This is a system of $2k$ equations of $(\alpha_{k-1,0}, \ldots, \alpha_{0,k-1})$ and $(\beta_{k-1,0}, \ldots, \beta_{0,k-1})$ and its solvability is equivalent to the full rank of the coefficient matrix. With a straightforward calculation one sees that the latter is true if and only if

$$
d \neq 0, \quad a_1d - b_1c \neq 0. \tag{3.17}
$$

Therefore under these conditions we can reduce X to the following normal form:

$$\begin{cases} \dot{x}_1 = i\omega x_1 + y_1 + x_1 \sum_{k=1}^{\infty}(a_k u + b_k v)v^{k-1} \\ \dot{x}_2 = -i\omega x_2 + y_2 - x_2 \sum_{k=1}^{\infty}(a_k u + b_k v)v^{k-1} \\ \dot{y}_1 = i\omega y_1 + y_1 \sum_{k=1}^{\infty}(a_k u + b_k v)v^{k-1} + x_1(cu + dv) \\ \dot{y}_2 = -i\omega y_2 - y_2 \sum_{k=1}^{\infty}(a_k u + b_k v)v^{k-1} + x_2(cu + dv) \end{cases} \tag{3.18}$$

The final step of normalization is to put (3.18) in the real form. This can be done by a linear change of coordinates $\tilde{x}_1 = x_1 + ix_2$, $\tilde{x}_2 = ix_1 + x_2$, $\tilde{y}_1 = y_1 + iy_2$, $\tilde{y}_2 = iy_1 + y_2$.

It is clear that all the normalizing transformations are within the centralizer of the involution φ.

4 Invariant Varieties

Any (1:1) reversible system X having normal form (1.7) is completely integrable. It has the following two first integrals:

$$I_1 = v, \quad I_2 = y_1^2 + y_2^2 - \frac{c}{2}r^4 - dr^2v - \beta\mu r^2. \tag{4.19}$$

Consequently, any orbit γ passing through an initial point $p_0 = (x_1^0, x_2^0, y_1^0, y_2^0)$ always lies in the intersection of two 3-dimensional manifolds, $\bigcap_{i=1}^{2}\{p \in \mathbb{R}^4 : I_i(p) = I_i(p_0)\}$. In the present section, we shall discuss the geometric properties of this intersection set according to all the possible values of the parameters c, μ and I_1 and I_2. First we define

$$\begin{aligned} \mathcal{F} : \mathbb{R}^4, 0 &\to \mathbb{R}^2, 0 \\ (x_1, x_2, y_1, y_2) &\mapsto (I_1, I_2), \end{aligned} \tag{4.20}$$

and denote

$$M_{c_1,c_2} = \{(x, y) : \mathcal{F}(x, y) = (c_1, c_2)\}. \tag{4.21}$$

$$S_{c_1,c_2} = \{(x, y) : \mathcal{F}(x, y) = (c_1, c_2),\ (c_1, c_2) \textit{ a critical value}\} \tag{4.22}$$

$$R_{c_1,c_2} = \{(x, y) : \mathcal{F}(x, y) = (c_1, c_2),\ (c_1, c_2) \textit{ a regular value}\} \tag{4.23}$$

Then S_{c_1,c_2} and R_{c_1,c_2} are X-invariant sets, and the latter is a 2-dimensional differentiable manifold.

4.1 The critical values

We shall in what follows consider the relation between the mapping $\mathcal{F}$ and the sets $S(\mathcal{F})$, $R(\mathcal{F})$, and $M(\mathcal{F})$. First we prove the following

Lemma 4.1 *Any given pair (I_1, I_2) is the critical value of $\mathcal{F}$ if and only if the following relations hold:*

$$\begin{cases} \rho^2 + r^2(cr^2 + dI_1 + \beta\mu) = 0 \\ \rho^2 - r^2(\frac{c}{2}r^2 + dI_1 + \beta\mu) = I_2 \\ r\rho = \pm I_1, \end{cases} \tag{4.24}$$

where $r^2 = x_1^2 + x_2^2$, $\rho^2 = y_1^2 + y_2^2$.

Proof. To prove the lemma we only need to find the conditions on (I_1, I_2) such that the preimage set $M(\mathcal{F})$ contains singular points only. According to the definition of $S_{I_1,I_2}(\mathcal{F})$, if $p \in S_{I_1,I_2}(\mathcal{F})$, then the rank of the matrix $\frac{\partial(I_1,I_2)}{\partial(x_1,x_2,y_1,y_2)}|_p$ is less than 2. With some calculation, one can see that it is equivalent to the following equalities

$$\begin{cases} x_1y_1 + x_2y_2 = 0 \\ y_1^2 + x_2^2(c(x_1^2 + x_2^2) + dI_1 + \beta\mu) = 0 \\ y_2^2 + x_1^2(c(x_1^2 + x_2^2) + dI_1 + \beta\mu) = 0. \end{cases} \tag{4.25}$$

Since $p \in S_{I_1,I_2}(\mathcal{F})$, therefore at the same time we have

$$\begin{cases} x_1y_2 - x_2y_1 = I_1 \\ y_1^2 + y_2^2 - \frac{c}{2}(x_1^2 + x_2^2)^2 - (I_1d + \beta\mu)(x_1^2 + x_2^2) = I_2. \end{cases} \tag{4.26}$$

Introduce the bi-polar coordinates

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1,\ y_1 = \rho\cos\theta_2,\ y_2 = \rho\sin\theta_2, \tag{4.27}$$

where $\rho \geq 0, r \geq 0$ and $0 \leq \theta_1, \theta_2 \leq 2\pi$. Then (4.25_1) is equivalent to $r\rho\cos(\theta_1 - \theta_2) = 0$ and $I_1 = r\rho\sin(\theta_1 - \theta_2)$. If $I_1 \neq 0$ then $r\rho \neq 0$, and it follows that to have singular points one has $\cos(\theta_1 - \theta_2) = 0$. Therefore $I_1 = \pm r\rho$. If $I_1 = 0$ then from $r\rho\sin(\theta_1 - \theta_2) = 0$ and $r\rho\cos(\theta_1 - \theta_2) = 0$, one has $r\rho = 0$. This also leads to $I_1 = \pm r\rho = 0$. On the other hand, if (4.24) holds, then it is clear that there exists singular set. Now the validity of the lemma follows from the above facts. □

Lemma 4.1 in fact says that if (I_1, I_2) satisfy (4.24) then $M_{I_1,I_2} = S_{I_1,I_2}$, otherwise, $M_{I_1,I_2} = R_{I_1,I_2}$. On the other hand it is not hard to see that the points (I_1, I_2) satisfying (4.24) form a curve C passing through the origin in the (I_1, I_2) plane. Therefore for generic (I_1, I_2), M_{I_1,I_2} is a 2-dimensional manifold.

4.2 The regular set

¿From above discussion, we know that for generic (c_1, c_2) the preimage set $M_{(c_1,c_2)}$ gives a 2-dimensional manifold. Below we shall consider this in more details according to possible combinations of the parameters in the normal form.

Case (1) $c < 0$. For simplicity we take $c = -1$ since this can be done by a suitable scaling. Then generically $M_{(c_1,c_2)}$ is diffeomorphic to a regular 2-dimensional torus. This fact comes from the relation $\rho^2 + \frac{1}{2}(r^2 - c_1 d - \beta\mu)^2 = c_2 + \frac{(c_1 d+\beta\mu)^2}{2}$ which gives a compact surface. By the Poincare-Hopf index theorem (see [6]) to the system (1.7), we know that the set $M_{(c_1,c_2)}$ defines a regular torus.

Case (2) $c > 0$. Under a scaling we can take $c = 1$. In this case, we shall show that generically $M_{(c_1,c_2)}$ can be a torus or a cylinder, depending on the values of (c_1, c_2). To show so, we distinguish two cases: $I_1 = 0$ and $I_1 \neq 0$.

If $I_1 = 0$ then in bi-polar coordinates this means that $r\rho\sin(\theta_1 - \theta_2) = 0$, $\rho^2 - \frac{1}{2}r^4 - \beta\mu r^2 = I_2$. Therefore if $\theta_1 = \theta_2$ and $\rho^2 + r^4 + \beta\mu r^2 \neq 0$ then M_{0,c_2} is a regular cylinder.

If $I_1 \neq 0$, then M_{c_1,c_2} defined by (4.21) gives a 2-dimensional compact manifold which consists of one to three pieces of tori.

In fact, given (c_1, c_2), we consider below the surface M_{c_1,c_2} defined by

$$r^2\rho^2 = \frac{c_1^2}{\sin^2(\theta_1 - \theta_2)}, \quad \rho^2 - \frac{1}{2}r^4 - (dc_1 + \beta\mu)r^2 = c_2. \tag{4.28}$$

In (r^2, ρ^2) plane, these equations give two curves, and in the first quadrant they intersect at least one point and at most three points. In particular, if

$$\frac{3}{2}r^4 + 2(dc_1 + \beta\mu)r^2 + c_2 = 0 \tag{4.29}$$

then (4.28) has one solution of multiplicity 2. If in addition of (4.29) the following relation holds

$$3r^2 + 2(dc_1 + \beta\mu) = 0 \tag{4.30}$$

then (4.28) has a root of multiplicity of 3.

If (4.28) has one real root (three different roots, resp.) then M_{c_1,c_2} consists of a torus (three tori, resp.). This is because that in this case (4.29) does not hold and therefore according to Lemma 4.1 M_{c_1,c_2} is a regular compact manifold. By the Poincare-Hopf index theorem we know it is a torus (three tori, resp.).

Notice that even if (4.29) or (4.30) hold but $\cos(\theta_1 - \theta_2) \neq 0$, M_{c_1,c_2} is also a regular compact manifold. This follows from $\sin(\theta_1 - \theta_2) \neq \pm 1$ and the violation of the last relation of (4.24). Thus M_{c_1,c_2} contains two or three tori, respectively.

If $\cos(\theta_1-\theta_2)=0$ and $\frac{3}{2}r^4+2(dc_1+\beta\mu)r^2+c_2=0$ then M_{c_1,c_2} contains singular points of the mapping $\mathcal{F}$. We shall consider this case in the next subsection.

4.3 The singular set and the periodic orbits

Lemma 4.1 can be used to determine the existence of symmetric periodic orbits. In fact, from the above discussion we know that any (I_1,I_2) satisfying (4.24) implies the relation $M_{I_1,I_2}=S_{I_1,I_2}$. To see the existence of symmetric periodic orbits, we parameterize the singular set S_{I_1,I_2}. One can check that it contains exactly those points $(r\cos\tau, r\sin\tau, \pm\rho\sin\tau, \mp\rho\cos\tau)$, where r and ρ satisfy (4.24), and τ is a real parameter. Now it is straightforward to prove the following result. The proof is omitted.

Proposition 4.1 *For any given critical value (c_1,c_2) system (1.7) has two 2-parameter families of periodic orbits:*

$$(r\cos(\sigma t+\theta_0), -r\sin(\sigma t+\theta_0), -\rho sin(\sigma t+\theta_0), -\rho\cos(\sigma t+\theta_0)) \tag{4.31}$$

and

$$(r\cos(\tilde{\sigma} t+\theta_0), -r\sin(\tilde{\sigma} t+\theta_0), \rho sin(\tilde{\sigma} t+\theta_0), \rho\cos(\tilde{\sigma} t+\theta_0)) \tag{4.32}$$

where θ is a parameter, r and ρ are parameters satisfying (4.24), and

$$\sigma=\omega+\frac{\rho}{r}+f(\mu,r^2,I_1), \quad \tilde{\sigma}=\omega-\frac{\rho}{r}+f(\mu,r^2,I_1). \tag{4.33}$$

It is easy to show that as the initial conditions approach the origin the periodic orbits shrink to 0 and the periods tend to $2\pi/\omega$. In fact, the first point follows from the following observation: as $(c_1,c_2)\to(0,0)$ along C, we have $r\to 0, \rho\to 0$, due to (4.24). Consequently, the orbits shrink to the origin due to the form (4.31). To prove the periods tend to $2\pi/\omega$, one needs only to show that both σ and $\tilde{\sigma}$ tend to ω as $(c_1,c_2)\to(0,0)$ along C. Doing a little calculation, one has

$$\frac{\rho^2}{r^2}=\frac{-(c_1d+\beta\mu)\pm\sqrt{4(c_1d+\beta\mu)^2-6cc_2}}{3}.$$

As a result, one has $\sigma\to\omega$ and $\tilde{\sigma}\to\omega$.

Below we consider two cases $c < 0$ and $c > 0$.

Case $c = -1$. Assume that the genericity conditions of (I_1, I_2) are violated. In the case $\beta\mu > 0$, this happens when $I_1 = 0$ and $I_2 = -\frac{(\beta\mu)^2}{2}$. It is easy to see that the singular points consist of $(\sqrt{\beta\mu}\cos\theta_1, \sqrt{\beta\mu}\sin\theta_1, 0, 0)$, which gives a closed curve. Since it is invariant therefore it is a periodic orbit. Moreover, for any fixed μ, these orbits do not approach the equilibrium, because the singular points exist only when $r^2 - \beta\mu \geq 0$. For more general I_1, if $M_{(I_1,I_2)}$ consists of singular points then necessarily $\cos(\theta_1 - \theta_2) = 0$. This means that in (I_1, I_2) plane there is a curve C such that any orbit with starting point on C is symmetric and periodic and has the form (4.31) and (4.32).

In the case $\beta\mu < 0$, by the same arguments one sees that if the genericity conditions are violated then in the neighborhood of the origin there also exist periodic solutions.

Case $c = 1$. Assume that $\beta\mu < 0$. If $I_1 \neq 0$, $\cos(\theta_1 - \theta_2) = 0$ and $\frac{3}{2}r^4 + 2(dc_1 + \beta\mu)r^2 + c_2 = 0$ then M_{c_1,c_2} contains singular points of the mapping $\mathcal{F}$. Therefore in (I_1, I_2) plane such pairs of (c_1, c_2) give a curve C. More precisely, This gives a piecewisely smooth closed curve C. Any orbit with starting point on C is symmetric and periodic.

If $I_1 = 0$ then one can show that there exist the following non-trivial symmetric periodic solutions

$$(\sqrt{-\beta\mu}\cos(\sigma t + \theta_0), -\sqrt{-\beta\mu}\sin(\sigma t + \theta_0), 0, 0)$$

where $\sigma = 1 + \alpha\mu$.

Finally in the case $c = 1$ and $\beta\mu > 0$ then M_{c_1,c_2} is always regular and consequently the dynamics on M_{c_1,c_2} is quasi-periodic.

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