Stationary Solutions for Generalized Boussinesq Models in Exterior Domains*

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Abstract

We establish the existence of a weak solution of a generalized Boussinesq model for thermally driven convection in exterior domains.

1 Introduction

We study stationary problem for the equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximations by assuming that viscosity and heat conductivity are temperature dependent in an exterior domain $\Omega \subset \mathbb{R}^3$. The equations are

$$\begin{aligned} -div(\nu(T)\nabla u) + u \cdot \nabla u - \alpha T g + \nabla p &= 0 \\ div u &= 0 \\ -div(\kappa(T)\nabla T) + u \cdot \nabla T &= 0. \end{aligned} \tag{1}$$

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Here $u(x) \in \mathbb{R}^3$ denotes the velocity of the fluid at a point $x \in \Omega$; $p(x) \in \mathbb{R}$ is the hydrostatic pressure; $T(x) \in \mathbb{R}$ is the temperature; $g(x)$ is the external force by unit of mass; $\nu(\cdot) > 0$ and $\kappa(\cdot) > 0$ are kinematic viscosity and thermal conductivity, respectively; and α is a positive constant associated to the coefficient of volume expansion. Without loss of generality, we have taken the reference temperature as zero. For a derivation of the above equations, see, for instance, Drazin and Reid [**?**].

The expressions $\nabla, \triangle$, and *div* denote the Gradient, Laplace, and Divergence operators, respectively, also we denote the Gradient by *grad*; the ith component of $u \cdot \nabla u$ is given by $(u \cdot \nabla u)_i = \sum_{j=1}^{3} u_j(\partial u_i/\partial x_i)$; $u \cdot \nabla T = \sum_{j=1}^{3} u_i(\partial T/\partial x_j)$.

The boundary and at infinity conditions are

$$u\,|_\Gamma = 0 \qquad \text{and} \qquad T\,|_\Gamma = T_0 > 0 \tag{2}$$

$$\lim_{|x|\to\infty} u(x) = 0 \qquad \text{and} \qquad \lim_{|x|\to\infty} T(x) = 0 \tag{3}$$

where Γ is the boundary of Ω.

The problem (1.1) was considered by Lorca and Boldrini[8] in a bounded domain with Dirichlet's conditions; while the reduced model, where ν and κ are positive constants, was studied by Morimoto[10](in a bounded domain) and recently by Oeda[11](in an exterior domain).

The evolution problem corresponding to (1.1) was analized by Lorca and Boldrini[9] in a bounded domain; when ν and κ are positive constants was discussed by many authors, see for instance, Korenev[6], Rojas-Medar and Lorca[14], [15](in a bounded domain) and Hishida[5], Oeda[12],[13] (in a exterior domain). In another publication we will study the evolution problem corresponding to (1.1).

2 Functions spaces and preliminaries

The functions in this paper are either $\mathbb{R}$ or $\mathbb{R}^3$-values and we will not distinguish these two situations in our notations. To which case we refer to will be clear from the context.

Now, we given the precise definition of the exterior domain Ω where our boundary-value problem associated to the problem (1)-(3) has been formulated.

Let K a subset compact of $\mathbb{R}^3$ whose boundary ∂K os of class C^2. The exterior domain Ω that we will consider is $\Omega = K^c$ and $\Gamma = \partial\Omega = \partial K$.

The extending domain method was introduced by Ladyzhenskaya [7] to study the Navier-Stokes equations in unbounded domain. As was observed by Heywood [3] the method is useful in certain class of unbouded domain, in this class certainly our domain is. The basic idea is the following: The exterior domain Ω can be approximated by interior domains $\Omega_m = B_m \cap \Omega$, where B_m is a ball with radius m and centre at 0, as $m \to \infty$.

In each interior domain Ω_m, we will prove the existence of weak solution, by using the Galerkin method together with the Brouwer's fixed point theorem as in Heywood [3]. Next, by using the estimates given in Ladyzhenskaya's book's[7] together with diagonal argument and Rellich's compactness theorem, we obtain the desirable weak solution to problem (1.1)-(1.3).

We use several function spaces. D denote Ω or Ω_m.

$$\begin{aligned}
W^{r,p}(D) &= \{u; D^\alpha u \in L^p(D), |\alpha| \le r\} \\
W_0^{r,p}(D) &= \text{ completion of } C_0^\infty(D) \text{ in } W^{r,p}(D) \\
C_{0,\sigma}^\infty(D) &= \{\varphi \in C_0^\infty(D); div\varphi = 0\} \\
J(D) &= \text{ completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\nabla\phi\| \\
H(D) &= \text{ completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\phi\|.
\end{aligned}$$

Here $\|\cdot\|$ is the L^2-norm; the L^p-norm we denoted by $\|\cdot\|_p$.

We note that $J(D)$ can be characterized as

$$J(D) = \left\{\phi \in W^{1,2}(D); \phi\,|_\Gamma = 0, div\phi = 0\right\}$$

as was proved by Heywood [3].

When $p = 2$, as it usual, we denoted $W^{r,p}(D) \equiv H^r(D)$ and $W_0^{r,p}(D) \equiv H_0^r(D)$.

We make use of some inequalities. Constants which appear in those inequalities depend only on the dimension and they are independent of domain (see, cap. I of [7]).

Lemma 1 *Suppose the space dimension is 3 and D is bounded or unbounded. Then* *(a) For* $u \in W_0^{1,2}(D)$ *(or* $J(D)$ *or* $H_0^1(D)$*), we have*

$$\|u\|_{L^6(D)} \leq C_L \|\nabla u\|_{L^2(D)}$$

where $C_L = (48)^{1/6}$.
(b) (Hölder's inequality). If each integral makes sense. Then we have

$$|((u \cdot \nabla)v, w)| \leq 3^{\frac{1}{p}+\frac{1}{r}} \|u\|_{L^p(D)} \|\nabla v\|_{L^q(D)} \|w\|_{L^r(D)}$$

where $p, q, r > 0$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

The following assumptions will be needed throughout the paper.
(S1) $w_0 \subset K$ (w_0 is a neighborhood of the origin 0) and $K \subseteq B = B(0, d)$ which is a ball with radius d and center at 0.
(S2) $\partial\Omega = \Gamma = \partial K \in C^2$.
(S3) $g(x)$ is a bounded and continuous vector function in $\mathbb{R}^3 \backslash w_0$. Moreover $g \in L^p(\Omega)$ for $p \geq 6/5$.

We suppose that the functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfy

$$\begin{aligned} 0 &< \nu_0(T_0) \leq \nu(\tau) \leq \nu_1(T_0) \\ 0 &< \kappa_0(T_0) \leq \kappa(\tau) \leq \kappa_1(T_0) \end{aligned}$$

for all $\tau \in \mathbb{R}$, where $\nu_0(T_0) = \inf\{\nu(t); |t| \leq \sup_{\partial\Omega} |T_0|\}/2, \nu_1(T_0) = \sup\{\nu(t); |t| \leq \sup_{\partial\Omega} |T_0|\}$ with analogous definitions for $\kappa_0(T_0)$ and $\kappa_1(T_0)$, and ν, κ, are continuous functions.

Here, in order to transform the boundary condition on T to a homogeneous one, we introduce an auxiliary function S (see Gilbarg and Trudinger [2] p. 137).

Lemma 2 *There exists a function* S *which satisfies the following properties (i)* $S(\Gamma) = T_0$. *(ii)* $S \in C_0^2(\mathbb{R}^3)$. *(iii) for any* $\epsilon > 0$ *and* $p \geq 1$, *we can retake* S, *if necessary, such that* $\|S\|_{L^p} < \epsilon$.

Now we make a change of variable: $\varphi = T - S$ to obtain

$$\begin{aligned} -div(\nu(\varphi + S)\nabla u) + u \cdot \nabla u - \alpha\varphi g - \alpha S g + \nabla p &= 0 \\ div\ u &= 0 \qquad (4) \\ -div(\kappa(\varphi + S)\nabla\varphi) + u \cdot \nabla\varphi - div(\kappa(\varphi + S)\nabla S) + u \cdot \nabla S &= 0 \text{ in } \Omega \end{aligned}$$

$$u = 0 \qquad \text{and} \qquad \varphi = 0 \qquad \text{on } \partial\Omega \tag{5}$$

$$\lim_{|x|\to\infty} u(x) = 0 \qquad ; \qquad \lim_{|x|\to\infty} \varphi(x) = 0 \tag{6}$$

Definition 2.1 $(u, \varphi) \in J(\Omega) \times H_0^1(\Omega)$ is called a stationary weak solution of (4)-(6) if it satisfies

$$\begin{array}{rcl} (\nu(\varphi + S)\nabla u, \nabla v) + B(u, u, v) - \alpha(\varphi g, v) - \alpha(Sg, v) & = & 0 \\ \text{for all } v & \in & J(\Omega) \\ (\kappa(\varphi + S)\nabla\varphi, \nabla\psi) + b(u, \varphi, \psi) + (\kappa(\varphi + S)\nabla S, \nabla\psi) + b(u, S, \psi) & = & 0 \\ \text{for all } \psi & \in & H_0^1(\Omega). \end{array} \tag{7}$$

Where $B(u, v, w) = (u \cdot \nabla v, w) = \int\limits_{\Omega} \sum_{i,j=1}^{3} u_j(x)(\partial v_i/\partial x_j)(x) w_i(x) dx$ and

$$b(u, \varphi, \psi) = (u \cdot \nabla\varphi, \psi) = \int\limits_{\Omega} \sum_{i,j=1}^{3} u_j(x)(\partial\varphi_i/\partial x_j)(x)\psi_i(x) dx.$$

Theorem 1 *(existence) Suppose assumptions (S1), (S2) and (S3) are satisfied. Then a stationary weak solution of (7) exists.*

3 Auxiliar problem.

According to the approach of the extending domain method, we first present a lemma which ensures the existence of weak solutions of interior problems (P_m) in domains $\Omega_m = B_m \cap \Omega$. The interior problem (P_m) is as follows:

$$(P_m) \left\{ \begin{array}{c} -div(\nu(\varphi + S)\nabla u) + u \cdot \nabla u - \alpha\varphi g - \alpha S g + \nabla p = 0 \qquad \text{in } \Omega_m \\ div\ u = 0 \\ -div(\kappa(\varphi + S)\nabla\varphi) + u \cdot \nabla\varphi - div(\kappa(\varphi + S)\nabla S) + u \cdot \nabla S = 0 \qquad \text{in } \Omega_m \\ u = 0,\ \varphi = 0 \text{ on } \partial\Omega_m = \partial\Omega \cap \partial B_m \end{array} \right.$$

Definition 2.2 $(u,\varphi) \in J(\Omega_m) \times H^1_0(\Omega_m)$ is called a stationary weak solution for (P_m) if it satisfies the following

$$\begin{aligned} (\nu(\varphi+S)\nabla u, \nabla v) + B(u,u,v) - \alpha(\varphi g, v) - \alpha(Sg, v) &= 0 \\ \text{for all } v &\in J(\Omega_m) \end{aligned} \tag{8}$$

$$\begin{aligned} (\kappa(\varphi+S)\nabla\varphi, \nabla\psi) + b(u,\varphi,\psi) + (\kappa(\varphi+S)\nabla S, \nabla\psi) + b(u,S,\psi) &= 0 \\ \text{for all } \psi &\in H^1_0(\Omega_m). \end{aligned}$$

Lemma 3 *Let assumptions (S1), (S2), and (S3) be satisfied. Then we can construct a weak solution* $(\overline{u}^m, \overline{\varphi}^m)$ *of* (P_m).

Proof Let m be arbitrary fixed. Let $\{v_j\}_{j=1}^{\infty} \subset J(\Omega_m)$ and $\{\psi_j\}_{j=1}^{\infty} \subset H^1_0(\Omega_m)$ be a sequence of functions, linearly independent and total in $J(\Omega_m)$ and $H^1_0(\Omega_m)$ respectively.

Since Ω_m is bounded, we can take them such that

$$(\nabla v_j, \nabla v_k) = \delta_{ik} \qquad ; \qquad (\nabla\psi_j, \nabla\psi_k) = \delta_{jk}$$

$$u^n(x) = \sum_{k=1}^{n} c_{n,k} v_k(x) \qquad ; \qquad \varphi^n(x) = \sum_{k=1}^{n} d_{n,k}\psi_k(x).$$

Then we consider the next system of equations:

$$\begin{aligned} &(\nu(\varphi^n+S)\nabla u^n, \nabla v_j) + B(u^n,u^n,v_j) - \alpha(\varphi^n g, v_j) - \alpha(Sg, v_j) = 0 \\ &(\kappa(\varphi^n+S)\nabla\varphi^n, \nabla\psi_j) + b(u^n,\varphi^n,\psi_j) + (\kappa(\varphi^n+S)\nabla S, \nabla\psi_j) + b(u^n,S,\psi_j) = 0 \end{aligned} \tag{9}$$

where $1 \le j \le n$. By using the representations of u^n, φ^n, we have

$$\begin{aligned} &\sum_{k=1}^{n} c_k(\nu(\varphi^n+S)\nabla v_k, \nabla v_j) + \sum_{k,l}^{n} c_k d_l B(v_k,v_l,v_j) \\ &- \sum_{k=1}^{n} \alpha d_k(g\psi_k, v_j) - \alpha(Sg, v_j) &= 0 \\ &\sum_{k=1}^{n} d_k(\kappa(\varphi^n+S)\nabla\psi_k, \nabla\psi_j) + \sum_{k,l}^{n} c_k d_l b(v_k,\psi_l,\psi_j) \\ &+(\kappa(\varphi^n+S)\nabla S, \nabla\psi_j) + \sum_{k=1}^{n} c_k b(v_k,S,\psi_j) &= 0 \end{aligned} \tag{10}$$

where $1 \le j \le n$.

We put $(c;d) = (c_1,...,c_n,d_1,...,d_n), P(c;d) = (P_1(c;d),...,P_{2n}(c;d))$. Then, of (10) we obtain

$$\begin{aligned}\sum_{k=1}^{n} c_k \nu_0(T_0)(\nabla v_k, \nabla v_j) &\le \left|\sum_{k,l} c_k d_l B(v_k,v_j,v_l)\right| + \left|\sum_k \alpha d_k (g\psi_k, v_j)\right| + |\alpha(Sg,v_j)| \\ \sum_{k=1}^{n} d_k \kappa_0(T_0)(\nabla \psi_k, \nabla \psi_j) &\le \left|\sum_{k,l} c_k d_l b(v_k,\psi_j,\psi_l)\right| + \kappa_1(T_0)\left|(\nabla S, \nabla\psi_j)\right| \qquad (11) \\ &\quad + \left|\sum_k c_k b(v_k,S,\psi_j)\right|\end{aligned}$$

thus

$$\begin{aligned}P_j(c;d) &\le \frac{1}{\nu_0(T_0)}\left\{\left|\sum_{k,l} c_k d_l B(v_k,v_j,v_l)\right| + \left|\sum_k \alpha d_k(g\psi_k,v_j)\right| + |\alpha(Sg,v_j)|\right\} \qquad (12) \\ P_{n+j}(c;d) &\le \frac{1}{\kappa_0(T_0)}\left\{\left|\sum_{k,l} c_k d_l b(v_k,\psi_j,\psi_l)\right| + \kappa_1(T_0)\left|(\nabla S,\nabla\psi_j)\right| + \left|\sum_k c_k b(v_k,S,\psi_j)\right|\right\}\end{aligned}$$

where $1 \le j \le n$. Then our problem is reduced to obtain a fixed point of $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Now we will use Brouwer's fixed point theorem. Namely, if all possible solutions $(c;d)$ of the equation $(c;d) = \lambda P(c;d)$ for $\lambda \in [0,1]$ stay in a same ball $\|(c;d)\| \le r$, then there exists a fixed point of P.

By multiplying $(11)_i$ (respectively. $(11)_{ii}$) by c_j (respectively. d_j), summing up with respect to j and noting $B(u^n,u^n,u^n) = 0, b(u^n,\varphi^n,\varphi^n) = 0$ we have

$$\begin{aligned}\nu_0(T_0)\sum_{j=1}^{n}|c_j|^2 &= \nu_0(T_0)\left|\nabla u^n\right|^2 = \nu_0(T_0)\lambda\sum_{j=1}^{n} P_j(c;d)c_j \\ &\le \lambda\alpha\left|(g\varphi^n,u^n)\right| + \left|(Sg,u^n)\right| \\ &\le \lambda\alpha\left\{|g|_{3/2}\left|\varphi^n\right|_6\left|u^n\right|_6 + |g|_{3/2}|S|_6\left|u^n\right|_6\right\} \\ &\le \lambda\alpha\left\{|g|_{3/2}\left(\left|\nabla\varphi^n\right| + |S|_6\right)\left|\nabla u^n\right|\right\}\end{aligned}$$

then

$$\left|\nabla u^n\right|^2 \le \frac{\lambda\alpha}{\nu_0(T_o)}|g|_{3/2}\left\{\left|\nabla\varphi^n\right| + |\nabla S|\right\}. \qquad (13)$$

In the same another, we find

$$|\nabla \varphi^n| \leq \frac{\lambda \kappa_1(T_0)}{\kappa_0(T_0)} |\nabla S| + \frac{\lambda}{\kappa_0(T_0)} |\nabla u^n| \, |S|_3 \tag{14}$$

by substituting the equation (14) into the equation (13), we obtain

$$|\nabla u^n| \leq \frac{\lambda \alpha}{\nu_0(T_o)} |g|_{3/2} \left\{ \frac{\lambda \kappa_1(T_0)}{\kappa_0(T_0)} |\nabla S| + \frac{\lambda}{\kappa_0(T_0)} |\nabla u^n| \, |S|_3 \right\} + \frac{\lambda \alpha}{\nu_0(T_o)} |g|_{3/2} |\nabla S|$$

thus,

$$\left(1 - \frac{\lambda^2 \alpha}{\nu_0(T_o)\kappa_0(T_0)} |g|_{3/2} |S|_3 \right) |\nabla u^n| \leq \frac{\lambda \alpha}{\nu_0(T_o)} |g|_{3/2} |\nabla S| \left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1 \right).$$

According to Lemma 2.2, with $p = 3$, we can choose an extension S of T_0 such that

$$\gamma \equiv \frac{\alpha}{\nu_0(T_o)\kappa_0(T_0)} |g|_{3/2} |S|_3 < 1/2$$

then we have

$$|\nabla u^n| \leq \frac{\lambda \alpha}{(1 - \lambda^2 \gamma)\nu_0(T_o)} |g|_{3/2} |\nabla S| \left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1 \right) \tag{15}$$

By substituting the previous inequality in the inequality (14), we obtain

$$|\nabla \varphi^n| \leq \frac{\lambda |\nabla S|}{\kappa_0(T_o)} \left(\kappa_1(T_0) + \frac{\lambda \alpha}{(1 - \lambda^2 \gamma)\nu_0(T_o)} |g|_{3/2} \left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1 \right) |S|_3 \right). \tag{16}$$

Since $0 \leq \lambda \leq 1$ and $\frac{1}{1 - \lambda^2 \gamma} \leq \frac{1}{1 - \gamma}$, we have from (15) and (16)

$$|\nabla u^n| \leq \frac{\alpha}{(1 - \gamma)\nu_0(T_o)} |g|_{3/2} |\nabla S| \left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1 \right) \equiv r_1 \tag{17}$$

$$|\nabla \varphi^n| \leq \frac{|\nabla S|}{\kappa_0(T_o)} \left(\kappa_1(T_0) + \frac{\lambda \alpha}{(1 - \gamma)\nu_0(T_o)} |g|_{3/2} \left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1 \right) |S|_3 \right) \equiv r_2 \tag{18}$$

Thus we have gotten uniform estimates on u^n and φ^n. Indeed, r_1 and r_2 are both independent of λ, n, m. Hence solutions of $(c; d) = \lambda P(c; d)$ for

$\lambda \in [0,1]$ lie in a $\mathbb{R}^{2n}-$ball $\left\{\sum_{j=1}^{n}\left(|c_j|^2+|d_j|^2\right)\le r_1^2+r_2^2\right\}$. Therefore, due to Brouwer's fixed point theorem, we have obtained a solution (u^n,φ^n) of the equations (8) with the property (after getting the fixed point, repeat the same calculation as $\lambda = 1$)

$$|\nabla u^n| \le r_1 \qquad ; \quad |\nabla \varphi^n| \le r_2. \tag{19}$$

Since $J(\Omega_m)$ (respectively. $H_0^1(\Omega_m)$) is compactly imbedded in $H(\Omega_m)$ (respectively. $L^2(\Omega_m)$) we can choose subsequences, which we again denote by (u^n,φ^n), and elements $\overline{u}^m \in J(\Omega_m)$, $\overline{\varphi}^m \in H_0^1(\Omega_m)$ such that $u^n \to \overline{u}^m$ weakly in $J(\Omega_m)$ and strongly in $H(\Omega_m)$ and also $\varphi^n \to \overline{\varphi}^m$ weakly in $H_0^1(\Omega_m)$, and strongly in $L^2(\Omega_m)$ and also everywhere in Ω_m.

Passing to the limit in (10) as $n \to \infty$, we find that $(\overline{u}^m,\overline{\varphi}^m)$ is a desired weak solution of (P_m).

Lemma 4 *Let us $(\overline{u}^m,\overline{\varphi}^m)$ be a weak solution for (P_m) obtained in previous lemma. By put*

$$u^m(x) = \begin{cases} \overline{u}^m(x) & \text{if } x \in \Omega_m \\ 0 & \text{if } x \in \Omega\backslash\Omega_m \end{cases}$$

$$\varphi^m(x) = \begin{cases} \overline{\varphi}^m(x) & \text{if } x \in \Omega_m \\ 0 & \text{if } x \in \Omega\backslash\Omega_m. \end{cases}$$

Then it holds that $(u^m,\varphi^m) \in J(\Omega)\times H_0^1(\Omega)$ and furthermore

$$|\nabla u^m| \le r_1 \quad ; \quad |\nabla \varphi^m| \le r_2 \tag{20}$$

where r_1 and r_2 be taken uniformly in m.

Proof It is easy to show $(u^m,\varphi^m) \in J(\Omega)\times H_0^1(\Omega)$. The estimates (20) are directly deduced from the estimates (19) and the lower semicontinuity of the norm.

4 Proof of Theorem

By using the previous lemma, applying Rellich's compactness theorem and using the diagonal argument, we can choose subsequences which we again denote by (u^m,φ^m) and $u \in J(\Omega), \varphi \in H_0^1(\Omega)$ such that

$$\begin{aligned} u^m &\to u \text{ weakly in } J(\Omega) \text{ and strongly in } L^2_{loc}(\Omega) \\ \varphi^m &\to \varphi \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2_{loc}(\Omega). \end{aligned}$$

Once we get such subsequences and limits, then we can show that (u, φ) become a stationary weak solution of (7). In fact, let us (ξ, ψ) be an arbitrary given test function. Then we find a bounded domain Ω' and a number m_0 such that *supp* ξ, *supp*$\psi \subset \Omega'$ and $\Omega' \subset \Omega_{m_0} \subset \Omega_m$ for all $m \geq m_0$. Then

$$\begin{aligned} & |(\nu(\varphi^m + S)\nabla\xi, \nabla u^m)_\Omega - (\nu(\varphi + S)\nabla\xi, \nabla u)_\Omega| \\ \leq \; & |((\nu(\varphi^m + S) - \nu(\varphi + S))\nabla\xi, \nabla u^m)_{\Omega'}| + |(\nu(\varphi + S)\nabla\xi, \nabla(u^m - u))_{\Omega'}| \\ \leq \; & |\nu(\varphi^m + S) - \nu(\varphi + S)|_\infty |\nabla\xi| \, |\nabla u^m| + |(\nu(\varphi + S)\nabla\xi, \nabla(u^m - u))_{\Omega'}| \end{aligned}$$

because the function ν is continuous and $\varphi^m \to \varphi$ strongly in $L^2_{loc}(\Omega)$, it is now immediate that $\nu(\varphi^m + S)$ converges strongly towards $\nu(\varphi + S)$. This, together with the weak convergence $u^m \to u$ in $J(\Omega)$, yields the following convergence

$$|(\nu(\varphi^m + S)\nabla\xi, \nabla u^m)_\Omega - (\nu(\varphi + S)\nabla\xi, \nabla u)_\Omega| \to 0$$

as $m \to \infty$. The other convergences are analogously established. Thus, we see (u, φ) is a stationary weak solution for (7)

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