RELATÓRIO DE PESQUISA

SOME REMARKS ON A SYSTEM OF QUASILINEAR ELLIPTIC EQUATIONS

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ABSTRACT - In this paper we study the functional

$$\Phi(u,v) = \frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q - \int F(x,u,v)$$

where the function F satisfies sets of conditions that imply that Φ is either coercive, or has a saddle point. Resonant cases are studied.

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Some remarks on a system of quasilinear elliptic equations

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1. Introduction. In this paper we study the functional

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \int_\Omega F(x,u,v). \tag{1.1}$$

where p and q are real numbers larger than $1, \Omega$ is some bounded domain in R^N, u and v are real-valued functions defined in $\overline{\Omega}$ and belonging to appropriate spaces of functions and F (sometimes refered as a *potential*) is a real-valued differentiable function with domain $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}$. Our aim is to study the geometry of this functional viewing to determining its critical points. Such critical points are the solutions of associated Euler-Lagrange equations, which in the present case is the system of quasilinear elliptic equations below

$$\begin{aligned} -\Delta_p u &= F_u(x,u,v) \\ -\Delta_q v &= F_v(x,u,v) \end{aligned} \tag{1.2}$$

where F_u designates the partial derivative of F with respect to u and Δ_p is the so-called p-Laplacian operator

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u).$$

The geometry of Φ is sort of similar to the one of the functional

$$\frac{1}{p}\int |\nabla u|^p - \int F(x,u),$$

which corresponds to a single quasilinear equation. However, some interesting features appear due to the coupling in the equations (1.2). Our theorems include and unify some previous results by Boccardo - Fleckinger de Thelin [BFT] and de Thélin-Vélin [VT].

Let us introduce the precise assumptions under which our problem is studied. Our functional Φ is to be defined in the Cartesian product of Sobolev spaces $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. For that matter, the following assumption on F has to be made, although stronger restriction will come timely:

(F_1) $$F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is } C^1 \text{ and}$$

$$|F(x,u,v)| \leq c(1 + |u|^{p^*} + |v|^{q^*}),$$

where $p^* = pN/(N-p)$ and c is some positive constant. Similarly q^*.

In this work, we assume that both p and q are less than N. If this is so, we have the continuous imbeddings $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$, which then tell us that Φ is well defined. In order to have it of class C^1, we require a stronger condition than (F_1), namely: F is C^1 and

(F_2) $$\begin{aligned} |F_u(x,u,v)| &\leq C\left(1 + |u|^{p^*-1} + |v|^{\frac{q^*(p^*-1)}{p^*}}\right) \\ |F_v(x,u,v)| &\leq C\left(1 + |v|^{q^*-1} + |u|^{\frac{p^*(q^*-1)}{q^*}}\right). \end{aligned}$$

It is easy to prove that, if (F_2) is satisfied, then also is (F_1). Under (F_2), it follows that the critical points of Φ are the weak solutions of system (1.2), subject to Dirichlet boundary conditions. For easy reference, we summarize the aforesaid as follows: • under hypothesis (F_2), with $E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, we have

$$\Phi : E \to \mathbb{R} \quad \text{is a } C^1\text{-functional.}$$

The geometry of Φ depends strongly on the values of r and s in the estimate below

(F_3) $$|F(x,u,v)| \leq c(1 + |u|^r + |v|^s),$$

where c is some positive constant, and $r \leq p^*, s \leq q^*$. We discuss three distinct cases

(I) $r < p$ and $s < q$. ("sublinear-like")
(II) $r > p$ or $s > q$, and $r < p^*, s < q^*$, ("superlinear-like")
(III) $r = p$ and $s = q$. ("of resonant-type").

The expressions in parenthesis are to remind us of similar terminology in the case $p = q = 2$. Of course, there are several other situations, which could be of interest to consider. Observe that we are considering only subcritical cases. The cases where either $r = p^*$ or $s = q^*$ or both equalities hold lead to a loss of compactness, and to problems which should be investigated.

Next we state the main results of this paper.

Theorem 1 (The coercive case). Assume (F_2) and (F_3) with r and s as in (I). Then Φ achieves a global minimum at some $(u_0, v_0) \in E$, which is then a weak solution of system (1.2).

If we are in the situation that

(F_4) $$F(x,0,0) = F_u(x,0,0) = F_v(x,0,0), \text{ for all } x \in \overline{\Omega}.$$

then $u \equiv 0$ and $v \equiv 0$ are a trivial solution of system (1.2). In this case the relevant question is obtaining a non-trivial solution of (1.2). This will be possible under appropriate assumptions on the function F, as it is stated in the next results.

Theorem 2 (The coercive case, non-trivial solution). Assume (F_2), (F_4) and (F_3) with r and s as in (I). Then Φ achieves a global minimum at a point $(u_0, v_0) \neq (0,0)$, provided there exist positive constant R and $\theta < 1$, and a continuous function $K : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

(F_5) $$F(x, t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) \geq t^\theta K(x,u,v), \text{ for } x \in \overline{\Omega}, |u|, |v| \leq R, \text{ and small } t > 0.$$

In this case a Palais-Smale condition holds (see Lemma 4.1) if we assume that there are numbers $R > 0$, θ_p and θ_q with

$$\frac{1}{p^*} < \theta_p < \frac{1}{p} \qquad \frac{1}{q^*} < \theta_q < \frac{1}{q}$$

such that

$$(F_6) \qquad 0 < F(x,u,v) \le \theta_p u F_u(x,u,v) + \theta_q v F_v(x,u,v)$$

for all $x \in \overline{\Omega}$ and $|u|, |v| \ge R$.

Theorem 3. Assume $(F_2), (F_4), (F_6)$ and (F_3) with r and s as in (II). Assume also that there are constants $c > 0$ and $\varepsilon > 0$ and numbers $\bar{r} > p, \bar{s} > q$, such that

$$(F_7) \qquad |F(x,u,v)| \le c(|u|^{\bar{r}} + |v|^{\bar{s}}), \text{ for } |u|, |v| \le \varepsilon, x \in \overline{\Omega}.$$

Then Φ has a non-trivial critical point.

Remark. Without loss of generality we may assume $\bar{r} < p^*$ and $\bar{s} < q^*$.

Next we study the situation when $(F3)$ holds with r and s as in (III), the case we called "of resonant type". In this case, it is quite adequate to assume a condition on F that implies that the functional Φ satisfies the so-called Cerami condition (see Section 4 for the definition). The assumption on F is: there are positive numbers c, R, μ and ν such that

$$(F_8) \qquad \frac{1}{p} u F_u + \frac{1}{q} v F_v - F \ge c(|u|^\mu + |v|^\nu)$$

for $|u|, |v| > R$.

This type of condition has been introduced by Costa-Magalhães [CM1], [CM2]. In order to avoid resonance we shall assume a condition on F involving an eigenvalue problem, which we introduce next. Let $G : R^2 \to [0, \infty)$ be a C^1 even function such that

$$(G_1) \qquad G(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) = t G(u,v)$$

$$(G_2) \qquad G(u,v) \le k(|u|^p + |v|^q).$$

Examples of such functions are

(i) $\quad G(u,v) = c_1|u|^p + c_2|v|^q$

(ii) $\quad G(u,v) = c|u|^\beta |v|^\gamma$, with $\frac{\beta}{p} + \frac{\gamma}{q} = 1$,

where c_1, c_2 and c are positive constants.

We shall prove in Section 3 that the eigenvalue problem

$$-\Delta_p u - a G_u = \lambda |u|^{p-2} u$$
$$-\Delta_q v - a G_v = \lambda |v|^{q-2} v$$

subject to Dirichlet boundary conditions, with $a \in L^\infty(\Omega)$, has an eigenvalue $\lambda_1(a)$, characterized variationally by

$$\frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q - \int a G(u,v) \ge \lambda_1(a)\left(\frac{1}{p}\int |u|^p + \frac{1}{q}\int |v|^q\right)$$

for all $(u,v) \in E$.

Now we introduce the following assumption:

$$(F_9) \qquad \lambda_1(a) > 0, \text{ where } \limsup_{|u|,|v| \to \infty} \frac{F(x,u,v)}{G(u,v)} \le a(x) \in L^\infty(\Omega)$$

and state the next results.

Theorem 4. Assume $(F_2), (F_8), (F_9)$ and (F_3) with r and s as in (III). Then the functional Φ is bounded below and the infimum is achieved.

Theorem 5. Assume $(F_2), (F_4), (F_8)$ and (F_3) with r and s as in (III). Suppose also that there are positive numbers R and ε, and $L^\infty(\Omega)$ functions $b(x)$ and $c(x)$ such that

$$(F_{10}) \qquad \lambda_1(b) < 0, \; F(x,u,v) \ge b(x) G(u,v), \; |u|, |v| \ge R$$
$$(F_{11}) \qquad \lambda_1(c) > 0, \; F(x,u,v) \le c(x) \hat{G}(u,v), \; |u|, |v| \le \varepsilon,$$

where G and $\hat{G}$ are functions satisfying the conditions $(G1)$ and $(G2)$. Then, the functional Φ possesses a non-trivial critical point.

2. Special classes of potentials F.

(i) The following class of potentials F (and its perturbations) have been considered by [dT], [VT], [FMT]:

$$F(x,u,v) = c(x)|u|^{\beta}|v|^{\gamma}$$

where $c(x) \in L^{\infty}(\overline{\Omega})$ and $\beta, \gamma \geq 1$. Using Young's inequality

$$|u|^{\beta}|v|^{\gamma} \leq \frac{1}{m}|u|^{\beta m} + \frac{1}{n}|v|^{\gamma n}$$

where $1/m + 1/n = 1$. Let $r = \beta m$ and $s = \gamma n$. So $\dfrac{\beta}{r} + \dfrac{\gamma}{s} = 1$. Consequently, for this class the three cases are

(I), $\qquad \dfrac{\beta}{p} + \dfrac{\gamma}{q} < 1.$

(II), $\qquad \dfrac{\beta}{p} + \dfrac{\gamma}{q} > 1$ and $\dfrac{\beta}{p^*} + \dfrac{\gamma}{q^*} < 1.$

(III), $\qquad \dfrac{\beta}{p} + \dfrac{\gamma}{q} = 1.$

In this example, condition (F_5) is precisely the inequality in (I) above. Condition (F_6) holds if β and γ are such that

$$\theta_p\beta + \theta_q\gamma \geq 1. \tag{2.1}$$

Observe that, if (2.1) holds with θ_p and θ_q as in the Introduction, then we are necessarily in case (II). That is, the problem is "superlinear-like".

The theorems stated in the Introduction contain and extend some of the results of the above mentioned papers.

(ii) In [BFT] the following system was studied

$$-\Delta_p u = a(x)|u|^{\alpha-2}u + b(x)|v|^{\beta-2}v + f$$
$$-\Delta_q v = c(x)|u|^{\gamma-2}u + d(x)|v|^{\delta-2}v + g$$

subject to Dirichlet boundary conditions. In this generality, the system is not variational. However, if $b(x) = c(x)$ and $\beta = \gamma = 2$, the above equations are the Euler Lagrange equations of a functional Φ as in (1.1) with

$$F(x,u,v) = a(x)|u|^{\alpha} + b(x)uv + d(x)|v|^{\delta} + fu + gv.$$

where we assume that $a, b, d \in L^{\infty}(\Omega)$, $\alpha, \delta \geq 1$ and $f \in L^{(p^*)'}(\Omega)$, $g \in L^{(q^*)'}(\Omega)$. Here $(p^*)' = \dfrac{pN}{p + N(p-1)}$ and a similar expression for $(q^*)'$. The fact that f and g are not necessarily in $L^{\infty}(\Omega)$ implies that the various pointwise estimates (F) cannot hold. However, since the terms where they appear are linear in u and v, their presence essentially do not change the proofs of the theorems. So, in this example, the three cases studied are:

$(\mathrm{I})_{ii}$ $\qquad \alpha < p,\ \delta < q,\ \dfrac{1}{p} + \dfrac{1}{q} < 1.$

$(\mathrm{II})_{ii}$ $\qquad \alpha > p$ or $\delta > q$ or $\dfrac{1}{p} + \dfrac{1}{q} > 1.$

$(\mathrm{III})_{ii}$ $\qquad \alpha = p,\ \delta = q,\ \dfrac{1}{p} + \dfrac{1}{q} = 1.$

We remark that case $(\mathrm{II})_{ii}$ was not considered in [BFT]. Our results for case $(\mathrm{III})_{ii}$ extend the ones in [BFT].

Remark. For the special examples above, condition (F5) can be replaced by (F5)′ there are positive constants c and ε such that

$$F(x, t^q, t^p) > ct^{pq} \qquad \text{for all} \quad x \in \overline{\Omega},\ 0 < t < \varepsilon.$$

3. The eigenvalue problem.

Let $G : R^2 \to [0,\infty)$ be a C^1 even function satisfying conditions (G1) and (G2) given in the Introduction.

Lemma 3.1. Given $a \in L^{\infty}(\Omega)$, there are a real number $\lambda_1(a)$ and $(u_0, v_0) \in E$, such that

$$\begin{cases} -\Delta_p u_0 - aG_u(u_0, v_0) = \lambda_1(a)u_0|u_0|^{p-2} \\ -\Delta_q v_0 - aG_v(u_0, v_0) = \lambda_1(a)v_0|v_0|^{q-2} \end{cases} \tag{3.1}$$

and

$$(3.2)\qquad \frac{1}{p}\int|\nabla u|^p + \frac{1}{q}\int|\nabla v|^q - \int aG(u,v) \geq \lambda_1(a)\left[\frac{1}{p}\int|u|^p + \frac{1}{q}\int|v|^q\right]$$

for all $(u,v)\in E$, with equality for (u_0,v_0).

Proof. Choose $M > k\|a\|_{L^\infty}$, where k is the constant in (G2). Then the functional

$$(3.3)\qquad J(u,v) = \frac{1}{p}\int|\nabla u|^p + \frac{1}{q}\int|\nabla v|^q - \int aG(u,v) + M\left[\frac{1}{p}\int|u|^p + \frac{1}{q}\int|v|^q\right]$$

is non-negative for $(u,v)\in E$. Let

$$(3.4)\qquad S = \left\{(u,v)\in E : \frac{1}{p}\int|u|^p + \frac{1}{q}\int|v|^q = 1\right\}$$

and let us look for $Inf\{J(u,v) : (u,v)\in S\}$. Let us denote this infimum by μ, and let us take a minimizing sequence $(u_n,v_n)\in S$. It follows that $\|u_n\|_{W^{1,p}}$ and $\|v_n\|_{W^{1,q}}$ are bounded. So we may choose subsequences (denoted again by (u_n) and (v_n)) such that (u_n) converges to u_0, weakly in $W_0^{1,p}$ and strongly in L^p. Similarly for (v_n). Passing to the limit

$$\frac{1}{p}\int|\nabla u_0|^p + \frac{1}{q}\int|\nabla v_0|^q - \int a + G(u_0,v_0) + M\left[\frac{1}{p}\int|u_0|^p + \frac{1}{q}\int|v_0|^q\right] \leq \mu$$

which indeed is an equality because $(u_0,v_0)\in S$. So the above infimum is achieved. If follows then that

$$(3.5)\qquad \begin{cases} -\Delta_p u_0 - aG_u(u_0,v_0) + Mu_0|u_0|^{p-1} = \mu_M u_0|u_0|^{p-1} \\ -\Delta_q v_0 - aG_v(u_0,v_0) + Mv_0|v_0|^{q-1} = \mu_M v_0|v_0|^{p-1} \end{cases}$$

where μ_M is the Lagrange multiplier. It follows from $(G1)$ that

$$(3.6)\qquad G(u,v) = \frac{1}{p}uG_u(u,v) + \frac{1}{q}vG_v(u,v)$$

for all $(u,v)\in R^2$. From the minimization above we have:

$$(3.7)\qquad \mu\left[\frac{1}{p}\int|u|^p + \frac{1}{q}\int|v|^q\right] \leq$$
$$\frac{1}{p}\int|\nabla u|^p + \frac{1}{q}\int|\nabla v|^q - \int aG(u,v) + M\left[\frac{1}{p}\int|u|^p + \frac{1}{q}\int|v|^q\right]$$

for all $(u,v)\in E$. It follows from (3.5), (3.6) and (3.7) that $\mu = \mu_M$. In this way we get (3.1) and (3.2) with $\lambda_1(a) = \mu - M$, and the eigenfunction pair (u_0,v_0) as obtained above. □

Remark. From the minimization argument above, it follows that both u_0 and v_0 can be taken ≥ 0 in Ω. As in [dT] it can be proved that u_0, v_0 are $C^1(\overline{\Omega})$ as consequence of regularity results of [T]. Then by the Vasquez Maximum Principle [V] for the p-Laplacian, we conclude that u_0 and v_0 are indeed > 0 in Ω.

Lemma 3.2. $\lambda_1(a)$ is a continuous function of a in the L^∞ norm.

Proof. Let us denote by J_a the functional defined in (3.3) and J_b the corresponding one with a replaced by b, where b is some other L^∞-function. Let $\lambda_1(b)$ be eigenvalue corresponding to b given in Lemma 3.1. We show that, given $\varepsilon > 0$ there is a $\delta > 0$ such that $|\lambda_1(a) - \lambda_1(b)| < \varepsilon$ provided $\|a-b\|_{L^\infty} \leq \delta$.

Indeed, given $\varepsilon > 0$, choose $(u_\varepsilon, v_\varepsilon)\in S_1$ such that

$$(3.8)\qquad J_a(u_\varepsilon,v_\varepsilon) \leq \lambda_1(a) + \frac{\varepsilon}{2}.$$

. Next (G2) implies

$$(3.9)\qquad G(u,v) \leq K\left(\frac{1}{p}|u|^p + \frac{1}{q}|v|^q\right)$$

where $K \geq \max\{kp, kq\}$. Using (3.9) we obtain

$$(3.10)\qquad |J_b(u_\varepsilon,v_\varepsilon) - J_a(u_\varepsilon,v_\varepsilon)| \leq \|b-a\|_{L^\infty}.K.$$

So, from (3.10) and (3.8)

$$\lambda_1(b) \leq J_b(u_\varepsilon, v_\varepsilon) \leq J_a(u_\varepsilon, v_\varepsilon) + K\|b-a\|_{L^\infty} \leq \lambda_1(a) + \frac{\varepsilon}{2} + K\|b-a\|_{L^\infty}$$

Choosing $\delta = \frac{\varepsilon}{2k}$ we have $\lambda_1(b) \leq \lambda_1(a) + \varepsilon$.

Reversing the roles of a and b, the result follows. □

4. Compactness conditions. We say that $\Phi : E \to I\!R$ satisfies the *(PS) condition* if, all $(u_n, v_n) \in E$ such that

$$(4.1) \qquad |\Phi(u_n, v_n)| \leq const \qquad \Phi'(u_n, v_n) \to 0$$

contains a convergent subsequence in the norm of E.

Lemma 4.1. Suppose that F satisfies $(F_2), (F_6)$ and (F_3) as in (II). Then the functional Φ defined in the Introduction satisfies the (PS) condition.

Proof. It follows from (4.1) that

$$\left|\frac{1}{p}\int |\nabla u_n|^p + \frac{1}{q}\int |\nabla v_n|^q - \int F(u_n, v_n)\right| \leq const,$$

$$\left|\int |\nabla u_n|^p - \int F_u(u_n, v_n)u_n\right| \leq \varepsilon_n \|u_n\|_{W^{1,p}}$$

and a similar one for v_n, where $\varepsilon_n \to 0$. Using these expressions we get

$$\left(\frac{1}{p} - \theta_p\right)\int |\nabla u_n|^p + \left(\frac{1}{q} - \theta_q\right)\int |\nabla v_n|^q - \int (F(u_n, v_n) - \theta_p u_n F_u - \theta_q v_n F_v)$$
$$\leq c + c(\|u_n\|_{W^{1,p}} + \|v_n\|_{W^{1,q}})$$

which implies, using (F6) that both $\|u_n\|_{W^{1,p}}$and $\|v_n\|_{W^{1,q}}$ are bounded. The existence of convergent subsequences follows in a standard way, since the growth of F is below the critical exponents p^* and q^*. □

We say that $\Phi : E \to I\!R$ satisfies the *Cerami condition*, (Ce) for short, if all $(u_n, v_n) \in E$ such that

$$(4.2) \qquad |\Phi(u_n, v_n)| \leq const \qquad (1 + \|u_n\|_{W^{1,p}} + \|v_n\|_{W^{1,q}})\Phi'(u_n, v_n) \to 0$$

constains a convergent subsequence in the norm of E.

Lemma 4.2. Suppose that F satisfies $(F_2), (F_8)$ and (F_3) as in (III). Then the functional Φ satisfies the (Ce) condition.

Proof. Take a sequence pair $(u_n, v_n) \in E$ satisfying (4.2). It suffices to prove that $\|u_n\|_{W^{1,p}}$ and $\|v_n\|_{W^{1,q}}$ are bounded, as remarked in the proof of the previous lemma. It follows readily (4.2) that

$$C' \geq \langle \Phi'(u_n, v_n), (\frac{1}{p}u_n, \frac{1}{q}v_n)\rangle - \Phi(u_n, v_n) = \int (\frac{1}{p}u_n F_u(u_n, v_n) + \frac{1}{q}v_n F_v(u_n, v_n) - F(u_n, v_n)).$$

Then using (F_8) we obtain

$$(4.3) \qquad \int |u_n|^\mu + \int |v_n|^\nu \leq const.$$

Next we use the following interpolation inequality, see [CM2]: let $0 < a < b < c$ and suppose that for some measurable function $u : \Omega \to R$ we have that

$$\int |u|^a < \infty \quad \text{and} \quad \int |u|^c < \infty$$

then

$$(4.4) \qquad \int |u|^b \leq \left(\int |u|^a\right)^{\frac{c-b}{c-a}} \cdot \left(\int |u|^c\right)^{\frac{b-a}{c-a}}$$

We use (4.4) for $0 < \mu < p < p^*$ and $0 < \nu < q < q^*$. So using (4.3) and (4.4) we get

$$\int |u_n|^p \leq C\left(\int |u_n|^{p^*}\right)^{\frac{p-\mu}{p^*-\mu}}, \quad \int |v_n|^q \leq C\left(\int |v_n|^{q^*}\right)^{\frac{q-\nu}{q^*-\nu}},$$

which implies by Sobolev imbedding

$$(4.5) \qquad \int |u_n|^p \leq c\|u_n\|^{\bar{p}}_{W^{1,p}}, \int |v_n|^q \leq c\|v_n\|^{\bar{q}}_{W^{1,q}}.$$

where

$$\tilde{p} = \frac{p-\mu}{p^*-\mu}p^*, \qquad \tilde{q} = \frac{q-\nu}{q^*-\nu}q^*$$

Using (F_3) as in (III) we get

$$\Phi(u_n, v_n) \geq \frac{1}{p}\int |\nabla u_n|^p + \frac{1}{q}\int |\nabla v_n|^q - c\int |u_n|^p - c\int |v_n|^q$$

Which estimate by (4.5) leads to

$$\Phi(u_n, v_n) \geq \frac{1}{p}||u_n||^p_{W^{1,p}} + \frac{1}{q}||v_n||^q_{W^{1,q}} - c||u_n||^{\tilde{p}}_{W^{1,p}} - c||v_n||^{\tilde{q}}_{W^{1,q}}$$

Since $\Phi(u_n, v_n)$ is bounded and $\tilde{p} < p$ and $\tilde{q} < q$, it follows the boundedness of $||u_n||_{W^{1,p}}$ and $||v_n||_{W^{1,q}}$. □

5. Proofs of Theorems completed.

i) Theorem 1. Condition (F_3) with r and s as in (I) implies that Φ is weakly lower semicontinuous and coercive in E. So, it assumes its infimum at a point $(u_0, v_0) \in E$. Condition (F_2) implies that this is a critical point Φ and consequently a weak solution of (1.2).

ii) Theorem 2. As in Theorem 1, Φ assumes its infimum at a point $(u_0, v_0) \in E$. To prove that this is not (0,0) it is enough to show that there is $(u_1, v_1) \in E$ such that $\Phi(u_1, v_1) < 0$.

Let φ be a first eigenfunction for the p-Laplacian

$$\begin{cases} -\Delta_p \varphi &= \lambda_1(p)|\varphi|^{p-2}\varphi, \quad \text{in } \Omega \\ \varphi &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

and ψ for the q-Laplacian. We know that $\varphi, \psi \in C^{1,\alpha}$, see [dB], [T]. So we can take $||\varphi||_{L^\infty}, ||\psi||_{L^\infty} \leq R$, where R is the constant in (F_5). So

$$\Phi(t^{\frac{1}{p}}\varphi, t^{\frac{1}{q}}\psi) \leq t\left\{\frac{1}{p}\lambda_1(p)\int |\varphi|^p + \frac{1}{q}\lambda_1(q)\int |\psi|^q\right\} - t^\theta \int K(x, \varphi, \psi).$$

which is negative for $t > 0$ small.

iii) Theorem 3. It is easy to see that in this case Φ has the geometry of the Mountain Pass Theorem. Indeed, it follows from (F_3) and (F_7) that

$$|F(x, u, v)| \leq c(|u|^{\bar{r}} + |v|^{\bar{s}} + |u|^r + |v|^s)$$

for all $x \in \overline{\Omega}$ and $(u, v) \in I\!R^2$, where $p < r, \bar{r} < p^*$ and $q < s, \bar{s} < q^*$. So by the Sobolev imbedding

$$\int F(x, u, v) \leq c(||u||^r_{W^{1,p}} + ||v||^s_{W^{1,q}} + ||u||^{\bar{r}}_{W^{1,p}} + ||v||^{\bar{s}}_{W^{1,q}})$$

for all $(u, v) \in E$. So, we can estimate Φ by

$$\Phi(u, v) \geq \frac{1}{p}||u||^p_{W^{1,p}} + \frac{1}{q}||v||^q_{W^{1,q}} - c(||u||^r_{W^{1,p}} + ||v||^s_{W^{1,q}} + ||u||^{\bar{r}}_{W^{1,p}} + ||v||^{\bar{s}}_{W^{1,q}})$$

which implies that there is an $\varepsilon > 0$ and $\rho > 0$ such that $\Phi(u, v) \geq \varepsilon$ for $||u||_{W^{1,p}} + ||v||_{W^{1,q}} = \rho$. On the other hand, using (F_6) we have

$$\frac{d}{dt}\{F(x, t^{\theta_p}u, t^{\theta_q}v)\} \geq \frac{1}{t}F(x, t^{\theta_p}u, t^{\theta_q}v)$$

which implies that $F(x, t^{\theta_p}u, t^{\theta_q}v) \geq tK(x, u, v)$ for some function K. Since

$$\frac{1}{p}\int |\nabla(t^{\theta_p}u)|^p + \frac{1}{q}\int |\nabla(t^{\theta_q}v)|^q = t^{\theta_p p}c_1 + t^{\theta_q q}c_2$$

and from the hypothesis (F_6), $\theta_p p < 1$ and $\theta_q q < 1$, we conclude that for any fixed

$$(u, v) \neq (0, 0), \quad \Phi(t^{\theta_p}u, t^{\theta_q}v) \to -\infty \quad \text{as} \quad t \to +\infty.$$

Since Φ satisfies the (PS) condition, as proved in Lemma 4.1, we may apply the Mountain Pass Theorem and conclude.

iv) Theorem 4. It follows from (F_9) that given $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$F(x, u, v) \leq (a(x) + \varepsilon)G(u, v) + C_\varepsilon, \quad \text{all} \quad x \in \overline{\Omega}, (u, v) \in R^2.$$

Hence

$$\Phi(u,v) \geq \frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q - \int (a+\varepsilon)G(u,v) - C_\varepsilon|\Omega|. \tag{5.1}$$

Now let $0 < \delta < 1$ be a real number to be chosen later, and write (5.1) as follows

$$\Phi(u,v) \geq \delta\left\{\frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q\right\} +$$

$$(1-\delta)\left\{\frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q - \int \frac{a+\varepsilon}{1-\delta}G(u,v)\right\} - c_\varepsilon|\Omega|$$

The expression in the second bracket is estimated from below by

$$\lambda_1\left(\frac{a+\varepsilon}{1-\delta}\right)\left(\frac{1}{p}|u|^p + \frac{1}{q}\int |v|^q\right).$$

Now choose ε and δ such that $\lambda_1\left(\frac{a+\varepsilon}{1-\delta}\right) > 0$, which is possible in view of the continuity of $\lambda_1(a)$ with respect to a (Lemma 3.2), in the L^∞-norm. So

$$\Phi(u,v) \geq \delta\left\{\frac{1}{p}\int \nabla u|^p + \frac{1}{q}\int |\nabla v|^q\right\} - C_\varepsilon|\Omega|$$

which shows that Φ is coercive. Since Φ is weakly lower semicontinuous, the result follows.

v) Theorem 5. It suffices to show that Φ in this case has the geometry of the Mountain Pass Theorem. The compactness condition (Cc) has already been proved in Lemma 4.2. First we prove that (0,0) is a local minimum. Indeed, it follows from (F_{11}) and (F_3) that there is a constant $c > 0$ such that

$$F(x,u,v) \leq c(x)\bar{G}(u,v) + c(|u|^{\bar r} + |v|^{\bar s}) \tag{5.2}$$

for all $x \in \bar\Omega$, $(u,v) \in R^2$, and $\bar r > p, \bar s > q$. Next using (5.2) and a $\delta \in (0,1)$ to be chosen later we have

$$\Phi(u,v) \geq \delta\left(\frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q\right) +$$

$$(1-\delta)\left\{\frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla u|^q - \int \frac{c(x)}{1-\delta}G(u,v)\right\}$$
$$-c||u||^{\bar r}_{W^{1,p}} - c||v||^{\bar s}_{W^{1,q}}$$

Choosing δ such that the expression in the bracket is positive, as a consequence of (F_{11}), we get

$$\Phi(u,v) \geq c_1||u||^p_{W^{1,p}} + c_2||v||^q_{W^{1,q}} - C||u||^{\bar r}_{W^{1,p}} - c||v||^{\bar s}_{W^{1,q}}$$

which shows that there are positive numbers ρ and ε such that $\Phi(u,v) > \varepsilon$ for $||u||_{W^{1,p}} + ||v||_{W^{1,q}} = \rho$. Finally we use (F_{10}) to estimate

$$\Phi(u,v) \leq \frac{1}{p}\int |\nabla u|^p + \frac{1}{q}\int |\nabla v|^q - \int b(x)G(u,v) + C \tag{5.3}$$

Choosing $u = t^{1/p}u_0$ and $v = t^{1/q}v_0$, where (u_0, v_0) is the eigenfunction pair corresponding to $\lambda_1(b)$ and using Lemma 3.1 we get from

$$\begin{aligned}\Phi(t^{1/p}u_0, t^{1/q}v_0) &\leq \frac{t}{p}\int |\nabla u_0|^p + \frac{t}{q}\int |\nabla v_0|^q - t\int bG(u_0,v_0) + C\\ &= t\lambda_1(b)\left\{\frac{1}{p}\int |u_0|^p + \frac{1}{q}\int |v_0|^q\right\} + C,\end{aligned}$$

which goes to $-\infty$ as $t \to +\infty$, in view of (F_{10}).

6. Final Comments.

There are many open problems connected with the study of the functional Φ depending on other assumptions on the potential F.

(i) Suppose that the growth of F with respect to u is like $|u|^{p^*}$ as $|u| \to \infty$. A similar question with respect to v. The scalar case has been studied by several people, Guedda-Veron [GV], Egnell [E]

(ii) Problems with Ω replaced by R^N. For example, in the scalar case see [0] and references there in.

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