Strong Solutions of the Equations for Nonhomogeneous Asymmetric Fluids*

by

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Abstract. We consider an initial boundary value problem for a system of equations describing nonstationary flows of nonhomogeneous incompressible asymmetric (polar) fluids. Under conditions similar to the ones for the usual Navier-Stokes equations, we prove the existence and uniqueness of strong solutions by the use of the spectral semi-Galerkin method. Several estimates for the solution and their approximations are given. These estimates can be used for the derivation of error bounds for the Galerkin approximations.

1. Introduction

In this paper we will study the equations for the motion of a nonhomogeneous viscous incompressible asymmetric fluid. These equations are considered in a bounded domain $\Omega \subset I\!R^3$, with boundary Γ, in a time interval $[0, T_*]$. To describe them let $u(x,t) \in I\!R^3$, $w(x,t) \in I\!R^3$, $\rho(x,t) \in I\!R$ and $p(x,t) \in I\!R$ denote, respectively, the unknown velocity, angular velocity of rotation of the fluid particles, the density and the pressure at a point $x \in \Omega$ and time $t \in [0, T_*]$. Then, the governing equations are

$$(1.1) \qquad \left\{ \begin{array}{l} \rho u_t + \rho(u \cdot \nabla)u - (\mu + \mu_r)\Delta u + \operatorname{grad} p = 2\mu_r \operatorname{rot} w + \rho f, \\ \operatorname{div} u = 0, \\ \rho w_t + \rho(u \cdot \nabla)w - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \operatorname{div} w \\ \qquad\qquad +4\mu_r w = 2\mu_r \operatorname{rot} u + \rho g, \\ \rho_t + (u \cdot \nabla)\rho = 0\ , \end{array} \right.$$

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together with the following boundary and initial conditions

$$\text{(1.2)} \qquad \begin{cases} u = 0, \quad w = 0, \quad \text{on} \quad \Gamma \times (0, T_*), \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad \rho(x,0) = \rho_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

where for simplicity of exposition we have taken homogeneous boundary conditions. Here $f(x,t)$ and $g(x,t)$ are respectively known external sources of linear and angular momentum of particles. The positive constants μ, μ_r, c_0, c_a, c_d characterize isotropic properties of the fluid; μ is the usual Newtonian viscosity; μ_r, c_0, c_a, c_d are new positive viscosities related to the asymmetry of the stress tensor, and in consequence related to the appearance of the field of internal rotation w; these constants satisfy $c_0 + c_d > c_a$. The expressions grad, Δ, div and rot denote the gradient, Laplacian, divergence and rotational operators, respectively (we also denote the gradient by ∇ and $\dfrac{\partial u}{\partial t}$ by u_t); the $\mathrm{i}^{\underline{\mathrm{th}}}$ component of $(u \cdot \nabla)u$ and $(u \cdot \nabla)w$ in cartesian coordinates are given by $[(u \cdot \nabla)u]_i = \displaystyle\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j}$ and $[(u \cdot \nabla)w]_i = \displaystyle\sum_{j=1}^{n} u_j \frac{\partial w_i}{\partial x_j}$ respectively; also $(u \cdot \nabla)\rho = \displaystyle\sum_{j=1}^{n} u_j \frac{\partial \rho}{\partial x_j}$.

For the derivation and physical discussion of equations (1.1) see Petrosyan [17] and Condiff, Dahler [5]. We observe that this model of fluid includes as a particular case the classical Navier-Stokes, which has been much studied (see, for instance, the classical books by Ladyzhenskaya [10] and Temam [20] and the references there in). It also includes the reduced model of the nonhomogeneous Navier-Stokes equations, which has been less studied than the previous case (see for instance Simon [19], Kim [9], Ladyzhenskaya and Solonnikov [11] and Salvi [18]).

Concerning the generalized model of fluids considered in this paper, Lukaszewicz [16] established the local existence of weak solutions for (1.1), (1.2) under certain assumptions by using linearization and an almost fixed point theorem. In that same paper Lukaszewicz remarked about the possibility of proving the existence of strong solutions (under the hyposthesis that the initial density is separated from zero) by using the techniques of [14] and [15] (linearization and fixed point theorems; [14] and [15] assume constant density). The properties of their solution are asserted to be similar to the ones in [14] and [15].

Since we are more interested in techniques more directly related with numerical applications, in this paper we will stablish the existence of strong of (1.1) and (1.2) by using the spectral semi-Galerkin method. We assume more regular initial data than the ones in [16], with initial density separated from zero, because we want to prove the existence of solutions that are stronger that the ones alluded in Lukaszewicz [16]. The reason for this is that the estimates we present in this paper are fundamental for obtaining error bounds for the Galerkin approximations constructed here. This is presented in another paper, [2].

2. Preliminaries and Results

In what follows we will assume Ω to be a bounded domain in $I\!R^n (n = 2 \text{ or } 3)$ of class $C^{1,1}$. The functions in this paper are either $I\!R$ or $I\!R^n$-valued, and sometimes we will not distinguish them in our notations. This being clear from the context. We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{ f \in L^q(D) / \quad \|\partial^\alpha f\|_{L^q(D)} < \infty, \quad (|\alpha| \leq m\}$$

for $m \in I\!N, 1 \leq q \leq \infty,\ D = \Omega$ or $D = \Omega \times (0, T_*), 0 < T_* \leq \infty$, with the usual norm. When $q = 2$, we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(D)$ in $H^m(D)$. If B is a Banach-space, we denote, by $L^q([0, T_*); B)$ the Banach space of B-valued functions defined in the interval $[0, T_*)$ that are L^q-integrables in the sense of Bochner. Let

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &= \{v \in C_0^\infty(\Omega) \,/\, \operatorname{div} v = 0 \ \text{ in } \Omega\}, \\ H &= \text{ closure of } \ C_{0,\sigma}^\infty(\Omega) \ \text{ in } \ (L^2(\Omega))^n, \\ V &= \text{ closure of } \ C_{0,\sigma}^\infty(\Omega) \ \text{ in } (H^1(\Omega))^n. \end{aligned}$$

Let P be the orthogonal projection from $L^2(\Omega)$ onto H obtained by the usual Helmholtz decomposition. Then the operator $A : H \to H$ gives by $A = -P\Delta$, with domain $D(A) = V \cap H^2(\Omega)$ is called the Stokes operator. It is well known that A is a positive definite self-adjoint operator and is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \ \text{ for all } \ w \in D(A),\, v \in V.$$

From now on, we denote the inner product in H (i.e., the L^2-inner product) by (,) with corresponding norm $\|\cdot\|$. The norm for other L^p-spaces will be denoted by $\|\cdot\|_{L^p}$.

The following assumptions on the initial data will hold throughouth this paper.

(A.1) The initial value for the density for belongs to $C^1(\Omega)$ and satisfies
$0 < \alpha \leq \rho_0(x) \leq \beta < +\infty$ in Ω,

(A.2) The initial value u_0 belong to $V \cap (H^2(\Omega))^n$, (A.3) The initial value w_0 belong to $(H_0^1(\Omega))^n \cap (H^2(\Omega))^n$.

Now, by using the properties of P, we can reformulate the problem (1.1)-(1.2) as follows: find $\rho \in C^1(\Omega \times (0,T_*))$ and $u \in C^1([0,T_*);H) \cap C((0,T_*);D(A))$, and $w \in C^1([0,T_*);(H_0^1(\Omega))^n) \cap C((0,T_*);D(B))$ such that

$$
(2.1)\begin{cases}
\rho_t + u\cdot\nabla\rho = 0 \quad \text{for} \quad (x,t)\in\Omega\times(0,T_*),\\[2ex]
(\rho u_t, v) + (\rho u\cdot\nabla u, v) + (\mu+\mu_r)(Au, v)\\
\qquad = 2\mu_r(\operatorname{rot} w, v) + (\rho f, v) \quad \text{for} \quad 0<t<T_*, \ \forall\, v\in V,\\[2ex]
(\rho w_t, \psi) + (\rho u\cdot\nabla w, \psi) - (c_a+c_d)(\Delta w, \psi) - (c_0+c_d-c_a)(\nabla \operatorname{div} w, \psi)\\
\quad +4\mu_r(w,\psi) = 2\mu_r(\operatorname{rot} u, \psi) + (\rho g, \psi) \quad \text{for} \quad 0<t<T_*, \forall \psi \in (H_0^1(\Omega))^n,\\[2ex]
u(x,0)=u_0(x), \quad \rho(x,0)=\rho_0(x,0)=\rho_0(x), \quad w(x,0)=w_0(x).
\end{cases}
$$

Actually, we will prove that the solution is better than what stated above. For this we need to recall some properties of the stokes operator $A=-P\Delta$. If Ω is bounded and $\partial\Omega$ is of class $C^{1,1}$, the mapping $A: V\cap H^2(\Omega)\to H$ is one-to-one and onto (Amrouche and Girault [1], Cattabriga [4] when $\partial\Omega$ is of class C^3). The inverse A^{-1} is complety continuous as a map $A^{-1}: H\to H$. Also, A is symmetric and, therefore, so is its inverse. Being complety continuous and symmetric, the operator A^{-1} posses an orthogonal sequence of eigenfunctions $\{\varphi^k(x)\}$ which is complete in its image $V\cap(H^2(\Omega))^n$. As the image contains $C_{0,\sigma}^\infty(\Omega)$, the eigenfunctions are also complete in H. They are also orthogonal and complete in V since

$$\int \nabla\phi\nabla\varphi^k dx = \lambda_k \int \phi\varphi^k dx$$

holds for $\phi\in V$, it λ_k is the k-th eigenvalue ($A\varphi^k = \lambda_k\varphi^k$). We take $\{\varphi^k(x)\}$ to be sequence of eigenfunctions, orthogonal in H. Therefore, the eigenfunctions $\{\varphi^k(x)/(\lambda_k)^{1/2}\}$ and $\{\varphi^k(x)/\lambda_k\}$ are complete and orthogonal in V (endowed with the inner product $(\nabla u, \nabla v)$, for $u,v\in V$) and $(H^2(\Omega))^n\cap V$ (endowed with the inner product (Au, Av), for $u,v\in D(A)$), respectively.

As well, we have that if $\partial\Omega$ is one $C^{k,m}$-manifold of $\mathbb{R}^n$ ($n=2$ or 3, $m=$), then the eigenfunctions $\varphi^k(x)$ belong to $H(\Omega)$.

In what follows we will also consider either the Laplace operator $B=-\Delta$, or the strongly uniformly elliptic operator

$$\mathcal{L} = -(c_a+c_d)\Delta - (c_0+c_d-c_a)\nabla\operatorname{div},$$

both with Dirichlet boundary conditions. Due to the condition $c_0 + c_d > c_a, \mathcal{L}$ is a positive definite operator. To simplify the notation, we will denote respectively by $\{\psi^k(x)\}$ and $\{\gamma_k\}$ the eigenfunctions and eigenvalues either of B or $\mathcal{L}$. From the context it will be clear in which case we are working with.

Let P_k the projection operator of $L^2(\Omega)$ onto the space V_k spanned by the k-th eigenfunctions $\langle\varphi^1(x), \ldots, \varphi^k(x)\rangle$ of A and let R_k the projection operator of $L^2(\Omega)$ onto the space W_k spanned by the k-th eigenfunctions $\langle\psi^1(x), \ldots, \psi^k(x)\rangle$ of either $B = -\Delta$ or $\mathcal{L}$ according to the context.

Then the solutions of problem (2.1) will can be obtained by using the semi-Galerkin approximation. That is, we consider the Galerkin approximations $u^k(x,t) = \sum^{k} c_{ik}(t)\varphi^i(x)$, $w^k(x,t) = \sum_{i=1}^{k} d_{ik}(t)\psi^k(x)$ for the velocity and rotation of particles, respectively, and an approximation $\rho^k(x,t)$ for the density satisfying the following equations:

$$
(2.2) \qquad \begin{cases} P_k(\rho^k u_t^k + \rho^k u^k \cdot \nabla u^k - \rho^k f - 2\mu_r \operatorname{rot} w^k) + (\mu + \mu_r) A u^k = 0, \\ R_k(\rho^k w_t^k + \rho^k u^k \cdot \nabla w^k - \rho^k g - 2\mu_r \operatorname{rot} u^k + 4\mu_r w^k) + J w^k = 0, \\ \rho_t + u^k \cdot \nabla \rho^k = 0, \\ u^k(0) = P_k u_0, \quad w^k(0) = R_k w_0, \quad \rho^k(0) = \rho_0, \end{cases}
$$

where $Jw^k = (c_a + c_d) B w^k - R_k(c_0 + c_d - c_a)\nabla \mathrm{div} w^k$ if we are working with the Laplace operator (recall that $BR_k = R_k B$ in this case), and $Jw^k = \mathcal{L} w^k$ if, we are working with the $\mathcal{L}$ operator (recall that $\mathcal{L} R_k = R_k \mathcal{L}$ in this case).

Equations (2.2) forms a coupled system of ordinary differential equations with a transport equation. By using the characteristics method for this last equation, it is possible to prove in a standard way that there is an unique solution (u^k, w^k, ρ^k) for (2.2) in an interval $[0, T_k)$, for all $k \in I\!N$. The a priori estimates that will prove will allow-us to take $T > 0$ such that $T \le T_k$ for all $k \in I\!N$. Thus, the approximate solutions (u^k, w^k, p^k) will be considered to be defined in a single interval $[0, T)$ for all $k \in I\!N$. Equations (2.2) are equivalente to the following

$$
(2.3) \qquad \begin{cases} (\rho^k u_t^k + \rho^k u^k \cdot \nabla u^k - \rho^k f - 2\mu_r \text{ rot } w^k, v) + (\mu + \mu_r)(A u^k, v) = 0, \\ (\rho^k w_t^k + \rho^k u^k \cdot \nabla w^k - \rho^k g - 2\mu_r \text{ rot } u^k, \psi) + (J w^k, \psi) = 0, \\ \forall\, v \in V_k, \quad \forall\, \psi \in W_k, \\ \rho_t + u^k \cdot \nabla \rho^k = 0, \\ u^k(0) = P_k u_0, \quad w^k(0) = R_k w_0, \quad \rho^k(0) = \rho_0. \end{cases}
$$

The result in this paper are the following:

Theorem 2.1 Let the initial values satisfy $u_0 \in V \cap (H^2(\Omega))^n$, $w_0 \in (H_0^1(\Omega))^n \cap (H^2(\Omega))^n, \rho_0 \in C^1(\overline{\Omega})$ and the external fields $f, g \in L^2(0, T_*; (H^1(\Omega))^n)$ with $f_t,\ g_t \in L^2(0, T_*; (L^2(\Omega))^n)$. Then, on a (possibly small) time interval $[0, T]$, $T \leq T_*$. Problem (1.1) and (1.2) has a unique strong solution (u, w, ρ). That is, there are functions u, w, ρ such that

$$P(\rho u_t + \rho u \cdot \nabla u - 2\mu_r \operatorname{rot} w - \rho f - (\mu + \mu_r)\Delta u) = 0 \text{ holds a.e. in } \Omega \times [0, T];$$

$$\rho w_t + \rho u \cdot \nabla w - 2\mu_r \operatorname{rot} u - \rho g - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \operatorname{div} w + 4\mu_r w = 0$$

holds a.e. in $\Omega \times [0, T]$;

$$\rho_t + u \cdot \nabla \rho = 0 \text{ holds in the } L^2(\Omega \times [0, T]) \text{ sense. Moreover,}$$

$$\rho \in C^1(\Omega \times [0, T]),$$

$$u \in C([0, T]; (H^2(\Omega))^n \cap V) \cap L^2([0, T]; (L^\infty(\Omega))^n)$$
$$\cap L^p([0, T]; (H^{3-\varepsilon}(\Omega))^n) \cap L^2((0, T]; (H^3(\Omega))^n) \cap L^\infty_{\text{Loc}}([0, T]; (H^3(\Omega))^n),$$

$$u_t \in C([0, T]; H) \cap L^2(0, T; (H^{2-\varepsilon}(\Omega))^n) \cap L^p([0, T]; (H^{1-\varepsilon}(\Omega))^n)$$
$$\cap L^2_{\text{Loc}}([0, T]; (H^2(\Omega))^n) \cap L^\infty_{\text{Loc}}((0, T]; (H^1(\Omega))^n),$$

$$u_{tt} \in L^2_{\text{Loc}}(0, T; H),$$

$$w \in C([0, T]; (H^2(\Omega))^n \cap (H_0^1(\Omega))^n) \cap L^2(0, T; (H^3(\Omega))^n) \cap L^2([0, T]; (L^\infty(\Omega))^n)$$
$$\cap L^p(0, T; (H^{3-\varepsilon}(\Omega))^n) \cap L^\infty_{\text{Loc}}((0, T]; (H^3(\Omega))^n),$$

$$w_t \in C([0, T]; L^2(\Omega) \cap (H_0^1(\Omega))^n) \cap L^2(0, T; (H^{2-\varepsilon}(\Omega))^n)$$
$$\cap L^p([0, T]; (H^{1-\varepsilon}(\Omega))^n) \cap L^2_{\text{Loc}}(0, T; (H^2(\Omega))^n) \cap L^\infty_{\text{Loc}}((0, T]; (H^1(\Omega))^n),$$

$$w_{tt} \in L^2_{\text{Loc}}(0, T; (L^2(\Omega))^n),$$

for all $\varepsilon > 0$ and $1 < p < +\infty$.

Remark. Actually it is possibly to prove that the strong solution of Theorem 2.1 is global either if $n = 2$ or if we take small enough initial data when $n = 3$.

The above result depends on certain estimates for the approximations (u^k, w^k, ρ^k); since these estimates will be necessary in a companion paper where we obtain error bounds, for future reference we summarize them in the following

Proposition 2.2 Let (u^k, w^k, ρ^k) be the solution of (2.3). Then, she satisfies

$$
\begin{aligned}
&\alpha \le \rho^k(x,t) \le \beta, \quad (0 < \alpha = \inf \rho_0, \quad \beta = \sup \rho_0) \\
&\|\nabla u^k(t)\|^2 + \|\nabla w^k(t)\|^2 \le F_1(t) \\
&\int_0^t \{\|\Delta w^k(s)\|^2 + \|P\Delta u^k(s)\|^2\} ds \le F_2(t), \\
&\int_0^t \{\|w_t^k(s)\|^2 + \|u_t^k(s)\|^2\} ds \le F_3(t), \\
&\|w_t^k(t)\|^2 + \|u_t^k(t)\|^2 + \int_0^t \{\|\nabla w_t^k(s)\|^2 + \|\nabla u_t^k(s)\|^2\} ds \le F_4(t), \\
&\|P\Delta u^k(t)\|^2 + \|\Delta w^k(t)\|^2 \le F_5(t), \\
&\|\nabla \rho^k(t)\|_{L^\infty}^2 \le F_6(t), \\
&\|\rho_t^k(t)\|_{L^\infty}^2 \le F_7(t), \\
&\int_0^t \{\|\nabla u^k(s)\|_{L^\infty}^2 + \|\nabla w^k(s)\|_{L^\infty}^2\} ds \le F_8(t), \\
&\int_0^t \{\|u^k(s)\|_{H^3}^2 + \|w^k(s)\|_{H^3}^2\} ds \le F_9(t), \\
&\int_0^t \sigma(s)\{\|u_{tt}^k(s)\|^2 + \|w_{tt}^k(s)\|^2\} ds + \sigma(t)\{\|\nabla u_t^k(t)\|^2 + \|\nabla w_t^k(t)\|\} \le F_{10}(t), \\
&\sigma(t)\{\|u^k(t)\|_{H^3}^2 + \|w^k(t)\|_{H^3}^2\} \le F_{11}(t), \\
&\sigma(t)\{\|\nabla u^k(t)\|_{L^\infty}^2 + \|\nabla w^k(t)\|_{L^\infty}^2\} \le F_{12}(t), \\
&\int_0^t \sigma(s)\{\|P\Delta u_t^k(s)\|^2 + \|\Delta w_t^k(s)\|^2\} ds \le F_{13}(t),
\end{aligned}
$$

Here, $\sigma(t) = \min\{1, t\}$. Moreover, the same estimates hold for (u, w, ρ) give in Theorem 2.1. The above results are true without any restriction if we the eigengunctions of $\mathcal{L}$ to build the approximations for w^k. If we use the eigenfunctions of $-\Delta$ then the above estimates are true if $\sigma = (c_0 + c_d - c_a)/(c_a + c_d)$ is sufficiently small.

Remark. The above estimates imply that the approximations (u^k, w^k, ρ^k) converges to the solution (u, w, ρ) in the senses indicated below

(i) $u^k \to u, \quad w^k \to w$	strongly in $L^p(0,T;H^{3-\varepsilon}(\Omega))$ and weakly -$\star$ in $L^\infty_{\text{Loc}}(0,T;H^3(\Omega))$
(ii) $u^k_t \to u_t, \quad w^k_t \to w_t$	weakly in $L^\infty_{\text{Loc}}(0,T;H^1(\Omega))$ and weakly in $L^2(0,T;H^{2-\varepsilon}(\Omega))$ in $L^p(0,T;H^{1-\varepsilon}(\Omega))$ and in $L^2_{\text{Loc}}(0,T;H^2(\Omega))$
(iii) $u^k_{tt} \to u_{tt}, \quad w^k_{tt} \to w_{tt}$	weakly in $L^2_{\text{Loc}}(0,T;L^2(\Omega))$
(iv) $\rho^k \to \rho$	strongly in $L^p(0,T;C^{0,\gamma}(\Omega))$ $0 \le \gamma < 1$
(v) $\nabla\rho^k \to \nabla\rho$	weakly -$\star$ in $L^\infty(\Omega \times [0,T])$
(vi) $\rho^k_t \to \rho_t$	weakly -$\star$ in $L^\infty(\Omega \times [0,T])$.

The above is true for all $\varepsilon > 0$ and $1 < p < +\infty$

Finally, we would like to say that as it is usual we will denote by C a generic constant depending at most on Ω and the fixed parameters in the problem (μ, μ_r, c_a, c_d, c_0 and the initial conditions, and also f, g and T). This will appear in most of the estimates to the be obtained. When for any reason we want to emphasize the dependence of a certain constant on a given parameter we will denote this constant with a subscript.

3. A Priori Estimates

We start by proving the estimates stated in Proposition 2.2. This will be done in several steps by combining variants of arguments used by Heywood [6], [7], Kim [9] and Boldrini and Rojas-Medar [3]. To fix the ideas, first we prove the estimates in the case where one uses the eigenfunctions of $-\Delta$ to approximate w; at the end of this section we explain the necessary changes to obtain the estimates when one uses eigenfunctions of $\mathcal{L}$.

Lemma 3.1 There is $0 < T \le T_*$ such that the approximations ρ^k, u^k, w^k satisfy for all $t \in [0,T)$.

(3.1) $$\alpha \le \rho^k(x,t) \le \beta,$$

$$\|u^k(t)\|^2 + \|w^k(t)\|^2 \leq L(t), \tag{3.2}$$

$$\int_0^t (\|\nabla u^k(s)\|^2 + \|\nabla w^k(s)\|^2) ds \leq H(t), \tag{3.3}$$

$$\|\nabla u^k(t)\|^2 + \|\nabla w^k(t)\|^2 \leq F_1(t), \tag{3.4}$$

$$\int_0^t (\|A u^k(s)\|^2 + \|\Delta w^k(s)\|^2) ds \leq F_2(t), \tag{3.5}$$

$$\int_0^t (\|u_t^k(s)\|^2 + \|w_t^k(s)\|^2) ds \leq F_3(t). \tag{3.6}$$

The functions on the right hand side of the above inequalities depend on α, β, Ω and the norms $\|\nabla u_0\|$, $\|\nabla w_0\|$. (3.5) and (3.6) depend also on $\|u_0\|_{H^2}$ and $\|w_0\|_{H^2}$. On the interval in question these functions can be assumed to be increasing and continuously differentiable with respect to t.

Proof. From the method of characteristics applied to the continuity equation (2.2) (iii), it follows immediately that whenever ρ^k exists it satisfies (3.1).

Now, by using $v = u^k$ and $\psi = w^k$ in (2.3) and working as in Lions [12], [13], one obtains

$$\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{\frac{1}{2}} u^k\|^2 + (\mu + \mu_r)\|\nabla u^k\|^2 = (\rho^k f, u^k) + 2\mu_r(\operatorname{rot} w^k, u^k),$$

$$\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{\frac{1}{2}} w^k\|^2 + (c_a + c_d)\|\nabla w^k\|^2 + (c_0 + c_d - c_a)\|\operatorname{div} w^k\|^2 + 4\mu_r\|w^k\|^2$$
$$= 2\mu_r(\operatorname{rot} u^k, w^k) + (\rho^k g, w^k).$$

By adding these two equations and working the terms in the right-hand side in a standard way, together with the use of (3.1), after an integration with respect to time from 0 to t, we are left with

$$\alpha\|u^k(t)\|^2 + \alpha\|w^k(t)\|^2 + (\mu + \mu_r)\int_0^t \|\nabla u^k(s)\|^2 ds + (c_a + c_d)\int_0^t \|\nabla w^k(s)\|^2 ds$$
$$+ (c_0 + c_d - c_a)\int_0^t \|\operatorname{div} w^k(s)\|^2 ds + 4\mu_r\int_0^t \|w^k(s)\|^2$$
$$\leq C_1 + C_2\int_0^t (\|f(s)\|^2 + \|g(s)\|^2) ds + C_3\int_0^t (\|u^k(s)\|^2 + \|w^k(s)\|^2) ds$$

where C_1, C_2 and C_3 are independent of k. (C_1 depends on the initial conditions).

Thus, by using Gronwall's Lemma, we obtain (3.2) and (3.3) for suitable $L(t)$ and $H(t)$.

Now, we take $v = u_t^k$ and subsequently $v = -\varepsilon A u^k$, with a suitable small $\varepsilon > 0$ in (2.3) (i). By adding the resulting equations and working as in Kim [9], one obtains

$$\alpha\|u_t^k\|^2 + (\mu+\mu_r)\frac{d}{dt}\|\nabla u^k\|^2 + C_4\|Au^k\|^2 \le C_5[\|f\|^2 + \|\nabla w^k\|^2 + \|u^k\cdot\nabla u^k\|^2], \tag{3.7}$$

with positive constants C_4 and C_5 independent of k (for instance, if we take $\varepsilon = \alpha(\mu+\mu_r)/9\beta^2,\ C_4 = \alpha(\mu+\mu_r)^2/81\beta^2$).

Now, we have to find a similar differential inequality for w^k. For this, we take $\psi = w_t^k$ in (2.3) (ii) to obtain

$$\begin{aligned}&\frac{\alpha}{2}\|w_t^k\|^2 + \frac{C_a+C_d}{2}\frac{d}{dt}\|\nabla w^k\|^2 + \frac{c_0+c_d-c_a}{2}\frac{d}{dt}\|\mathrm{div} w^k\|^2 + 2\mu_r\frac{d}{dt}\|w^k\|^2\\ &\quad\le C_5[\|g\|^2 + \|\nabla u^k\|^2 + \|u^k\cdot\nabla w^k\|^2],\end{aligned}$$

with $C_5 > 0$ independent of k.

Now, we take $\psi = -\Delta w^k$ in (2.3)(ii). This, with the remark that

$$\mathcal{L}w = -(c_a+c_d)\Delta w - (c_0+c_d-c_a)\nabla\mathrm{div}\, w$$

is a strongly elliptic operator, and thus

$$(\mathcal{L}w^k, -\Delta w^k) \ge C^*\|\Delta w^k\|^2 - C^{**}\|\nabla w^k\|^2,$$

where $C^* > 0$ and $C^{**} \ge 0$ depend on $c_a + c_d, c_0 + c_d - c_a$ and Γ, will furnish

$$\frac{c_a+c_d}{2}\|\Delta w^k\|^2 + 4\mu_r\|\nabla w^k\|^2 \le C_7[\|g\|^2 + \|\nabla u^k\|^2 + \|w_t^k\|^2 + \|u^k\cdot\nabla w^k\|^2]$$

with $C_7 > 0$ independent of k.

Now, we add the above inequalities, but with this last one multiplied by $\dfrac{\alpha}{4C_7}$. We obtain

$$\begin{aligned}&\alpha\|w_t^k\|^2 + 2(c_a+c_d)\frac{d}{dt}\|\nabla w^k\|^2 + 2(c_0+c_d-c_a)\frac{d}{dt}\|\mathrm{div}\, w^k\|^2 + 4\mu_r\frac{d}{dt}\|w^k\|^2\\ &\quad+C_8\|\Delta w^k\|^2 + C_9\|\nabla w^k\|^2 \le C_{10}[\|g\|^2 + \|\nabla u^k\|^2 + \|u^k\cdot\nabla w^k\|^2],\end{aligned} \tag{3.8}$$

with positive constants C_8, C_9 and C_{10}.

Now, we observe that by standard interpolation and Sobolev inequalities,

$$\|u^k \nabla u^k\|^2 \le \|u^k\|_{L^6}^2 \|\nabla u^k\|_{L^3}^2 \le C\|\nabla u^k\|^2 \|Au^k\| \le C_\varepsilon \|\nabla u^k\|^6 + \varepsilon \|Au^k\|^2,$$

for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$.

Analogously, we have

$$\|u^k \cdot \nabla w^k\|^2 \le C_\varepsilon \|\nabla u^k\|^4 \|\nabla w^k\|^2 + \varepsilon \|\Delta w^k\|^2.$$

By adding (3.7) and (3.8) and using the above inequalities with suitable small ε, we conclude that there is a positive constant C such that

$$\begin{aligned} & \frac{d}{dt}\theta(t) + \psi(t) \le \phi(t) + C\theta^3(t) \\ \text{with } \theta(t) &= (\mu + \mu_r)\|\nabla u^k(t)\|^2 + (c_a + c_d)\|\nabla w^k\|^2 + (c_0 + c_d - c_a)\|\operatorname{div} w^k\|^2 \\ & \quad +4\mu_r \|w^k\|^2, \\ \psi(t) &= \alpha \|u_t^k\|^2 + \frac{C_4}{2}\|Au^k\|^2 + \frac{C_8}{2}\|\Delta w^k\|^2 + C_9 \|\nabla w^k\|^2, \\ \phi(t) &= C(\|f\|^2 + \|g\|^2). \end{aligned}$$

By making use of Lemma 3 in Heywood [8], p. 656, we conclude that there is $0 < T \le T_*$ such that on the interval $[0, T]$ (3.4) - (3.6) hold with suitable $F_i(t), i = 1, 2, 3$. ■

Lemma 3.2 For all $t \in [0, T]$, the approximations (u^k, w^k) satisfy

$$\|u_t^k(t)\|^2 + \|w_t^k(t)\|^2 + \int_0^t (\|\nabla u_t^k(s)\|^2 + \nabla w_t^k(s)\|^2) ds \le F_4(t), \tag{3.9}$$

$$\|Au^k(t)\|^2 + \|\Delta w^k(t)\|^2 \le F_5(t). \tag{3.10}$$

The functions on the right hand side of the above inequalities depend on α, β, Ω, the norms $\|u_0\|_{H^2}, \|w_0\|_{H^2}$ and the functions given in the Lemma 3.1. On the interval in question the functions can be assumed to be increasing and continuously differentiable with respect to t.

Proof. By differentiating (2.3)(i) and (ii) with respect to t and setting $v = u_t^k, \psi = w_t^k$ and working analogously as in Boldrini and Rojas-Medar [3] (use the fact that $\rho_t^k = -\operatorname{div}(\rho^k u^k)$), we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{1/2} u_t^k\|^2 + \frac{1}{9}(\mu + \mu_r)\|\nabla u_t^k\|^2$$

$$\leq C_{10}\|u_t^k\|^2\{\|Au^k\|^2+\|\nabla u^k\|^4+\|\nabla u^k\|^8+1\}+C_{10}\|\nabla u^k\|^4\|Au^k\|^2$$
$$+\varepsilon\|\nabla w_t^k\|^2+C_{10}\|f_t\|^2+C_{10}\|f\|_{H^1}^2\{\|\nabla u^k\|^2+1\},$$
$$\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{1/2}w_t^k\|^2+\frac{1}{5}(c_a+c_d)\|\nabla w_t^k\|^2+(c_0+c_d-c_a)\|\text{div}\, w_t^k\|^2+4\mu_r\|w_t^k\|^2$$
$$\leq C_{11}\|u_t^k\|^2\{\|Au^k\|^2+\|\Delta w^k\|^2+1\}+C_{11}\|w_t^k\|^2\{\|Au^k\|^2+1\}$$
$$+C_{11}\{\|Au^k\|^2+\|\Delta w^k\|^2\}+\delta\|\nabla u_t^k\|^2+\varepsilon\|\nabla w_t^k\|^2$$
$$+C_{11}\|g_t\|^2+C_{11}\|g\|_{H^1}^2\{\|\nabla u^k\|^2+1\}$$

for any $\varepsilon,\delta>0$ and suitable $C_{10}(\varepsilon), C_{11}(\varepsilon,\delta)$.

By taking $\delta=(\mu+\mu_r)/10$ and $\varepsilon=(c_a+c_d)/12$, by adding and integrating in time the above two inequalities, we obtain the integral inequality

$$\frac{(c_a+c_d)}{30}\int_0^t\|\nabla w_t^k(s)\|^2ds+\frac{(\mu+\mu_r)}{90}\int_0^t\|\nabla u_t^k(s)\|^2ds+\frac{1}{2}\|(\rho^k)^{1/2}(t)u_t^k(t)\|^2$$
$$+\frac{1}{2}\|(\rho^k)^{1/2}(t)w_t^k(t)\|^2+4\mu_r\int_0^t\|w_t^k(s)\|^2ds+(c_0+c_d-c_a)\int_0^t\|\text{div}\, w_t^k(s)\|^2ds$$
$$\leq M(t)+C_{12}\int_0^t(\|Au^k(s)\|^2+\|\Delta w^k(s)\|^2)ds$$
$$+C_{12}\int_0^t\|u_t^k(s)\|^2\{\|Au^k(s)\|^2+\|\Delta w^k(s)\|^2+1\}ds$$
$$+C_{12}\int_0^t\|w_t^k(s)\|^2\{\|Au^k(s)\|^2+1\}ds$$
$$+\frac{1}{2}\|(\rho_0^k)^{1/2}u_t^k(0)\|^2+\frac{1}{2}\|(\rho_0^k)^{1/2}w_t^k(0)\|^2, \tag{3.11}$$

with a suitable constant $C_{12}>0$ and

$$M(t)=C\int_0^t(\|g_t(s)\|^2+\|g(s)\|_{H^1}^2+\|f_t(s)\|^2+\|f(s)\|_{H^1}^2)ds$$

where we have used the estimates in Lemma 3.1

Now, by taking $t=0$ and $v=u_t^k(0)$ in (2.3)i, we obtain

$$\|u_t^k(0)\|^2\leq\frac{1}{\alpha}\{\|(\rho_0^k)^{1/2}u_0^k\cdot\nabla u_0^k\|+(\mu+\mu_r)\|Au_0^k\|+\|\rho_0^kf(0)\|+2\mu_r\|\text{rot}\, w_0^k\|\}\|u_t^k(0)\|,$$

and consequently,

$$\|u_t^k(0)\|\leq\frac{1}{\alpha}\{C\beta\|f_0\|+2\mu_rC\|\nabla w_0\|\}+\{C\beta\|\nabla u_0\|+(\mu+\mu_r)\}\frac{\|Au^k\|}{\alpha}\leq C,$$

with C independent of k, because $u^k(0) \to u_0$ in $H^2(\Omega)$ as $k \to \infty$.

We can bound $\|w_t^k(0)\| \le C$ in the same way. Thus, by using this we obtain

$$\frac{(c_a+c_d)}{15}\int_0^t \|\nabla w_t^k(s)\|^2 ds + \frac{(\mu+\mu_r)}{45}\int_0^t \|\nabla u_t^k(s)\|^2 ds + \alpha\{\|u_t^k(t)\|^2 + \|w_t^k(t)\|^2\}$$
$$+8\mu_r\int_0^t \|w_t^k(s)\|^2 ds + 2(c_0+c_d-c_a)\int_0^t \|\mathrm{div}\, w_t^k(s)\|^2 ds$$
$$\le M(t) + CF_2(t) + C$$
$$+C\int_0^t (\|u_t^k(s)\|^2 + \|w_t^k(s)\|^2)\{\|P\Delta u^k(s)\|^2 + \|\Delta w^k(s)\|^2 + 1\} ds.$$

Now applying of Gronwall's Lemma, we get (3.9) with

$$F_4(t) = \gamma^{-1}(M(t) + CF_2(t) + C)\exp(F_2(t) + Ct),$$

where $\gamma = \min\{\frac{(c_a+c_d)}{15}, \frac{(\mu+\mu_r)}{45}, \alpha\} > 0$.

The second estimate follows, from the first one by observing that by taking $v = P\Delta u^k$ in (2.3)i, we obtain

$$(\mu+\mu_r)\|P\Delta u^k\| \le \|\rho^k u_t^k\| + 2\mu_r\|\mathrm{rot}\, w^k\| + \|\rho^k f\| + \|\rho^k u^k\cdot\nabla u^k\|.$$

Now, we observe that

$$\begin{aligned}\|\rho^k u^k\cdot\nabla u^k\| &\le \beta\|u^k\|_{L^4}\|\nabla u^k\|_{L^4} \le \beta\|\nabla u^k\|\ \|\nabla u^k\|^{1/4}\ \|P\Delta u^k\|^{3/4}\\ &\le C_\varepsilon\beta\|\nabla u^k\|^5 + \varepsilon\|P\Delta u^k\|,\end{aligned}$$

schoosing $\varepsilon = (\mu+\mu_r)/2$, we obtain

$$\begin{aligned}\|P\Delta u^k\| &\le \frac{2}{\mu+\mu_r}[C\beta\|\nabla u^k\|^5 + \beta\|u_t^k\| + 2\mu_r C\|\nabla w^k\| + \beta\|f\|]\\ &\le \frac{2}{\mu+\mu_r}[C\beta F_1(t)^{5/2} + \beta F_4(t)^{1/2} + CF_1(t)^{1/2} + \beta\|f\|] \equiv F_5(t).\end{aligned}$$

We can treat $\|\Delta w^k\|$ in the same way, and this complete the proof of Lemma 3.2. ■

Lemma 3.3 The approximations ρ^k, u^k satisfy for all $t \in [0,T]$,

(3.12) $$\int_0^t \|u^k(s)\|^2_{W^{2,6}} ds \le G_1(t),$$

(3.13) $$\int_0^t \|\nabla u^k(s)\|^2_{L^\infty} ds \le G_2(t),$$

(3.14) $$\|\nabla\rho^k(t)\|_{L^\infty} \le F_6(t),$$

(3.15) $$\|\rho_t^k(t)\|_{L^\infty} \le F_7(t).$$

The functions on the righ hand side of the above inequalities depend only on $\alpha, \beta, \Omega, \|u_0\|_{H^2}, \|w_0\|_{H^2}$ and the functions $F_i, i = 1, 2, 3, 4, 5$ of the above Lemmas.

These functions on continuous on the interval in question.

Proof. We observe that for any $\phi \in C^\infty_{0,\sigma}(\Omega)$, we have

$$-(\mu + \mu_r)(\nabla u^k, \nabla \phi) = (P_k(\rho^k u^k_t + \rho^k u^k \cdot \nabla u^k - \rho^k f - 2\mu_r \mathrm{rot}\, w^k), \phi) \equiv (\chi^k, \phi).$$

From the previous estimates, we have $\chi^k \in L^2(0,T; L^6(\Omega))$ uniformly in k whence, by Amrouche and Girault results for the Stokes operator, we get (3.12). From usual Sobolev's embedding results, we then have

$$\int_0^t \|\nabla u^k(s)\|^2_{L^\infty} ds \le G_2(t), \qquad \forall t \in [0,T].$$

Hence, from the formula of Ladyzhenkaya and Solonnikov [11, Lemma 1.3], we conclude

$$\|\nabla \rho^k(t)\|_{L^\infty} \le F_6(t) \qquad \text{and} \qquad \|\rho^k_t(t)\|_{L^\infty} \le F_7(t).$$

■

Lemma 3.4 In the case of $J = B$ we have that if $\sigma = (c_0 + c_d - c_a)/(c_a + c_d)$ is small enough, the approximations w^k satisfy for all $t \in [0,T]$

$$\int_0^t \|w^k(s)\|^2_{W^{2,6}} ds \le G_3(t), \tag{3.16}$$

$$\int_0^t \|\nabla w^k(s)\|^2_{L^\infty} ds \le G_4(t). \tag{3.17}$$

The functions on the righ hand sides of the above inequalities depend only on α, β, Ω the norm $\|w_0\|_{H^2}$ and the functions of the above Lemmas. These functions one continuous in the interval in question.

Proof. We have for any $\psi \in C^\infty_0(\Omega)$,

$$-(c_a + c_d)(\Delta w^k, \psi) = (\eta^k, \psi) + (c_0 + c_d - c_a)(\nabla \mathrm{div}\, w^k, \psi),$$

where $\eta^k = -\rho^k w^k_t - \rho^k u^k \cdot \nabla w^k + \rho^k g + 2\mu_r \mathrm{rot}\, u^k - 4\mu_r w^k$, consequently

$$(c_a + c_d)\|\Delta w^k\|_{L^6} \le \|\eta^k\|_{L^6} + (c_0 + c_d - c_a)\|\nabla \mathrm{div}\, w^k\|_{L^6}. \tag{3.18}$$

By other hand side, there exists positive constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|\Delta w^k\|^2_{L^6} \ge k_1 \|w^k\|^2_{W^{2,6}} \qquad \text{and} \qquad \|\nabla \mathrm{div}\, w^k\|^2_{L^6} \le k_2 \|w^k\|^2_{W^{2,6}}.$$

Consequently in (3.16), we obtain

$$k_1(c_a + c_d)\|w^k\|^2_{W^{2,6}} \leq \|\eta^k\|^2_{L^6} + k_2(c_0 + c_d - c_a)\|w^k\|^2_{W^{2,6}}$$

thus

$$(k_1(c_a + c_d) - k_2(c_0 + c_d - c_a))\|w^k\|^2_{W^{2,6}} \leq \|\eta^k\|^2_{L^6}.$$

Since σ is small enough, we have $k_1(c_a + c_d) - k_2(c_0 + c_d - c_a) > 0$; also by the previous estimates $\eta^k \in L^2(0,T;L^6(\Omega))$ is bounded uniformly in k. We conclude that $w^k \in L^2(0,T;W^{2,6}(\Omega))$ uniformly in k and, by using Sobolev's embedding $\nabla w^k \in L^2(0,T;L^\infty(\Omega))$ also uniformly in k. ■

Now, we consider the eigenfunctions of the operator $\mathcal{L}w = -(c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \mathrm{div}\, w$ as basis for Galerkin approximations of w. In this case, the approximate equation for w is

$$(\mathcal{L}w^k, \psi) + (\rho^k w^k_t + \rho^k u^k \cdot \nabla w^k + 4\mu_r w^k - \rho^k g - 2\mu_r \mathrm{rot}\, u^k, \psi) = 0 \tag{3.19}$$

for all $\psi \in W_k$.

We observe that two first estimates obtained in Lemma 3.1 remains valid. For the second estimate we proceed as follows: We take $\psi = \mathcal{L}w^k$ in (3.19), we get

$$\|\mathcal{L}w^k\|^2 = (\rho^k g + 2\mu_r \mathrm{rot}\, u^k - \rho^k w^k_t - \rho^k u^k \cdot \nabla w^k - 4\mu_r w^k, \mathcal{L}w^k).$$

By using the Hölder and Young inequalities, we obtain

$$\begin{aligned}\|\mathcal{L}w^k\|^2 &\leq C\|\rho^k\|^2_{L^\infty}\|g\|^2 + C\|\nabla u^k\|^2 + C\|\rho^k\|^2_{L^\infty}\|w^k_t\|^2 \\ &\quad + C\|\rho^k\|^2_{L^\infty}\|u^k \cdot \nabla w^k\|^2 + C\|w^k\|^2 \\ &\leq C\|g\|^2 + C\|\nabla u^k\|^2 + C\|w^k_t\|^2 + C\|u^k \cdot \nabla w^k\|^2 + C\|w^k\|^2\end{aligned}$$

by the estimate (3.1) in the Lemma 3.1.

Taking $\psi = w^k_t$ in (3.19), we have

$$\begin{aligned}&\frac{\alpha}{2}\|w^k_t\|^2 + \frac{c_a + c_d}{2}\frac{d}{dt}\|\nabla w^k\|^2 + \frac{c_0 + c_d - c_a}{2}\frac{d}{dt}\|\mathrm{div}\, w^k\|^2 + 2\mu_r\frac{d}{dt}\|w^k\|^2 \\ &\quad \leq C(\|g\|^2 + \|\nabla u^k\|^2 + \|u^k \cdot \nabla w^k\|^2).\end{aligned}$$

Now, we observe that

$$\begin{aligned}\|u^k \cdot \nabla w^k\|^2 &\leq C\|u^k\|^2_{L^4}\|\nabla w^k\|^2_{L^4} \leq C\|u^k\|^2_{L^4}\|\nabla w^k\|^{1/2}\|w^k\|^{3/2}_{H^2} \\ &\leq C\|\nabla u^k\|^2\|\nabla w^k\|^{1/2}\|\mathcal{L}w^k\|^{3/2} \leq C_\varepsilon\|\nabla u^k\|^8\|\nabla w^k\|^2 + \varepsilon\|\mathcal{L}w^k\|^2.\end{aligned}$$

The rest of analysis is exactly equal to the one in Lemma 3.1 to obtain the estimate (3.5), in this case, we obtain

$$\int_0^t \{\|Au^k(s)\|^2 + \|\mathcal{L}w^k(s)\|^2\} ds \leq \widetilde{F}_2(t).$$

We observe also that the estimate for w_t^k is done exactly as in Lemma 3.1. Therefore, the Lemm 3.1 remains valid if we consider the $\mathcal{L}$ operator instead of the Laplacian operator. The Lemmas 3.2 and 3.3 are proved exactly equals. The analogous to the Lemma 3.4 in this case is the following

Lemma 3.5 In the case that $J = \mathcal{L}$, the approximations w^k satisfy the following estimates for any $t \in [0, T]$

$$\int_0^t \|w^k(s)\|^2_{W^{2,6}} ds \leq \widetilde{G}_3(t), \tag{3.20}$$

$$\int_0^t \|\nabla w^k(s)\|^2_{L^\infty} ds \leq \widetilde{G}_4(t). \tag{3.21}$$

Proof. We have, for any $\psi \in C_0^\infty(\Omega)$,

$$(\mathcal{L}w^k, \psi) = (\eta^k, \psi),$$

where $\eta^k = \rho^k g + 2\mu_r \mathrm{rot}\, u^k - 4\mu_r w^k - \rho^k w_t^k - \rho^k u^k \cdot \nabla w^k$.

We observe that $\eta^k \in L^2(0, T; L^6(\Omega))$ is bounded uniformly in k. We conclude that $w^k \in L^2(0, T; W^{2,6}(\Omega))$ uniformly in k and, by using Sobolev's embedding $\nabla w^k \in L^2(0, T; L^\infty(\Omega))$ also uniformly in k. ■

Now, by taking $F_8(t) = G_2(t) + G_4(t)$, the estimates in the last Lemmas prove the ninth estimate in Proposition 2.2.

Lemma 3.6 The approximations (ρ^k, u^k, w^k) satisfy for all $t \in [0, T]$

$$\int_0^t (\|u^k(s)\|^2_{H^3} + \|w^k(s)\|^2_{H^3}) ds \leq F_9(t). \tag{3.22}$$

Proof. We observe that (2.2)i is equivalent to

$$\rho^k u_t^k + \rho^k u^k \cdot \nabla u^k - \rho^k f - 2\mu_r \,\mathrm{rot}\, w^k + (\mu + \mu_r)\Delta u^k + \nabla p^k + \phi^k = 0, \tag{3.23}$$

where $\phi^k \in C^\infty(0, T; V)$, $p^k \in C^\infty(0, T; H^2(\Omega))$ with $\phi^k(t) \in V_k^\perp$, $\nabla p^k \in V^\perp$ for each $t \in [0, T]$, where $S^\perp$ denote the orthogonal of the subspace S in $L^2(\Omega)$.

Differentiating the above identity with respect to x_i,i=1,...,n and taking the L^2-inner product with $A\frac{\partial u^k}{\partial x_i}$, after of adding over i, we obtain

$$\begin{aligned}\|u^k(s)\|^2_{H^3} \quad \leq \quad & C\{\|w^k(s)\|^2_{H^2} + \|\rho^k\|^2_{L^\infty}\|f\|^2_{H^1} + \|\nabla\rho^k\|^2_{L^\infty}\|f\|^2 \\ & +\|\rho^k\|^2_{L^\infty}\|u^k\|^2_{L^\infty}\|u^k\|^2_{H^2} + \|\rho^k\|^2_{L^\infty}\|u^k\|^4_{H^2} \\ & +\|\nabla\rho^k\|^2_{L^\infty}\|u^k\|^2_{L^\infty}\|\nabla u^k\|^2 + \|\rho^k\|^2_{L^\infty}\|\nabla u^k_t\|^2 + \|\nabla\rho^k\|^2_{L^\infty}\|u^k_t\|^2\end{aligned}$$

since,

$$\sum(\frac{\partial\phi^k}{\partial x_i}, A\frac{\partial u^k}{\partial x_i}) = \sum\int_{\partial\Omega}\phi^k A\frac{\partial u^k}{\partial x_i} + \int_\Omega \phi^k\frac{\partial}{\partial x_i}A\frac{\partial u^k}{\partial x_i} = 0,$$
$$\sum(\nabla\frac{\partial p^k}{\partial x_i}, A\frac{\partial u^k}{\partial x_i}) = \sum(P\nabla\frac{\partial p^k}{\partial x_i}\triangle\frac{\partial u^k}{\partial x_i}) = 0.$$

Now, we integrate (3.23) with respect to t and using the above's estimates, we obtain the desired result. Analogously, we prove the result for w.

The following remark will be necessary for the following estimates.

Remark. Let $f \in L^1(a,b)$ be a positive function. Then there is a sequence $\varepsilon_n \to a_+$ such that $\varepsilon_n f(\varepsilon_n) \to 0$ as $n \to \infty$. ∎

Now we shall study higher order estimates for the approximations,

Lemma 3.7 Under the assumed hypotheses, there hold

i) $\displaystyle\int_0^t \sigma(s)(\|u^k_{tt}(s)\|^2 + \|w^k_{tt}(s)\|^2)ds + \sigma(t)(\|\nabla u^k_t(t)\|^2 + \|\nabla w^k_t(t)\|^2) \leq F_{10}(t),$

ii) $\sigma(t)(\|u^k(t)\|^2_{H^3} + \|w^k(t)\|^2_{H^3}) \leq F_{11}(t),$

iii) $\sigma(t)(\|\nabla u^k(s)\|^2_{L^\infty} + \|\nabla w^k(s)\|^2_{L^\infty}) \leq F_{12}(t),$

where $\sigma(t) = \min\{1,\, t\}$. The functions on the right hand sides depend on their argument t, and in addition on $T \leq T_*,\ \alpha > 0,\ \beta > 0, \Gamma$ and the norms $\|u_0\|_{H^2},\ \|w_0\|_{H^2}$. On the interval in question these functions are continuous in the variable t.

Sketch of Proof. Differentiating (2.3)i with respect to t, and multiplying by u^k_{tt} and integrating in Ω, we get

$$c_1\|u^k_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^k_t\|^2 \leq c_2 + c_3\|\nabla u^k_t\|^2 + c_4\|\nabla w^k_t\|^2, \tag{3.24}$$

where c_i are constants depend on the sup in t for the functions done in the above Lemmas, the regularity on Γ, the initial data and independ of k.

Multiplying (3.24) by $\sigma(t)$ and integrating in (ε, t), we get

$$\begin{aligned} c_1 \int_\varepsilon^t \sigma(s)\|u_{tt}^k(s)\|^2 ds + \frac{1}{2}\int_\varepsilon^t \sigma(s)\frac{d}{dt}\|\nabla u_t^k(s)\|^2 ds \\ \leq c_2 \int_\varepsilon^t \sigma(s)ds + c_3\int_\varepsilon^t \sigma(s)\|\nabla u_t^k(s)\|^2 ds + c_4\int_\varepsilon^t \sigma(s)\|\nabla w_t^k(s)\|^2 ds \\ \leq c_2 t + c_3\int_0^t \|\nabla u_t^k(s)\|^2 ds + c_4 \int_0^t \|\nabla w_t^k(s)\|^2 ds. \end{aligned}$$

A continuation, we observe that

$$\begin{aligned} c_1\int_\varepsilon^t \sigma(s)\frac{d}{dt}\|\nabla u_t^k(s)\|^2 ds &= \sigma(t)\|\nabla u_t^k(t)\|^2 - \sigma(\varepsilon)\|\nabla u_t^k(\varepsilon)\|^2 \\ &\quad + \int_\varepsilon^t \sigma'(s)\|\nabla u_t^k(s)\|^2 ds \qquad \text{a.e. in} \quad t. \end{aligned} \tag{3.25}$$

Bearing in mind (3.9) and the above Remark, we have passing to the limit $\varepsilon \to 0^+$, (i); (ii) follows of (i), by using the inequality (3.25). (iii) follows imediately of (ii). The arguments for w are analogous. This completes the proof. ∎

Analogously, we can prove.

Lemma 3.8 Under the hypotheses done , we have

$$\int_0^t \sigma(s)(\|P\Delta u_t^k(s)\|^2 + \|\Delta w_t^k(s)\|^2)ds \leq F_{13}(t).$$

4. Existence of Solutions

By the estimates given in tke Lemma 3.2, we can choose a subsequence of $\{u^k\}$ still denote by $\{u^k\}$ such that $u^k \to u$ weak - $\star$ in $L^\infty(0,T;H^2(\Omega))$ and $u_t^k \to \theta$ weak - $\star$ in $L^\infty(0,T;L^2(\Omega))$. By standar arguments $\theta = u_t$. Analogously, we can proved for angular velocity. Truly, we can strenghten the convergence of u^k and w^k using the Aubin-Lions Lemma , we get $u^k \to u$ and $w^k \to w$ strongly in $L^p(0,T;H^1(\Omega))$ for every p finite.

Also, we have by estimates given in the Lemma 3.3, $\rho^k \to \rho$ weak - $\star$ in $L^\infty(0,T;C^1(\Omega))$ and $\rho_t^k \to \rho_t$ weak - $\star$ in $L^\infty(0,T;L^p(\Omega))$ for every $p \in (1,\infty]$. Thus, $\rho^k \to \rho$ in $\mathcal{D}'(0,T;L^q(\Omega))$ whence $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Likewise, we observe that

$\rho_t^k \to \chi$ weak - $\star$ in $L^\infty(0,T;L^\infty(\Omega))$ thanks to the estimate (3.15). Thus $\rho_t^k \to \chi$ weak - $\star$ in $L^\infty(0,T;L^p(\Omega))$ with $1 \leq \rho < \infty$; immediately $\chi = \rho_t^k$. Therefore, we can to streng then the convergence to the density by using the Aubin-Lions Lemma, we get $\rho^k \to \rho$ in $L^{p_0}(0,T;W^{r,l}(\Omega))$ whence $0 \leq r < 1,\ 1 < p_0 < \infty,\ 1 < l < \infty$. It follows by using the Sobolev embedding, for l large enough, we have $r - n/l > 0 \quad (r = 2$ or 3) and for so much the convergence, $\rho^k \to \rho$ in $L^{p_0}(0,T;C^{0,\gamma}(\Omega))$ $(0 \leq \gamma < 1)$.

A continuation we show that

$$\int_0^T \langle \rho^k u_t^k,\ v(x)\phi(t)\rangle dt \longrightarrow \int_0^T \langle \rho u_t,\ v(x)\phi(t)\rangle dt, \tag{4.1}$$

$$\int_0^T \langle \rho^k w_t^k,\ z(x)\zeta(t)\rangle dt \longrightarrow \int_0^T \langle \rho w_t,\ z(x)\zeta(t)\rangle dt, \tag{4.2}$$

whence $k \longrightarrow \infty$, for every $v(x) \in C^3(\Omega),\ \phi(t) \in \mathcal{D}(0,T);\ z(x) \in C_0^3(\Omega),\ \zeta(t) \in \mathcal{D}\,(0,T)$, respectively.

We have

$$\begin{aligned} |\int_0^T \langle \rho^k u_t^k,\ v(x)\phi(t)\rangle dt| \ &\leq\ |\int_0^T \langle (\rho^k - \rho) u_t^k,\ v(x)\phi(t)\rangle dt| \\ &+ |\int_0^T \langle \rho^k (u_t^k - u_t),\ v(x)\phi(t)\rangle dt|. \end{aligned}$$

We observe that

$$|\int_0^T \langle (\rho^k - \rho) u_t^k,\ v(x)\phi(t)\rangle dt| \leq \sup |v(x)\phi(t)| \int_0^T \|\rho^k - \rho\|\ \|u_t^k\| dt$$

consequently,

$$|\int_0^T \langle (\rho^k - \rho) u_t^k,\ v(x)\phi(t)\rangle dt| \longrightarrow 0 \qquad \text{whence} \quad k \longrightarrow \infty.$$

Therefore,

$$|\int_0^T \langle \rho(u_t^k - u_t),\ v(x)\phi(t)\rangle dt| = |\int_0^T \langle u_t^k - u_t,\ \rho v(x)\phi(t)\rangle dt|$$

and bearing in mind that $u_t^k \longrightarrow u_t$ weakly, we have

$$\int_0^T \langle \rho(u_t^k - u_t),\ v(x)\phi(t)\rangle dt \longrightarrow 0 \qquad \text{as} \quad k \longrightarrow \infty.$$

Thus, we proved (4.1).(4.2) is proved similary.

Now we show that

$$\int_0^T \langle \rho^k u^k \cdot \nabla u^k,\ v(x)\phi(t)\rangle dt \longrightarrow \int_0^T \langle \rho u \cdot \nabla u,\ v(x)\phi(t)\rangle dt| \tag{4.3}$$

$$\int_0^T \langle \rho^k u^k \cdot \nabla w^k,\ z(x)\zeta(t)\rangle dt \longrightarrow \int_0^T \langle \rho u \cdot \nabla w,\ z(x)\zeta(t)\rangle dt \tag{4.4}$$

whence $k \longrightarrow \infty$, for every $v(x) \in C^3(\Omega),\ \phi(t) \in \mathcal{D}(0,T),\ z(x) \in C_0^3(\Omega), \zeta(t) \in \mathcal{D}(0,T)$, respectively.

We show (4.4), (4.3) is make similary. Then we have

$$\begin{aligned}&\int_0^T \langle \rho^k u^k \cdot \nabla w^k,\ z(x)\zeta(t)\rangle dt \\ &\quad = \int_0^T \langle (\rho^k - \rho) u^k \cdot \nabla w^k,\ z(x)\zeta(t)\rangle dt + \int_0^T \langle \rho(u^k - u) \cdot \nabla w^k,\ z(x)\zeta(t)\rangle dt \\ &\qquad + \int_0^T \langle \rho u \cdot \nabla(w^k - w),\ z(x)\zeta(t)\rangle dt.\end{aligned}$$

A continuation we observe that the firsts integral convergence to zero, enough to apply of Schwarz inequality with respect to space variavel and observe that $\rho^k \to \rho$ in $L^2(\Omega \times (0,T))$ and that

$$\int_0^T \int_\Omega |u^k \cdot \nabla w^k| dx\ dt \le \int_0^T \|u^k\|_{L^\infty}^2 \|\nabla w^k\|^2 dt \le c,$$

thanks to the Lemmas 3.1 and 3.2. In the second integral, we have

$$\begin{aligned}\int_0^T \int_\Omega \rho(u^k - u) \cdot \nabla w^k z(x)\zeta(t) dx dt &= \int_0^T \int_\Omega (u^k - u) \cdot \nabla w^k \rho z(x)\zeta(t) dx dt \\ &\le \sup |\rho z(x)\phi(t)| \int_0^T \int_\Omega |u^k - u|\ |\nabla w^k| dx dt \\ &\le C \int_0^T \left\{ \int_\Omega |u^k - u|^2 dx \right\}^{1/2} \left\{ \int_\Omega |\nabla w^k|^2 dx \right\}^{1/2} dt \\ &\le C \int_0^T \|u^k - u\|^2 dt,\end{aligned}$$

we observe that its integral convergence to zero, thanks to (3.2).

The third integral is treated analogously.

Passage to the Limit in the Approximated Equation

Thus,

$$\int_0^T \langle \rho^k u_t^k + \rho^k u^k \cdot \nabla u^k - \rho^k f - 2\mu_r \operatorname{rot} w^k - (\mu + \mu_r)\Delta u^k,\ v\rangle \phi(t) dt = 0$$

for every $\phi \in L^\infty(0,T)$ and passing to the limit for $k \to \infty$, by standard way we obtain

$$\int_0^T \langle \rho u_t + \rho u \cdot \nabla u - \rho f - 2\mu_r \operatorname{rot} w - (\mu + \mu_r)\Delta u,\ v\rangle \phi(t) dt = 0$$

for every $\phi \in L^\infty(0,T)$. Now with help to the Du Bois-Reymond's Theorem we obtain

$$\langle \rho u_t + \rho u \cdot \nabla u - \rho f - 2\mu_r \operatorname{rot} w - (\mu + \mu_r)\Delta u,\ v\rangle = 0$$

a.e. in Ω, for every $v \in L^2(\Omega)$. This

$$P(\rho u_t + \rho u.\nabla u - \rho f - 2\mu_r \text{ rot } w - (\mu + \mu_r)\Delta u) = 0$$

a.e. in Ω.

The passing to the limit in the equation for w^k is similary. For the density, we observe that

$u^k \longrightarrow u$ strong in $L^2(\Omega \times (0,T))$,
$\rho_t^k \longrightarrow \rho_t$ weak in $L^2(\Omega \times (0,T))$ and
$\nabla \rho^k \longrightarrow \nabla\rho$ weak in $L^2(\Omega \times (0,T))$.

Thus we have passing to the limit for $k \longrightarrow \infty$, in the continuity equation approximed:

$$\rho_t + u \cdot \nabla \rho = 0 \text{ in the } L^2(\Omega \times (0,T)) \text{ sense.}$$

Next, we prove the continuous assumption of the initial data, we have

Proposition 4.1 Under the hypotheses done, we have

$$(i) \quad \lim_{t\to 0^+} \|u(x,t) - u(x,0)\| = 0,$$

$$(ii) \quad \lim_{t\to 0^+} \|w(x,t) - w(x,0)\| = 0,$$

$$(iii) \quad \lim_{t\to 0^+} \|\nabla u(x,t) - \nabla u(x,0)\| = 0,$$

$$(iv) \quad \lim_{t\to 0^+} \|\nabla u(x,t) - \nabla u(x,0)\| = 0,$$

i.e., the solution u, w assumes the initial data continuously in the $H^1(\Omega)$-norm.

Proof. We prove only (i) and (iii), (ii) and (iv) are analogously proved. We chosen approximation u^k, to satisfy the conditions $u^k(0) \longrightarrow u(0)$, in the $L^2(\Omega)$ sense (strong). We have

$$u^k(x,t) - u^k(x,0) = \int_0^t u_t^k(x,s)ds$$

for every $k = 1, 2, \ldots$. Thus,

$$\|u^k(x,t) - u^k(x,0)\| \leq \int_0^t \|u_t^k(s)\|ds \leq Ct$$

in virtue of Lemma 3.2. Now, we have passing to the limit for $k \longrightarrow \infty$

$$\|u(x,t) - u(x,0)\| \leq Ct$$

Finally, if $t \longrightarrow 0^+$, we obtain (i).

Considerer now (iii). One easily concludes that

$$\lim_{t\to 0^+} \sup \|\nabla u(t)\| \leq \|\nabla u_0\|.$$

Thus, $u(t) \longrightarrow u_0$ strongly in V if $u(t) \longrightarrow u_0$ weakly in V; and to established the latter we need only show

$$\int_\Omega \nabla(u(t) - u_0)\nabla\varphi^l dx \longrightarrow 0 \qquad \text{as} \quad t \longrightarrow 0^+,$$

for each basis function φ^l. This requires several observations. First, notice that

$$\begin{aligned} (4.5) \qquad & |\int_\Omega \nabla(u^k(t) - u^k(0))\nabla\varphi^l dx| = |\int_0^t \frac{d}{dt}(\nabla u^k,\ \nabla\varphi^l)ds| \\ & = |-\int_0^t (u_t^k,\ P\Delta\varphi^l)ds| \leq \frac{1}{2}\int_0^t \|u_t^k\|^2 ds + \frac{1}{2}\int_0^t \|P\Delta\varphi^l\|^2 ds \leq Ct \end{aligned}$$

thanks to the lemmas. Consequently,

$$(4.6) \qquad \int_\Omega \nabla(u^k(t) - u^k(0))\nabla\varphi^l dx \longrightarrow 0 \qquad \text{as} \qquad t \longrightarrow 0^+.$$

Next, observe that for any fixed $t \in (0,T)$,

$$\int_\Omega \nabla(u(t) - u^k(t))\nabla\varphi^l dx \longrightarrow 0 \qquad \text{as} \qquad k \longrightarrow \infty,$$

because if we set $\chi^k = u - u^k$ and let $h(s)$ be a smooth function, wich vanishes for $s \leq t/2$ and equals one for $s \geq t$, then

$$\begin{aligned}(4.7)\qquad \int_\Omega \nabla^k \chi \nabla \varphi^l dx &= \int_0^t \frac{d}{ds} \int_\Omega h(s) \nabla \chi^k \nabla \varphi^l dx ds \\ &= \int_0^t \int_\Omega \{h_s \nabla \chi^k \nabla \varphi^l - h \chi_t^k P \Delta \varphi^l\} dx ds \longrightarrow 0\end{aligned}$$

as $k \longrightarrow \infty$. Here we are appealing to the weak convergence $\nabla \chi^k, \chi_t^k \rightarrow 0$ in $L^2(0,T;L^2(\Omega))$. Finally, we note

$$\int_\Omega \nabla(u^k(0) - u_0) \nabla \varphi^l dx = 0 \quad \text{for} \quad k \geq l,$$

is just another way of starting the condition used to determine the initial value $c_{l,k}(0)$. Cleary (4.5), (4.6) and (4.7) together imply (iii). This complete the proof. ■

Proposition 4.2 Under the hypotheses done, we have

(i) $\lim\limits_{t \to 0^+} \|P\Delta u(x,t) - P\Delta u(x,0)\| = 0,$
i.e., the initial velocity is assumed strongly in $H^2(\Omega)$.

(ii) $\lim\limits_{t \to 0^+} \|u_t(x,t) - u_t(x,0)\| = 0.$

Proof. To prove (i), it is sufficient to show

$$\lim_{t \to 0^+} \sup \|P\Delta u(.,t)\| \leq \|P\Delta u_0\|,$$

as we already know $u(.,t) \rightarrow u_0$ in $H^1(\Omega)$. Multiplying (2.3)i by $P\Delta u_t^k$ and integrating in Ω, we get

$$\begin{aligned}\|P\Delta u^k\|^2 &\leq \|P\Delta u_0\|^2 + 2(\mu+\mu_r)^{-1}\{(\rho^k u^k \nabla u^k - \rho^k f - 2\mu_r \text{ rot}\, w^k,\ P\Delta u^k) \\ &\quad + (\rho_0 u_0^k \nabla u_0^k - \rho_0 f_0 - 2\mu_r \text{ rot } w_0^k,\ P\Delta u_0^k)\} + Nt\end{aligned}$$

uniformly in k. From this, we conclude

$$\begin{aligned}\|P\Delta u(t)\|^2 &\leq \|P\Delta u_0\|^2 + 2\{(\rho u \nabla u - \rho f - 2\mu_r \text{rot}\, w, P\Delta u) - (\rho_0 u_0 \nabla u_0 \\ &\quad - \rho_0 f_0 - 2\mu_r \text{rot}\, w_0,\ P\Delta u_0)\} + Nt.\end{aligned}$$

Since $\rho u \nabla u \rightarrow \rho_0 u_0 \nabla u_0$ in $L^2, \rho f \rightarrow \rho_0 f_0$ in $L^2,$ $\text{rot}\, w \rightarrow \text{rot}\, w_0$ in L^2 and $P\Delta u \rightarrow P\Delta u_0$ weakly in L^2 as $t \rightarrow 0^+$, we obtain the desired result. Cleary, now, (ii) follows from (i). This complete the proof of the Proposition. ■

Analogously, for the angular velocity, we have the

Proposition 4.3 Under the hypotheses done, we have

(i) $\lim_{t \to 0^+} \|\Delta w(x,t) - \Delta w(x,0)\| = 0,$
i.e., the initial angular velocity is assumed strongly in $H^2(\Omega)$.

(ii) $\lim_{t \to 0^+} \|w_t(x,t) - w_t(x,0)\| = 0,$ ■

Remark. The argument used in the propositions truly can be make for all $t = t_0 > 0$ instead of $t = 0$. This we will give the continuity to the right in the spaces adequate. The same type of analysis we give the ontinuity to the left for $t = t_0 > 0$. For the same reason it is obtain the continuity indicate in the enunciate of Theorem 2.1.

5. Uniqueness

We consider now the question of uniqueness the solution. Let

$$\begin{aligned}
&\Sigma_1 = \{v \,/\, v \in L^2(0,T_1; H^3(\Omega) \cap V),\ v_t \in L^2(0,T_1;V)\},\\
&\Sigma_2 = \{u \,/\, u \qquad \text{satisfy the conclusions of Theorem 2.1}\},\\
&\mathcal{H}_1 = \{\psi \,/\, \psi \in L^2(0,T_1; H^2(\Omega) \cap H^1_0(\Omega)), \psi_t \in L^2(0,T_1;H^1(\Omega))\}\\
&\mathcal{H}_2 = \{w \,/\, w \qquad \text{safisfy the conditions of Theorem 2.1}\}.
\end{aligned}$$

With this notations we can enunciate the

Theorem 5.1 Assumed that (σ, v, ψ) is any one solution of the problem (1.1) - (1.3) in $C^1(\Omega \times [0,T]) \times \sum_1 \times \mathcal{H}_1$. Then, we have

$$\rho = \sigma,\ u = v \qquad \text{and} \qquad w = \psi$$

in $[0,T_2]$, where $T_2 = \min\{T, T_1\}$, where T is the time give in Theorem 1 and (ρ, u, w) is the solution of the problem (1.1) - (1.3) obtained in $C^1(\Omega \times [0,T]) \times \sum_2 \times \mathcal{H}_2$.

Proof. Let $\pi = \rho - \sigma,\ \eta = u - v$ and $\xi = w - \psi$. Then these variables satisfy the following equations

$$
(5.1)\begin{cases}
\pi_t + u\cdot\nabla\pi = -\eta\nabla\sigma,\\
\pi(0)=0,\\
P(\rho\eta_t) + (\mu+\mu_r)A\eta(t) = P(\pi f) + 2\mu_r P(\operatorname{rot}\xi) - P(\pi v_t) - P(\pi u\cdot\nabla u)\\
\qquad\qquad\qquad\qquad\qquad -P(\sigma\eta\cdot\nabla u) - P(\sigma v\cdot\nabla\eta),\\
\eta(x,0)=0,\\
\rho\xi_t - (c_a+c_d)\Delta\xi - (c_0+c_d-c_a)\nabla\ \operatorname{div}\xi + 4\mu_r\xi\\
\qquad\qquad = \pi g + 2\mu_r\ \operatorname{rot}\eta - \pi\psi_t - \pi u\cdot\nabla w - \sigma\eta\cdot w - \sigma v\cdot\nabla\eta,\\
\xi(x,0)=0.
\end{cases}
$$

Multiplying (5.1)iii by η and integrating over Ω we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\eta\|^2 + (\mu+\mu_r)\|\nabla\eta\|^2 &= (\pi f,\eta) + (\pi v_t,\eta) + 2\mu_r(\operatorname{rot}\xi,\eta) - (\pi u\cdot\nabla u,\eta)\\
&\quad -(\sigma\eta\cdot\nabla u,\eta) - (\sigma v\cdot\nabla\eta,\eta) + \frac{1}{2}(\rho_t\eta,\eta).
\end{aligned}
$$

Now, estimating as it is usual in the above identity, we obtain the following integral inequality

$$
\begin{aligned}
&\|\rho^{1/2}(t)\eta(t)\|^2 + (\mu+\mu_r)\int_0^t \|\nabla\eta(s)\|^2 ds\\
&\quad\le C\int_0^t (\|f(s)\|_{H^1}^2 + \|\nabla v_t(s)\|^2 + \|Au(s)\|^4)\|\pi(s)\|^2 ds + C\int_0^t \|\xi(s)\|^2 ds\\
&\qquad +C\int_0^t (\|Au(s)\|^2 + \|A\sigma(s)\|^2 + \|\rho_t(s)\|_{L^\infty})\|\eta(s)\|^2 ds.
\end{aligned}
$$

(δ will be chosen suitably).

Multiplying (5.1)i by π and integrating over Ω, after of integrate over $[0,T]$ we obtain

$$
\begin{aligned}
\|\pi(t)\|^2 &\le C\int_0^t \|\eta(s)\|\|\nabla\sigma(s)\|_{L^\infty}\|\pi(s)\|ds\\
&\le C\left(\int_0^t \|\eta(s)\|^2 ds + \int_0^t \|\pi(s)\|^2 ds\right). \qquad (5.2)
\end{aligned}
$$

Multiplying (5.1)v by ξ and integrating over Ω one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\xi\|^2 + (c_0+c_d)\|\nabla\xi\|^2 + (c_0+c_d-c_a)\|\operatorname{div}\xi\|^2 + 4\mu_r\|\xi\|^2\\
&\quad = (\pi g,\xi) + 2\mu_r(\operatorname{rot}\eta,\xi) - (\pi\psi_t,\xi) - (\pi u\cdot\nabla w,\xi)\\
&\qquad -(\sigma\eta\cdot\nabla w,\xi) - (\sigma v\cdot\nabla\xi,\xi) - \frac{1}{2}(\rho_t\xi,\xi). \qquad (5.3)
\end{aligned}
$$

Estimating as it is usual in the above identity, we obtain the following integral inequality

$$
\begin{aligned}
&\|\rho^{1/2}(t)\xi(t)\|^2 + (c_a + c_d)\int_0^t \|\nabla\xi(s)\|^2 ds + (c_0 + c_d - c_a)\int_0^t \|\operatorname{div}\xi(s)\|^2 ds \\
&+4\mu_r \int_0^t \|\xi(s)\|^2 ds \\
&\quad \leq C\int_0^t (\|g(s)\|_{H^1}^2 + \|\nabla\psi_t(s)\|^2 + \|Au(s)\|^2\|\Delta w(s)\|^2)\, ds \\
&\qquad +C\int_0^t \|\eta(s)\|^2(1 + \|\sigma(s)\|_{L^\infty}^2\|\Delta w(s)\|^2)\, ds \\
&\qquad +C\int_0^t \|\xi(s)\|^2(\|\rho_t(s)\|_{L^\infty} + \|\sigma(s)\|_{L^\infty}^2\|Av(s)\|^2)\, ds. \qquad (5.4)
\end{aligned}
$$

Adding (5.2), (5.3) and (5.4), one has

$$
\|\xi(s)\|^2 + \|\eta(s)\|^2 + \|\pi(s)\|^2 \leq \int_0^t h(s)(\|\pi(s)\|^2 + \|\xi(s)\|^2 + \|\eta(s)\|^2) ds
$$

$$
\begin{aligned}
\text{where} \quad h \quad = \quad & C(1 + \|f\|_{H^1}^2 + \|g\|_{H^1}^2 + \|\nabla v_t\|^2 + \|\nabla\psi_t\|^2 + \|Au\|^4 + \|Au\|^2\|\Delta w\|^2 \\
& +\|Au\|^2 + \|Av\|^2 + \|\rho_t\|_\infty + \|\sigma\|_{L^\infty}^2\|\Delta w\|^2 + \|\sigma\|_{L^\infty}^2\|Av\|^2).
\end{aligned}
$$

We observe that $h(s) \geq 0$ and $h(\cdot)$ is a integrable function, consequently applying the Gronwall's Lemma, we get

$$
\|\xi(t)\|^2 + \|\eta(t)\|^2 + \|\pi(t)\|^2 = 0,
$$

thus we obtain $\sigma = \rho,\ u = v$ and $w = \psi$. ■

6. Results the Pressure

We can also obtain now informations on the pressure.

Proposition 6.1 Under the hypothesis to the Theorem 2.1, there is $p \in C(\varepsilon, T; H^1(\Omega)/\mathbb{R})$, for any $\varepsilon > 0$ such that together to the solution (u, w, ρ) given the Theorem 2.1 satisfy

$$
\begin{aligned}
&\rho u_t + \rho u \cdot \nabla u - (\mu + \mu_r)\Delta u + \nabla p = \rho f + 2\mu_r \operatorname{rot} w, \\
&\operatorname{div} u = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\rho w_t + \rho u \cdot \nabla w - (c_0 + c_d)\Delta w + (c_0 + c_d - c_a)\nabla \operatorname{div} w + 4\mu_r w \\
&\qquad\qquad = \rho g + 2\mu_r \ \operatorname{rot} u, \\
&\rho_t + u \cdot \nabla \rho = 0, \\
&u|_{\partial\Omega} = 0,\ w|_{\partial\Omega} = 0,\ u(0) = u_0,\ w(0) = w_0,\ \rho(0) = \rho_0.
\end{aligned}
$$

Proof. We have

$$-(\mu + \mu_r)\Delta u + \nabla p = j$$

where $j = \rho(f - u_t - u \cdot \nabla u) + 2\mu_r \operatorname{rot} w$. We observe that the Theorem 2.1 implies that $j \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, applying the estimates for the Stokes problem (Amrouche and Girault [1]), we have $p \in L^\infty(0, T; H^1(\Omega)/I\!R) \cap L^2(0, T; H^2(\Omega)/I\!R)$. Therefore, we have

$$-(\mu + \mu_r)\Delta u_t + \nabla p_t = j_t,$$

where $j_t = p_t(f - u_t - u \cdot \nabla u) + 2\mu_r \operatorname{rot} w_t + \rho(f_t - u_{tt} - u_t \cdot \nabla u - u \cdot \nabla u_t) \in L^\infty(\varepsilon, T; L^2(\Omega))$. Thus, newly by the estimates for the Stokes problem, we get $p_t \in L^\infty(\varepsilon, T; H^1(\Omega)/I\!R)$, for any $\varepsilon > 0$, consequently, we have

$$p \in C(\varepsilon, T' H^1(\Omega)/I\!R),\ \forall\, \varepsilon > 0.$$

Remark. In order to obtain informations in $t = 0$ are necessary certain conditions of compatibility over the datum. This is done of the same manner as in the case of the Navier-Stokes equations and for this is very instructive the discussion make in the paper of Heywood and Rannacher [8].

7. Remark on the Global Existence

We present three Theorems on global existence in time of strong solutions for problem (1.1) - (1.3). By using the thechnicality of the above section together with the arguments of the work [3], we can proved easily in the case $n = 3$

Theorem 7.1 ($n = 3$). Let the initial values satisfy $u_0 \in V \cap (H^2(\Omega)^3)$, $w_0 \in (H_0^1(\Omega)^3) \cap (H^2(\Omega)^3)$, $\rho_0 \in C^1(\overline{\Omega})$ and the external fields $f, g \in L^\infty([0, \infty); (H^1(\Omega))^3)$ with $f_t, g_t \in L^\infty([0, \infty); (L^2(\Omega))^3)$. If $\|u_0\|_{H^1}, \|w_0\|_{H^1}$ and $\|f\|_{L^\infty([0,\infty); L^2(\Omega))}$ and $\|g\|_{L^\infty([0,\infty); L^2(\Omega))}$ are sufficiently small, then the solution (ρ, u, w) of problem (1.1) and (1.2) exists globally in time and satisfies

$$u \in C([0, \infty); V \cap H^2(\Omega)), \quad w_0 \in C([0, \infty);\ H_0^1(\Omega) \cap H^2(\Omega))$$

$\rho \in C^1(\overline{\Omega} \times [0,T])$ for any $T > 0$. Moreover, for any $\gamma > 0$ there exists some finite positive constants M and C such that

$$\sup_{t\geq 0} \|\nabla u(t)\| = M, \qquad \sup_{t\geq 0} \|\nabla w(t)\| = M,$$
$$\sup_{t\geq 0} \|u_t(t)\| \leq C, \qquad \sup_{t\geq 0} \|w_t(t)\| \leq C,$$
$$\sup_{t\geq 0} \|Au(t)\| \leq C, \qquad \sup_{t\geq 0} \|\Delta w(t)\| \leq M,$$
$$\sup_{t\geq 0} e^{-\gamma t} \int_0^t e^{\gamma s} (\|\nabla u_t(s)\|^2 + \|\nabla w_t(s)\|^2) ds \leq C,$$
$$\sup_{t\geq 0} e^{-\gamma t} \int_0^t e^{\gamma s} (\|u(s)\|^2_{W^{2,6}} + \|w(s)\|^2_{W^{2,6}}) ds \leq C,$$
$$\sup_{t\geq 0} e^{-\gamma t} \int_0^t e^{\gamma s} (\|\nabla u(s)\|^2_{C(\Omega)} + \|\nabla w(s)\|^2_{C(\Omega)}) ds \leq C.$$

Also the same kind of estimates hold uniformly in k for the semi-Galerkin approximations. ■

In the case two-dimensional, we have

Theorem 7.2 ($n = 2$). Suppose that the initial values satisfy $u_0 \in V \cap (H^2(\Omega))^2, w_0 \in (H_0^1(\Omega))^2 \cap (H^2(\Omega))^2, \rho_0 \in C^1(\overline{\Omega})$ and the external fields $f, g \in L^\infty([0,\infty); (H^1(\Omega))^2), f_t, g_t \in L^\infty([0,\infty); (L^2(\Omega))^2)$ then the solution (ρ, u, w) of problem (1.1) and (1.2) exists globally in time and satisfies $u, w \in C([0,\infty); V \cap (H^2(\Omega))^2)$, $\rho \in C^1(\overline{\Omega} \times [0,T])$ for any $T > 0$. Moreover, the estimates given in Theorem 7.1 are true for any $\gamma > 0$.

Theorem 7.3 Suppose that $n = 2$ or 3, that

$$u_0 \in V \cap (H^2(\Omega))^n, \quad w_0 \in (H_0^1(\Omega))^n \cap (H^2(\Omega))^n,$$

$\rho_0 \in C^1(\overline{\Omega})$ and that for some constant $\overline{\gamma} > 0$,

$$e^{\overline{\gamma} t}(f + g) \in L^\infty([0,\infty); (H^1(\Omega))^n), \quad e^{\overline{\gamma} t}(f_t + g_t) \in L^\infty([0,\infty); (L^2(\Omega))^n).$$

Under these conditions if $n = 2$, or with the additional condition that

$$\|u_0\|_{H^1(\Omega)}, \; \|w_0\|_{H^1(\Omega)}, \; \|e^{\overline{\gamma} t} f\|_{L^\infty([0,\infty);(L^2(\Omega))^n)}$$

and $\|e^{\overline{\gamma} t} g\|_{L^\infty([0,\infty);(L^2(\Omega))^n)}$ are small enough if $n = 3$, then there is a global solution (ρ, u, w) of problem (1.1) - (1.2). Moreover, there is a positive constant $\gamma^* \leq \overline{\gamma}$ such that for any $0 \le \theta < \gamma^*$ there hold the following estimates

$$\sup e^{\gamma^* t}(\|\nabla u(t)\|^2 + \|\nabla w(t)\|^2) < +\infty$$
$$\sup_{t\ge 0} e^{\theta t}(\|u_t(t)\|^2 + \|w_t(t)\|^2 + \|Au(t)\|^2 + \|\Delta w(t)\|^2) < +\infty$$
$$\sup_{t\ge 0} \int_0^t e^{\theta s}(\|\nabla u_t(s)\|^2 + \|\nabla w_t(s)\|^2)ds < +\infty$$
$$\sup_{t\ge 0} \int_0^t e^{\theta s}(\|u(s)\|^2_{W^{2,6}} + \|w(s)\|^2_{W^{2,6}})ds < +\infty,$$
$$\sup_{t\ge 0} \int_0^t e^{\theta s}(\|\nabla u(s)\|^2_{C(\Omega)} + \|\nabla w(s)\|^2_{C(\Omega)})ds < +\infty$$
$$\sup_{t\ge 0}(\|\nabla \rho(t)\|_{L^\infty} + \|\rho_t(s)\|_{L^\infty}) < +\infty$$
$$\sup_{t\ge 0} \sigma(t)(\|\nabla u_t(t)\|^2 + \|\nabla w_t(t)\|^2) < +\infty$$
$$\sup_{t\ge 0} \int_0^t \sigma(s)(\|u_{tt}(s)\|^2 + \|w_{tt}(s)\|^2)ds < +\infty$$
$$\sup_{t\ge 0} \int_0^t \sigma(s)(\|Au_t(s)\|^2 + \|\Delta w_t(s)\|^2)ds < +\infty$$

In the last three estimates $\sigma(t) = \min\{1, t\}e^{\theta t}$ the same kind of estimates hold uniformly in $k \in I\!N$ for the semi-Galerkin approximations. ■

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