RELATÓRIO DE PESQUISA

CHARACTERIZATIONS OF RADON SPACES

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ABSTRACT - Assuming hypothesis only on the σ-algebra $\mathcal{F}$, we characterize (via Radon spaces) the class of measurable spaces $(\Omega, \mathcal{F})$ that admits regular conditional probability for all probabilities on $\mathcal{F}$.

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Characterizations of Radon Spaces

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Abstract

Assuming hypothesis only on the σ-algebra $\mathcal{F}$, we characterize (via Radon spaces) the class of measurable spaces $(\Omega, \mathcal{F})$ that admits regular conditional probability for all probabilities on $\mathcal{F}$.

Key words: Regular conditional probability property, perfect probability, compact probability, Marczewski characteristic function, Radon space.

1 Introduction

Since the seminal paper of Kolmogorov (1933) a myriad of papers discussing several aspects of the delicate concept of conditional probability has appeared in the specialized literature (see, for instance, Pachl (1978), Faden (1985) and the references therein). Of particular interest in this scenario is the notion of regular conditional probability. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ a measurable space, a transition probability from E to $(\Omega, \mathcal{F})$ is a function $\nu : E \times \mathcal{F} \to [0, 1]$ which satisfies the following two conditions:

i. $\nu(x, \cdot)$ is a probability on $\mathcal{F}$, for all $x \in E$;

ii. $\nu(\cdot, A)$ is a measurable function on $(E, \mathcal{E})$, for all $A \in \mathcal{F}$.

Given a measurable function $T : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$, a **regular conditional probability (RCP)** with respect to T (when it exists) is a transition probability ν such that

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$$P\left[A \cap T^{-1}(B)\right] = \int_B \nu(x, A) \quad T.P(dx) \quad ; \quad A \in \mathcal{F} \text{ and } B \in \mathcal{E} \tag{1}$$

where $T.P$ is the image probability of P under T.

It is well known at least since Halmos (1950, pp. 210) that it is not always possible to construct a RCP, even if the probability space $(\Omega, \mathcal{F}, P)$ is countably generated. Despite the difficulty, progress has been made and sufficient conditions has been obtained. One of the simplest of those conditions is the following: assume that Ω is a Hausdorff topological space, $\mathcal{F}$ is a separable σ-algebra in Ω, and P a regular probability on $\mathcal{F}$, *i.e.*, for all $A \in \mathcal{F}$,

$$P(A) = \sup \{P(K) : K \text{ compact}, \ K \subset A, \ K \in \mathcal{F}\};$$

under this hypothesis there exists the RCP for all measurable function T.

By eliminating non-essential topological aspects of the definition of regular probability, Marczewski (1954) introduced the concept of compact probability. He applied this concept to obtain an abstract formulation for the Kolmogorov's Extension Theorem. In the same year Jirina established the existence of RCP using compact probability and separable (countably generated) σ-algebras. Pachl (1978) proved that a probability space is compact if and only if it admits countably additive disintegration (a weaker property than RCP). Another important concept in this scenario, which was devised by Gnedenko and Kolmogorov (1949), is the one of perfect probability, it was applied to establish the existence of RCP by Sazonov (1965). Furthermore, Faden (1985) proved that if $\mathcal{F}$ is separable, perfect probability is equivalent to the existence of RCP.

In all these papers, in order to guarantee the existence of RCP hypothesis on the probability space are imposed. Here, we shall concentrate on hypothesis concerning exclusively the measurable space $(\Omega, \mathcal{F})$, in such a way that the desired result holds for any probability P defined on $\mathcal{F}$.

In the next section we present some definitions and basic results which include the well known Marczewski characteristic function that provides, for a suitable class of measurable spaces, a measurable isomorphism between $(\Omega, \mathcal{F})$ and a subset U of the $[0,1]$ endowed with the Borel σ-algebra β, Marczewski (1938). In Section 3 we establish that all probability P on $(\Omega, \mathcal{F})$ is compact if and only if U is a universally measurable set. Using this fact, we characterize a class of measurable spaces which has the regular conditional probability property via Radon spaces (measurable spaces isomorphic to a universally measurable subset of a compact metrizable space endowed with its Borel σ-algebra).

2 Definitions and Notations

Let $(\Omega, \mathcal{F})$ be a measurable space. The atoms of $\mathcal{F}$ are the equivalence classes in Ω for the equivalence relation given by $\omega \sim \omega'$ if and only if

$$\mathbb{I}_A(w) = \mathbb{I}_A(w') \quad ; \quad \forall A \in \mathcal{F}$$

The measurable space $(\Omega, \mathcal{F})$ is called Hausdorff if the atoms of $\mathcal{F}$ are the points of Ω. If there exists a sequence of elements of $\mathcal{F}$ which generates $\mathcal{F}$, we say that $(\Omega, \mathcal{F})$ is separable. Two measurable spaces are said to be isomorphic if there exists a bijection between them, which is measurable and has measurable inverse. Such a bijection is a measurable isomorphism. Let $(\Omega, \mathcal{F})$ be a separable Hausdorff measurable space and $\{E_n\}$ a sequence of measurable sets which generates $\mathcal{F}$. Marczewski (1938) established that the characteristic function of the sequence of measurable sets $\{E_n\}$, defined by

$$\Psi(w) = 2 \cdot \sum_{n=1}^{\infty} \left(\frac{\mathbb{I}_{E_n}(w)}{3^n} \right)$$

is a measurable isomorphism from $(\Omega, \mathcal{F})$ to a measurable space (U, β_U), where

$$U = \Psi(\Omega) \subset [0,1] \quad ; \quad \beta_U = \{A \cap U : A \in \beta\}$$

and $\mathbb{I}_{E_n}$ denotes the characteristic function of the set E_n and β stands for the Borel σ-algebra on $[0,1]$.

Given an abstract space Ω with a class of subsets $\mathcal{C}$ which contains the empty set and Ω itself, we say that $\mathcal{C}$ is a compact class if for every sequence $\{C_n : n \geq 1\}$ in $\mathcal{C}$ with intersection $\cap_{n \geq 1} C_n = \emptyset$ there exists an integer n_0 such that $\cap_{n \leq n_0} C_n = \emptyset$. In a probability space $(\Omega, \mathcal{F}, P)$, we say that P is **compact** if there exists a compact class $\mathcal{C} \subset \mathcal{F}$, such that

$$P(A) = \sup\{P(C) : C \subset A, C \in \mathcal{C}\}$$

for each $A \in \mathcal{F}$. For example, if P is a regular probability on a topological space then P is compact.

We say that a probability P on the measurable space $(\Omega, \mathcal{F})$ is **perfect** if for any measurable function $f : \Omega \to \mathbb{R}$ there exists a Borel set B such that $B \subset f(\Omega)$ and $P(f^{-1}(B)) = 1$. If the σ-algebra $\mathcal{F}$ is separable, it is well known that perfectness and compactness are equivalent, see Tortrat (1977, Prop. 3).

Given another measurable space $(E, \mathcal{E})$, we denote by $\mathcal{E}_\lambda$ the completion of $\mathcal{E}$ with respect to a probability measure λ on this space; the **universal σ-algebra** $\overline{\mathcal{E}}$ is defined by:

$$\overline{\mathcal{E}} = \bigcap_{\lambda} \mathcal{E}_\lambda$$

where the intersection is taken over all probabilities λ on $(E, \mathcal{E})$. The relevancy of the universal σ-algebra is going to be pointed out in the sequel.

Let Y be a compact metrizable space endowed with its Borel σ-field β_Y. We say that $(\Omega, \mathcal{F})$ is a **Radon space** if there exists a measurable isomorphism $\phi : (\Omega, \mathcal{F}) \to (U, \beta_U)$, where $U \in \overline{\beta_Y}$ and $\beta_U = \{A \cap U : A \in \beta_Y\}$, the trace of β_Y on U. It is easy to see that β_U is the Borel σ-field on U, then, it follows that

$$\overline{\beta_U} = \left\{B \cap U : B \in \overline{\beta_Y}\right\}$$

Once $U \in \overline{\beta_Y}$, we have that $\overline{\beta_U} \subset \overline{\beta_Y}$.

Remark: Radon spaces defined in a similar way has been used by several authors, see *e.g.* the appendix in Sharpe (1988) and the references therein.

The following lemma is useful in the proof of the main theorem of this paper.

Lemma 2.1 (Sazonov (1965), Lemma 3) *If $X \subset \mathbb{R}$, every probability on (X, β_X) is perfect if and only if X is universally measurable.*

We say that the measurable space $(\Omega, \mathcal{F})$ has the **regular conditional probability property (RCP property)**, if for all probability P on $(\Omega, \mathcal{F})$ and any measurable space $(E, \mathcal{E})$ with a measurable function T from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$, there exists a transition probability ν from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ satisfying equation (1).

Finally we introduce another concept which is closely related to the RCP property: the probability space $(\Omega, \mathcal{F}, P)$ has the **product regular conditional probability property** if for any probability space of the form $(\Omega \times E, \mathcal{F} \times \mathcal{E}, \lambda)$, with $(E, \mathcal{E})$ a measurable space and $(\pi_\Omega)_* \lambda = P$, there exists a transition probability ν from $(E, \mathcal{E})$ to $(\Omega, \mathcal{F})$ which satisfies:

$$\lambda(A \times B) = \int_E \nu(x, A) \quad (\pi_\Omega)_* \lambda(dx),$$

with $A \in \mathcal{F}$, $B \in \mathcal{E}$. Note that if λ is a product probability then the equation above is easily verified.

Lemma 2.2 (Faden (1985), Thm. 6) *Given a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F}$ separable, the following are equivalents:*

1. *$(\Omega, \mathcal{F}, P)$ has the product regular conditional probability property;*
2. *$(\Omega, \mathcal{F}, P)$ has a regular conditional probability for any real valued measurable function T on $(\Omega, \mathcal{F})$;*
3. *P is perfect.*

3 Main Results

First, we shall characterize Radon spaces via compact probabilities on Hausdorff separable measurable spaces.

Theorem 3.1 *The measurable space $(\Omega, \mathcal{F})$ is Radon if and only if it is separable, Hausdorff measurable space such that every probability on $(\Omega, \mathcal{F})$ is compact.*

Remark: By the result of Tortrat (1977) mentioned above, compactness in this case is equivalent to perfectness.

Proof: Firstly assume that $(\Omega, \mathcal{F})$ is a Radon space. There exists a measurable isomorphism $\phi : (\Omega, \mathcal{F}) \to (U, \beta_U)$, where $U \in \overline{\beta_Y}$. Since (U, β_U) is a separable Hausdorff measurable space, $(\Omega, \mathcal{F})$ is also separable and Hausdorff. We only have to prove that every probability P on $(\Omega, \mathcal{F})$ is compact. Let P be a probability on $(\Omega, \mathcal{F})$ and $\lambda = (\phi)_* P$ the corresponding image probability on (U, β_U). We can extend λ on $(U, \overline{\beta_U})$, where $\overline{\beta_U}$ is the universally measurable σ-field on U. Then,

$$Q(A) = \lambda[A \cap U] \; ; \; A \in \overline{\beta_Y}$$

defines a probability on $(Y, \overline{\beta_Y})$ which is compact. Since,

$$\beta_U \subset \overline{\beta_Y}$$

the restriction of Q on β_U , *i.e.*, the probability λ, is compact (see Pachl (1978), Prop. 4.1). By the isomorphism ϕ the probability P is also compact.

Now let $(\Omega, \mathcal{F})$ be a separable Hausdorff measurable space such that every probability P is compact. The Marczewski characteristic function defines a measurable isomorphism Ψ from $(\Omega, \mathcal{F})$ to (U, β_U), $U \subset [0, 1]$ with $\beta_U = \{A \cap U : A \in \beta\}$. We have to prove that $U \in \overline{\beta}$. From the measurable isomorphism Ψ all probabilities on (U, β_U) are compacts. Since compactness and perfectness are equivalent in this case it follows that U is universally measurable by Lemma 2.1.

□

In the course of the proof of the theorem above, we have also proved the following corollary.

Corollary 3.1 *The measurable space $(\Omega, \mathcal{F})$ is Radon if and only if the Marczewski characteristic Function is a measurable isomorphism from $(\Omega, \mathcal{F})$ to (U, β_U), where U is a universally measurable subset of $[0, 1]$.*

The next theorem gives another characterization in terms of regular conditional probability property, the proof is a direct consequence of Theorem 3.1 and Lemma 2.2.

Theorem 3.2 *The measurable space $(\Omega, \mathcal{F})$ is Radon if and only if it is a separable Hausdorff measurable space which has the regular conditional probability property.*

Proof: Suppose that $(\Omega, \mathcal{F})$ is a Radon space. Given an arbitrary probability P on $\mathcal{F}$, P is compact (Theorem 3.1) hence it is perfect. For any measurable space $(E, \mathcal{E})$ and measurable function $T : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$, let λ be the probability induced on $(\Omega \times E, \mathcal{F} \times \mathcal{E})$ by the mapping $w \to (w, T(w))$, such that

$$\lambda(A \times B) = P[A \cap T^{-1}(B)] \ ; \ A \in \mathcal{F} \text{ and } B \in \mathcal{E}.$$

By Lemma 2.2 there exists a transition probability $\nu : E \times \mathcal{F} \to [0,1]$, such that

$$P[A \cap T^{-1}(B)] = \lambda(A \times B) = \int_B \nu(x, A) \quad (\pi_E).\lambda(dx) \ ; \ A \in \mathcal{F} \text{ and } B \in \mathcal{E}.$$

It follows by definition that ν is a regular conditional probability. Since P is arbitrary, $(\Omega, \mathcal{F})$ has the regular conditional probability property.

Reciprocally, let $(\Omega, \mathcal{F})$ be a separable Hausdorff measurable space which has the regular conditional probability property. Then, given a probability P on $\mathcal{F}$ and a measurable function $T : (\Omega, \mathcal{F}) \to (\mathbb{R}, \beta_R)$, there exists a regular conditional probability $\nu : \mathbb{R} \times \mathcal{F} \to [0,1]$, such that

$$P\left[A \cap T^{-1}(B)\right] = \int_B \nu(x, A) \quad T.P(dx) \ ; \ A \in \mathcal{F} \text{ and } B \in \mathcal{E}.$$

Again, from Lemma 2.2 it follows that P is perfect. Since P is arbitrary, the measurable space $(\Omega, \mathcal{F})$ is Radon.

□

Dellacherie and Meyer (1978) used subclasses of Radon spaces (Lusin, Souslin and Co-Souslin spaces) to present some topics concerning the general theory of stochastic processes. We are particularly interested in showing the adequacy of Radon space to deal with regularity of trajectories of stochastic processes, a paper in this direction is under preparation.

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