

RELATÓRIO DE PESQUISA

LINES OF CURVATURE ON
SURFACES IMMERSED
IN \mathbb{R}^4

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ABSTRACT – We give the differential equation of the *lines of curvature* for immersions of surfaces in \mathbb{R}^4

$$(*) \quad 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0$$

and conversely, we prove that for any given analytic functions $a, b : U \rightarrow \mathbb{R}$ defined on an open neighborhood $U \subset \mathbb{R}^2$ of a point p , there exists an immersion $f : V \rightarrow \mathbb{R}^4$, where $V \subset U$ is some small open neighborhood of p , such that the differential equation of the lines of curvature of f is given by $(*)$ and the coordinates (u, v) are isothermic.

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Lines Of Curvature On Surfaces Immersed In \mathbb{R}^4

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Abstract. We give the differential equation of the *lines of curvature* for immersions of surfaces in \mathbb{R}^4

$$(*) \quad 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0$$

and conversely, we prove that for any given analytic functions $a, b : U \rightarrow \mathbb{R}$ defined on an open neighborhood $U \subset \mathbb{R}^2$ of a point p , there exists an immersion $f : V \rightarrow \mathbb{R}^4$, where $V \subset U$ is some small open neighborhood of p , such that the differential equation of the lines of curvature of f is given by $(*)$ and the coordinates (u, v) are isothermic.

Introduction. Let $f : M \rightarrow \mathbb{R}^4$ be a smooth immersion of a surface M . Let α be the second fundamental form of the immersion f . The set $\varepsilon_p = \{\alpha(X, X); |X| = 1, X \in T_p M\}$ is an ellipse called the *ellipse of curvature* of the immersion f at p . Every semi-axis of this ellipse determine a field of pairs of lines or a 4-cross field (see [3]) in the tangent space. The integral lines of these smooth cross field are called *Lines of Curvature* of the immersion f . This is a natural extension of the classical idea for surfaces in \mathbb{R}^3 .

Let U be an open neighborhood $U \subset M$ with isothermic coordinates (u, v) , Let $z = u + iv$ and $\lambda = |\partial_u| = |\partial_v|$ where $\partial_u = \frac{\partial}{\partial u}$ and $\partial_v = \frac{\partial}{\partial v}$.

Let us introduce the two Wirtingen operators

$$(1.1) \quad \partial_z = \frac{1}{\sqrt{2}}(\partial_u - i\partial_v), \quad \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_u + i\partial_v).$$

If we denote

$$(1.2) \quad A = \alpha(\partial_z, \partial_z), \quad B = \alpha(\partial_z, \partial_{\bar{z}}), \\ a = Re(\langle A, A \rangle), \quad b = 2Im(\langle A, A \rangle)$$

where \langle , \rangle is a bilinear complex extension of the inner product of $T^\perp M$ to $T^\perp M \otimes \mathbb{C}$, and $T^\perp M$ denotes the normal bundle. We prove the following theorem.

Theorem. Let $f : M \rightarrow \mathbb{R}^4$ be a smooth immersion of a surface M . In isothermic coordinates $(u, v) : M \rightarrow \mathbb{R}^2$, the differential equation of the lines of curvature of f is given by

$$(1.3) \quad 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0$$

where $a = a(u, v)$ and $b = b(u, v)$ are real valued functions given by (1.2).

Conversely, for any given analytic functions $a, b : U \rightarrow \mathbb{R}$ defined on an open neighborhood $U \subset \mathbb{R}^2$ of a point p , there exists an immersion $f : V \rightarrow \mathbb{R}^4$, where $V \subset U$ is some small open neighborhood of p , such that the differential equation of the lines of curvature of f is given by (1.3) and the coordinates (u, v) are isothermic.

2. Preliminaries. Using the notation (1.2) we have

Gauss Equation

$$(2.1) \quad \langle R(\partial_z, \partial_{\bar{z}})\partial_z, \partial_{\bar{z}} \rangle = \langle \alpha(\partial_z, \partial_z) - \alpha(\partial_{\bar{z}}, \partial_{\bar{z}}), \partial_z \rangle - |\alpha(\partial_z, \partial_{\bar{z}})|^2$$

Ricci Equation

$$(2.2) \quad R^\perp(\partial_z, \partial_{\bar{z}})v = \alpha(A_v \partial_{\bar{z}}, \partial_z) - \alpha(A_v \partial_z, \partial_{\bar{z}}), \quad v \in T^\perp M$$

Codazzi Equation

$$(2.3) \quad (\nabla_{\partial_z}^\perp \alpha)(\partial_{\bar{z}}, \partial_z) = (\nabla_{\partial_{\bar{z}}}^\perp \alpha)(\partial_z, \partial_z).$$

Let $\{e_3, e_4\}$ be a normal frame and let η be a smooth function such that

$$(2.4) \quad \begin{aligned} \nabla_{\partial_u}^\perp e_3 &= \eta e_4 & , \quad \nabla_{\partial_u}^\perp e_4 &= -\eta e_3 \\ \nabla_{\partial_v}^\perp e_3 &= 0 & , \quad \nabla_{\partial_v}^\perp e_4 &= 0. \end{aligned}$$

If we denote

$$(2.5) \quad A_\beta = \langle A, e_\beta \rangle \quad , \quad B_\beta = \langle B, e_\beta \rangle \quad , \quad \beta = 3, 4$$

then the Gauss, Ricci and Codazzi equations we can re-write, respectively as:

$$(2.6) \quad -\lambda^2 \Delta \log \lambda = |A|^2 - |B|^2 \quad (\text{Gauss equation}).$$

Proof. We have

$$(2.7) \quad \begin{aligned} R(\partial_z, \partial_{\bar{z}})\partial_z &= \nabla_{\partial_z} \nabla_{\partial_{\bar{z}}} \partial_z - \nabla_{\partial_{\bar{z}}} \nabla_{\partial_z} \partial_z \\ &= -\nabla_{\partial_{\bar{z}}} \left(\frac{2}{\lambda} \frac{\partial \lambda}{\partial z} \right) \partial_z \\ &= -\nabla_{\partial_{\bar{z}}} \left(2 \frac{\partial \log \lambda}{\partial z} \right) \partial_z \\ &= -\Delta \log \lambda \partial_z . \end{aligned}$$

This implies that

$$(2.8) \quad \langle R(\partial_z, \partial_{\bar{z}})\partial_z, \partial_{\bar{z}} \rangle = -\lambda^2 \Delta \log \lambda.$$

Hence, from (1.2), (2.1) and (2.8) follows (2.6)

$$(2.9) \quad \frac{\partial \eta}{\partial v} = -\frac{2}{\lambda^2} \operatorname{Im}(\bar{A}_3 A_4) \quad (\text{Ricci equation}).$$

Proof. Using (2.2) and (2.5) we obtain

$$\begin{aligned} R^\perp(\partial_z, \partial_{\bar{z}})e_3 &= \alpha(A_{e_3} \partial_{\bar{z}}, \partial_z) - \alpha(A_{e_3} \partial_z, \partial_{\bar{z}}) \\ &= \frac{1}{\lambda^2} [\alpha(\bar{A}_3 \partial_z + B_3 \partial_{\bar{z}}, \partial_z) + \alpha(A_3 \partial_{\bar{z}} + B_3 \partial_z, \partial_{\bar{z}})] \\ &= \frac{1}{\lambda^2} (\bar{A}_3 A - A_3 \bar{A}) \\ &= 2 \frac{i}{\lambda^2} \operatorname{Im}(\bar{A}_3 A). \end{aligned}$$

Hence, follows

$$(2.10) \quad \langle R^\perp(\partial_z, \partial_{\bar{z}})e_3, e_4 \rangle = 2 \frac{i}{\lambda^2} \operatorname{Im}(\bar{A}_3 A_4).$$

Also, we obtain

$$\begin{aligned} R^\perp(\partial_z, \partial_{\bar{z}})e_3 &= \nabla_{\partial_z}^\perp \nabla_{\partial_{\bar{z}}}^\perp e_3 - \nabla_{\partial_{\bar{z}}}^\perp \nabla_{\partial_z}^\perp e_3 \\ &= \nabla_{\partial_z}^\perp \left(\frac{\eta}{\sqrt{2}} e_4 \right) - \nabla_{\partial_{\bar{z}}}^\perp \left(\frac{\eta}{\sqrt{2}} e_4 \right) \\ &= \frac{1}{\sqrt{2}} \left(\frac{\partial \eta}{\partial z} - \frac{\partial \eta}{\partial \bar{z}} \right) e_4 \\ &= -i \frac{\partial \eta}{\partial v} e_4 \end{aligned}$$

and hence we get

$$(2.11) \quad \langle R^\perp(\partial_z, \partial_{\bar{z}})e_3, e_4 \rangle = -i \frac{\partial \eta}{\partial v}.$$

Hence, from (2.10) and (2.11) follows (2.9).

$$\begin{aligned} \frac{\partial A_3}{\partial z} - A_4 \eta &= \frac{\partial B_3}{\partial z} - B_4 \eta - 2 \frac{\partial \log \lambda}{\partial z} B_3 \\ (2.12) \quad &\qquad \qquad \qquad (\text{Codazzi equations}) \\ \frac{\partial A_4}{\partial z} + A_3 \eta &= \frac{\partial B_4}{\partial z} + B_3 \eta - 2 \frac{\partial \log \lambda}{\partial z} B_4 \end{aligned}$$

Proof. Using the equation (2.3) we have

$$(2.13) \quad \nabla_{\bar{z}}^{\perp} A = \nabla_{\bar{z}}^{\perp} B - 2 \frac{\partial \log \lambda}{\partial z} B.$$

Also, we find that

$$(2.14) \quad \nabla_{\bar{z}}^{\perp} A = \left(\frac{\partial A_3}{\partial \bar{z}} - A_4 \eta \right) e_3 + \left(\frac{\partial A_4}{\partial \bar{z}} + A_3 \eta \right) e_4$$

and

$$(2.15) \quad \nabla_{\bar{z}}^{\perp} B = \left(\frac{\partial B_3}{\partial \bar{z}} - B_4 \eta \right) e_3 + \left(\frac{\partial B_4}{\partial \bar{z}} + B_3 \eta \right) e_4.$$

Hence, from (2.13), (2.14) and (2.15) follows (2.12).

Now, if we denote

$$(2.16) \quad A_{\beta 1} = Re(\langle A, e_{\beta} \rangle), \quad A_{\beta 2} = Im(\langle A, e_{\beta} \rangle); \quad \beta = 3, 4$$

then the equations (2.6), (2.9) and (2.12) we can re-write, respectively as:

$$(2.17) \quad \lambda_{uv} = \frac{1}{\lambda} (-A_{31}^2 - A_{32}^2 - A_{41}^2 - A_{42}^2 + B_3^2 + B_4^2 + \lambda_u^2 + \lambda_v^2 - \lambda \lambda_{uu})$$

$$(2.18) \quad (\eta)_v = \frac{2}{\lambda^2} (A_{41} A_{32} - A_{31} A_{42})$$

$$(A_{32})_v = (A_{31})_u - (B_3)_u - \eta A_{41} + \eta B_4 + \frac{2}{\lambda} \lambda_u B_3$$

$$(A_{31} + B_3)_v = -(A_{32})_u + \eta A_{42} + \frac{2}{\lambda} \lambda_v B_3$$

$$(A_{42})_v = (A_{41})_u - (B_4)_u + \eta A_{31} - \eta B_3 + \frac{2}{\lambda} \lambda_u B_4$$

$$(A_{41} + B_4)_v = -(A_{42})_u - \eta A_{32} + \frac{2}{\lambda} \lambda_v B_4$$

3. Proof of the Theorem. First, observe that an easy computation shows that the differential equation of the lines of curvature of f is given by $Im(\langle A, A \rangle dz) = 0$ (See [1], Prop. 5.1) or (1.3).

Now, we need to prove only that for any given analytic functions a, b , there exists an immersion f such that the differential equation of the lines of curvature of f is given by (1.3) and the coordinates (u, v) are isothermal.

First, we observe that if we can find A_{32}, A_{42} then we can get A_{31} and A_{41} depending on a and b .

In fact if A_{42} is positive and sufficiently large and A_{32} sufficiently small from

(3.1)

$$a = A_{31}^2 - A_{32}^2 + A_{41}^2 - A_{42}^2 \\ b = 2(A_{31} A_{32} + A_{41} A_{42})$$

we can get

(3.2)

$$A_{31} = \frac{b A_{32} \pm A_{42} c}{2(A_{32}^2 + A_{42}^2)}, \quad A_{41} = \frac{b A_{42} \mp A_{32} c}{2(A_{32}^2 + A_{42}^2)}$$

where

$$c^2 = 4(A_{32}^2 + A_{42}^2)(4a + A_{32}^2 + A_{42}^2) - b^2.$$

So we set convenient initial conditions for A_{32} and A_{42} .

By using Cauchy-Kowalewsky Theorem (See [4]) we can find smooth functions C_β , $\beta = 3, 4, U_1, U_2, U_3, \eta, A_{32}$ and A_{42} satisfying

$$\frac{\partial U_1}{\partial v} = U_2$$

$$\frac{\partial U_2}{\partial v} = \frac{1}{U_1} \left(C_3^2 - 2A_{31}C_3 - 2A_{41}C_4 - A_{42}^2 + U_2^2 + U_3^2 - U_1 \frac{\partial U_3}{\partial u} \right)$$

$$\frac{\partial U_3}{\partial v} = \frac{\partial U_2}{\partial u}$$

$$\frac{\partial \eta}{\partial v} = \frac{2}{U_1^2} (A_{41}A_{32} - A_{31}A_{42})$$

$$(3.3) \quad \frac{\partial A_{32}}{\partial v} = 2 \frac{\partial A_{31}}{\partial u} - \frac{\partial C_3}{\partial u} - 2\eta A_{41} + \eta C_4 + \frac{2}{U_1} U_2 (C_3 - A_{31})$$

$$\frac{\partial C_3}{\partial v} = -\frac{\partial A_{32}}{\partial u} + \eta A_{42} + \frac{2}{U_1} U_2 (C_3 - A_{31})$$

$$\frac{\partial A_{42}}{\partial v} = 2 \frac{\partial A_{41}}{\partial u} - \frac{\partial C_4}{\partial u} + 2\eta A_{31} - \eta C_3 + \frac{2}{U_1} U_3 (C_4 - A_{41})$$

$$\frac{\partial C_4}{\partial v} = -\frac{\partial A_{42}}{\partial u} - \eta A_{32} + \frac{2}{U_1} U_2 (C_4 - A_{41}).$$

Now, it will be proved that there is an analytic function $\lambda = \lambda(u, v) > 0$ defined in a disk with center $O \in \mathbb{R}^2$ and analytic functions $A_{\beta i}, B_\beta$; $i = 1, 2$, $\beta = 3, 4$ which satisfy (2.17), (2.18) and (2.19).

In fact set $C_\beta = A_{\beta i} + B_\beta$; $\beta = 3, 4$ and $\lambda = U_1$, $\lambda_v = U_2$.

To have $\lambda > 0$ we include among the initial conditions $U_1(u, 0) = 1$. Observe

that an easy computation shows that $U_3 = \frac{\partial U_1}{\partial u} = \lambda_u$ (See [2], Prop. 5.1c).

Now the existence of the immersion f follows from the theorem of existence and unicity of immersions.

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