

3232

RT - BIMECC
3134

MINIMAL IMMERSIONS OF SURFACES
INTO n -DIMENSIONAL
SPACE FORMS

Irwen Valle Guadalupe

Março

RP 13/94

Relatório de Pesquisa

**Instituto de Matemática
Estatística e Ciência da Computação**



**UNIVERSIDADE ESTADUAL DE CAMPINAS
Campinas - São Paulo - Brasil**

0 6 JAN 1995

ABSTRACT - Using the motion of the ellipse of curvature we study minimal immersions of surfaces into n -dimensional space forms. In this paper we obtain an extension of theorem 2 of [9]. Also, we obtain some inequalities relating the integral of the normal curvature with topological invariants.

IMECC - UNICAMP
Universidade Estadual de Campinas
CP 6065
13081-970 Campinas SP
Brasil

O conteúdo do presente Relatório de Pesquisa é de única responsabilidade do(s) autor(es).

Março - 1994

I. M. E. C. C.
B I B L I O T E C A

Minimal Immersions of Surfaces Into n -Dimensional Space Forms

Irwen Valle Guadalupe

UNICAMP-IMECC

Universidade Estadual de Campinas

Caixa Postal 6065

13081-970, Campinas, SP, Brazil

Abstract

Using the motion of the ellipse of curvature we study minimal immersions of surfaces into n -dimensional space forms. In this paper we obtain an extension of theorem 2 of [9]. Also, we obtain some inequalities relating the integral of the normal curvature with topological invariants.

1. **Introduction.** Let M be an oriented surface which is isometrically immersed into an orientable n -dimensional space form $Q^n(c)$, $n \geq 4$, where $Q^n(c)$ stands for the sphere $S^n(c)$ of radius $1/c$, the Euclidean space \mathbb{R}^n or the hyperbolic space $H^n(c)$, according to c is positive, zero or negative. If the normal curvature tensor R^\perp of the immersion is nowhere zero, then exists an orthogonal bundle splitting $NM = (NM)^* \oplus (NM)^0$ of the normal bundle NM of the immersion, where $(NM)^0$ consists of the normal directions that annihilate R^\perp and $(NM)^*$ is a 2-plane subbundle of NM .

Let K and K_N be the Gaussian and the normal curvature of M . Let K^* be the intrinsic curvature of $(NM)^*$.

We shall make use of the *curvature ellipse* of $x : M \rightarrow Q^n(c)$, which is, for each p in M the subset of N_pM given by

$$\varepsilon_p = \{B(X, X) \in N_pM; X \in T_pM \text{ and } \|X\| = 1\}$$

where B is the second fundamental form of the immersion. The first result of this paper is an extension of theorem 2 of Rodriguez-Guadalupe [9] to the case when M is not homeomorphic to the sphere S^2 .

Theorem 1. Let $x : M \rightarrow S^n(1)$ be a minimal immersion of a complete oriented surface M into the unit sphere $S^n(1)$ with $R^\perp \neq 0$ and $K \geq 0$. If $2K \geq K^*$ at every point, then K^* , the normal curvature K_N and the Gaussian curvature K of M are constant.

Remarks. (1) If $K > 0$, then we obtain a minimal S^2 of constant curvature in $S^n(1)$. These were classified by Do Carmo-Wallach [4]. Itoh [6] and Asperti-Ferus-Rodriguez [1] have a similar theorem.

(2) For $K = 0$ we obtain a "flat" minimal torus. These were studied by Kenmotsu [7], [8].

The second result of this paper is the following.

Theorem 2. Let $x : M \rightarrow S^n(1)$ be a minimal immersion of a complete oriented surface M into the unit sphere $S^n(1)$. If $K \geq 0$ at every point, then either $K \equiv 0$ or the ellipse is a circle.

The following theorem relates an inequality between the integral of the

normal curvature with topological invariants.

Theorem 3. Let $x : M \rightarrow S^n(1)$ be a minimal immersion of a compact oriented surface M into an oriented n -dimensional space form $Q^n(c)$ of constant curvature c with $R^\perp \neq 0$. Then we have

$$(1.1) \quad \int_M K_N dM \geq 4\pi\mathcal{X}(M)$$

the equality holds if and only if $(M \sim S^2)$ $n = 4$.

Corollary 1. Let $x : M \rightarrow S^n(1)$ be a minimal immersion of a compact oriented surface M into the unit sphere $S^n(1)$ with $R^\perp \neq 0$. Then we have

$$(1.2) \quad \text{Area}(M) \geq 6\pi\mathcal{X}(M)$$

the equality holds if and only if $(M \sim S^2)$ $n = 4$.

Remark. Of course (1.2) has interest only when $M \sim S^2$, otherwise $\mathcal{X}(M) \leq 0$ and (1.2) becomes trivial.

The proofs of the above results are presented in section 4.

I want to thank Professor Asperti for bringing [2] and [3] for my attention.

2. Preliminaries. Let M be a surface immersed in a Riemannian manifold Q^n . For each p in M , we use T_pM , TM , N_pM and NM to denote the tangent space of M at p , the tangent bundle of M , the normal space of M at p and the normal bundle of M , respectively. We choose a local field of orthonormal frames e_1, e_2, \dots, e_n in Q^n such that restricted to M , the vectors e_1, e_2 are in T_pM and e_3, \dots, e_n are in N_pM .

We shall make use the following convention on the ranges of indices:

$$1 \leq A, B, C \dots \leq n, \quad 1 \leq i, j, k \leq 2$$

$$3 \leq \alpha, \beta, \gamma \dots \leq n$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of Q^n chosen above, let $\omega^1, \omega^2, \dots, \omega^n$ be the field of dual frames. Then the structure equations of Q^n are given by.

$$(2.1) \quad d\omega_A = -\sum \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0$$

$$(2.2) \quad d\omega_{AB} = -\sum \omega_{AC} \wedge \omega_{CB}, \quad \phi_{AB} = \sum K_{ABCD} \omega_C \wedge \omega_D$$

$$K_{ABCD} + K_{ABDC} = 0$$

If we restrict these forms to M . Then

$$(2.3) \quad \omega_\alpha = 0$$

since $0 = d\omega_\alpha = -\sum \omega_{\alpha i} \wedge \omega_i$, by Cartan's lemma we may write

$$(2.4) \quad \omega_{i\alpha} = \sum h_{ij}^\alpha \omega_j, \quad h_{ji}^\alpha = h_{ji}^\alpha$$

From these formulas, we obtain

$$(2.5) \quad d\omega_i = -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0$$

$$(2.6) \quad d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l$$

$$(2.7) \quad R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha)$$

$$(2.8) \quad d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta\ell\epsilon} \omega_\ell \wedge \omega_\epsilon$$

$$(2.9) \quad R_{\alpha\beta\ell\epsilon} = K_{\alpha\beta\ell\epsilon} + \sum_i (h_{ik}^\alpha h_{i\ell}^\beta - h_{i\ell}^\alpha h_{ik}^\beta)$$

The Riemannian connection of M is defined by (ω_{ij}) . The form $(\omega_{\alpha\beta})$ defines a connection ∇^\perp in the normal bundle of M . We call

$$(2.10) \quad B = \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \omega_j e_\alpha$$

the *second fundamental form* of M . The *mean curvature vector* is given by

$$(2.11) \quad H = \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha$$

An immersion is said to be minimal if $H = 0$.

Let R^\perp be the curvature tensor associated with ∇^\perp . Let $\{e_1, e_2\}$ be a tangent frame, if we denote $B_{ij} = B(e_i, e_j); i, j = 1, 2$ then it is easy to see that

$$(2.12) \quad R^\perp(e_1, e_2) = (B_{11} - B_{22}) \wedge B_{12}$$

An interesting notion in the study of surfaces in higher codimension is that of the *ellipse of curvature* defined as $\{B(X, X) \in N_p M : \langle X, X \rangle = 1\}$. To see that it is an ellipse, we just have to look at the following formula, for

$$X = \cos \theta e_1 + \sin \theta e_2$$

$$(2.13) \quad B(X, X) = H + \cos 2\theta u + \sin 2\theta v$$

where $u = \frac{B_{11} - B_{22}}{2}$, $v = B_{12}$ and $\{e_1, e_2\}$ is a tangent frame. So we see that, as X goes once around the unit tangent circle, $B(X, X)$ goes twice around the ellipse. Of course this ellipse could degenerate into a line segment or a point. Everywhere the ellipse is not a circle we can choose $\{e_1, e_2\}$ orthonormal such that u and v are perpendicular. When this happens they will coincide with the semi-axes of the ellipse.

From (2.12) it follows that if $R^\perp \neq 0$ then u and v are linearly independent and we can define a 2-plane subbundle $(NM)^*$ of the normal bundle NM . This plane

inherits a Riemannian connection from that NM . Let R^* be its curvature tensor and define its curvature K^* by

$$(2.14) \quad d\omega_{34} = -K^* \omega_1 \wedge \omega_2$$

Now, if ξ is perpendicular to $(NM)^*$, then from (2.12), $R^\perp(e_1, e_2)\xi = 0$. Hence, it makes sense to define the normal curvature as

$$(2.15) \quad K_N = \langle R^\perp(e_1, e_2)e_4, e_3 \rangle$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are orthonormal oriented bases of T_pM and N_pM , respectively. If TM and $(NM)^*$ are oriented, then K_N is globally defined. In codimension 2, $NM = (NM)^*$ and K_N has a sign. In higher codimension, if $R^\perp \neq 0$, $(NM)^*$ is globally defined and oriented if TM is. In this case, it is shown in [1] that $\mathcal{X}(NM)^* = 2\mathcal{X}(M)$, where $\mathcal{X}(NM)^*$ denote the Euler characteristic of the plane bundle $(NM)^*$ and $\mathcal{X}(M)$ denote the Euler characteristic of the tangle bundle TM .

3. Minimal Immersions with $R^\perp \neq 0$

In this section we assume that M has non-zero normal curvature tensor R^\perp . Also if M is orientable, then we will always choose orientations in TM and in $(NM)^*$ such that K_N is positive. We have

Proposition 1.1 Let $x : M \rightarrow Q^n(c)$ be a minimal immersion of an oriented surface M into an orientable n -dimensional space form $Q^n(c)$ of constant curvature c . Then we have

$$(3.1) \quad \Delta(\log |K_N - K + c|) = 2(2K - K^*)$$

if $(K - c)^2 - K_N^2 > 0$, and consequently

$$(3.2) \quad \Delta(\log |K_N + K - c|) = 2(2K + K^*)$$

Proof. By Itoh [6] there exists isothermal coordinates $\{x_1, x_2\}$ such that putting $X_i = \frac{\partial}{\partial x_i}$, $i = 1, 2$ then $u = B(X_1, X_1) = -B(X_2, X_2)$ and $v = B(X_1, X_2)$ are the semi-axes of the ellipse at every point where $(K - c)^2 - K_N^2 \neq 0$. Moreover we observe that $|X_i|^2 = E = ((K - c)^2 - K_N^2)^{-1/4}$, $i = 1, 2$. If we denote $\lambda = \langle u, u \rangle^{1/2}$ and $\mu = \langle v, v \rangle^{1/2}$ and following the same arguments that [10] we have

$$(3.3) \quad \lambda^2 - \mu^2 = 1$$

$$(3.4) \quad \lambda^2 + \mu^2 = -(K - c)E^2$$

$$(3.5) \quad 2\lambda\mu = K_N E^2$$

If $(K - c)^2 - K_N^2 > 0$ from (2.19) and (2.20) we obtain

$$(3.6) \quad \lambda + \mu = (K_N - K + c/K_N + K - c)^{1/4}$$

Let $e_3 = \lambda^{-1}u$ and $e_4 = \mu^{-1}v$ an oriented frame in $(NM)^*$. Now, following the same computations that [10] we get

$$(3.7) \quad \omega_{34}(X_1) = -X_2(f)$$

$$(3.8) \quad \omega_{34}(X_2) = X_1(f)$$

where $f = \log |\lambda + u|$.

Hence, we have

$$(3.9) \quad \omega_{34} = -X_2(f)dX_1 + X_1(f)dX_2.$$

Deriving (3.9) and using (2.14) we get

$$\begin{aligned}
 (3.10) \quad -K^* \omega_1 \wedge \omega_2 &= d\omega_{34} E^{-1} \\
 &= (-X_2 X_2(f) dX_2 \wedge dX_1 + X_1 X_1(f) dX_1 \wedge dX_2) E^{-1} \\
 &= (X_1 X_1(f) + X_2 X_2(f)) E^{-1} dX_1 \wedge dX_2 \\
 &= \tilde{\Delta}(f) E^{-1} \omega_1 \wedge \omega_2
 \end{aligned}$$

where $\tilde{\Delta}$ denotes de Laplacian of the "flat" metric. We know $\tilde{\Delta}(f) = E\Delta(f)$, where Δ is the Laplacian of the surface. Hence, from (2.18) and (2.22) we get

$$(3.11) \quad \Delta(\log |K_N - K + c/K_N + K - c|) = -4K^*$$

Using $E = (K - c)^2 - K_N^2)^{-1/4}$ and the Gaussian curvature K given by the equation

$$(3.12) \quad K = -\frac{1}{2} E^{-1} \tilde{\Delta} \log E$$

we obtain

$$(3.13) \quad \Delta(\log |K_N - K + c|) + \Delta(\log |K_N + K - c|) = 8K$$

From (3.11) and (3.13) we get the equations (3.1) and (3.2).

Corollary 1. Let $x : M \rightarrow Q^n(c)$ be a minimal immersion with $K^* > 0$. Then the ellipse is a circle.

Proof. Suppose that the ellipse is not a circle then from (3.11) $\Delta(\log |K_N - K + c/K_N + K - c|) < 0$. So $K_N > 0$ implies that $\log \frac{|K_N - K + c|}{|K_N + K - c|}$ is subharmonic and bounded from below. Therefore $\log \frac{|K_N - K + c|}{|K_N + K - c|}$ is constant and this implies that $K^* \equiv 0$. This is a contradiction.

Corollary 2. Let $x : M \rightarrow Q^n(c)$ be a minimal immersion of a compact surface M with $2K > K^*$. Then the ellipse is a circle.

Proof. Suppose that the ellipse is not a circle then from (3.1) $\Delta(\log |K_N - K + c|) > 0$. So we have that $\log |K_N - K + c|$ is subharmonic and bounded from above and therefore is constant. This implies that $2K = K^*$. This is a contradiction.

4. Proof of Theorems.

Proof of Theorem 1. First we consider the case when the ellipse is not a circle, i.e., $(K - 1)^2 - K_N^2 > 0$. Now if $2K \geq K^*$ then from (3.1) follows that $\Delta(\log |K_N - K + 1|) \geq 0$. So we have that $\log |K_N - K + 1|$ is subharmonic and bounded from above. Then

$$(4.1) \quad K_N - K + 1 = \text{constant}$$

and $2K = K^*$. On the other hand, from (3.2) we get $\Delta(\log |K_N + K - 1|) = 2(2K + K^*) = 8K \geq 0$. Similarly from above we have

$$(4.2) \quad K_N + K - 1 = \text{constant.}$$

From (4.1) and (4.2) follows that K^* , K_N and K are constant.

In the case that ellipse is a circle the theorem follows by Rodriguez-Guadalupe [9]. This completes the proof of theorem.

Proof of Theorem 2. Suppose that the ellipse is not a circle. From (3.13) we obtain

$$(4.3) \quad \Delta(\log |K_N - K + 1/K_N + K - 1|) = 8K \geq 0$$

From (3.5), Rodriguez-Guadalupe ([9], p. 9) and $K \geq 0$ implies $2\lambda\mu E^{-2} = K_N \leq 1$. So from (3.4) we have $(\lambda^2 + \mu^2)E^{-2} = 1$. Therefore we get

$$\begin{aligned} 0 &\leq (\lambda + \mu)^2 E^{-2} = K_N - K + 1 \\ &= (\lambda^2 + \mu^2)E^{-2} + 2\lambda\mu E^{-2} \leq 2 \end{aligned}$$

This implies that $|K_N - K + 1|$ is bounded from above. Similarly $|K_N + K - 1|$ is bounded from above, too. Then $\log(|K_N - K + 1|/|K_N + K - 1|)$ is subharmonic and bounded from above and therefore is constant. From (4.3) follows that $K \equiv 0$.

Proof of Theorem 3. From Asperti ([2], prop. 3.6) we have

$$(4.4) \quad K^* = K_N - \frac{\|B^2\|^2}{2K_N}$$

where B^2 is the 3-th. fundamental form of M . From (4.4) we obtain

$$(4.5) \quad K_N \geq K^*$$

Integrating (4.5) over M and applying Ferus-Rodriguez-Asperti ([1], th. 1) we get

$$(4.6) \quad \begin{aligned} \int_M K_N dM &\geq \int_M K^* dM = 2\pi\mathcal{X}(NM)^* \\ &= 4\pi\mathcal{X}(M) \end{aligned}$$

If $\int_M K_N dM = 4\pi\mathcal{X}(M)$ then $K_N = K^*$ and from (4.4) $B^2 \equiv 0$. By Erbacher [5] the codimension is two and $n = 4$.

Proof of Corollary 1. If $\text{Area}(M) = 6\pi\mathcal{X}(M)$ then $\mathcal{X}(M) > 0$ and, actually $\text{Area}(M) = 12\pi$. It follows from Asperti ([3], p. 60) that

$$(4.7) \quad 12\pi \geq 2\pi(s+1)(s+2)$$

where s is such that $n = 2 + 2s$. It is clear now from (4.7) that $s = 1$ and $n = 4$.

On the other hand, if $n = 4$ and $x : M \rightarrow S^4(1)$ is a minimal two sphere with $R^\perp \neq 0$, then $K_N = K^*$ and by theorem 3 above and Corollary 1 of Rodriguez-Guadalupe ([9]) we have

$$\begin{aligned}\text{Area}(M) &= 2\pi\mathcal{X}(M) + 2\pi\mathcal{X}(NM) \\ &= 2\pi\mathcal{X}(M) + \int_M K_N dM \\ &= 6\pi\mathcal{X}(M)\end{aligned}$$

References

- [1] A. C. Asperti, D. Ferus and L. Rodriguez: *Surfaces with non-zero Normal Curvature Tensor*, Atti Accad. Naz. Zincei, 73 (1982), 109-115.
- [2] A. C. Asperti: *Generic Minimal Surfaces*, Math. Z., 200 (1989), 181-186.
- [3] A. C. Asperti: *Superfícies mínimas genéricas e elipses de curvatura generalizadas*, Thesis de Livre-Docência (1986), Universidade de São Paulo.
- [4] M. do Carmo and N. Wallach: *Representations of compact groups and minimal immersions of spheres into spheres*, J. Diff. Geometry, 4 (1970), 91-104.
- [5] J. A. Erbacher: *Reduction of the codimension of an isometric immersion*, J. Diff. Geometry, 5 (1971), 333-340.
- [6] T. Itoh: *Minimal surfaces in 4-dimensional Riemannian manifolds of constant curvature*, Kodai Math. Sem. Rep., 22 (1971), 451-458.

- [7] K. Kenmotsu: *On a parametrization of minimal immersions of \mathbb{R}^2 into S^5* , Tôhoku Math. J., 27 (1975), 83-90.
- [8] K. Kenmotsu: *On minimal immersions of \mathbb{R}^2 into S^n* , J. Math. Soc. Japan, 28 (1976), 182-191.
- [9] L. Rodriguez and I. V. Guadalupe: *Normal curvature of surfaces in space forms*, Pacific J. Math., 106 (1983), 95-103.
- [10] R. Tribuzy and I. V. Guadalupe: *Minimal immersions of surfaces into 4-dimensional space forms*, Rend. Sem. Mat. Univ. Padova, 73 (1985), 1-13.

RELATÓRIOS DE PESQUISA — 1994

- 01/94 Stability of the Sucker Rod's Periodic Solution — *Aloisio Freiria Neves*.
- 02/94 A New Strategy for Solving Variational Inequalities in Bounded Polytopes — *Ana Friedlander, José Mario Martínez and Sandra Augusta Santos*.
- 03/94 Prime and Maximal Ideals in Polynomial Rings — *Miguel Ferrero*.
- 04/94 A New Globalization Strategy for the Resolution of Nonlinear Systems of Equations — *Ana Friedlander, Márcia A. Gomes-Ruggiero, José Mario Martínez and Sandra Augusta Santos*.
- 05/94 An Ambrosetti-Prodi Type Result for a System of Elliptic Equations via Leray-Schauder Degree — *Daniel Cordeiro de Moraes Filho*.
- 06/94 On Attractivity of Discontinuous Systems — *Marco Antonio Teixeira*.
- 07/94 Weakly Elliptic Systems of Variational Inequalities: a 2×2 Model Problem with Obstacles in both Components — *D. R. Adams and H. J. Nussenzweig Lopes*.
- 08/94 A New Method for Large-Scale Box Constrained Convex Quadratic Minimization Problems — *Ana Friedlander, José Mario Martínez and Marcos Raydan*.
- 09/94 Weak - Strong Continuity of Multilinear Mappings Pełczyński - Pitt Theorem — *Raymundo Alencar and Klaus Floret*.
- 10/94 Removable Singularities Theorems for the Yang-Mills Functional — *Antonella Marini*.
- 11/94 Magneto-Micropolar Fluid Motion: Existence and Uniqueness of Strong Solution — *Marko A. Rojas-Medar*.
- 12/94 Absolutely Summing Analytic Operators and the Generalized Khintchine Inequality — *Mário C. Matos*.