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EXISTENCE OF PRIMITIVES FOR
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EXISTENCE OF PRIMITIVES FOR ANALYTIC MAPPINGS BETWEEN BANACH SPACES.

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I. INTRODUCTION

We use notations, concepts and results of the theory of analytic mappings between Banach spaces as they appear, for instance, in [1] and [2]. Nevertheless we start by fixing some notations, definitions and results that we are going to use in this work.

If E and F are complex Banach spaces and $n \in \mathbb{N}$, we indicate by $\mathcal{L}(^nE;F)$ (respectively, $\mathcal{L}_s(^nE;F)$) the Banach space of all continuous (respectively, symmetric continuous) n-linear mappings from E^n into F under the norm:

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$$\|T\| := \sup_{\substack{x_j \in B_E \\ j=1,\dots,n}} \|T(x_1,\dots,x_n)\|$$

(Here B_E denotes the closed unit ball of E centered at 0). The corresponding Banach space of the continuous n-homogeneous polynomials from E into F is given by the vector space

$$\mathcal{P}(^nE;F) = \{\hat{T} : T \in \mathcal{L}(^nE;F)\}$$

(where $\hat{T}(x) = T(x,\dots,x)$ for $x \in E$) endowed with the norm

$$\|P\| = \sup_{x \in B_E} \|P(x)\| \qquad (\forall \in \mathcal{P}(^nE;F)).$$

There is a topological isomorphism $P \in \mathcal{P}(^nE;F) \mapsto \check{P} \in \mathcal{L}_s(^nE;F)$ with

$$\check{P}(x_1,\dots,x_n) = \frac{1}{n!2^n} \sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\dots,n}} \varepsilon_1 \dots \varepsilon_n P\left(\sum_{j=1}^{n} \varepsilon_j x_j\right)$$

for $x_j \in E, j = 1,\dots,n$. In this case $\hat{\check{P}} = P$ and $\|P\| \le \|\check{P}\| \le \frac{n^n}{n!}\|P\|$. We denote by $\mathcal{P}(^0E;F)$ the Banach space of the constant mappings from E into F identified to the Banach space F in a natural way.

If A is an open subset of E and $a \in A$, a mapping f from A into F is said to be *analytic at a* if there is a sequence $(P_n)_{n=0}^{\infty}$ of elements $P_n \in \mathcal{P}(^nE;F)$ such that

$$f(a+h) = \sum_{n=0}^{\infty} P_n(h)$$

uniformly for h in some open ball $B_\rho(0)$ of center 0 and radius $\rho > 0$ with $a + B_\rho(0) = B_\rho(a) \subset A$. There is unicity of representation of f by this series and it is usual to denote P_n and $\check{P}_n$ respectively by $\frac{1}{n!}\hat{d}^n f(a)$ and $\frac{1}{n!}d^n f(a)$ for all $n \in \mathbb{N} \cup \{0\}$. If f is analytic at every point of A, it is said that f is *analytic on* A and we use the notation $\mathcal{H}(A;F)$ to indicate the vector space of all such mappings.

It is kwnow (see [2]) that, if $g \in \mathcal{H}(A;F)$ and $dg(x)$ denotes the Fréchet diffeential of g at the point $x \in A$, then dg is in $\mathcal{H}(A;\mathcal{L}(E;F))$. Hence we have the natural question:

PROBLEM - If $f \in \mathcal{H}(A;\mathcal{L}(E;F))$ is there $g \in \mathcal{H}(A;F))$ such that $dg = f$ on A?

We shall see soon that if $\dim(E) \ge 2$ the answer is no in general. Hence the next natural question is to characterize those analytic mappings having primitives in the sense of the above problem.

II. PRIMITIVES OF POLYNOMIALS

It is well known that the mapping

$$\psi : \mathcal{L}(^kE;\mathcal{L}(E;F)) \longrightarrow \mathcal{L}(^{k+1}E;F)$$

given by

$$\Gamma(x_1,\dots,x_{k+1}) = T(x_1,\dots,x_k)(x_{k+1})$$

for $x_j \in E, j = 1,\dots,k+1$, is a linear isometry. Next exemple shows that for $\dim(E) \ge 2$ it is possible to have $\psi(\mathcal{L}_s(^kE;\mathcal{L}(E;F)) \not\subset \mathcal{L}_s(^{k+1}E;F)$.

1. EXAMPLE - For $E = \mathbb{C}^2$ we consider T from E into $\mathcal{L}(E;\mathbb{C}) = E' = \mathbb{C}^2$ given by $T((x_1,x_2))((y_1,y_2)) = x_1y_2$ for all $(x_1,x_2),(y_1,y_2) \in \mathbb{C}^2$. Hence T is linear but $\psi T \notin \mathcal{L}_s(^2E;\mathbb{C})$, since

$$\psi T((x_1,x_2),(y_1,y_2)) = x_1y_2$$
$$\psi T((y_1,y_2),(x_1,x_2)) = y_1x_2$$

Next proposition shows that the symmetry of $\psi(\check{P})$ has a strong relation with the polynomials P having primitives.

2. PROPOSITION. - For $P \in \mathcal{P}(^kE;\mathcal{L}(E;F))$ the following conditions are equivalent:

(1) $v(\dot{P}) \in \mathcal{L}_s(^{k+1}E;F)$

(2) There is $Q \in \mathcal{P}(^{k+1}E;F)$ such that $dQ(x) = P(x)$ or every $x \in E$.

PROOF - In order to show that (1) implies (2) we consider $Q = \frac{1}{k+1}\widehat{v(\dot{P})} \in \mathcal{P}(^{k+1}E;F)$ and evaluate its differential:

$$dQ(x)(h) = (k+1)\frac{1}{k+1}v(\dot{P})(x^k,h) = \dot{P}(x^k)(h) = P(x)(h)$$

for all $x, h \in E$ (Here, when we write x^k we are meaning $x,\ldots,x$ k times).

Now to prove the other implication we assume (2). Hence it follows that

$$\dot{P}(x^k)(h) = dQ(x)(h) = (k+1)\dot{Q}(x^k,h) \qquad (\forall x, h \in E)$$

and $v(\dot{P})(x_1,\ldots,x_{k+1}) = \dot{P}(x_1,\ldots,x_k)(x_{k+1}) =$

$$= \frac{1}{k!2^k}\sum_{\substack{\varepsilon_j=\pm1\\ j=1,\ldots,k}} \varepsilon_1\ldots\varepsilon_k\dot{P}\Big(\big(\sum_{j=1}^{k}\varepsilon_jx_j\big)^k\Big)(x_{k+1}) =$$

$$= \frac{1}{k!2^k}\sum_{\substack{\varepsilon_j=\pm1\\ j=1,\ldots,k}} \varepsilon_1\ldots\varepsilon_k(k+1)\dot{Q}\Big(\big(\sum_{j=1}^{k}\varepsilon_jx_j\big)^k, x_{k+1}\Big) =$$

$$= (k+1)\dot{Q}(x_1,\ldots,x_k,x_{k+1})$$

for $x \in E, j = 1,\ldots,k+1$. Since $\dot{Q}$ is symmetric, it follows that $v(\dot{P})$ is symmetric.

■

3. COROLLARY - The set of all $P \in \mathcal{P}(^kE;\mathcal{L}(E;F))$ such that $v(\dot{P})$ is symmetric is a closed vector subspace of $\mathcal{P}(^kE;\mathcal{L}(E;F))$.

PROOF - That the mentioned set is a vector subspace follows from the linearity of the differential and Proposition 2. Now we consider a sequence $(P_j)_{j=1}^{\infty}$ convergent to P in $\mathcal{P}(^kE;\mathcal{L}(E;F))$ with $v(\dot{P}_j)$ symmetric for every $j \in \mathbb{N}$. Thus

$\lim_{j\to\infty} v(\dot{P}_j) = v(\dot{P})$ in $\mathcal{L}(^{k+1}E;F)$. Since $v(\dot{P}_j)$ is symmetric for every j we have $v(\dot{P}) \in \mathcal{L}_s(^{k+1}E;F)$. ■

4. DEFINITION - If $P \in \mathcal{P}(^kE;\mathcal{L}(E;F))$ is such that $v(\dot{P})$ is symmetric (or equivalently: there is $Q \in \mathcal{P}(^{k+1}E;F)$ such that $dQ = P$), it is called an *exact differential on E*.

5. EXEMPLE - For fixed $\varphi \in E'$ and $b \in F$ we consider $\varphi \otimes b$ in $\mathcal{L}(E;F)$ given by $(\varphi \otimes b)(x) = \varphi(x)b$, for each $x \in E$. Hence we have $P \in \mathcal{P}(^kE;\mathcal{L}(E;F))$ defined by $P(x) = [\varphi(x)]^k\varphi \otimes b$ for $x \in E$. Thus $\dot{P}(x_1,\ldots,x_k) = \varphi(x_1)\ldots\varphi(x_k)\varphi\otimes b$ and $v(\dot{P})(x_1,\ldots,x_{k+1}) = \varphi(x_1)\ldots\varphi(x_{k+1})b$ is symmetric. Then P is an exact differential on E. As a consequence of this example and Corollary 3, for $\varphi_1,\ldots,\varphi_m \in E'$ and $b_1,\ldots,b_m \in F$ fixed

$$S(x) = \sum_{j=1}^{m}[\varphi_j(x)]^k\varphi_j \otimes b_j \qquad (\forall x \in E)$$

defines an exact differential on E. Of course every k-homogeneous polynomial from E^k into $\mathcal{L}(E;F)$ that is a limit in the norm of $\mathcal{P}(^kE;\mathcal{L}(E;F))$ of a sequence of polynomials of the form of S is an exact differential on E.

III EXISTECE OF LOCAL PRIMITIVES FOR ANALYTIC MAPPINGS

In this section A is a non-empty open subset of E.

We recall that for $f \in \mathcal{H}(A;\mathcal{L}(E;F))$ and $a \in A$ we have

$$f(a+h) = \sum_{k=0}^{\infty}\frac{1}{k!}d^kf(a)(h)$$

uniformly for $h \in \overline{B}_\rho(0)$ with $0 < \rho < r_f(a)$, where

$$r_f(a) = \min\left\{d(a, E\setminus A), \frac{1}{\overline{\lim}_{k\to\infty}\|\frac{1}{k!}d^kf(a)\|^{1/k}}\right\} > 0$$

1. THEOREM - If $f \in \mathcal{H}(A; \mathcal{L}(E;F))$ and $a \in A$ are such that $\hat{d}^k f(a)$ is an exact differential on E for every $k \in \mathbb{N}$, then there is $g \in \mathcal{H}(B_{\frac{1}{e}r_f(a)}(a); F)$ such that $dg(x) = f(x)$ for all $x \in B_{\frac{1}{e}r_f(a)}(a)$.

PROOF - We denote $P_k = \frac{1}{k!}\hat{d}^k f(a)$ for $k \in \mathbb{N}$ and consider

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{k+1}\widehat{\psi(\check{P}_k)}(x-a)$$

Since

$$\left\|\frac{1}{k+1}\widehat{\psi(\check{P}_k)}\right\| \le \frac{1}{k+1}\|\psi(\check{P}_k)\| = \frac{1}{k+1}\|\check{P}_k\| \le \frac{k^k}{k!(k+1)}\|P_k\|$$

we have

$$\overline{\lim_{k\to\infty}}\left\|\frac{1}{k+1}\widehat{\psi(\check{P}_k)}\right\|^{\frac{1}{k+1}} \le e\ \overline{\lim}_{k\to\infty}\|P_k\|^{\frac{1}{k}}$$

and it follows that g is well defined and analytic on $B_{\frac{1}{e}r_f(a)}(a)$. Since, as we saw before,

$$d\left(\frac{1}{k+1}\widehat{\psi(\check{P}_k)}\right) = P_k \qquad (\forall k \in \mathbb{N})$$

we have

$$dg(x)(t) = \sum_{k=0}^{\infty} P_k(x-a)(t) = f(x)(t) \qquad (\forall x \in B_{\frac{1}{e}r_f(a)}(a))$$

$$(\forall t \in E). \qquad \blacksquare$$

2. THEOREM - If $f \in \mathcal{H}(A; \mathcal{L}(E;F))$ and $a \in A$ are such that there are $r > 0$ and $g \in \mathcal{H}(B_r(a); F)$ satisfying $B_r(a) \subset A$ and $dg(x) = f(x)$ for all $x \in B_r(a)$, then $\hat{d}^k f(a)$ is an exact differential on E for every $k \in \mathbb{N}$.

PROOF - Of course we may take $r = \min\{r_f(a), r_g(a)\}$. Thus we can write:

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{d}^n g(a)(x-a) \qquad (\forall x \in B_r(a))$$

and

$$\begin{aligned} dg(x) &= \sum_{n=0}^{\infty} d\left[\frac{1}{n!}\hat{d}^n g(a)\right](x-a) \\ &= \sum_{n=1}^{\infty} n\frac{1}{n!}d^n g(a)\,((x-a)^{n-1}, \cdot) \qquad (\forall x \in B_{\frac{r}{e}}(a)). \end{aligned}$$

Hence for $t \in E$ and $x \in B_{\frac{r}{e}}(a)$ we have:

$$\begin{aligned} dg(x)(t) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!}d^n g(a)\,((x-a)^{n-1}, t) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}d^{k+1} g(a)\,((x-a)^k, t) \end{aligned}$$

and

$$\begin{aligned} f(x)(t) &= \sum_{k=0}^{\infty} \frac{1}{k!}d^k f(a)\,(x-a)^k(t) \\ &= \sum_{k=0}^{\infty} \psi(d^k f(a))((x-a)^k, t) \end{aligned}$$

By the uniqueness of the power series representation of $dg(x)(t) = f(x)(t)$, it follows that for $k \in \mathbb{N}$

$$d^k f(a)(u^k)(t) = \psi(d^k f(a))(u^k, t) = d^{k+1} g(a)(u^k, t) \qquad (*)$$

for all $u, t \in E$ (in fact we have it for all $t \in E$ and u in $B_{r/e}(0)$, but, by k-homogeneity, our statement follows). Now, since $d^k f(a)$ and $d^{k+1} g(a)$ are symmetric, we may write through the use of $(*)$:

$$\psi(d^k f(a))(y_1, \ldots, y_k, t) = d^k f(a)(y_1, \ldots, y_k)(t) =$$

$$= \frac{1}{k!2^k}\sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\ldots,n}} \varepsilon_1 \ldots \varepsilon_n d^k f(a)\left(\sum_{j=1}^{k} \varepsilon_j y_j\right)^k (t) =$$

$$= \frac{1}{k!2^k} \sum_{\substack{\varepsilon_j = \pm 1 \\ j=1,\dots,n}} \varepsilon_1 \dots \varepsilon_n d^{k+1} g(a) \left(\left(\sum_{j=1}^{k} \varepsilon_j y_j \right)^k, t \right) =$$

$$= d^{k+1} g(a)(y_1, \dots, y_k, t) \qquad (\forall y_j \in E, j = 1, \dots, k, t \in E).$$

Hence $\psi(d^k f(a))$ is symmetric and $d^k f(a)$ is an exact differential on E for all $k \in I\!N$. ■

Now it is natural to introduce the following

3. DEFINITION - It is said that $f \in \mathcal{H}(A; \mathcal{L}(E; F))$ is a *local exact differential at the point* $a \in A$ if $d^k f(a)$ is an exact differential on E for every $k \in I\!N$ (or equivalently, by 1. and 2., if there are $r > 0$ and $g \in \mathcal{H}(B_r(a); F)$ such that $B_r(a) \subset A$ and $dg(x) = f(x)$ for every $x \in B_r(a)$).

4. THEOREM - If A is connected and $f \in \mathcal{H}(A; \mathcal{L}(E; F))$ is a local exact differential at some point $a \in A$, then f is a local exact differential at every point of A.

PROOF - We consider

$$B = \{b \in A : f \ \text{ is a local exact differential at } b\}.$$

By hypothesis $B \neq \phi$. By 1 and 2 it follows that if $b \in B$, then $B_{\frac{1}{e} r_f(b)}(b) \subset B$. Thus B is open in A. Now we consider $b \in \overline{B} \cap A$ and a sequence $(b_j)_{j=1}^{\infty}$ in B such that $b = \lim_{j \to \infty} b_j$. Since $K = \{b\} \cup \{b_j; j \in I\!N\}$ is a compact subset of A and r_f is a strictly positive continuous function on A, there is $\rho > 0$ such that $r_f(t) \geq \rho$ for every $t \in K$. If $j_0 \in I\!N$ is such that

$$j \geq j_0 \ \Rightarrow \ \|b - b_j\| < \frac{\rho}{2e},$$

since $\rho \leq r_f(b_{j_0})$, it follows that $b \in B_{\frac{1}{2} r_f(b_{j_0})}(b_{j_0}) \subset B$. Thus B is closed in A. Since A is connected, $A = B$. ■

IV - A CAUCHY TYPE THEOREM AND THE EXISTENCE OF GLOBAL PRIMITIVES FOR ANALYTIC MAPPINGS.

In this section A is a connected open subset of E.

1. DEFINITION - A *path* γ *in* A is a continuous mapping from $[a, b] \subset I\!R$ into A, with $a < b$. The point $\gamma(a)$ is called the *origin* of γ and $\gamma(b)$ the *final point* of γ. If $\gamma(a) = \gamma(b)$ then γ is a *closed path* in A. If $\gamma([a, b]) = \{x_0\}$, it is said that γ is a *path reduced to the point* x_0 of A. The path γ_- given by $\gamma_-(t) = \gamma(a + b - t)$ for $t \in [a, b]$ is called the *opposite path* to γ. If $\gamma_1 : [b, c] \longrightarrow A$ is a path such that $\gamma_1(b) = \gamma(b)$ and γ_2 is defined by $\gamma_2(t) = \gamma(t)$ for $t \in [a, b]$, $\gamma_2(t) = \gamma_1(t)$ for $t \in [b, c]$, then the path γ_2 is called the *justapositon of* γ *and* γ_1 and denoted by $\gamma \vee \gamma_1$. The path γ is said to be *regular* if there are $M \geq 0$ and a finite or denumerable subset D of $[a, b]$ such that γ' exists and is continuous on $[a, b] \setminus D$ with $\sup_{t \in [a, b] \setminus D} |\gamma'(t)| \leq M$.

Two regular paths $\gamma_j : [a_j, b_j] \mapsto A$, $j = 1, 2$ are said to be *equivalent* if there is a bijection φ from $[a_1, b_1]$ onto $[a_2, b_2]$, $\varphi(a_1) = a_2$, $\varphi(b_1) = b_2$ such that φ and φ^{-1} are regular paths and $\gamma_1 = \gamma_2 \circ \varphi$. This gives an equivalence relation in the set of the regular paths in A. If $\gamma_1 : [a_1, b_1] \mapsto A$ is a regular path and $a_2 < b_2$ are given in $I\!R$, then there is a regular path $\gamma_2 : [a_2, b_2] \mapsto A$ equivalent to γ_1. It is enough to take $\varphi : [a_2, b_2] \mapsto [a_1, b_1]$ of the form $\varphi(t) = at + b$ with $\varphi(a_2) = a_1$ e $\varphi(b_2) = b_1$ and then define $\gamma_2 = \gamma_1 \circ \varphi$.

2. DEFINITION -If $\varphi : [a, b] \mapsto A$ is a regular path and $f : \gamma([a, b]) \mapsto \mathcal{L}(E; F)$ is continuous, the Riemann integral

$$\int_a^b f(\gamma(t))(\gamma'(t))dt,$$

that exists and is in F, is called the *integral of* f *over* γ and is denoted

$$\int_\gamma f(x)dx.$$

The following properties of this integral can be easily proved.

(1) $\int_{\gamma_1} f(x)dx = \int_{\gamma_2} f(x)dx$ for equivalent regular paths γ_1, γ_2.

(2) $\int_{\gamma} f(x)dx = -\int_{\gamma_-} f(x)dx$ for every regular path γ.

(3) If $\gamma = \gamma_1 \vee \gamma_2$ then $\int_{\gamma} f(x)dx = \int_{\gamma_1} f(x)dx + \int_{\gamma_2} f(x)dx$.

(4) If $\gamma : [a, b] \mapsto A$ is a closed regular path and $c \in]a, b[$, then $\gamma_c : [c, c+(b-a)] \mapsto A$ given by $\gamma_c(t) = \gamma(t)$ for $t \in [c, b]$ and $\gamma_c(t) = \gamma(t-b+a)$ for $t \in [b, c+b-a]$, is a closed regular path with image equal to the image of γ and

$$\int_{\gamma_c} f(x)dx = \int_{\gamma} f(x)dx.$$

3. DEFINITION - A *homotopy of a closed path* $\gamma_1 : [a, b] \mapsto A$ *into a closed path* $\gamma_2 : [a, b] \mapsto A$ is a continuous mapping $h : [a, b] \times [0, 1] \mapsto A$ such that $h(t, 0) = \gamma_1(t)$, $h(t, 1) = \gamma_2(t)$ for $t \in [a, b]$ and $h(a, \theta) = h(b, \theta)$ for each $\theta \in [0, 1]$. It is usual to say that γ_1 and γ_2 are *homotopic*.

It is easy to show that homotopy is an equivalence relation in the set of all closed paths in A.

Now we are ready to state and prove the following Cauchy type theorem.

4. THEOREM - If $f \in \mathcal{H}(A; \mathcal{L}(E; F))$ is a local exact differential at one point $a \in A$ (hence at every point A, by III.4) and γ_1, γ_2 are homotopic closed regular paths in A, then

$$\int_{\gamma_1} f(x)dx = \int_{\gamma_2} f(x)dx.$$

PROOF - We consider γ_1 and γ_2 defined on $[a, b]$ and the homotopy h from $[a, b] \times [0, 1]$ into A between γ_1 and γ_2. By the compacteness of $L = h([a, b] \times [0, 1])$, the analyticity of f on A and the fact that f is a local exact differential at each point of A, it is possible to find $x_1, \ldots, x_n \in L$ such that

$$L \subset \bigcup_{k=1}^{n} B_{\frac{1}{2}r_f(x_k)}(x_k) \subset A.$$

As we know from previous results there is $g_k \in \mathcal{H}(B_{\frac{1}{2}r_f(x_k)}(x_k); F)$ such that $dg_k = f_k$ on $B_{\frac{1}{2}r_f(x_k)}(x_k)$, $k = 1, \ldots, n$. If we take $\rho = \min\left\{\frac{1}{2e} r_f(x_k) : k = 1, \ldots, n\right\} > 0$, then for each $x \in L$, there is some $k \in \{1, \ldots, n\}$ such that $B_\rho(x) \subset B_{\frac{1}{2}r_f(x_k)}(x_k)$. Hence f is the sum of its power series around x on $B_\rho(x)$ for each $x \in L$. By the uniform continuity of h on $[a, b] \times [0, 1]$, there is $\varepsilon > 0$ such that

$$|t - t'| \leq \varepsilon \text{ and } |\theta - \theta'| \leq \varepsilon \Rightarrow ||h(t, \theta) - h(t', \theta')|| \leq \frac{\rho}{4}.$$

Now we can obtain partitions $\alpha_0 = a < \alpha_1 < \cdots < \alpha_{p-1} < \alpha_p = b$ and $\theta_0 = 0 < \theta_1 < \cdots < \theta_{q-1} < \theta_q = 1$ such that $|\alpha_{k+1} - \alpha_k| \leq \varepsilon$, $|\theta_{j+1} - \theta_j| \leq \varepsilon$ for $k = 0, \ldots, p-1$ and $j = 0, \ldots, q-1$. We define

$$\sigma_j(t) = h(\alpha_k, \theta_j) + \frac{t - \alpha_k}{\alpha_{k+1} - \alpha_k}[h(\alpha_{k+1}, \theta_j) - h(\alpha_k, \theta_j)]$$

for $t \in [\alpha_k, \alpha_{k+1}]$, $k = 0, 1, \ldots, p-1$ and $j = 1, \ldots, q-1$, and $\sigma_0 = \gamma_1, \sigma_q = \gamma_2$. These $\sigma_j, j = 0, \ldots, q$, are closed regular paths and our theorem will be proved if we show that

$$(*) \qquad \int_{\sigma_j} f(x)dx = \int_{\sigma_{j+1}} f(x)dx \quad \text{for } j = 0, \ldots, q-1$$

The choices of φ_k and θ_j were such that $\sigma_j(t), \sigma_{j+1}(t)$ are in $B_\rho(h(\alpha_k, \theta_j))$ for $t \in [\alpha_k, \alpha_{k+1}]$. Now we consider $g_{k,j} \in \mathcal{H}(B_\rho(h(\alpha_k, \theta_j)); F)$ such that $dg_{k,j} = f$ on $B_\rho(h(\alpha_k, \theta_j))$. Hence we can write:

$$\int_{\sigma_j} f(x)dx = \sum_{k=0}^{p-1} \int_{\alpha_k}^{\alpha_{k+1}} f(\sigma_j(t))(\sigma_j'(t))dt =$$

$$\sum_{k=0}^{p-1} \int_{\alpha_k}^{\alpha_{k+1}} dg_{k,j}(\sigma_j(t))(\sigma_j'(t))dt =$$

$$\sum_{k=0}^{p-1} [g_{k,j}(\sigma_j(\alpha_{k+1})) - g_{k,j}(\sigma_j(\alpha_k))].$$

Thus $(*)$ is equivalent to prove that

$$\sum_{k=0}^{p-1} [g_{k,j}(\sigma_j(\alpha_{k+1})) - g_{k,j}(\sigma_j(\alpha_k))] =$$

$$= \sum_{k=0}^{p-1} [g_{k,j}(\sigma_{j+1}(\alpha_{k+1})) - g_{k,j}(\sigma_{j+1}(\alpha_k))]$$

and this can be written

$$(**) \qquad \sum_{k=0}^{p-1} [g_{k,j}(\sigma_j(\alpha_{k+1})) - g_{k,j}(\sigma_{j+1}(\alpha_{k+1})) - g_k(\sigma_j(\alpha_k)) + g_{k,j}(\sigma_{j+1}(\alpha_k))] = 0$$

Now we note that $B_\rho(h(\alpha_{k-1}, \theta_j)) \cap B_\rho(h(\alpha_k, \theta_j)) \neq \phi$ and $g_{k-1,j} - g_{k,j}$ is constant on this set since its differential vanishes on it. As $\sigma_j(\alpha_k), \sigma_{j+1}(\alpha_k)$ belong to this set, we can write:

$$g_{k,j}(\sigma_j(\alpha_k)) - g_{k,j}(\sigma_{j+1}(\alpha_k)) = g_{k-1,j}(\sigma_j(\alpha_k)) - g_{k-1,j}(\sigma_{j+1}(\alpha_k)).$$

Hence $(**)$ may be written:

$$(***) \qquad g_{p-1,j}(\sigma_j(\alpha_p)) - g_{p-1,j}(\sigma_{j+1}(\alpha_p)) - g_{0,j}(\sigma_j(\alpha_0)) + g_{0,j}(\sigma_{j+1}(\alpha_0)) = 0$$

Since σ_j and σ_{j+1} are closed paths, $\alpha_0 = a$, $\alpha_p = b$, we have $\sigma_j(\alpha_0) = \sigma_j(\alpha_p), \sigma_{j+1}(\alpha_0) = \sigma_{j+1}(\alpha_p)$ and these points are in $B_\rho(h(\alpha_0, \theta_j)) \cap B_\rho(h(\alpha_{p-1}, \theta_j))$ where $g_{p-1,j} - g_{0,j}$ is constant, since its differential vanishes on it. Thus $(***)$ is true and the theorem is proved. ■

Of course the above proof of Theorem 4 is an adaptation of the usual proof of the Cauchy Theorem for analytic functions of one variable.

5. DEFINITION - The open connected subset A of E is *simply connected* if each closed path in A is homotopic to a closed path reduced to a point of A.

6. THEOREM - If A is simply connected and $f \in \mathcal{H}(A; \mathcal{L}(E; F))$ is a local exact differential at some point of A, then is $g \in \mathcal{H}(A; F)$ such that $dg = f$ on A.

PROOF - By III.4 f is a local exact differential at each point of A. We fix an $a \in A$. For each $z \in A$ we consider two regular paths γ_1 and γ_2 in A with origin a and final point z. Thus $\gamma_1 \vee \gamma_{2-}$ is a closed regular path in A, hence homotopic to a path reduced to a point of A. By theorem 4 we have

$$\int_{\gamma_1 \vee \gamma_{2-}} f(x)dx = 0$$

and then

$$\int_{\gamma_1} f(x)dx = \int_{\gamma_2} f(x)dx.$$

Therefore $h : A \mapsto F$ given by

$$h(z) = \int_{\gamma_z} f(x)dx,$$

where γ_z is a regular path in A with origin a and final point z, is well defined and does not depend on the particular γ_z considered. We want to whow that $dh = f$ and h is analytic on A. It is enough show that for each $z_0 \in A$ there is $r > 0$ with $B_r(z_0) \subset A, dh = f$ on $B_r(z_0)$ and h analytic on $B_r(z_0)$. By our hypotheses we know that we can find $r > 0$, such that $B_r(z_0) \subset A$, and a $g \in \mathcal{H}(B_r(z_0), F)$ such that $dg = f$ on $B_r(z_0)$. By adding a constant if necessary we may suppose that $g(z_0) = h(z_0)$. For each $z \in B_r(z_0)$ we consider $\sigma_z(t) = z_0 + t(z - z_0)$ for $t \in [0, 1]$. Hence

$$\int_{\sigma_z} f(x)dx = \int_0^1 f(z_0 + t(z - z_0))(z - z_0)dt =$$

$$= \int_0^1 dg(z_0 + t(z - z_0))(z - z_0)dt = g(z) - g(z_0) = g(z) - h(z_0)$$

If γ_{z_0} is regular path in A with origin a and final point z_0 we have that $\sigma_z \vee \gamma_{z_0}$ is a regular path in A with origin a and final point z. Thus

$$
\begin{aligned}
h(z) &= \int_{\sigma_z \vee \gamma_{z_0}} f(x)dx = \int_{\sigma_z} f(x)dx + \int_{\gamma_{z_0}} f(x)dx = \\
&= g(z) - h(z_0) + h(z_0) = g(z)
\end{aligned}
$$

for each $z \in B_\rho(z_0)$. ■

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