$\mathbb{Z}_2$–FIXED SETS OF STATIONARY POINT FREE $\mathbb{Z}_4$–ACTIONS

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Abstract: In this work we consider the question: Which classes in the unoriented bordism group of free $\mathbb{Z}_2$–actions can be realized as the $\mathbb{Z}_2$–fixed set of stationary point free $\mathbb{Z}_4$–action on a closed manifold with $\mathbb{Z}_2$–fixed point set having constant codimension k?

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1. Introduction

In [3], Capobianco has studied the fixed sets of involutions. He shows that the set of classes in the Thom bordism group $\mathcal{N}_n$ formed by manifolds which can be realized as the fixed set of an involution with the fixed set having codimension k is $\mathcal{N}_n$ if k is even, and is the subgroup of classes in $\mathcal{N}_n$ with zero Euler characteristic if k is odd, where $2 \leq k \leq n$.

A stationary point free Z_4-action is a Z_4-action with every isotropy subgroup being either Z_2 or the unit subgroup. Given a closed manifold with a stationary point free Z_4-action, one can consider the fixed point set of the action restricted to Z_2 with the action induced by the Z_4-action on it. So, one obtains an element in the unoriented bordism group of free Z_2-actions, and it will be called the Z_2-fixed point set of the Z_4-action.

In this work we consider the question: Which classes in the unoriented bordism group of free Z_2-actions can be realized as the Z_2-fixed set of stationary point free Z_4-action on a closed manifold with Z_2-fixed point set having constant codimension k?

Denote by C_n^k the set of classes in the n-dimensional bordism group of Z_2-free actions that can be realized as the

Z_2-fixed point set of a stationary point free Z_4-action on a closed $(n+k)$-manifold. The main result is the following:

<u>Theorem</u>.

(a) $C_n^1 = \{0\}$;

(b) $C_n^k = \mathcal{N}_n^{Z_2}(\{\{1\}\})$ if k is even and $n \geq 0$;

(c) $C_n^k = \chi_n[S^0,-1]+\chi_{n-1}[S^1,-1]$ if k odd and $2 < k \leq n-1$,

where $\mathcal{N}_n^{Z_2}(\{\{1\}\})$ is the n-dimensional group of Z_2-free actions and χ_* is the set of classes in $\mathcal{N}_*$ with zero Euler characteristic.

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2. <u>Even Codimension</u>

Let $\mathcal{N}_*^{Z_4}(\{\{1\},Z_2\})$ be the unoriented bordism group of stationary point free Z_4-actions and $\mathcal{N}_*^{Z_2}(\{\{1\}\})$ the unoriented bordism group of free Z_2-actions.

There is the restriction homomorphism

$$\rho\colon \mathcal{N}_*^{Z_4}(\{\{1\},Z_2\}) \to \mathcal{N}_*^{Z_2}(\{\{1\},Z_2\}),$$

which assigns to $[M,T]$ the class $[M,T^2]$. The Z_2-fixed point set of $[M,T]$ in $\mathcal{N}_*^{Z_4}(\{\{1\},Z_2\})$ is the free involution $[F_{T^2}M,T']$, where $F_{T^2}M$ is the fixed point set of $\rho([M,T]) = [M,T^2]$ and $T' \equiv T|F_{T^2}M$.

Next, consider the $\mathcal{N}_*$-module homomorphism

$$F_{Z_2} : \mathcal{N}_*^{Z_4}(\{\{1\},Z_2\}) \to \mathcal{N}_*^{Z_2}(\{\{1\}\}),$$

which assigns to $[M,T]$ the class of the Z_2-fixed point set of $[M,T]$. The objective of this work is to study the homomorphism F_{Z_2}. We denote the set of classes in $\mathcal{N}_n^{Z_2}(\{\{1\}\})$ that can be realized as the Z_2-fixed set of a stationary point free Z_4-action $[V^{n+k},T]$ by C_n^k.

There is a Z_4-action on a sphere $[S^{r-1+2j},T]$, where T is given by $T(x_1,\dots,x_r,z_1,\dots,z_j) = (-x_1,\dots,-x_r,iz_1,\dots,iz_j)$ with $i = \sqrt{-1}$, whose Z_2-fixed point set is $[S^{r-1},-1]$. It is well known that the classes $[S^{r-1},-1]$ form a basis for $\mathcal{N}_*^{Z_2}(\{\{1\}\})$ as $\mathcal{N}_*$-module, therefore if $k = 2j$ even we have that the image of F_{Z_2} is $\mathcal{N}_n^{Z_2}(\{\{1\}\})$, and $C_n^k = \mathcal{N}_n^{Z_2}(\{\{1\}\})$ for all k even. Thus, we are reduced to the case of k being odd.

3. Codimension One

Let $\mathcal{N}_*^{Z_4}(\{\{1\},Z_2\},\{\{1\}\})$ be the relative bordism group of Z_4-actions with isotropy group $\{1\}$ or Z_2 on manifolds with boundary for which the action is free on the boundary. There is an exact sequence

$$\mathcal{N}_*^{Z_4}(\{\{1\},Z_2\}) \to \mathcal{N}_*^{Z_4}(\{\{1\},Z_2\},\{\{1\}\}) \xrightarrow{\partial} \mathcal{N}_*^{Z_4}(\{\{1\}\})$$

and an isomorphism

$$F: \mathcal{N}_*^{Z_4}(\{\{1\},Z_2\},\{\{1\}\}) \to \bigoplus_{k=0}^{*} \mathcal{N}_{*-k}^{Z_2}(\{\{1\}\})(BO_k(C^\infty))$$

with

$$\mathcal{N}_{*-k}^{Z_2}(\{\{1\}\})(BO_k(C^\infty)) \cong \mathcal{N}_{*-k}(BO_k(C^\infty) \times_{Z_2} EZ_2),$$

where

$$BO_k(C^\infty) \times_{Z_2} EZ_2 \cong BSO_k \times BZ_4 \quad \text{for } k \text{ odd}$$

(see [1; p. 85]).

The boundary homomorphism ∂ sends the $k = 1$ summand isomorphically to $\mathcal{N}_*^{Z_4}(\{\{1\}\})$. This says that for $k = 1$, $C_n^1 = (0)$. Therefore, we may assume $k > 1$ and odd.

4. A Construction

Being given a closed manifold with a free Z_4-action $[N^p,T]$ and an involution $[W^q,t]$ one can form a quotient $(N^p \times W^q)/(T^2 \times t)$ with the induced Z_4-action $T\times 1$. The Z_2-fixed point set is $(N/T^2)\times F_tW$ with involution $T\times 1$. If W is closed, so is $(N\times W)/(T^2\times t)$, and if W has boundary on which t is free, then $T\times 1$ acts freely on the boundary $\partial((N\times W)/(T^2\times t)) = (N\times\partial W)/(T^2\times t)$.

Lemma 4.1. The map

$$\varphi: \mathcal{N}_*^{Z_4}(\{\{1\}\}) \otimes_{\mathcal{N}_*} \mathcal{N}_*(BSO_k) \to \mathcal{N}_*^{Z_2}(\{\{1\}\})(BO_k(C^\infty)),$$

which assigns to $[N,T]\times[P,\xi]$ the class of $[(N\times D\xi)/(T^2\times -1), T\times 1]$ is an isomorphism for all k odd.

Proof. For all k odd, we have

$$\begin{aligned}\mathcal{N}_*^{Z_2}(\{\{1\}\})(BO_k(C^\infty)) &\cong \mathcal{N}_*(BO_k(C^\infty) \times_{Z_2} EZ_2) \\ &\cong \mathcal{N}_*(BSO_k \times BZ_4) \text{ (see [1; p. 86])} \\ &\cong \mathcal{N}_*(BSO_k) \otimes_{\mathcal{N}_*} \mathcal{N}_*(BZ_4),\end{aligned}$$

by Kunneth theorem.

Next, consider the homomorphism

$$F_C: \mathcal{N}_*^{Z_4}(\{\{1\}\}) \to \mathcal{N}_*^{Z_2}(\{\{1\}\}),$$

which sends $[N,T]$ to $[N/T^2, T]$. Thus we have

Theorem 4.2. The image of the homomorphism F_C is the $\mathcal{N}_*$-submodule generated by the free involutions $[RP(2n)][S^0,-1]$ and $[CP(n)][S^1,-1]$, where n runs through the non-negative integers.

<u>Proof</u>. First, we recall that $\mathcal{N}_*^{Z_4}(\{\{1\}\})$ is freely generated as an $\mathcal{N}_*$-module by extensions of the antipodal actions on even-dimensional spheres, $y_{2n} = [S^{2n} \times_{Z_2} Z_4, 1\times i]$, and $y_{2n+1} = [S^{2n+1}, i]$, where $i = \sqrt{-1}$.

Now, calculating the image of F_c on the generators of $\mathcal{N}_*^{Z_4}(\{\{1\}\})$, we have

$$F_c([S^{2n} \times_{Z_2} Z_4, 1\times i]) = [(S^{2n} \times_{Z_2} Z_4)/(1\times -1), 1\times i]$$
$$= [RP(2n)[S^0, -1],$$

and

$$F_c([S^{2n+1}, i]) = [S^{2n+1}/-1, i] = [RP(2n+1), i].$$

Next, to see that $[RP(2n+1), i] = [\mathbb{C}P(n)][S^1, -1]$, consider $f\colon RP(2n+1) \to BZ_2$ classifying the Z_2-bundle $S^{2n+1} \to RP(2n+1)$ and $g\colon RP(2n+1)/Z_2 \to BZ_4$ classifying the Z_4-bundle $S^{2n+1} \to RP(2n+1)/Z_2$, where Z_4 acts on S^{2n+1} by multiplication by $i = \sqrt{-1}$. Let $p\colon RP(2n+1) \to RP(2n+1)/Z_2$ be the canonical projection. Thus, we have the commutative diagrams

$$\begin{array}{ccc} RP(2n+1) & \xrightarrow{f} & BZ_2 \\ p\downarrow & & \downarrow \\ RP(2n+1)/Z_2 & \xrightarrow{g} & BZ_4 \end{array},$$

and

$$\begin{array}{ccc} H^*(RP(2n+1);Z_2) & \xleftarrow{f^*} & H^*(BZ_2;Z_2) \\ p^*\uparrow & & \uparrow \\ H^*(RP(2n+1)/Z_2;Z_2) & \xleftarrow{g^*} & H^*(BZ_4;Z_2) \end{array}$$

Therefore, since f^* and g^* are isomorphisms in dimensions $\leq 2n+1$, we have $H^*(RP(2n+1)/Z_2;Z_2) \cong Z_2[x_1,x_2]/(x_1^2 = 0,\ x_2^{n+1} = 0)$, where $x_1 = g^*(\alpha)$ and $x_2 = g^*(\beta)$ being α,β the generators of $H^*(BZ_4;Z_2) \cong Z_2[\alpha,\beta]/(\alpha^2 = 0,\ \beta^{n+1} = 0)$.

Now, considering the map

$$RP(2n+1)/Z_2 \xrightarrow{g} BZ_4 \to BZ_2$$

classifying the involution $[RP(2n+1),i]$, we see that the characteristic class of this involution is $c = x_1$ and $c^j = 0$ for all $j > 1$.

Next, let ξ be the linear bundle over complex projective 2n-space $\mathbb{C}P(n)$. Thus, S^{2n+1} can be identified with the total space of the sphere bundle of ξ, i.e., $S^{2n+1} \equiv S(\xi)$. In the same way, we have $RP(2n+1) \equiv S(\xi\otimes\xi)$ and $RP(2n+1)/Z_2 \equiv S(\xi\otimes\xi\otimes\xi\otimes\xi)$. The tangent bundle of $S(\xi\otimes\xi\otimes\xi\otimes\xi)$ is equivalent to $\pi^*(\tau(\mathbb{C}P(n))) \oplus \pi^*(\xi\otimes\xi\otimes\xi\otimes\xi)$, where $\pi\colon RP(2n+1)/Z_2 \to \mathbb{C}P(n)$ is the projection. Thus, the Stiefel-Whitney class of $RP(2n+1)/Z_2$ is

$$\begin{aligned} w(RP(2n+1)/Z_2) &= \pi^*(w(\mathbb{C}P(n)))\pi^*(w(\xi\otimes\xi\otimes\xi\otimes\xi)) \\ &= (1+x_2)^{n+1}(1+4x_2) \\ &= (1+x_2)^{n+1}. \end{aligned}$$

On the other hand, considering the involution $[\mathbb{C}P(n)][S^1,-1]$, the characteristic class of this involution is given by $c' = 1\times\alpha_1$, where α_1 is the generator of $H^1(RP(1);Z_2)$ and the Stiefel-Whitney class of $\mathbb{C}P(n)\times RP(1)$ is

$$w(\mathbb{C}P(n)\times RP(1)) = \sum_{i=0}^{n} \binom{n+1}{i} \alpha_2^i \times 1,$$

where α_2 is the generator of $H^2(\mathbb{C}P(n);Z_2)$.

Therefore, it is easy to see that all of the involutions numbers of the two involutions are the same. Hence the theorem follows.

5. An Upper Bound.

Consider the $\mathcal{N}_*$-module homomorphism

$$\overline{F}_{Z_2} : \mathcal{N}_*^{Z_4}(\{\{1\},Z_2\},\{\{1\}\}) \to \mathcal{N}_*^{Z_2}(\{\{1\}\})$$

mapping the class of $[M,T]$ into the class of $[M,T^2]$, and recall that

$$\mathcal{N}_*^{Z_4}(\{\{1\},Z_2\},\{\{1\}\}) \cong \bigoplus_{k=0}^{*} \mathcal{N}_{*-k}^{Z_2}(\{\{1\}\})(BO_k(C^\infty)).$$

Now, considering $\bigoplus\limits_{\substack{k=1\\ k\ \text{odd}}}^{*} \mathcal{N}_{*-k}^{Z_2}(\{\{1\}\})(BO_k(C^\infty))$, we have

Theorem 5.1.

$$\overline{F}_{Z_2}\Big(\bigoplus_{\substack{k=1\\ k\ \text{odd}}}^{*} \mathcal{N}_{*-k}^{Z_2}(\{\{1\}\})(BO_k(C^\infty))\Big) = \mathcal{N}_*[S^0,-1]+\mathcal{N}_{*-1}[S^1,-1].$$

Proof. Using the isomorphism of the lemma (4.1), we may calculate the image of $\overline{F}_{Z_2}$ on the elements $[D\xi^k \times_{Z_2} N, 1\times t]$, where

$[N,t]$ runs through a set of generators of $\mathcal{N}_*^{Z_4}(\{\{1\}\})$. We have then that

$$\bar{F}_{Z_2}([D\xi^k \times_{Z_2}(S^{2n}\times_{Z_2} Z_4), 1\times 1\times i]) = [P][(S^{2n}\times_{Z_2} Z_4)/(1\times -1), 1\times i]$$

$$= [P][RP(2n)][S^0,-1],$$

where P is the base space of ξ^k, and

$$\bar{F}_{Z_2}([D\xi^k \times_{Z_2} S^{2n+1}, 1\times i]) = [P][RP(2n+1), i]$$

$$= [P][\mathbb{C}P(n)][S^1,-1],$$

as in the proof of the theorem (4.2). Thus, it follows that $\mathcal{N}_*[S^0,-1]+\mathcal{N}_{*-1}[S^1,-1]$ contains the image.

Now, taking the free Z_4-actions $[S^0\times_{Z_2} Z_4, 1\times i]$, $[S^1, i]$ and the bundle $[P,\xi^k]$ with the base space P consisting of a single point, we see that

$$\bar{F}_{Z_2}([D\xi^k \times_{Z_2}(S^0\times_{Z_2} Z_4), 1\times 1\times i]) = [S^0,-1],$$

and

$$\bar{F}_{Z_2}([D\xi^k \times_{Z_2} S^1, 1\times i]) = [S^1,-1].$$

Therefore, we have the result.

<u>Note</u>: By the theorem above, we see that

$$C_n^k \subset \mathcal{N}_n[S^0,-1]+\mathcal{N}_{n-1}[S^1,-1]$$

for all n,k and $k > 1$ odd.

<u>Theorem 5.2</u>. $\chi_n[S^0,-1]+\chi_{n-1}[S^1,-1] \subset C_n^k \subset \mathcal{N}_n[S^0,-1]+\mathcal{N}_{n-1}[S^1,-1]$ for all $2 < k \leq n-1$ and k odd, where χ_* is the set of classes in $\mathcal{N}_*$ with zero Euler characteristic.

<u>Proof</u>. Let M^n and N^{n-1} be in χ_n and χ_{n-1}, respectively. By Capobianco [3], there are involutions $[V_1^{n+k},T_1]$ and $[V_2^{n-1+k},T_2]$ such that the fixed point sets are M^n and N^{n-1}, respectively, for all $2 < k \leq n-1$ and k odd. Thus, the stationary point free Z_4-action

$$[(V_1\times Z_4)/(T_1\times -1),1\times i] + [(V_2\times S^1)/(T_2\times -1),1\times i]$$

has Z_2-fixed point set $[M^n][S^0,-1] + [N^{n-1}][S^1,-1]$.

6. <u>Euler Characteristics</u>

Let $g_*\colon \mathcal{N}_*(BO_{k-1}) \to \mathcal{N}_*(BSO_k)$ be the map given by $g_*([M,\xi^{k-1}]) = [M,\xi^{k-1} \oplus \det \xi^{k-1}]$. This map is well defined. In fact, calculating the Stiefel-Whitney class of the bundle $\xi^{k-1} \oplus \det \xi^{k-1}$, we have

$$\begin{aligned} w(\xi^{k-1} \oplus \det \xi^{k-1}) &= w(\xi^{k-1})w(\det \xi^{k-1}) \\ &= (1+w_1+\cdots+w_{k-1})(1+w_1) \\ &= 1+(w_1+w_1)+(w_2+w_1^2)+\cdots+w_{k-1}w_1 \\ &= 1+(w_2+w_1^2)+\cdots+w_{k-1}w_1, \end{aligned}$$

where w_i are the Stiefel-Whitney classes of the bundle ξ^{k-1}. Thus, the first Stiefel-Whitney class of the bundle

$\xi^{k-1} \oplus \det \xi^{k-1}$ is zero and $\xi^{k-1} \oplus \det \xi^{k-1}$ is orientable, i.e., the given class is in $\mathcal{N}_*(BSO_k)$.

Now, recall that $H^*(BSO_k, Z_2) = Z_2[v_2, \ldots, v_k]$ and $H^*(BO_{k-1}; Z_2) = Z_2[v_1', v_2', \ldots, v_{k-1}']$, where $v = 1+v_2+\cdots+v_k$ and $v' = 1+v_1'+\cdots v_{k-1}'$ are the total universal Whitney classes in $H^*(BSO_k; Z_2)$ and $H^*(BO_{k-1}; Z_2)$, respectively. Next, let $g^*: H^*(BSO_k; Z_2) \to H^*(BO_{k-1}; Z_2)$ be the induced map given by $g^*(v) = v'(1+v_1')$. Since g^* is monic, then [4; 17.3] implies that g^* is epic.

Now, taking $J = (j(1), j(2), \ldots, j(k-1))$ a $(k-1)$-tuple of non-negative integers with $j(1) \geq j(2) \geq \cdots \geq j(k-1)$, and considering ξ^J the bundle

$$p_1^*(\xi_{j(1)}) \oplus p_2^*(\xi_{j(2)}) \oplus \cdots \oplus p_{k-1}^*(\xi_{j(k-1)})$$
$$\oplus\ (p_1^*(\xi_{j(1)}) \otimes p_2^*(\xi_{j(2)}) \otimes \cdots \otimes p_{k-1}^*(\xi_{j(k-1)}))$$

over $RP^J = RP(j(1)) \times RP(j(2)) \times \cdots \times RP(j(k-1))$, where $\xi_{j(i)}$ is the canonical line bundle over the projective space $RP(j(i))$ and $p_i: RP^J \to RP(j(i))$ is the projection onto the i-th factor, we have:

Lemma 6.1. The bundles ξ^J, $J = (j(1), j(2), \ldots, j(k-1))$ with $j(1) \geq j(2) \geq \cdots \geq j(k-1) \geq 0$ constitute a set of generators for $\mathcal{N}_*(BSO_k)$.

Proof. The result follows by the above remarks and [5; 3.4.2].

Theorem 6.2. The kernel of the homomorphism

$$G: \bigoplus_{s=0}^{n} \mathcal{N}_s^{Z_4}(\{\{1\}\}) \otimes_{\mathcal{N}_*} \mathcal{N}_{n-s}(BSO_k) \xrightarrow{\partial\circ\varphi} \mathcal{N}_{n+k-1}^{Z_4}(\{\{1\}\}) \xrightarrow{\rho} \mathcal{N}_{n+k-1}^{Z_2}(\{\{1\}\})$$

is contained in the set of classes $[\alpha]$ such that $\overline{F}_{Z_2}([\alpha])$ belongs to $\chi_n[S^0,-1]+\chi_{n-1}[S^1,-1]$, for all n, k odd and $k > 1$.

Proof. It is sufficient to verify the result for all the generators of $\mathcal{N}_*^{Z_4}(\{\{1\}\})$ and $\mathcal{N}_*(BSO_k)$. Considering the even 2ℓ-dimensional generators of $\mathcal{N}_*^{Z_4}(\{\{1\}\})$ and being given the bundle $[P,\xi]$ an element in $\mathcal{N}_{n-2\ell}(BSO_k)$, we have $G([S^{2\ell}\times_{Z_2}Z_4,1\times i],[P,\xi]) = [S^{2\ell}\times Z_2,1\times -1][S(\xi),-1] = 0$, since $[S^{2\ell}\times Z_2,1\times -1]$ is boundary in $\mathcal{N}_*^{Z_2}(\{\{1\}\})$; and $\overline{F}_{Z_2}([S^{2\ell}\times_{Z_2}Z_4,1\times i],[P,\xi]) = (RP(2\ell)\times P)[S^0,-1]$ with $\chi(RP(2\ell)\times P) \equiv 0$, since the dimension of $RP(2\ell)\times P$ is $2\ell+(n-2\ell) = n$ odd.

Now, considering $[RP^J,\xi^J]$ a generator of $\mathcal{N}_{n-2\ell-1}(BSO_k)$ and odd $(2\ell+1)$-dimensional generators of $\mathcal{N}_*^{Z_4}(\{\{1\}\})$, we have

$$G([S^{2\ell+1},i],[RP^J,\xi^J]) = [S^{2\ell+1},-1][S(\xi^J),-1],$$

and taking the isomorphism $\overline{F}\colon \mathcal{N}_*^{Z_2}(\{\{1\}\}) \xrightarrow{\cong} \mathcal{N}_*RP^\infty)$, we see that

$$\overline{F}([S^{2\ell+1},-1][S(\xi^J),-1]) = [RP(2\ell+1)\to RP^\infty][RP(\xi^J)\to RP^\infty]$$
$$= [f\colon RP(2\ell+1)\times RP(\xi^J)\to RP^\infty],$$

by [7], where the map f classifies the bundle $[RP(2\ell+1)\times RP(\xi^J), \gamma^1 \otimes \gamma^2]$ with γ^1 the line bundle over $RP(2\ell+1)$ and γ^2 the line bundle over $RP(\xi^J)$. Calculating the Whitney number $\langle cw_{2k+n-1}, \sigma_{2k+n}\rangle$ of the map f, where $c = \alpha_{2\ell+1}\times 1$ and $\alpha_{2\ell+1}$ is the generator of $H^1(RP(2\ell+1);Z_2)$, we have

$$\langle cw_{2k+n-1}, \sigma_{2k+n}\rangle = \langle (\alpha_{2\ell+1}\times 1) w_{2k+n-1}(RP(2\ell+1)\times RP(\xi^J)), \sigma_{2k+n}\rangle$$

$$= \langle (\alpha_{2\ell+1}\times 1)(\binom{2\ell+2}{2\ell}\alpha_{2\ell+1}^{2\ell} \times \chi(RP(\xi^J)), \sigma_{2k+n}\rangle.$$

On the other hand,

$$\bar{F}_{Z_2}([S^{2\ell+1}, i], [RP^J, \xi^J]) = (\mathbb{C}P(\ell)\times RP^J)\cdot[S^1, -1],$$

with

$$\chi(\mathbb{C}P(\ell)\times RP^J) = \binom{\ell+1}{\ell}\beta^\ell \times \chi(RP^J),$$

where β is the generator of $H^2(\mathbb{C}P(\ell);Z_2)$. Therefore, we conclude that $\chi(\mathbb{C}P(\ell)\times RP^J) \equiv \langle cw_{2k+n-1}, \sigma_{2k+n}\rangle$. Thus, if $\chi(\mathbb{C}P(\ell)\times RP^J) \neq 0$, we see that $([S^{2\ell+1}, i], [RP^J, \xi^J])$ isn't in the kernel of G.

<u>Theorem 6.3</u>.

(a) $C_n^1 = (0)$;

(b) $C_n^k = \mathcal{N}_n^{Z_2}(\{\{1\}\})$ for all $n \geq 0$ and k even;

(c) $C_n^k = \chi_n[S^0,-1]+\chi_{n-1}[S^1,-1]$ for $2 < k \leq n-1$ and k odd.

<u>Proof</u>. Considering k odd and n even, let $[M,t] = A[S^0,-1] + B[S^1,-1]$ be in $C_n^k \subset \mathcal{N}_n[S^0,-1] + \mathcal{N}_{n-1}[S^1,-1]$ and $[V^{n+k},T]$ a stationary point free Z_4-action such that $[M^n,t]$ is the Z_2-fixed

point set. Then, $n+k$ is odd, so $\chi(V) \equiv 0$. Since the Z_4-action T free on $V-M$, we have that $\chi(M) \equiv 0 \bmod 4$. Then, $M/t = A+(B\times RP(1))$ in $\mathcal{N}_n$, $\chi(M/t) \equiv \bmod 2$ and $\chi(B\times RP(1)) \equiv 0$, since the dimension of $B\times RP(1)$ is $n-1$ odd, imply that $\chi(A) \equiv 0$, i.e., A belongs to χ_n. One has $\chi_{n-1} = \mathcal{N}_{n-1}$, since $n-1$ is odd, therefore we have that $C_n^k = \chi_n[S^0,-1]+\chi_{n-1}[S^1,-1]$ for k odd and n even.

Next, for k odd and n odd, we have the exact sequence and commutative diagram

$$\begin{tikzcd}
\mathcal{N}_*^{Z_4}(\{\{1\},Z_2\}) \arrow[r] \arrow[dr, "F_{Z_2}"'] & \mathcal{N}_*^{Z_4}(\{\{1\},Z_2\},\{\{1\}\}) \arrow[r, "\partial"] & \mathcal{N}_{*-1}^{Z_4}(\{\{1\}\}) \\
& \mathcal{N}_*^{Z_2}(\{\{1\}\}) \arrow[u, leftarrow, "\overline{F}_{Z_2}"'] &
\end{tikzcd}$$

Thus,

$$\chi_n[S^0,-1]+\chi_{n-1}[S^1,-1] \subset C_n^k \subset (\mathcal{N}_n[S^0,-1]+\mathcal{N}_{n-1}[S^1,-1]) \cap \overline{F}_{Z_2}(\ker \partial)$$

by the exactness of the sequence and Theorem (5.2). Further, since $\overline{F}_{Z_2}(\ker \partial) \subset \chi_n[S^0,-1]+\chi_{n-1}[S^1,-1]$ by the theorem (6.2), we conclude that

$$\chi_n[S^0,-1]+\chi_{n-1}[S^1,-1] \subset C_n^k \subset \chi_n[S^0,-1]+\chi_{n-1}[S^1,-1],$$

that is, $C_n^k = \chi_n[S^0,-1]+\chi_{n-1}[S^1,-1]$ for k odd and n odd.

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