RELATÓRIO DE PESQUISA

SOME RESULTS ON PARTITIONS WITH DIFFERENCE CONDITIONS

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ABSTRACT - In this paper we prove a few theorems on partitions including one result previously given by Gordon. To do this we use some of Slater's identities.

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SOME RESULTS ON PARTITIONS WITH DIFFERENCE CONDITIONS

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Abstract: *In this paper we prove a few theorems on partitions including one result previously given by Gordon. To do this we use some of Slater's identities.*

Many of the identities given by Slater [5] have been used to help in proving several combinatorial results in partitions. Here we used identities 34, 36, 53 and 57 that are listed, in this order, below:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{8n-1})(1-q^{8n-7})(1-q^{8n})}{(1-q^{2n})} \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{8n-3})(1-q^{8n-5})(1-q^{8n})}{(1-q^{2n})} \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{4n^2}}{(q^4;q^4)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{12n-5})(1-q^{12n-7})(1-q^{12n})}{(1-q^{4n})} \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n+1} q^{4n(n+1)}}{(q^4;q^4)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1+q^{12n-1})(1+q^{12n-11})(1-q^{12n})}{(1-q^{4n})} \tag{4}$$

The results proved in this paper were done in a way that is similar to the proofs given by Andrews in [1] for some Theorems of Göllnitz.

Theorem 1 The number of partitions of n in an even number of parts, $2s$, such that $a_{2j-1} - a_{2j} = 1$ or 2 and $a_{2s} \geq 1$ is equal to the number of partitions of n in parts $\equiv \pm 3, 4 \pmod{8}$.

Proof.: Let $A(n)$ denote the number of partitions described in the first part of the theorem, i.e, partitions of the form $a_1 + a_2 + \cdots + a_{2s}$, where $a_{2j-1} - a_{2j} = 1$ or 2 and $a_{2s} \geq 1$.

We define, for $\lambda = 1$ or $2, g_\lambda(2s, n)$ as the number of partitions of the type enumerated by $A(n)$ with the added restriction that the number of parts is exactly $2s$ and each part $\geq \lambda$. For $n = s = 0$ we define $g_\lambda(2s, n) = 1$ and $g_\lambda(2s, n) = 0$ if $n < 0$ or $s < 0$ or $n = 0$ and $s > 0$.

In what follows we prove two identities for $g_\lambda(2s, n)$.

(i) $g_1(2s, n) = g_2(2s-2, n-2s-2) + g_1(2s-2, n-2s-1) + g_1(2s, n-2s)$

(ii) $g_2(2s, n) = g_1(2s, n-2s)$.

To prove the first one we split the partitions enumerated by $g_1(2s, n)$ into two classes: (a) those partitions in which 1 is a part and (b) those in which 1 is not a part.

The ones in class (a) can have as the two smallest parts either "3+1" or "2+1". In the first case if we remove the "3+1" and subtract 1 from each of the remaining parts we are left with a partition of $n-(3+1)-(2s-2) = n-2s-2$ into 2s-2 parts each ≥ 2 and these are the partitions enumerated by $g_2(2s-2, n-2s-2)$. From those in class (a) having "2+1" as smallest parts we remove the "2+1" and subtract 1 from each of the remaining parts. In doing this we are left with a partition of $n - (2+1) - (2s-2) = n - 2s - 1$ into 2s-2 parts each ≥ 1 and these are the partitions enumerated by $g_1(2s-2, n-2s-1)$. It is important to observe that after doing these operations the restrictions on the difference between parts is not changed.

Now we consider the partitions from class (b). Since 1 is not a part we can subtract 1 from each part obtaining a partitions of $n - 2s$ in 2s parts and these are

the ones enumerated by $g_1(2s, n-2s)$ which completes the proof of identity (i).

The proof for (ii) follows by the reason we have just mention in the consideration of class (b).

Next we define

$$G_\lambda(z, q) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_\lambda(2s, n) z^{2s} q^n.$$

Using, now, identity (i) we have

$$\begin{aligned}
G_1(2s, n) &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s, n) z^{2s} q^n \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(g_2(2s-2, n-2s-2) + g_1(2s-2, n-2s-1) + \\
&\quad g_1(2s, n-2s)) z^{2s} q^n \\
&= z^2 q^4 \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s-2, n-4s)(zq^2)^{2s-2} q^{n-4s} + \\
&\quad z^2 q^3 \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s-2, n-2s-1)(zq)^{2s-2} q^{n-2s-1} + \\
&\quad \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s, n-2s)(zq)^{2s} q^{n-2s} \\
&= z^2 q^4 G_1(zq^2, q) + z^2 q^3 G_1(zq, q) + G_1(zq, q) \qquad (5)
\end{aligned}$$

where, on the first sum, we used (ii).

Now, comparing coefficients of z^{2n} in (5) after making the substitution

$$\begin{aligned}
G_1(z, q) &= \sum_{n \geq 0} \gamma_n z^{2n}, \text{ we have} \\
\gamma_n &= q^{4n}\gamma_{n-1} + q^{2n+1}\gamma_{n-1} + q^{2n}\gamma_n.
\end{aligned}$$

Therefore

$$\gamma_n = q^{2n+1}\frac{(1+q^{2n-1})}{(1-q^{2n})}\gamma_{n-1} \qquad (6)$$

and, observing that $\gamma_0 = 1$, we may iterate this $n-1$ times to get:

$$\gamma_n = \frac{(-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_n}.$$

Then

$$G_1(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} z^{2n}$$

and the theorem follows since

$$\begin{aligned}\sum_{n=0}^{\infty} A(n)q^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s,n)q^n = G_1(1,q)\\ &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n}\\ &\stackrel{(1)}{=} \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n-3})(1-q^{8n-4})(1-q^{8n-5})} . \qquad (7)\end{aligned}$$

We observe, now, that the partitions of n with an even number of parts $2s$ where $a_{2j-1} - a_{2j} = 1$ or $2, a_{2s} \geq 1$ can be transformed into partitions of n in s parts just by adding the parts $a_{2j-1} + a_{2j}$, i.e., the partition

$$a_1 + a_2 + \cdots + a_{2j-1} + a_{2j} + a_{2j+1} + \cdots + a_{2s}$$

is transformed in

$$b_1 + b_2 + \cdots + b_j + \cdots + b_s$$

where $b_j = a_{2j-1} + a_{2j}$, $b_j - b_{j+1} \geq 2$ and $b_j - b_{j+1} \geq 3$ if b_{j+1} is even with the restriction $b_s \geq 3$.

This operation can be reversed, easily, in the following way:
if b_j is even we write it as $a_{2j-1} + a_{2j}$ where $a_{2j-1} = \frac{b_j}{2} + 1$ and $a_{2j} = \frac{b_j}{2} - 1$,
if b_j is odd we write it as $a_{2j-1} + a_{2j}$ where $a_{2j-1} = \frac{b_j+1}{2}$ and $a_{2j} = \frac{b_j-1}{2}$.

With this transformation we get the original one

$$a_1 + a_2 + \cdots + a_{2s}$$

with exactly the same restrictions, i.e., $a_{2j-1} - a_{2j} = 1$ or 2 and $a_{2s} \geq 1$.

To ilustrate this we list the partitions of 16 as described in the theorem and the ones obtained by the transformation given above.

16	⟷	9 + 7
13 + 3	⟷	7 + 6 + 2 + 1
12 + 4	⟷	7 + 5 + 3 + 1
11 + 5	⟷	6 + 5 + 3 + 2
10 + 6	⟷	6 + 4 + 4 + 2
9 + 7	⟷	5 + 4 + 4 + 3
8 + 5 + 3	⟷	5 + 3 + 3 + 2 + 2 + 1

By this transformation we have proved the following theorem:

"The number of partitions of n of the form

$$n = b_1 + b_2 + \cdots + b_s, \quad \text{where } b_j - b_{j+1} \geq 2, b_j - b_{j+1} \geq 3 \text{ if } b_{j+1} \text{ is even and } b_s \geq 3 \qquad (8)$$

is equal to the number of partitions of n into parts $\equiv \pm 3, 4 (\mathrm{mod} 8)$."

This Theorem was proved by Gordon in [3] (Theorem 3, page 741) and can be, also, proved in a similar way as we did for Theorem 1.

The proof is as follows:

Let $B(n)$ denote the number of partitions of n as in (8).

We define $g(s,n)$ as the number of partitions of the type enumerated by $B(n)$ with the added restriction that the number of parts is exactly s. The following identity is verified by $g(s,n)$:

$$g(s,n) = g(s-1, n-2s-1) + g(s-1, n-4s) + g(s, n-2s). \qquad (9)$$

To see this we split the partitions enumerated by $g(s,n)$ into three classes:

(a) the ones in which 3 is a part

(b) the ones in which 4 is a part

(c) the ones with parts > 4.

For the partitions in class (a) we drop the part "3" and subtract 2 from each of the remaining parts. In doing so we are left with a partition of $n - 2s - 1$ into exactly $s - 1$ parts satisfying (8) and these are the partitions enumerated by

$g(s-1,n-2s-1)$. In class (b) we drop the part "4" and subtract 4 from each of the remaining parts obtaining partitions that are enumerated by $g(s-1,n-4s)$ and finally for class (c) we just subtract 2 from each part obtaining the partitions enumerated by $g(s,n-2s)$ which completes the proof of (9).

Next we define

$$G(z,q) := \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g(s,n)z^s q^n.$$

Therefore using (9) we get

$$\begin{aligned} G(z,q) &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g(s,n)z^s q^n = \\ & zq^3 \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(g(s-1,n-2s-1)(zq^2)^{s-1}q^{n-2s-1} + \\ & zq^4 \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g(s-1,n-4s)(zq^4)^{s-1}q^{n-4s} + \\ & \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g(s,n-2s)(zq^2)^s q^{n-2s} \\ &= zq^3 G(zq^2,q) + zq^4 G(zq^4,q) + G(zq^2,q) \end{aligned} \quad (10)$$

If $G(z,q) = \sum_{n=0}^{\infty}\gamma_n z^n$ we get, by comparing coefficient in both sides of (10) that

$$\gamma_n = q^{2n+1}\gamma_{n-1} + q^{4n}\gamma_{n-1} + q^{2n}\gamma_n$$

i.e.,

$$\gamma_n = q^{2n+1}\frac{(1+q^{2n-1})}{(1-q^{2n})}\gamma_{n-1}$$

which is (6) and, therefore

$$G(z,q) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n}z^n$$

From this we have

$$\sum_{n=0}^{\infty} B(n)q^n = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g(s,n)q^n = G(1,q) = \sum\frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n}$$

which is (7) completing the proof.

Theorem 2 of Gordon in [3] can be, also, obtained in the way we have just proved his Theorem 3 or by a bijection after proving the following theorem:

Theorem 2. The number of partitions of n in an even number of parts , $2s$, such that $a_{2j-1} - a_{2j} = 1$ or 2 and $a_{2s} \geq 0$ is equal to the number of partitions of n in parts $\equiv \pm 1, 4 \pmod 8$.

We omit the proof since it follows in exactly the same way of our Theorem 1 making use of identity (2) which is identity 36 in Slater [5].

In our next theorem we used identities 53 and 57 of Slater.

Theorem 3. The number of partitions of n in distinct parts of the form $n = a_1 + a_2 + \cdots + a_s$ where $a_s \equiv 1$ or $2 \pmod 4)$, $a_j \equiv 3$ or $4 \pmod 4$ if $a_{j+1} \equiv 1$ or $4 \pmod 4$ and $a_j \equiv 1$ or $2 \pmod 4$ if $a_{j+1} \equiv 2$ or $3 \pmod 4$ is equal to the number of partitions of n in parts that are distinct odd $\equiv \pm 5 \pmod{12}$ or even $\equiv \pm 4 \pmod{12}$ plus the number of partitions of "$n-1$" in distinct odd $\equiv \pm 1 \pmod{12}$ or even $\equiv \pm 4 \pmod{12}$.

Proof.: Let $C(n)$ denote the number of partitions described in the first part of the theorem. We define $f(s,n)$ as the number of partitions of the type enumerated by $C(n)$ with the added restriction that the number of parts is exactly s and each part ≥ 1. The following identity is true for $f(s,n)$:

$$f(s,n) = f(s-1,n-2s+1) + f(s-1,n-4s+2) + f(s,n-4s). \quad (11)$$

To prove this we split the partitions enumerated by $f(s,n)$ into three classes: (a) those in which 1 is a part, (b) those in which 2 is a part and (c) those with all parts > 2.

If in those from class (a) we drop the part "1" and subtract 2 from each of the remaining parts we are left with a partition of $n-1-2(s-1) = n-2s+1$ in exactly

$s-1$ parts each ≥ 1 and these are the ones enumerated by $f(s-1, n-2s+1)$. From those in class (b) we drop the part "2" and subtract 4 from each of the remaining parts obtaining partitions that are enumerated by $f(s-1, n-4s+2)$ and for the ones in class (c) we subtract 4 from each part obtaining the partitions enumerated by $f(s, n-4s)$.

Now we define

$$F(z,q) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n) z^s q^n$$

and using (11) we get:

$$\begin{aligned} F(z,q) = & \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(f(s-1,n-2s+1)+f(s-1,n-4s+2)+f(s,(n-4s)z^s q^n \\ &= zq \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s-1,n-2s+1)(zq^2)^{s-1} q^{n-2s+1} + \\ & zq^2 \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s-1,n-4s+2)(zq^4)^{s-1} q^{n-4s+2} + \\ & \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n-4s)(zq^4)^s q^{n-4s} \\ &= zq\, F(zq^2,q) + zq^2 F(zq^4,q) + F(zq^4,q). \end{aligned} \tag{12}$$

If $F(z,q) = \sum_{n=0}^{\infty} \gamma_n z^n$ we may compare coefficients of z^n in (12) obtaining

$$\gamma_n = \gamma_{n-1} q^{2n-1} + \gamma_{n-1} q^{4n-4} + \gamma_n q^{4n}.$$

Therefore

$$\gamma_n = q^{2n-1} \frac{(1+q^{2n-1})}{(1-q^{4n})} \gamma_{n-1} \tag{13}$$

and observing that $\gamma_0 = 1$ we may iterate (13) to get

$$\gamma_n = \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n}.$$

From this

$$F(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2} z^n}{(q^4;q^4)_n}$$

and now we can finish the proof since

$$\begin{aligned} \sum_{n=0}^{\infty} C(n) q^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n) q^n = F(1,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} \\ &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n} q^{4n^2}}{(q^4;q^4)_{2n}} + q \sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n+1} q^{4n^2+4n}}{(q^4;q^4)_{2n+1}} \\ &= \prod_{n=1}^{\infty} \frac{(1+q^{12n-5})(1+q^{12n-7})(1-q^{12n})}{(1-q^{4n})} + \\ & q \prod_{n=1}^{\infty} \frac{(1+q^{12n-1})(1+q^{12n-11})(1-q^{12n})}{(1-q^{4n})} \end{aligned}$$

where we have used identities (3) and (4) after replacing in (3) "q" by "-q" which completes the proof.

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