RELATÓRIO DE PESQUISA

HARMONIC SEQUENCES OF HARMONIC 2-SPHERES
IN GRASSMANN MANIFOLDS

Xiaohuan Mo

and

Caio J. C. Negreiros

Fevereiro

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ABSTRACT – We shown in this paper that the maximum of ∂' $\left(\text{or } \frac{\partial}{\partial Z}\right)$-order of harmonic maps $\phi : S^2 \to G_k(\mathbb{C}^n)$ is equal to $n-1$ where $G_k(\mathbb{C}^n)$ is the Grassmann manifold of k-planes in $\mathbb{C}^n$ equipped with its natural Fubini-Study metric.

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13083-970 Campinas SP
Brasil

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Harmonic sequences of harmonic 2-spheres in Grassmann manifolds

Xiaohuan Mo
(University of Peking and IMECC-UNICAMP)
and
Caio J.C. Negreiros
(IMECC-UNICAMP)

Abstract

We shown in this paper that the maximum of $\partial'\left(\text{or } \frac{\partial}{\partial Z}\right)$-order of harmonic maps $\phi : S^2 \to G_k(\mathbb{C}^n)$ is equal to $n-1$ where $G_k(\mathbb{C}^n)$ is the Grassmann manifold of k-planes in $\mathbb{C}^n$ equipped with its natural Fubini-Study metric.

§0 Introduction

The study of harmonic maps of compact Riemann surfaces into homogeneous spaces is receiving a large attention in Geometry and Physics.

In the late 1960's Chern [4] and Calabi [3] published several works on minimal immersions into spheres or more generally real projective spaces, which are in the spirit of this paper.

The problem was reexamined by physicists Din-Zakarewski [6] and Glaser and Storal [9] which complexify it. Inspired by these ideas Eells and Wood [8] gave a complete classification for harmonic 2-spheres in $\mathbb{C}P^n$, and also some important partial results for the higher genus cases in terms of holomorphic data.

A number of related results have appeared including Burstall and Wood [2], Chern and Wolfson [5] and Uhlenbeck [14]. These authors have studied harmonic 2-spheres into complex Grassmannians and [14] also studies the symmetric space case in general via the use of a Cartan's theorem.

We also want to mention that the study of critical points of the energy functional (which are the harmonic maps!) is linked with the study of Yang-Mills-Higgs fields in 3 dimensions, Yang-Mills in 4 dimensions or more generally with the Seiberg-Witten equations.

We will now state the main result in this paper (3.1 Theorem). Let $G_k(\mathbb{C}^n)$ the complex Grassmann manifold consisting of k-planes in $\mathbb{C}^n$ and $k = 0, 1, \ldots, n-1$. We define Ω_k as the set of harmonic 2-spheres in $G_k(\mathbb{C}^n)$ where $S^2(\approx \mathbb{C}P^1)$ and $G_k(\mathbb{C}^n)$ are equipped with its natural Fubini-Study metric. Let $\Omega = \bigcup_{k=1}^{n-1} \Omega_k$.

We define the function $\tau : \Omega \to \mathbb{N} \cup \{0\}$ by $\tau(\phi) =$ the $\partial'\left(\text{or } \frac{\partial}{\partial Z}\right)$- order of ϕ. Hence:

0.1. Theorem ([8]):

a) $\max_{\phi \in \Omega_1} \tau(\phi) = n - 1$

b) The set $\{\phi \in \Omega_1; \tau(\phi) = n-1\}$ is equal to the set of full holomorphic 2-spheres in $G_1(\mathbb{C}^n)(\approx \mathbb{C}P^{n-1})$.

We also have:

0.2. Theorem ([2], [5] and [14]):

$$\tau(\phi) < \infty, \ \forall \phi \in \Omega$$

From these very fundamental theorems two natural questions are raised:

0.3. Question: Is it true that

$$\max_{\phi \in \Omega_k} \tau(\phi) < \infty \text{ for } k = 2, 3, \ldots, n-1 \ ?$$

0.4. Question: If 0.3 Question is true what is $\max_{\phi \in \Omega_k} \tau(\phi)$ where k and n vary in such way that for a given n we have $k = 0, 1, \ldots, n-1$?

Our main result answers both questions. It says the following:

3.1. Theorem: $\max_{\phi \in \Omega_k} \tau(\phi) = n - 1$ where $k = 1, 2, \ldots, n-1$.

It is a nice open question to describe the set $\{\phi \in \Omega_k; \tau(\phi) = n-1$ for $k = 2, 3, \ldots, n-1\}$. We know from 0.1. Theorem that the set $\{\phi \in \Omega_1; \tau(\phi) = n-1\}$ is equal to the set of non-degenerate holomorphic 2-spheres in $G_1(\mathbb{C}^n)(\approx \mathbb{C}P^{n-1})$ but we will show that this fact is not true for $k = 2, 3, \ldots, n-1$ because the $u \in \Omega_k$ such that $\tau(u) = n-1$ which we will find in the proof of 3.1 Theorem is neither holomorphic nor antiholomorphic.

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§1. Preliminaries

We recall that $G_k(\mathbb{C}^n)$ is the set formed by k-dimensional complex subspaces of $\mathbb{C}^n$. Then the tautologously k-dimensional vector bundle T defined on $G_k(\mathbb{C}^n)$ has as fiber on $V \in G_k(\mathbb{C}^n)$ the same set V seeing as a k-dimensional complex subspace of $\mathbb{C}^n$.

Hence we identify a smooth map $\phi : M^2 \to G_k(\mathbb{C}^n)$ with a subbundle $\underset{\sim}{\phi}$ of $\underset{\sim}{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$ of rank k which has fibre at $x \in M$ given by $\underset{\sim}{\phi}_x = T_{\phi(x)}$ i.e. $\underset{\sim}{\phi} = \phi^*(T)$ and M^2 is an arbitrary compact Riemann surface without boundary.

Any subbundle $\underset{\sim}{\phi}$ of $\underset{\sim}{\mathbb{C}}^n$ inherits a metric denoted by $\langle\ ,\ \rangle_\phi$ and connection denoted by D_ϕ from the flat metric and connection ∂ on $\underset{\sim}{\mathbb{C}}^n$. More explicitly we have: $\langle V, W\rangle_\phi = \langle V(x), W(x)\rangle$ for any $V, W \in \underset{\sim}{\phi}_x, x \in M^2$ and $(D_\phi)_Z W = \pi_{\underset{\sim}{\phi}}(\partial_Z W)$ where $Z \in T(M)' = T(M)^{(1,0)}$ and $W \in \Gamma(\underset{\sim}{\phi})$. Here $\pi_{\underset{\sim}{\phi}} : \underset{\sim}{\mathbb{C}}^n \to \underset{\sim}{\phi}$ denotes the Hermitian projection in the subbundle $\underset{\sim}{\phi}$. We will denote $\pi_{\underset{\sim}{\phi}}$ by V throught this paper.

The $\partial'\left(\text{or } \frac{\partial}{\partial Z}\right)$-second fundamental form of $\underset{\sim}{\phi}$ in $\underset{\sim}{\mathbb{C}}^n$ is the vector bundle morphism $A'_\phi : \Gamma(\underset{\sim}{\mathbb{C}}^n) \to \Gamma(\underset{\sim}{\phi}^\perp)$ where $A'_\phi = V^\perp \circ \frac{\partial}{\partial Z} \circ V$. Similarly we define $A''_\phi = V^\perp \circ \frac{\partial}{\partial \overline{Z}} \circ V$.

Given holomorphic vector bundles E, F over M^2 and a holomorphic section $s \in \Gamma(\bigotimes^\ell (M^{(1,0)})^* \otimes L(E, F))$, they determine in a unique way holomorphic subbundles of F and E such that: $(\underline{Im}\, s)_x = \mathrm{Im}(s(x))$ and $(\underline{ker}\, s)_x = \ker(s(x))$ for $x \in M$.

1.1. Definition: Let $\phi : M^2 \to G_k(\mathbb{C}^n)$ harmonic. $\underline{Im} A'_\phi$ is a holomorphic subbundle of $\underset{\sim}{\phi}^\perp$ called the ∂'-Gauss bundle of ϕ (we will denote it by $G'(\phi)$).

1.2. Theorem ([2], [5] or [14]) $G'(\phi))$ is harmonic.

We can iterate the construction above, hence we have:

1.3. Definition Let $\phi : M^2 \to G_k(\mathbb{C}^n)$ harmonic we define $\phi^i (i \in \mathbb{N} \cup \{0\})$ by $\phi^0 = \phi, \phi^i = G'(\phi^{i-1})$.

1.4. Remark We of course can do the same with respect to ∂''.

1.5. Definition We say that a harmonic map $\phi : M^2 \to G_k(\mathbb{C}^n)$ has ∂'-order r if $\phi^r \neq 0$ but $\phi^{r+1} = 0$

Let $\mathbb{C}_n$ the set formed by $n \times n$ matrices with complex coefficients. We know that $U(n) = \{A \in \mathbb{C}_n, A^{-1} = \overline{A}^t\}$ and its Lie algebra $u(n) = \{M \in \mathbb{C}_n, M + \overline{M}^t = 0\}$

Let $\mu \in \Gamma(T^*(U(n)) \otimes u(n))$ be the Maurer-Cartan form of $U(n)$ i.e. for any $X \in T(U(n))_a$ $[u(X)]_a = X$

1.6.Lemma If $u(n) \cong T(U(n))_I$ then $\mu(X) = a^{-1}X$ for any $X \in (T(U(n)))_a$.

Proof: According to its definition we have: $\mu(X) = [\mu(X)]_I = (dL_{a^{-1}})[\mu(X)]_a = (dL_{a^{-1}})(X)$. Then by the definition of the derivative we can consider $\gamma = \gamma(t) \subseteq U(n), \gamma(0) = a$ and $\gamma'(0) = X$ and so we have:

$$\mu(X) = (dL_{a^{-1}})(X) = \frac{d}{dt}(a^{-1}\gamma)\Big|_{t=0} = a^{-1}\frac{d\gamma}{dt}\Big|_{t=0} = a^{-1}X.$$

We know that the complex structure of M^2 determines a natural splitting $(T(M)^*)^{\mathbb{C}} = (T(M)')^* \oplus (T(M)'')^*$. So if $\psi M^2 \to U(n)$ is an arbitrary smooth map, $\alpha = \psi^* u$ is equal to $\psi^{-1}d\psi$ by 1.6. Lemma and by making use of the this natural splitting we have that $\alpha = \alpha' + \alpha''$ where $\alpha' \in \Gamma((T(\mathbb{C}P^1)')^* \otimes u(n))$ and $\alpha'' \in \Gamma((T(\mathbb{C}P^1)'')^* \otimes u(n))$

For each $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$ we define

$$\alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha''$$

then $\alpha_\lambda \in \Gamma[((T(\mathbb{C}P^1))^*)^{\mathbb{C}} \otimes u(n)]$. Since $u(n)^{\mathbb{C}} = g\ell(n, \mathbb{C})$ then $U(n)^{\mathbb{C}} = G\ell(n, \mathbb{C})$

1.7. Definition Let $\Psi : \mathbb{C}^* \times M^2 \to G\ell(n, \mathbb{C})$. Ψ is called an extended solution of ψ if $\forall \lambda \in \mathbb{C}^* \Psi_\lambda = \Psi(\lambda,) : M^2 \to G\ell(n, \mathbb{C})$ satisfies $\Psi_\lambda^{-1} d\Psi_\lambda = \alpha_\lambda$ for any $\psi : M^2 \to U(n)$.

The fundamental observation of Uhlenbeck is:

1.8. Theorem ([14]) $\psi : S^2 \to G_k(\mathbb{C}^n)$ is harmonic if and only if $(\Psi_\lambda)_{\lambda \in \mathbb{C}^*}$ is integrable

Proof: See [14].

§2 On extended solutions

We will consider from now on the set $ES(\psi)$ where $\Psi \in ES(\psi)$ if and only if Ψ is an extended solution of ψ.

2.1. Theorem ([14]) Let $\psi : S^2 \to U(n)$ harmonic. Then there exists $\Psi \in ES(\psi)$ such that:

(i) $\Psi_1 = I$ (where $I(x) = I \ \ \forall x \in S^2$)

(ii) $\Psi_\lambda = \sum_{u=0}^{m} T_u \lambda^u, T_m \not\equiv 0$ where $T_u : S^2 \to g\ell(n) = \text{End}\ (\mathbb{C}^n)$ or equivalently $T_u : S^2 \times \mathbb{C}^n \to \mathbb{C}^n$

(iii) span $I_m(T_0) = \mathbb{C}^n$

(iV) $\sum_{u=0}^{m} T_u T_{u+V} = \delta_{V0} I, V = 0, \pm 1, ..., \pm m$

and $T_u = 0$ for $u < 0$ or $u > m$.

Proof See [14]. We can also use Theorem 4.2 and 4.3 in [13] keeping in mind that the definitions of harmonic and pluri-harmonic maps are equivalent if $\phi : M^2 \to (N^n, h)$ is an arbitrary smooth map.

We also have:

2.2. Lema Suppose $\psi : S^2 \to U(n)$ is harmonic and $\Psi \in ES(\psi)$. Then there exists $\psi' : S^2 \to U(n)$harmonic and $\Psi'_\lambda \in ES(\psi')$ where $\Psi'_\lambda = \lambda^{-1}\Psi_\lambda(V + \lambda V^\perp), \psi' = \psi(V^\perp - V)$. (As usual V stands for $\underline{Im} A'_\psi$

Proof: Using [14] we can show that V is an instanton (or a flag factor) of ψ. Therefore

$$V^{\perp}\alpha' V = 0$$

and $V^{\perp}(\partial V + \frac{1}{2}\alpha'' V) = 0$

where $\overline{\partial} V = dV|_{V_0 S^2}$

Therefore by using again [14] we have that $\Psi_{-1}(V - V^{\perp}) : S^2 \to U(n)$ is harmonic then $\Psi_\lambda = a\psi$ hence $-\psi' : S^2 \to U(n)$ is harmonic (up to a constant $a \in U(n)$) and $\lambda\Psi'_\lambda \in ES(\Psi_{-1}(V - V^{\perp})) = ES(-\psi')$

Therefore if we fix $\lambda \in \mathbb{C}^*$ we have:

$(\Psi'_\lambda)^{-1} d\Psi'_\lambda = (\lambda\Psi'_\lambda)^{-1} d(\lambda\Psi'_\lambda)$. But $(\psi')^{-1} d\psi' = (-\psi')^{-1} d(-\psi')$ so $\psi' : S^2 \to U(n)$ is harmonic and $\Psi'_\lambda \in ES(\psi')$.

Our next result is the following:

2.3. Lemma: Let $\phi : M^2 \to U(n)$ be a harmonic map. Then:

a) $\underline{ker} A''_{G'(\phi)} = 0$

b) $\underline{ker} A''_\phi \oplus G'(\phi^{\perp}) = \underset{\sim}{\phi}$

Proof: a) Let $\zeta \in \Gamma(G'(\phi))$ such that $A''_{G'(\phi)}(\zeta) = 0$.

Hence $0 = \langle \eta, A''_{G'(\phi)}(\zeta)\rangle = -\langle A'_{G'(\phi)^{\perp}}\eta, \zeta\rangle$ for any $\eta \in \Gamma(G'(\phi)^{\perp})$ therefore $\underset{\sim}{\phi} \subset G'(\phi)^{\perp}$ and $\langle A'_\phi \eta, \zeta\rangle = \langle A'_{G'(\phi)^{\perp}}\eta, \zeta\rangle = 0$ for any $\eta \in \Gamma(\underset{\sim}{\phi})$.

If we choose $\eta_0 \in \Gamma(\underset{\sim}{\phi})$ such that $A'_\phi \eta_0 = \zeta$ then we have that $\zeta = 0$

b) Let $\zeta \in \Gamma(\underset{\sim}{\phi})$ such that $A''_\phi(\zeta) = 0$ and $\eta \in \Gamma(G'(\phi^{\perp}))$ that is there exists $\theta \in \Gamma(\underset{\sim}{\phi}^{\perp})$ such that $A'_{\phi^{\perp}} + \theta = \eta$

Hence $\langle \eta, \zeta\rangle = \langle A'_{\phi^{\perp}} + \theta, \zeta\rangle = \langle \theta, A''_\phi(\zeta)\rangle = 0$. Therefore $\underline{ker} A''_\phi + G'(\phi^{\perp})$

But $A''_\phi = -(A'_{\phi^{\perp}})^*$ therefore rank $(G''(\phi))$ = rank $G'(\phi^{\perp})$. Then finally we have: rank $(\underline{ker} A''_\phi \oplus G'(\phi^{\perp}))$ = rank $(\underline{ker} A''_\phi)$+ rank $(G'(\phi^{\perp}))$ = rank $(\underset{\sim}{\phi})$− rank $(G''(\phi))$+ rank $(G'(\phi^{\perp}))$ = rank $(\underset{\sim}{\phi})$.

We will denote by $c : G_k(\mathbb{C}^n) \to U(n)$ $(V \mapsto V - V^{\perp})$ which is predicted by Cartan's theorem. Let $\psi' = \psi(V^{\perp} - V)$ the map whose existence is predicted in 2.2 Lemma. We have:

2.4. Lemma Let $\phi \in \Omega_k$ and $\psi = c \circ \phi$. If $\underline{ker} A''_\phi = 0$ then $\psi' = c \circ \phi'$

Proof. Let $\alpha = \psi^{-1} d\psi = A^\psi_Z + A^\psi_{\overline{Z}} d\overline{Z}$ where as usual $A^\psi_Z = \psi^{-1}\frac{\partial \psi}{\partial Z} =$

$(\phi - \phi^{\perp})^{-1}\frac{\partial(\phi - \phi^{\perp})}{\partial Z} = (\phi - \phi^{\perp})\left(\frac{\partial \phi}{\partial Z} - \frac{\partial \phi^{\perp}}{\partial Z}\right) = \phi\frac{\partial \phi}{\partial Z} - \phi^{\perp}\frac{\partial \phi}{\partial Z} - \phi\frac{\partial \phi^{\perp}}{\partial Z} + \phi^{\perp}\frac{\partial \phi^{\perp}}{\partial Z} = \frac{1}{2}\frac{\partial}{\partial Z}[\phi^2 + (\phi^{\perp})^2] - \phi^{\perp}\frac{\partial \phi}{\partial Z} - \phi\frac{\partial \phi^{\perp}}{\partial Z} = \frac{1}{2}\frac{\partial}{\partial Z}(I) - (A'_\phi + A'_{\phi^{\perp}}) = -(A'_\phi + A'_{\phi^{\perp}})$ (remember that we are idetifying ϕ with $\underset{\sim}{\phi}$)

But by hypothesis $\underline{ker} A''_\phi = 0$ so according to the previous lemma $G'(\phi^{\perp}) = \underset{\sim}{\phi}$ so $V = Im A^\psi_Z = G'(\phi) \oplus G'(\phi^{\perp}) = G'(\phi) \oplus \phi$.

Therefore $\psi' = \psi(V^{\perp} - V) = (\phi - \phi^{\perp})(V^{\perp} - \phi \oplus G'(\phi)) = -\phi^{\perp}V^{\perp} - \phi^2 + G'(\phi) = \phi - (\phi')^{\perp} = c \circ \phi'$.

See [10], [11] or [12] for related material.

§3 Proof of Main theorem

We will prove in this section the following result:

3.1. Theorem $\max_{\phi \in \Omega_k} \tau(\phi) = n - 1$ where $k = 1, 2, \ldots, n - 1$.

Proof: If $\phi \in \Omega_k$ is constant then $G'(\phi) = 0$ so $\tau(\phi) = 0$.

Then without loss of generality we can assume that ϕ is not constant $c : G_k(\mathbb{C}^n) \to U(n)$ the Cartan's embbeding is known to be totally geodesic and an embbedding. Therefore $\psi = c \circ \phi$ is non-constant and a harmonic map from S^2 to $U(n)$. Then according to 2.1 Theorem there exists $\Psi \in ES(\psi)$ such that (i)-(iv) of this Theorem are true.

Case 1. We will suppose that $\underline{ker} A''_\phi = \underset{\sim}{0}$

By combining 2.2 Lemma with 2.4 Lemma we have: $\Psi'_\lambda = \lambda^{-1}\Psi_\lambda(V + \lambda V^{\perp}) \in ES(\psi') = ES(c \circ \phi)$ (1), hence $\Psi \in ES(\psi)$.

Hence $\Psi_\lambda^{-1} d\Psi_\lambda = \alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})\alpha' + \frac{1}{2}(1 - \lambda)\alpha''$ (2), therefore $\partial\Psi_\lambda = \frac{1}{2}(1 - \lambda^{-1})\Psi_\lambda\alpha'$ (3) and $\overline{\partial}\Psi_\lambda = \frac{1}{2}(1 - \lambda)\Psi_\lambda\alpha''$ (4).

By above $\sum_{u=0}^{m}(\partial T_u)\lambda^u = \frac{1}{2}(1 - \lambda^{-1})\sum_{u=0}^{m} T_u\lambda^u\alpha'$ (5). Similarly $\sum_{u=0}^{m}(\overline{\partial} T_i)\lambda^u = \frac{1}{2}(1 - \lambda)\sum_{u=0}^{m} T_u\lambda^u\alpha''$ (6).

Comparing coefficients in the above expressions we obtain:

$T_0\alpha' \overset{(5)}{=} 0$ (7), $\partial T_0 \overset{(5)}{=} \frac{1}{2}(T_0 - T_1)\alpha'$ (8), $\overline{\partial} T_0 \overset{(6)}{=} \frac{1}{2}T_0\alpha''$ (9).

Therefore:

$$\Psi'_\lambda = \lambda^{-1}\sum_{u=0}^{m} T_u\lambda^u(V+\lambda V^\perp) = \sum_{u=1}^{m} T_u\lambda^{u-1}V +$$

$$+\sum_{u=0}^{m} T_u\lambda^u V^\perp = \sum_{u=0}^{m-1} T_{u+1}\lambda^u V + \sum_{u=0}^{m} T_u\lambda^u V^\perp =$$

$$= \sum_{u=0}^{m} T'_u\lambda^u \quad (10)$$

where $T'_i = T_uV^\perp + T_{u+1}V, T_{m+1} \equiv 0$ (11).

Hence $T'_0V^\perp = (T_0V^\perp + T_1V)V^\perp = T_0V^\perp \overset{(7)}{=} T_0V^\perp + T_0V = T_0$ (12). Therefore Im (T_0) = Im $(T'_0V^\perp) \subseteq$ Im (T'_0) (13) (so rank $(T_0) \leq$ rank (T'_0)).

We now consider $T_0 : S^2 \times \mathbb{C}^n \to \mathbb{C}^n$ (14). We know according 2.1 Theorem, that $\mathbb{C}^n =$ span Im T'_0 (15).

Let $t_0 = \max_{x\in S^2}$ rank $(T_0)_x$ and $t'_0 = \max_{x\in S^2}$ rank $(T'_0)_x$

Let $X = \{x \in S^2;$ rank $(T_0)_x = t_0$ and rank $(T'_0)_x = t'_0\}$. Then for example arguing like in the proof of Lemma 6.3 of [13] we can show that X is a connected, open and dense subset of $\mathbb{C}P^1(\approx S^2)$

Claim $0 < t_0 < t'_0$

Proof: if $t_0 = 0$ then $\forall x \in S^2$, rank $T_0 = 0$ i.e. $(T_0)_x = \underset{\sim}{0}$ Therefore span Im $(T_0) = 0$ which contradicts (15).

On the other hand if $t_0 = t'_0$ then on X Im $(T_0) \overset{(13)}{=}$ Im (T'_0). Therefore Im $(T_1V) \overset{(11)}{=}$ Im $(T'_0 - t_0V) \overset{(13)}{\subseteq}$ Im (T_0). Then if we apply (8) we see that Im $(T_0|_X) \subseteq X \times \mathbb{C}^n$ is anti-holomorphic. By (9) there exists a subspace W of $\mathbb{C}^n$ such that Im $(T_0|_X) = X \times W$. Then we see that $\bigcup_{x\in S^2}(x,$ Im $(T_0)_x) \subseteq S^2 \times W$. Hence $\mathbb{C}^n =$ span Im $(T_0) = W$ then $t_0 = n$. Since ψ is not constant, there exists $m > 0$ such that $T_0T_m = 0$ then $T_m = 0$ on X then $T_m \equiv 0$ which is impossible.

Suppose now $\tau(\phi) = r$. Then $G^{(1)}(\phi) \neq 0, ..., G^r(\phi) \neq 0$ but $G^{(r+1)}(\phi) = \underset{\sim}{0}$

We have following diagram:

$$\begin{array}{ccccccccc}
\phi & \to & \phi' & \to & \phi^2 & \to & \dots & \to & \phi^{r+1} \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\psi & \to & \psi' & \to & \psi^2 & \to & \dots & \to & \psi^{r+1} \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\Psi & \to & \Psi' & \to & \Psi^2 & \to & \dots & \to & \Psi^{r+1} \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 < t_0 & < & t'_0 & < & t_0^2 & < & \dots & < & t_0^{r+1}
\end{array}$$

Therefore $t_0^{r+1} = \max_{x\in S^2}(T_0^{r+1})_x \leq n$. So $r+2 \leq n$ then $r \leq n-2$ (*)

Case 2 We will suppose that $\underline{ker}\ A''_\phi \neq 0$

By (1) we know that $\underline{ker}\ A''_{G'(\phi)} = \underset{\sim}{0}$. Hence if we apply (*) we have that $\tau(G'(\phi)) \leq n-2$. Therefore $\tau(\phi) = \tau(G'(\phi)) + 1 \leq n-1$.

It only remains now to find $u \in \Omega_k$ such that $\tau(u) = n-1$. For this purpose we can take any $h : S^2 \to \mathbb{C}P^{n-1}$ holomorphic and non-degenerate (for example we can consider the Veronese maps $h : \mathbb{C} \cup \{\infty\} \approx S^2 \to \mathbb{C}P^{n-1}$ where $h(Z) = [1, Z, ..., Z^{n-1}]$ and $h(\infty) = [(0,0,...0,1)]$

We can now apply the notation and results in [2] and [16].

Let the harmonic sequence $h = h^0, h^1, ...$ in the notation of [16] can be written:

$$h^0 = h \to h^1 \to h^2 \dots h^{n-2} \to h^{n-1}$$

We define:

$$H^k = h^0 \oplus h^{n-k+1} \oplus ... \oplus h^{n-1}$$

According to [1] we have that $A'_{H^k} = (H^k)^\perp \circ \frac{\partial}{\partial Z} \circ H^k = (h^1 + ... + h^{n-k}) \circ \frac{\partial}{\partial Z} \circ (h^0 + h^{n-k+1} + ... + h^{n-1}) = h' \circ \frac{\partial}{\partial Z} \circ h = A'_h$.

Therefore $G'(H^k) = h^1$, and in general $G^{(j)}(H^k) = h^j$ hence $\tau(H^k) = n-1$. So $u = H^k : S^2 \to G_k((\mathbb{C}^n)$ it is the wanted map in Ω_k such that $\tau(u) = n-1$ since u is harmonic according for example the results in [2] or [16].

3.2. Remark a) The $u \in \Omega_k$ above founded is not holomorphic since $G''(u) = G''(H^k) = h^{n-k}$. Such u is usually called a mixed pair

b) Combining 0.1 Theorem and the result showed in this paper we have:

$$[\tau^{-1}(n-1) \cap \Omega_k] - Hol(S^2, G_k(\mathbb{C}^n)) = \begin{cases} \phi & \text{if } k = 1 \\ \neq \phi & \text{for } k > 1 \end{cases}$$

The remark above suggest us naturally the following question:

3.3. Question: Describe precisely the set $\{\phi \in \Omega_k; \tau(\phi) = n-1$ for $k = 2, 3, ..., n-1\}$.

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