Stationary solutions of magneto-micropolar fluids equations in exterior domains

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Abstract

We establish the existence of a weak solution for the equations of motion of magneto-micropolar fluid in exterior domains. Also we discuss the uniqueness of weak solutions.

1. Introduction

In this work we study existence of weak solutions for the equations that describes the motion of a viscous incompressible magneto-micropolar fluid in a exterior domain $\Omega \subseteq I\!R^3$. Such equation are given by (see [1], for instance):

$$\begin{aligned} & u \cdot \nabla u - (\mu + \chi)\Delta u + \nabla(p + \frac{r}{2} h \cdot h) = \chi \operatorname{rot} w + r h \cdot \nabla h + f, \\ & j u \cdot \nabla w - \gamma \Delta w + 2\chi w - (\alpha + \beta)\nabla \operatorname{div} w = \chi \operatorname{rot} u + g, \\ & -\nu \Delta h + u \cdot \nabla h - h \cdot \nabla u = 0, \\ & \operatorname{div} u = 0, \quad \operatorname{div} h = 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{1.1}$$

Here, $u(x) \in I\!R^3$ denotes the velocity of the fluid at point $x \in \Omega$; $w(x) \in I\!R^3$, $h(x) \in I\!R^3$ and $p(x) \in I\!R$ denote, respectively, the microrotational velocity, the magnetic field and the hydrostatic pressure; the constants $\mu,\ \chi,\ r,\ \alpha,\ \beta,\ \gamma,\ j$ and ν are constants associated to properties of the material. From physical reasons, these constants satisfy $\min\{\mu,\ \chi,\ r,\ j,\ \gamma,\ \nu,\ \alpha+\beta+\gamma\} > 0$; $f(x)$ and $g(x)\ \in I\!R^3$ are given external fields.

Work performed under the auspices of FAPESP(Fundação de Amparo à Pesquisa do Estado de São Paulo, Brazil), Project Number: 1997/3711-0, and CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil), Project Number: 300116/93-4(RN)

1991 Mathematics Subject Classification: 35 Q35, 76 D05

We assume that on the boundary $\partial\Omega$ of Ω, the following conditions hold

$$u(x) = w(x) = h(x) = 0, \quad x \in \partial\Omega \tag{1.2}$$

and the following conditions at infinity

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} w(x) = \lim_{|x|\to\infty} h(x) = 0. \tag{1.3}$$

Equation (1.1)(i) has the familiar form of the Navier-Stokes equations but it is coupled with equation (1.1)(ii), which essentially describes the motion inside the macro-volumes as they undergo microrotational effects represented by the microrotational velocity vector w. For fluids with no microstructure this parameter vasnishes. For Newtonian fluids, equations (1.1)(i) and (1.1)(ii) decouple since $\chi = 0$.

It is now appropriate to cite some earlier works on the initial boundary-value problem (1.1)-(1.3), which are related to ours and also to locate our contribution therein. When the magnetic field is absent ($h \equiv 0$), the reduced problem was studied by Lukaszewicz [8], Abid [2]. Lukaszewicz [8] established existence of weak solutions for (1.1) - (1.3) in bounded domains under certain assumptions by using linearization and the Leray-Schauder principle. In the same paper, by using the regurality of the Stokes equations and the elliptic systems proved the regularity of solutions, he also show conditions under to which the uniqueness holds. Again when $h \equiv 0$, Abid [2] established results similar to the ones of Lukaszewicz [8] in exterior domains by using the results of Girault and Sequeira [4] for Navier-Stokes equations .

In this work, we use "the extending domain method" as in Ladyzhenskaya[7] and Heywood [6], to prove the existence of weak solutions , we also discuss the uniqueness of solutions.

We reach in this way, for weak solutions, basically the same level of knowledge as in the case of the classic Navier-Stokes equations.

Finally, the paper is organized as follows: in Section 2 we state the basic assumptions and results that to used later on in the paper; we also rewrite (1.1) - (1.3) in a more suitable weak form; we describe the approximation method and state our results (Theorems 2.3 and 2.4). Each one of the following sections will be devoted to their proofs.

2. Functions spaces and preliminaires

The functions in this paper are either $I\!R$ or $I\!R^3$-values and we will not distinguish these two situations in our notations. To which case we refer to will be clear from the context.

Now, we give the precise definition of the exterior domain Ω where our boundary-value problem associated to the problem (1.1)-(1.3) has been formulated.

Let K a subset compact of $I\!R^3$ whose bondary ∂K is of classe C^2. The exterior domain Ω that we will consider is $\Omega = K^c$ and $\partial\Omega = \partial K$.

The extending domain method was introduced by Ladyzhenkaya [7] to study the Navier-Stokes equations in unbounded domains. As was observed by Heywood [6] the method is useful in certain class of unbounded domain, in this class certainly our domain is.

The principal ideia is the following: the exterior domain Ω can be approximated by interior domain $\Omega_k = B_k \cap \Omega$ (B_k is a ball with radius k and center at 0) as $k \to \infty$.

In each interior domain Ω_k, we will prove the existence of weak solution, to do is, we will use the Galerkin method together with the Brouwer's Fixed Point Theorem as in Heywood [6]. Next, by using the estimates given in Ladyzhenskaya's book [7] (we recall these estimates later) together with diagonal argument and Rellich's compactness theorem, we obtain the desirable weak solution to problem (1.1)-(1.3). Properties of regularity and uniqueness are also studied.

We use several functions spaces. D denotes Ω or Ω_k.

$$\begin{aligned} W^{r,p}(D) &= \{u \; ; \; D^\alpha u \in L^p(D), |\alpha| \le r\}, \\ W_0^{r,p}(D) &= \text{Completion of } C_0^\infty(D) \text{ in } W^{r,p}(D), \\ C_{0,\sigma}^\infty(D) &= \{\varphi \in C_0^\infty(D) \; ; \; \operatorname{div}\varphi = 0\}, \end{aligned}$$

$$
\begin{aligned}
J(D) &= \text{Completion of} C_{0,\sigma}^{\infty}(D) \text{ in norm } \|\nabla\phi\| \\
H(D) &= \text{Completion of} C_{0,\sigma}^{\infty}(D) \text{ in norm } \|\phi\|
\end{aligned}
$$

Here $\|\cdot\|$ is the L^2-norm; the L^p-norm we denote by $\|\cdot\|_p$.

We observe that $J(D)$ is equivalent to

$$
\{\phi \in W^{1,2}(D) \ ; \ \phi|_{\partial\Omega} = 0, \ \operatorname{div}\phi = 0\},
$$

as was proved by Heywood [5].

When $p = 2$, as it usual, we denote $W^{r,p}(D) \equiv H^r(D)$ and $W_0^{r,p}(D) \equiv H_0^r(D)$.

The following inequalities are in Ladyzhenskaya [7].

Lemma 2.1. Let $D \subseteq I\!R^3$ bounded or unbounded then
(a) For $u \in W_0^{1,2}(D)$ (or $J(D)$ or $H_0^1(D)$), we have

$$
\|u\|_{L^6(D)} \le C_L \|\nabla u\|_{L^2(D)},
$$

where $C_L = (48)^{1/6}$.
(b) (Hölder's inequality). If each integral makes sense, then we have

$$
|((u\cdot\nabla)v, w)| \le 3^{\frac{1}{p}+\frac{1}{r}} \|u\|_{L^p(D)} \|\nabla v\|_{L^q(D)} \|w\|_{L^r(D)}
$$

where $p, q, r > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$.

We make several assumptions on the boundary $\partial\Omega$ and the external forces:

(S_1) $O_0 \subseteq$ int K (O_0 is a neighbourhood of the origen 0) and $K \subseteq B(0,d)$ which is a ball with radius d and center at 0.

(S_2) $\partial\Omega = \partial K \in C^2$.

(S_3) $f \in J(\Omega)^*$, $g \in H^{-1}(\Omega)$,
where $J(\Omega)^*$ is the topological dual of $J(\Omega)$.

Let us denote

$$
a(v,w) = \sum_{i,j=1}^{3} \int_D \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} dx, \quad b(u,v,w) = \sum_{i,j=1}^{3} \int_D u_j \frac{\partial v_i}{\partial x_j} w_i dx
$$

which we define for all vector-valued functions $u,\, v,\, w$ for which the integrals are well defined.

We can now define a notion of weak solution for (1.1)-(1.3).

Definition 2.2. We will say that a triple of functions (u, w, h) defined on Ω is a weak solution of (1.1)-(1.3) if only if the functions $u,\, w,\, h$ satisfy

$$u,\, h \in J(\Omega), \quad w \in H_0^1(\Omega)$$

and also satisfy the following equations

$$\begin{aligned}
&(\mu+\chi)a(u,\varphi) - b(u,\varphi,u) + rb(h,\varphi,h) = (f,\varphi) + \chi(w,\ \operatorname{rot}\varphi),\\
&\gamma a(w,\phi) + (\alpha+\beta)(\operatorname{div} w,\ \operatorname{div}\phi) - jb(u,\phi,w) + 2\chi(w,\phi) = (g,\phi) + \chi(u,\operatorname{rot}\phi),\\
&\nu a(h,\psi) - b(u,\psi,h) + b(h,\psi,u) = 0
\end{aligned}$$

for all $\varphi, \psi \in C_{0,\sigma}^\infty(\Omega)$ and $\phi \in C_0^\infty(\Omega)$.

Remark. If $u,\, h \in J(\Omega)$ and $w \in H_0^1(\Omega)$, then

$$u|_{\partial\Omega} = h|_{\partial\Omega} = w|_{\partial\Omega} = 0$$

and moreover por (a) in Lemma

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} w(x) = \lim_{|x|\to\infty} h(x) = 0.$$

Our results are:

Theorem 2.3 (Existence). Under the hypotheses (S_1), (S_2) and (S_3) the problem (1.1)-(1.3) has a stationary weak solution.

Theorem 2.4 (Uniqueness). Under the hypotheses (S_1), (S_2) and (S_3) if there exists a stationary weak solution satisfying the following conditions

$$\begin{aligned}
&\frac{3C_L}{2\mu}\left(\|u\|_3 + \|w\|_3 + (1+r)\,\|h\|_3\right) < 1,\\
&\frac{3C_L}{2r\nu}(\|u\|_3 + (1+r)\,\|h\|_3) < 1,\\
&\frac{3C_L}{2\gamma}\|w\|_3 < 1
\end{aligned}$$

where $C_L = (48)^{1/6}$, then the weak solution is unique.

3. The interior problem.

In this section we consider the following interior problem (P_k) in domains $\Omega_k = B_k \cap \Omega$ $(k \in I\!N)$.

$$(P_k) \quad \begin{cases} -(\mu+\chi)\Delta u + (u\cdot\nabla)u + \nabla(p + \frac{r}{2}h\cdot h) = \chi \mathrm{rot}\, w + r(h\cdot\nabla)h + f, \\ -\gamma\Delta w - (\alpha+\beta)\nabla \mathrm{div}\, w + j(u\cdot\nabla)w + 2\chi w = \chi \mathrm{rot} u + g, \\ -\nu\Delta h + (u\cdot\nabla)h - (h\cdot\nabla)u = 0, \\ \mathrm{div}\ u = 0, \quad \mathrm{div}\ h = 0, \\ u = 0,\ w = 0,\ h = 0 \ \text{ on } \ \partial\Omega_k = \partial\Omega\cap\partial B_k. \end{cases}$$

The notion of weak solution for $(P_k)$is completely similar to the ones for (1.1)-(1.3).

Proposition 3.1. The problem (P_k) has a weak solution $(\tilde{u}^k, \tilde{w}^k, \tilde{h}^k) \in J(\Omega_k) \times H_0^1(\Omega_k) \times J(\Omega_k)$.

To prove the existence of weak solution of the system (P_k) we will use the Galerkin method together with Brouwer's Fixed Point Theorem as in Fujita [3] (see also Heywood [6]).

First, we will prove a priori estimates for weak solution of (P_k).

Lemma 3.2. Let $(\tilde{u}^k, \tilde{w}^k, \tilde{h}^k)$ a weak solution of (P_k). Then, they satisfy the following estimates

$$\mu\, a(\tilde{u}^k, \tilde{u}^k) + \gamma\, a(\tilde{w}^k, \tilde{w}^k) + 2r\nu\, a(\tilde{h}^k, \tilde{h}^k) \le \frac{1}{\mu}\|f\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\|g\|^2_{H^{-1}(\Omega)}. \tag{3.1}$$

Proof. Multiplying $(P_k)i$, $(P_k)ii$ and $(P_k)iii$ by $\tilde{u}^k, \tilde{w}^k$ and $r\tilde{h}^k$, respectively, after of integrate on Ω_k, we obtain

$$\begin{aligned} &(\mu+\chi)\, a(\tilde{u}^k, \tilde{u}^k) = \chi(\mathrm{rot}\, \tilde{w}^k, \tilde{u}^k) + rb(\tilde{h}^k, \tilde{h}^k, \tilde{u}^k) + (f, \tilde{u}^k), \\ &\gamma\, a(\tilde{w}^k, \tilde{w}^k) + (\alpha+\beta)\|\mathrm{div}\, \tilde{w}^k\|^2 + 2\chi\|\tilde{w}^k\|^2 = \chi(\mathrm{rot}\, \tilde{u}^k, \tilde{w}^k) + (g, \tilde{w}^k), \\ &r\nu\, a(\tilde{h}^k, \tilde{h}^k) = rb(\tilde{h}^k, \tilde{u}^k, \tilde{h}^k). \end{aligned}$$

Adding the above equalities, we get

$$
\begin{aligned}
&(\mu+\chi)\, a(\widetilde{u}^k,\widetilde{u}^k)+\gamma\, a(\widetilde{w}^k,\widetilde{w}^k)+r\nu\, a(\widetilde{h}^k,\widetilde{h}^k)+(\alpha+\beta)\|\operatorname{div}\widetilde{w}^k\|^2+2\chi\|\widetilde{w}^k\|^2\\
&\qquad = 2\chi(\widetilde{w}^k,\operatorname{rot}\widetilde{u}^k)+(f,\widetilde{u}^k)+(g,\widetilde{w}^k) \qquad (3.2)
\end{aligned}
$$

since $r\, b(\widetilde{h}^k,\widetilde{h}^k,\widetilde{u}^k)+r\, b(\widetilde{h}^k,\widetilde{u}^k,\widetilde{h}^k)=0$.

We estimate of the right-hand side of the equality (3.2) as follows

$$
2\chi(\widetilde{w}^k,\ \text{rot}\ \widetilde{u}^k)\le 2\chi\|\widetilde{w}^k\|\ \|\text{rot}\widetilde{u}^k\|\le 2\chi\|\widetilde{w}^k\|\ \|\nabla\widetilde{u}^k\|\le \chi\|\widetilde{w}^k\|^2+\chi a(\widetilde{u}^k,\widetilde{u}^k),
$$

since $\|\operatorname{rot}\widetilde{u}^k\|=\|\nabla\widetilde{u}^k\|$. Also,

$$
\begin{aligned}
(f,\widetilde{u}^k) &\le \|f\|_{J(\Omega_k)^*}\|\nabla\widetilde{u}^k\|\le \frac{1}{2\mu}\|f\|^2_{J(\Omega)^*}+\frac{\mu}{2}a(\widetilde{u}^k,\widetilde{u}^k),\\
(g,\widetilde{w}^k) &\le \|g\|_{H^{-1}(\Omega_k)}\|\nabla\widetilde{w}^k\|\le \frac{1}{2\gamma}\|g\|^2_{H^{-1}(\Omega)}+\frac{\gamma}{2}a(\widetilde{w}^k,\widetilde{w}^k).
\end{aligned}
$$

Consequently, using the above estimates in (3.2), we get

$$
\begin{aligned}
&\mu a(\widetilde{u}^k,\widetilde{u}^k)+\gamma a(\widetilde{w}^k,\widetilde{w}^k)+2r\nu a(\widetilde{h}^k,\widetilde{h}^k)+2\chi\|\widetilde{w}^k\|^2+2(\alpha+\beta)\|\operatorname{div}\widetilde{w}^k\|^2\\
&\qquad\le \frac{1}{\mu}\|f\|^2_{J(\Omega)^*}+\frac{1}{\gamma}\|g\|^2_{H^{-1}(\Omega)}.
\end{aligned}
$$

This estimates imply immediately (3.1).

Remark. We observe that estimate (3.1) is independent of k.

Now, we prove the existence of solution $(\widetilde{u}^k,\widetilde{w}^k,\widetilde{h}^k)$ for (P_k).

As mth approximate solution of eq. (P_k), we choose functions

$$
u^m(x)=\sum_{j=1}^m c_{mj}\varphi^j(x),\quad w^m(x)=\sum_{j=1}^m d_{mj}\phi^j(x)\quad\text{and}\quad h^m(x)=\sum_{j=1}^m e_{mj}\varphi^j(x),
$$

satisfying the equations

$$
(\mu+\chi)\, a(u^m,\varphi^j)+b(u^m,u^m,\varphi^j)-r\, b(h^m,h^m,\varphi^j)
$$

$$= \chi(\operatorname{rot} w^m, \varphi^j) + (f, \varphi^j), \tag{3.3}$$

$$\gamma\, a(w^m, \phi^j) + (\alpha+\beta)(\operatorname{div} w^m, \operatorname{div}\phi^j) + j b(u^m, w^m, \phi^j) + 2\chi(w^m, \phi^j)$$

$$= \chi(\operatorname{rot} u^m, \phi^j) + (g, \phi^j), \tag{3.4}$$

$$\nu\, a(h^m, \varphi^j) + b(u^m, h^m, \varphi^j) - b(h^m, u^m, \varphi^j) = 0, \tag{3.5}$$

for $1 \le j \le m$.

First we assume the existence of (u^m, w^m, h^m) for any $m \in N$. Note that solutions (u^m, w^m, h^m) must satisfy estimate (3.1). In fact, the identity (3.1) for u^m is obtained by multiplying (3.3) by c_{mj} and summing over j from 1 to m. Similary, we have identities (3.1) for w^m and h^m.

Estimate (3.1) follow from eqs. (3.3), (3.4) and (3.5) as in Lemma 3.2. Therefore the sequence (u^m, w^m, h^m) is bounded in $J(\Omega_k) \times H_0^1(\Omega_k) \times J(\Omega_k)$.

Since $J(\Omega_k)$ (respectively $H_0^1(\Omega_k)$) is compactly imbedded in $H(\Omega_k)$ (respectively $L^2(\Omega_k)$) we can choose subsequences which we again denote by (u^m, w^m, h^m) and elements $\tilde{u}^k \in J(\Omega_k)$, $\tilde{w}^k \in H_0^1(\Omega_k)$ and $\tilde{h}^k \in J(\Omega_k)$ such that

$$\left.\begin{array}{r} u^m \to \tilde{u}^k \\ h^m \to \tilde{h}^k \end{array}\right\} \text{ weakly in } J(\Omega_k) \text{ and strongly in } H(\Omega_k),$$

$$w^m \to \tilde{w}^k \quad \text{weakly in } H_0^1(\Omega_k) \text{ and strongly in } L^2(\Omega_k).$$

This is enough to take the limit as m goes to ∞ in (3.3), (3.4) and (3.5). Therefore, $(\tilde{u}^k, \tilde{w}^k, \tilde{h}^k)$ is a required weak solution to problem (P_k).

In order to prove the solvability of system (P_k) for any $k \in I\!N$, we follow Heywood [6] in applying Brouwer's Fixed Point Theorem.

Let V_m be the subespace of $J(\Omega_k)$ spanned by $\{\varphi^1, ..., \varphi^m\}$, and let M_m be the subespace of $H_0^1(\Omega_k)$ spanned by $\{\phi^1, ..., \phi^m\}$. For every $(v, \xi, \theta) \in V_m \times M_m \times V_m$ we consider the unique solution $L(v, \xi, \theta) = (u, w, h) \in V_m \times M_m \times V_m$ of the linearized equations

$$(\mu+\chi)a(u, \varphi^j) + b(v, u, \varphi^j) - r b(\theta, h, \varphi^j) - \chi(\operatorname{rot} w, \varphi^j) - (f, \varphi^j) = 0, \tag{3.6}$$

$$\gamma a(w, \phi^j) + (\alpha+\beta)(\operatorname{div} w, \operatorname{div}\phi^j) + j b(v, w, \phi^j) + 2\chi(w, \phi^j)$$

$$-\chi(\operatorname{rot} u, \phi^j) - (g, \phi^j) = 0, \tag{3.7}$$

$$\nu a(h, \varphi^j) + b(v, h, \varphi^j) - b(\theta, u, \varphi^j) = 0, \tag{3.8}$$

for $1 \leq j \leq m$. This is a system of $3m$ linear equations for the coefficients in the expansions $u = \sum_{j=1}^{m} c_j \varphi^j, \quad w = \sum_{j=1}^{m} d_j \phi^j, \quad h = \sum_{j=1}^{m} e_j \varphi^j$.

Equations (3.6), (3.7) and (3.8) have a unique solution because the associated homogeneous system ($f = 0$, $g = 0$) has an unique solution. In fact, if (u, w, h) is a solution of the homogeneous system, proceeding as before, we multiply (3.6) by c_j, (3.7) by d_j and (3.8) by re_j and sum over j from 1 to m, to obtain

$$\begin{aligned}
&(\mu + \chi))\|\nabla u\|^2 = \chi(\operatorname{rot} w, u) + rb(\theta, h, u),\\
&\gamma\|\nabla w\|^2 + (\alpha + \beta)\|\operatorname{div} w\|^2 + 2\chi\|w\|^2 = \chi(\operatorname{rot} u, w),\\
&r\nu\|\nabla h\|^2 = rb(\theta, u, h).
\end{aligned}$$

Adding the above identities, we obtain

$$\begin{aligned}
&(\mu + \chi)\|\nabla u\|^2 + \gamma\|\nabla w\|^2 + r\nu\|\nabla h\|^2 + 2\chi\|w\|^2 + (\alpha + \beta)\|\operatorname{div} w\|^2\\
&\quad = 2\chi(\operatorname{rot} u, w) \leq 2\chi\|\nabla u\|\|w\| \leq \chi\|\nabla u\|^2 + \chi\|w\|^2.
\end{aligned}$$

Consequently,

$$\mu\|\nabla u\|^2 + \gamma\|\nabla w\|^2 + r\nu\|\nabla h\|^2 + \chi\|w\|^2 + (\alpha + \beta)\|\operatorname{div} w\|^2 \leq 0.$$

Hence $u = 0$, $w = 0$ and $h = 0$. The continuity of L follows by applying arguments that are similar to the ones used to takes the limit in (3.3), (3.4) and (3.5).

We also have the estimate

$$\mu\|\nabla u\|^2 + \gamma\|\nabla w\|^2 + 2r\nu\|\nabla h\|^2 \leq \frac{1}{\mu}\|f\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\|g\|^2_{H^{-1}(\Omega)},$$

which are shown in exactly the same way as was done for a solution (u^m, w^m, h^m) of (3.3), (3.4), (3.5).

Then,

$$\|\nabla u\|^2 \leq \frac{1}{\mu^2}\|f\|^2_{J(\Omega)^*} + \frac{1}{\gamma\mu}\|g\|^2_{H^{-1}_{(\Omega)}} \equiv \ell_1^2 \tag{3.9}$$

$$\|\nabla w\|^2 \leq \frac{1}{\mu\gamma}\|f\|^2_{J(\Omega)^*} + \frac{1}{\gamma^2}\|g\|^2_{H^{-1}_{(\Omega)}} \equiv \ell_2^2 \tag{3.10}$$

$$\|\nabla h\|^2 \leq \frac{1}{2r\nu\mu}\|f\|^2_{J(\Omega)^*} + \frac{1}{2r\nu\gamma}\|g\|^2_{H^{-1}_{(\Omega)}} \equiv \ell_3^2 \tag{3.11}$$

Thus, (3.9), (3.10), (3.11) define a continuous mapping L from the closed and convex set

$$S = \{(v, \xi, \theta) \in V_m \times M_m \times V_m \ ; \ \|\nabla v\| \leq \ell_1, \quad \|\nabla \xi\| \leq \ell_2, \quad \|\nabla \theta\| \leq \ell_3\}$$

into itself. Using Brouwer's Fixed Point Theorem, we conclude that the map L has at least one fixed point, which is a solution of (3.6), (3.7), (3.8). Thus, the existence of weak solution $(\tilde{u}^k, \tilde{w}^k, \tilde{h}^k)$ of (P_k) is complete.

Lemma 3.2. Let $(\tilde{u}^k, \tilde{w}^k, \tilde{h}^k)$ be a weak solution for (P_k) obtained in Proposition 3.1. Put

$$u^k(x) = \begin{cases} \tilde{u}^k(x) & \text{if } x \in \Omega_k, \\ 0 & \text{if } x \in \Omega \setminus \Omega_k, \end{cases}$$

$$w^k(x) = \begin{cases} \tilde{w}^k(x) & \text{if } x \in \Omega_k, \\ 0 & \text{if } x \in \Omega \setminus \Omega_k, \end{cases}$$

$$h^k(x) = \begin{cases} \tilde{h}^k(x) & \text{if } x \in \Omega_k, \\ 0 & \text{if } x \in \Omega \setminus \Omega_k. \end{cases}$$

Then it holds that

$$(u^k, w^k, h^k) \in J(\Omega) \times H_0^1(\Omega) \times J(\Omega)$$

and furthermore

$$\|\nabla u^k\| \leq \ell_1, \quad \|\nabla w^k\| \leq \ell_2, \quad \|\nabla h^k\| \leq \ell_3$$

where ℓ_1, ℓ_2 e ℓ_3 be taken uniformly in k.

Proof. It is easy to show $(u^k, w^k, h^k) \in J(\Omega) \times H_0^1(\Omega) \times J(\Omega)$. The estimates are directly deduced from the estimates (3.9)-(3-11) and the lower semicontinuity of the norm.

4. Proof of Theorem of Existence

From estimates given in Lemma, we get by using the Rellich's compactness theorem and the diagonal argument, that there exist subsequences which we again denote by (u^k, w^k, h^k) and elements u, $h \in J(\Omega)$ and $w \in H_0^1(\Omega)$ such that

$$\left.\begin{array}{r} u^k \to u \\ h^k \to h \end{array}\right\} \text{ weakly in } J(\Omega) \text{ and strongly in } L^2_{loc}(\Omega), \tag{4.1}$$

$$w^k \to w \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2_{loc}(\Omega). \tag{4.2}$$

Once we obtain these convergences and limits we can show that (u, w, h) is a desirable stationary weak solution for (P). In fact, let (φ, ϕ, ψ) be an arbitrary given tests functions, then we find a bounded domain Ω' and k_0 such that supp φ, supp ϕ, supp $\psi \subseteq \Omega' \subseteq \Omega_{k_0} \subseteq \Omega_k$ for all $k \geq k_0$. Then,

$$\begin{aligned} &|((u^k \cdot \nabla)\varphi, w^k)_\Omega - ((u \cdot \nabla)\varphi, w)_\Omega| \\ &\quad \leq |((u^k - u)\nabla\varphi, w)_{\Omega'}| + |((u^k \cdot \nabla)\varphi, w - w^k)_{\Omega'}| \\ &\quad \leq 3\|u^k - u\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^3(\Omega')}\|w\|_{L^6(\Omega')} \\ &\qquad +3\|u^k\|_{L^6(\Omega'}\|\nabla\varphi\|_{L^3(\Omega')}\|w - w^k\|_{L^2(\Omega')} \\ &\quad \leq 3C\sqrt{\ell_2}\|u^k - u\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^3(\Omega')} + 3C\sqrt{\ell_1}\|w - w^k\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^3(\Omega')} \end{aligned}$$

thanks you to Lemmas 2.1 and 3.2 and the convergences (4.1) and (4.2) show that

$$|((u^k.\nabla)\varphi, w^k)_\Omega - ((u.\nabla)\varphi, w)_\Omega| \to 0$$

as $k \to \infty$. The other convergences are analogously established. Thus, we see

(u, w, h) is a stationary weak solution for (P).

5.Proof of Theorem of Uniqueness

Let (u^1, w^1, h^1), (u^2, w^2, h^2) be a weak solutions of (1.1), (1.2), (1.3). Put $u = u^1 - u^2,\ w = w^1 - w^2,\ h = h^1 - h^2$. Then, they satisfy

$$
\begin{aligned}
&(\mu+\chi)\ (\nabla u, \nabla\varphi) + (u\cdot\nabla u^1, \varphi) + (u^2\cdot\nabla u, \varphi) = \chi(\operatorname{rot} w, \varphi) + r(h\cdot\nabla h^1, \varphi)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +(h\cdot\nabla h, \varphi),\\
&\gamma(\nabla w, \nabla\phi) + (\alpha+\beta)(\operatorname{div} w, \operatorname{div}\phi) + 2\chi(w,\phi) + j(u\cdot\nabla w^1, \phi) + j(u^2\cdot\nabla w, \phi)\\
&\qquad\qquad\qquad\qquad\qquad\qquad = \chi(\operatorname{rot} u, \phi),\\
&\nu(\nabla h, \nabla\psi) + (u\cdot\nabla h^1, \psi) + (h^2\cdot\nabla h, \psi) - (h\cdot\nabla u^1, \psi) - (h^2\cdot\nabla u, \psi) = 0.
\end{aligned}
$$

We take $\varphi = u, \phi = w$ and $\psi = rh$ in these last inequalities, thus obtaining

$$
\begin{aligned}
(\mu+\chi)\|\nabla u\|^2 &= \chi(\operatorname{rot} w, u) - (u\cdot\nabla u^1, u) + r(h\cdot\nabla h^1, u)\\
&\quad + r(h^2\cdot\nabla h, u), \qquad (5.1)
\end{aligned}
$$

$$\gamma\|\nabla w\|^2 + (\alpha+\beta)\|\operatorname{div} w\|^2 + 2\chi\|w\|^2 = \chi(\operatorname{rot} u, w) - j(u\cdot\nabla w^1, w), \quad (5.2)$$

$$r\nu\|\nabla h\|^2 = r(h\cdot\nabla u^1, h) + r(h^2\cdot\nabla u, h) - r(u\cdot\nabla h^1, h). \quad (5.3)$$

By using the Lemma 2.1, we get

$$
\begin{aligned}
|(u\cdot\nabla u^1, u)| = |(u\cdot\nabla u, u^1)| &\le 3\|u\|_6\|\nabla u\|\ \|u^1\|_3 \le 3C_L\|\nabla u\|^2\|u^1\|_3,\\
|r(h\cdot\nabla h^1, u)| = |r(h\cdot\nabla u, h^1)| &\le 3r\|h\|_6\|\nabla u\|\ \|h^1\|_3\\
&\le 3C_L\|\nabla h\|\ \|\nabla u\|\ \|h^1\|_3,\\
|\chi(\operatorname{rot}w, u)| = |\chi(w, \operatorname{rot}u)| &\le \chi\|w\|\ \|\nabla u\| \le \chi\|w\|^2 + \chi\|\nabla u\|^2\\
|(u\cdot\nabla w^1, w)| = |+(u\cdot\nabla w, w^1)| &\le 3\|u\|_6\|\nabla w\|\ \|w^1\|_3\\
&\le 3C_L\|\nabla u\|\ \|\nabla w\|\ \|w^1\|_3\\
|r(h\cdot\nabla u^1, h)| = |r(h\cdot\nabla h, u^1)| &\le 3r\|h\|_6\|\nabla h\|\ \|u^1\|_3 \le 3rC_L\|\nabla h\|^2\|u^1\|_3\\
|r(u\cdot\nabla h^1, h)| = |r(u\cdot\nabla h, h^1)| &\le 3r\|u\|_6\|\nabla h\|\ \|h^1\|_3\\
&\le 3rC_L\|\nabla u\|\ \|\nabla h\|\ h^1\|_3.
\end{aligned}
$$

Consequently, adding the equalities (5.1)-(5.3) and using the above estimates, we obtain

$$
\begin{aligned}
\mu\|\nabla u\|^2 + \gamma\|\nabla w\|^2 + (\alpha+\beta)\|\operatorname{div} w\|^2 + r\nu\|\nabla h\|^2 + \chi\|w\|^2 \\
\leq \frac{3}{2}C_L(\|u^1\|_3 + \|w^1\|_3 + (1+r)\,\|h^1\|_3)\|\nabla u\|^2 \\
+\frac{3C_L}{2}(\|u^1\|_3 + (1+r)\,\|h^1\|_3)\|\nabla h\|^2 + \frac{3C_L}{2}\|w^1\|_3\|\nabla w\|^2 \\
= \frac{3C_L}{2\mu}(\|u^1\|_3 + \|w^1\|_3 + (1+r)\,\|h^1\|_3)\,\mu\|\nabla u\|^2 \\
+\frac{3C_L}{2r\nu}(\|u^1\|_3 + (1+r)\,\|h^1\|_3)\,r\nu\|\nabla h\|^2 + \frac{3C_L}{2\gamma}\|w^1\|_3\|\nabla w\|^2.
\end{aligned}
$$

Thus, by hypotheses, we obtain

$$\|\nabla u\| = 0, \quad \|\nabla w\| = 0, \quad \|\nabla h\| = 0.$$

Therefore, we find $u =$ const., $w =$ const. and $h =$ const. But $u|_{\partial\Omega} = w|_{\partial\Omega} = h|_{\partial\Omega} = 0$, hence $u = 0$, $w = 0$ and $h = 0$. Thus we have prove the uniqueness theorem.

Acknowledgements. The first author would like to express his deepest thanks to FAPESP (Project N$^{\underline{o}}$ 1997/11228-8) for support during the author's stay in the Departement of Applied Mathematics of the UNICAMP in January and February of 1998 where this paper was completed.

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