QUASI–INVARIANCE OF PRODUCT MEASURES UNDER LIE GROUP PERTURBATIONS: FISHER INFORMATION AND L^2–DFFERENTIABILITY

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Abstract. A sequence of measures on a topological space is perturbed by a sequence of elements of a Lie group acting on that space. Criteria are given for the singularity and equivalence of the corresponding product measures. These criteria extend the results of Shepp (1965) and Steele (1986). In particular Fisher information come into the scene and its role is further clarified.

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Introduction

This work investigates the Lebesgue decomposition between two product measures, one with a fixed marginal and the other with each marginal perturbed by an element of a Lie group. For general product measures the problem was studied by Kakutani (1948), who proved the following dichotomy: "two product measures are either equivalent on singular" and his criterion is as follows. Let $\mu_1, \mu_2, \dots, \tilde{\mu}_1, \tilde{\mu}_2$ be measures and

$$\mu = \Pi_{n=1}^{\infty} \mu_n, \quad \tilde{\mu} = \Pi_{n=1}^{\infty} \tilde{\mu}_n$$

be their product measures. Let H denote the Hellinger product between measures (see LeCam (1970)), Kakutani showed that:

i) μ and $\tilde{\mu}$ are singular ($\mu \perp \tilde{\mu}$) if and only if $\Pi_{n=1}^{\infty} H(\mu_n, \tilde{\mu}_n) = 0$

ii) If μ_n and $\tilde{\mu}_n$ are equivalent ($\mu_n \sim \mu_n$), for all n, i e, they are mutually absolutely continuous, them $\mu \sim \tilde{\mu}$ if and only if $\Pi_{n=1}^{\infty} H(\mu_n, \tilde{\mu}_n) > 0$.

More specific results on equivalence and singularity of product measures where considered by Feldman (1961), Shepp (1965), Renyi (1967), LeCam (1970), Chatterji and Mandrekar (1977), Steele (1986) and Marques (1987). In particular the problem of translates of product measures in $\mathbb{R}^{\infty}$ was settled by Shepp, who brought the role of Fisher information to the scene. Extensions of Shepp's result to the groups of rigid motions and affine transformations in Euclidean spaces were obtained by Steele and Marques respectively.

Here we will show that Shepp's results holds under a more general scenario. We will extend the results to Lie group perturbation acting on certain topological spaces and the role of Fisher information into the problem is further clarified.

Apart from facts in probability theory, our techniques of proofs involve mainly differential calculus of maps into Hilbert spaces and some representation theory of Lie groups. The differential calculus and its relation to Fisher information is discussed in Section 2. Concerning group representations we refer to the books of Warner (1972) or Bourbaki (1963). Let us however recall here some basic notions of this theory.

A representation of a group G on a vector space E is a homomorphism $g \to U(g)$ of G into the group of invertible linear maps of E. In case E is a topological vector space and G a Lie group (or even a topological group), the representation U is said to be continuous in case the map $(g, v) \in G \times E \to U(g)v \in E$ is continuous. In the body of the paper only unitary representations of Lie groups will appear. These are representations in which E is a Hilbert space and each $U(g), g \in G$ is an unitary operator. For representations of Lie groups on Hilbert (or Banach) spaces the above continuity condition is equivalent to the weaker one which states that every $\nu \in E$ is a continuous vector i. e., the map $g \in G \to U(g)\nu \in E$ is continuous (c. f. Bourbaki (1963) ch. VIII § n$^{\text{o}}$ 1).

If U_i, $i = 1, 2$, are representations on E_i, $i = 1, 2$, of the same Lie group, a continuous operator $A : E_1 \to E$ is said to be an intertwining operator for U_1 and U_2 if $A_0 U_1(g) = U_2(g)_0 A$ for all $g \in G$. The representations U_1 and U_2 are equivalent in case an intertwining operator which is a bicontinuous bijection exists. Of course, if U_1 and U_2 are equivalent, they can be treated as the same representation.

A vector v in the representation space E (assumed to be a Banach space) of a Lie group G is a C^∞ (C^k, continuous, analytic, etc ...) vector in case the map $\psi_v : g \in G \to \psi_v(g) = U(g)v \in E$ is C^∞ (C^k,etc...). The set E_∞ of C^∞ - vectors in E is a dense subspace (c. f. Bourbaki (1963) § 4.4.1). If $v \in E_\infty$ and X is an elementof the Lie algebra $\mathfrak{g}$ of G, it makes sense to define

$$U_\infty(X)v = \lim_{t \to 0} \frac{U(\exp tX)v - v)}{t}$$

The vector $U_\infty(X)v$ is also C^∞ and the assignment $v \in E_\infty \to U_\infty(X)v$ defines an operator of E which is in general unbounded if E is infinite dimensional. Clearly, the derivative of $\psi_v(g) = U(g)v$ at the identity in the direction of X is $U_\infty(X)v$. Also, if $v \in E_\infty$ then $U(g)v \in E_\infty$, all $g \in G$ and if X is regarded as a left invariant vector field, $(d\psi_v)_g(X(g)) = \frac{d}{dt}(U(ge^{tX})v)_t = U(g)U_\infty(X)v$ and $(d\psi_v)_g(X(g)) = \frac{d}{dt}(U(e^{tX}g)v)_{t=0} = U_\infty(X)U(g)v$ in case X is regarded as a right invariant vector field.

Since the image of $U_\infty(X), X \in \mathfrak{g}$ is contained in E_∞, it is possible to take compositions and consider higher order operators, i.e., linear combination of operators of the form $U_\infty(X_1) \circ \ldots \circ U_\infty(X_k)$, $X_i \in \mathfrak{g}$. These are also densely defined operators in E. Latter in Section 5, a second order operator of this kind will appear.

2. Immersions and Fisher information

We start with some elementary considerations on maps into Banach spaces. Let U be an open subset of $\mathbb{R}^d$ and $\varphi : U \to E$ a continuous injection into the Banach space $(E, || \ ||)$. Define on U the distance $d_E(x, y) = ||\varphi(x) - \varphi(y)||$. Assume that at some $x \in U$ φ is differentiable at x and that its differential $d\varphi_x : \mathbb{R}^d \to E$ is injective. Then for $v \in \mathbb{R}^d$, $\varphi(x+v) = \varphi(x) + d\varphi_x(v) + o(v)$ with $\lim_{v\to 0} \frac{o(v)}{|v|} = 0$, where $|\cdot|$ stands for the Euclidian norm in $\mathbb{R}^d$. Putting $M = \sup\{d\varphi_x(v) : |v| = 1\}$ and $m = \inf\{d\varphi_x(v) : |v| = 1\}$, we have $m > 0$ and for small $v \in \mathbb{R}^d$ and $\epsilon > 0$, $(m-\epsilon)|v| \leq d_E(x, x+v) \leq (M+\epsilon)|V|$. Thus if φ is differentiable d_E is locally equivalent to the Euclidian distance $d(x,y) = |x-y|$ restricted to U. So that if $(y_n)_{n\geq 1}$ is a sequence in U with $y_n \to x$ then $\sum_{n\geq 1} d(x, y_n)^2 < \infty$, if and only if $\sum_{n\geq 1} d_E(x, y_n)^2 < \infty$.

Shifting to manifolds, suppose that M is a finite dimensional smooth manifold endowed with some smooth Riemannian metric $\langle\ ,\ \rangle$. Let $\varphi = M \to E$ be a continuous injection. Through the arc length of curves joining points in M, a Riemannian metric defines a distance $d(x,y)$ in M. Localizing around $x \in M$ and taking some coordinate system, the existence of the exponential maps guarantees the equivalence of this Riemannian distance with the Euclidian distance of the open subset of $\mathbb{R}^d$ where the coordinate system is defined. We are thus led to the previous situation so that $d(x,y)$ becomes locally equivalent to $d_E(x,y) = ||\varphi(x) - \varphi(y)||$. Again we have that if φ is differentiable with $d\varphi_x$ one–one, then for sequences $y_n \to x$, $\sum_{n\geq 1} d(x,y_n)^2 < \infty$ if and only if $\sum_{n\geq 1} d_E(x,y_n)^2 < \infty$. This statement can be slightely improved in case φ is a one–one immersion of M into E. In fact, by a simple compactness argument one sees that the distances $d(\cdot,\ \cdot)$ and $d_E(\cdot,\ \cdot)$ are equivalent on compact subsets of M. Therefore, if φ is an immersion, $\sum_{n\geq 1} d(x,y_n)^2 < \infty$ if and only if $\sum_{n\geq 1} d_E(x,y_n)^2 < \infty$, in case it is now that y_n does not leave a compact.

Now, let $\{\mu_x\}_{x\in M}$ be a dominated model parameterized on M. Each μ_x, $x \in M$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$ and μ_x is absolutely continuous with respect to a basic measure μ. We let $p(x,\omega) = \frac{d\mu_x}{d\mu}(\omega)$ stand for the set of Radon–Nykodim derivatives. For $x \in M$, $p(x,\cdot)$ is μ–integrable so $x \to p(x,\cdot)$ defines a map $M \to L^1(\mu)$. Also, since $p(x,\cdot)$ is $\mu.a.s$ positive, $q(x,\omega) = p(x,\omega)^{1/2}$ makes sense and $x \to \varphi(x) = q(x,\cdot)$ defines a map from M into the Hilbert space $L^2(\mu)$. We say that the model is L^2–continuous, L^2–differentiable, etc ... in case such a property is satisfied by φ.

Suppose the model is L^2–differentiable at $x \in M$ and define the Fisher inner product $\langle\ ,\ \rangle_x$ on the tangent space T_xM at x by putting $\langle u, v\rangle_x = \langle d\varphi_x(u), d\varphi_x(v)\rangle$ with $u, v \in T_xM$ and $\langle\ ,\ \rangle$ on the right hand side standing for the inner product in $L^2(\mu)$. Clearly, $\langle\ ,\ \rangle_x$ is positive semi–definite and it is positive definite if and only if $d\varphi_x$ is injective. We call $\langle\ ,\ \rangle_x$ the Fisher inner product because when it is expressed in coordinate systems, its matrix is anything but the Fisher information matrix as in Ibragimov and Has'minskii (1981). Note that in the setting adopted here, the very definition of Fisher information requires differentiability of φ. However the usual way of dealing with Fisher information is by assuming the differentiability of $q(x,\omega)$ as a function of x and putting

$$\langle u, v\rangle_x = \int D_u q(x,\omega) D_v q(x,\omega)\tilde{\mu}(d\omega)$$

provided the integral exists (here $D_v q(x,\omega)$ means directional derivative). In this case. under some regularity conditions, usually called Cramer–Wald and Hajek's conditions, the existence of $\langle u, v\rangle_x$, $u, v \in T_x M$, implies the existence of $d\varphi_x$ (see LeCam (1970)). So in order to normalize terminology, we understand that the existence of Fisher information at x means the same as the existence of $d\varphi_x$.

Finally, putting $d_H(x;y) = \|\varphi(x) - \varphi(y)\|_2$, with $\|\cdot\|_2$ meaning norm in $L_2(\mu)$, $d_H(\cdot,\ \cdot)$ is the so called Hellinger distance between μ_x and μ_y, that can also be expressed as $d_H(x,y) = 2(1 - H(\mu_x, \mu_y))$. So taking into account Kakutani's theorem and the previous remarks, we have.

Proposition 2.1: In a dominated model $\{\mu_x\}_{x\in M}$ with M a smooth manifold with a distance given by any smooth Riemannian metric, suppose that $\varphi : M \to L^2(\mu)$ (notation as above) is one–one and differentiable at $x \in M$. Suppose that $y_n \to x$. Then

$$\Pi_n \mu_{y_n} \sim \Pi_n \mu_x$$

if and only $\sum_{n\geq 1} d(x, y_n)^2 < \infty$. In case φ is an injective immersion the same statement holds by assuming that y_n is contained in some compact subset.

Remarks: 1) In case φ is a one–one smooth immersion, the Fisher information defines a smooth Riemannian metric. In this case Fisher information itself can be used to measure the intrinsic distance in M.
2) Smoothness of M and $\langle\ ,\ \rangle$ is not essential, C^2 would do.

3. Lie group perturbation of measures

From now on we consider only dominated models obtained by acting Lie groups on quasi-invariant probability measures. Let G be a connected finite dimensional Lie group and X a topological space which is assumed to be locally compact and satisfying the second axiom of enumerability. Let $(g, x) \to g\ x$ or $g(x)$ denote a continous action $G \times X \to X$ of G on X and lift it to the space $M(X)$ of Borel measures on X by putting $(g\mu)(A) = \mu(g^{-1}A)$ for any Borel subset $A \subset X$ and $\mu \in M(X)$. The mapping $(g,\mu) \to g\mu$ defines in fact on action of G on $M(X)$, which is moreover continuous when $M(X)$ is considered whith the vague topology.

We recall that a G quasi-invariant measure is a measure $\mu \in M(X)$ for which $g\mu$ is equivalent to μ for every $g \in G$. Thus if μ is a quasi-invariant probability we have defined a dominated model $\{g\mu\}_{g\in G}$. In this model a dominant measure may be taken to be μ itself. This is the "combined transformational model" in Barndorff–Nielsen (1987). In the sequel we shall put

$$p(g,x) = \frac{dg\mu}{d\mu}(x) \quad \text{and} \quad q(g,x) = p(g,x)^{1/2}; \quad g \in G, \quad x \in X \tag{3.1}$$

from which the map $\psi : g \in G \rightarrow q(q,.) \in L^2(\mu)$ is defined. The parameter manifold of this model is G. Note however that defined this way the model need not to be one-one. So in order to avoid duplication of parameters it is convenient to quotientize G by a subgroup. Set

$$H = \{g \in G : g\mu = \mu\}.$$

Since $(g,\mu) \rightarrow g\mu$ is a continuous action, H is a closed subgroup of G. It is thus a Lie subgroup of G so that coset space G/H has a structure of an analytic manifold on which G acts analitically. Since for $h \in H, (gh)\mu = g\mu$, we have that $q(gh,x) = q(g,x)\mu - a.s.$. Hence the map $\psi : G \rightarrow L^2(\mu)$ defines a one-one map $\varphi : gH \in G/H \rightarrow \psi(g) \in L^2(\mu)$. We take G/H as our parameter manifold. For the mapping φ assumptions like that appearing in proposition 2.1 are acceptable.

In G/H take an arbitrary smooth Riemannian metric $<,>$ with associated Riemmanian distance $d(.,.)$. The set of sequences $(g_n)_{n\geq 1} \subset G$ which are square summable in G/H is defined to be

$$l^2_H = \{(g_n)_{n\geq 1} \subset G : \sum_{n\geq 1} d(g_n H, H)^2 < \infty\}. \tag{3.2}$$

We put

$$E(\mu) = \{(g_n)_{n\geq 1} \subset G : \prod_n g_n\mu \sim \prod_n \mu\}.$$

and

$$E_o(\mu) = \{(g_n)_{n\geq 1} \in E(\mu); \; d(g_n H, \; H) \rightarrow 0\}$$

It is easily checked that $E(\mu)$ is a subgroup of the group G^{N} of sequences in G.

Now the main result can be stated.

Theorem 3.1 : Whith G and X as above, let μ be a quasi-invariant probability in $M(X)$ and H the closed subgroup which fixes μ. Then

a) If φ is continuous then $E_o(\mu) \subset l^2_H$

b) If φ is differentiable at the origin $H \in G/H$ then $E_o(\mu) = l^2_H$.

c) If $E_o(\mu) = l^2_H$ then φ is continuous and differentiable on G/H.

Remark : In c) it is not necessary to assume apriori that μ is quasi-invariant. In fact, if $E_o(\mu) \supset l^2_H$ them $g\mu \sim \mu$ for every $g \in G$, i. e., μ is quasi-invariant.

The proof of this theorem shall be made subsequently. Part b) is essentialy section 2. above. It involves only the notion of differentiability and will be clarified soon. Parts a) and c) are more delicate and their proofs appear in later sections. By now we introduce a representation of G associated whith $q(g, x)$, from which we derive our main techniques.

An alternative way of defining $g\mu$ is through

$$\int_X f(x) g\mu(dx) = \int_X f(gx)\mu(dx)$$

for integrable f. From this equality one gets quickly that if $\mu_1 \sim \mu_2$ then $g\mu_1 \sim g\mu_2$ and

$$\frac{dg\mu_1}{dg\mu_2}(x) = \frac{d\mu_1}{d\mu_2}(g^{-1}x) \qquad \mu_2 - a.s.$$

So that if we put as before $p(g,x) = \dfrac{dg\mu}{d\mu}(x)$ then p satisfies (for every $g, h \in G$ and $\mu - a.a\ x$)

$$p(gh, x) = p(h, g^{-1}x)p(g, x) \tag{3.3}$$

that is to say p is a cocycle on G over X. If H is the subgroup which fixes μ then p is H - invariant in the sense that if $h \in H$ $p(h, x) = 1$ (this and other equalities appearing below are to be taken μ - a.s.). Moreover $p(h, x) = 1$ iff $h \in H$. Note that H - invariance and (3.2) imply that if $h \in H$ then $p(gh, x) = p(g, x)$, as was already remarked before.

Clearly by putting $q(g, x) = p(g, x)^{1/2}$, q also becomes a cocycle over X. This fact permits the introduction of the following representation of G: For $g \in G$ and $f \in L^2(\mu)$ define the function

$$(U(g)f)(x) = q(g, x)f(g^{-1}x) \tag{3.4}$$

From

$$\|U(g)f\|_2 = \int_X q(g,x)^2\, |f(g^{-1}x)|^2\, \mu(dx) = \int_X |f(g^{-1}x)|^2 g\mu(dx) = \|f\|_2$$

we see that $f \to U(g)f$ defines a unitary operator in $L^2(\mu)$. Moreover, the cocycle condition (3.1) for q leads to $U(gh) = U(g)U(h), g, h \in G$, so that the mapping $g \to U(g)$ becomes a unitary representation of G on $L^2(\mu)$.

Note that if we denote by 1 the constant function $1(x) = 1$, then $\psi(g) = U(g)1$,

i. e., the family $q(g, x)$ is anything but the orbit of 1 under the action of G on $L^2(\mu)$ defined by the representation U.

At this juncture we can complete the proof of **b)** in Theorem 3.1. Suppose that φ is differentiable at the origin of G/H. By Section 2., it is enough to show that its differential is injective. In terms of the representation U, differentiability of φ at the origin means differentiability of the vector $1 \in L^2(\mu)$, that is, differentiability at the identity of G of the mapping $\psi(g) = U(g)1$. As it is known, this is enough to assure that ψ is everywhere differentiable. In fact it happens that $d\psi_g = U(g)_0 d\psi_1$. Let $A \in \mathbf{g}$ be such that $d\psi_1(1) = 0$, i. e., such that $\frac{d}{dt}U(e^{tA})(1)/_{t=0} = 0$. Then

$$\frac{d}{dt}U(e^{tA})(1) = U(e^{tA})(\frac{d}{ds}U(e^{sA})(1)/_{s=0}) = 0$$

So that $U(e^{tA})(1) = 1$ for all $t \in \mathbb{R}$ and therefore $A \in \mathbf{h}$, the Lie algebra of H. This is enough to show that the differential of φ at the origin of G/H is injective thus completing the proof of **b)**.

We give now some cases in which the continuity of φ or ψ can be taken for sure, thus clarifying the condition in **a)**. Firstly, note that if U is a continuous representation then since $\psi(g) = U(g)1, \psi$ and hence φ is continuous. Actually, the continuity of ψ is equivalent to the continuity of U, as is shown by .

Proposition 3.2: $\psi : G \to L^2(\mu)$ **is continuous if and only if the representation** U **is continuous.**

proof: Suppose ψ is continuous and let $f \in L^2(\mu)$. To show the continuity of U it is enough to show that $g \to U(g)f$ is continuous when f is assumed to be continuous whith compact support (c.f Bourbaki (1963) chapter VIII § 2 n$^{\underline{o}}$2). It is also enough to show the continuity at the identity of G. In this case

$$\begin{aligned} \|U(g)f - f\|_2 &= \|q(g,x)f(g^{-1}x) - f(x)\|_2 \\ &\leq \|(q(g,x)-1)f(x)\|_2 + \|f(g^{-1}x) - f(x)\|_2 \\ &+ \|q(g,x)(f(g^{-1}x) - f(x))\| \end{aligned}$$

which converges to zero as $g \to 1$.

Proposition 3.3: If either
a) $q(g, x)$ **is jointly continuous in** (g, x), **or**
b) $q(g, x)$ **is bounded**
Then ψ **is continuous.**

proof: a) is Proposition 8 and **b)** Proposition 9 in Bourbaki (1963) ch VIII §2 n$^{\underline{o}}$ 5.

Another case where the continuity of U can be assured is when G is transitive on X.

Proposition 3.4: If G act transitively on X, i. e., $X = G/L$ for some closed subgroup L, then the representation U is continuous.

proof: It is known (c.f. Bourbaki (1963) ch. VII §2 n° 5) that two quasi-invariant measures on homogeneous spaces are equivalent. It is also known (see for instance Bruhat's Lemma A.1.1 in Warner (1972)) that in G/L a quasi-invariant measure ν with C^∞ cocycle $\frac{dg\nu}{d\nu}(x) = s(g,x), g \in G, x \in G/L$ exists. Put $\rho(x) = \frac{d\mu}{d\nu}$. Then $p > 0$ $\nu - a.s.$ and

$$\begin{aligned} p(g,x) &= \frac{dg\mu}{d\mu}(x) = \frac{dg(\rho\nu)}{d\rho\nu}(x) = \frac{\rho(g^{-1}x)}{\rho(x)}\frac{dg\nu}{d\nu}(x) \\ &= \frac{\rho(g^{-1}x)}{\rho(x)}s(g,x) \end{aligned} \tag{3.5}$$

Let $A : L^2(\mu) \to L^2(\nu)$ be the operator defined by $A(f) = \rho^{1/2}f$. It is easy to check that A is an isometry. Moreover,

$$\begin{aligned} (A \circ U(g)f)(x) &= \rho^{1/2}(x)q(g,x)f(g^{-1}x) \\ &= s(g,x)^{1/2}(\rho^{1/2}f)(g^{-1}x) \\ &= (\tilde{U}(g) \circ A(f))(x) \end{aligned}$$

where $(\tilde{U}f)(x) = s(g,x)^{1/2}f(g^{-1}x)$ is the representation associeted whit the cocycle $s(g,x)^{1/2}$. Therefore A intertwines the representation U and $\tilde{U}$. Hence U is continuous if and only if $\tilde{U}$ is. The continuity of $\tilde{U}$ follows from Proposition 3.3 a) above.

Finally, let us draw some comments about the Fisher information of the model. As was remarked at the end of the last section in case φ is smooth, the metric measuring intrinsic distances on the parameters manifold could be taken to be Fisher information itself. For the model $\{g\mu\}_{g\in G}$ we have been considering, the Fisher information metric on G/H - which is necessarily non degenerate - might become very convenient. A reason for this comes from the formula $d\psi_g = U(g) \circ d\psi_1$ from which one shows that Fisher information is G - invariant , i. e., $< dg_\xi(v), dg_\xi(w) >_{g\xi} = \langle v, w\rangle_\xi$ where $< \, , \, >_\xi$ is Fisher information at $\xi \in G/H$, $v, w \in T_\xi(G/H)$ and dg is the differential of the mapping $g : G/H \to G/H$ induced by $g \in G$.

It is worth to compare this way of dealing with the notion of Fisher information and the form employed by Steele (1986). There Lebesgue measure on $\mathbb{R}^d$ is taken as the dominant measure for the models. These are parameterized by the group of rigid motion on $\mathbb{R}^d$. The representation to be taken is then the representation of the group G of rigid motions on the L^2 - space of Lebesgue measure on $\mathbb{R}^d$, say $L^2(\mathbb{R}^d)$. This is given by $(U(g)f)(x) = f(gx), x \in \mathbb{R}^d, g \in G, f \in L^2(\mathbb{R}^d)$. In Steele (1986) it is said that a density f has finite Fisher information provided $Lf \in L^2(\mathbb{R}^d)$ for certain operators defined from this representation. In our earlier notation, these operators are extensions of $U_\infty(X), X \in \mathfrak{g}$, from E_∞ to a bigger space.

Now, it is known (c.f. Ibragimov and Has'minshii (1981) p. 65) that the existence of Fisher information does not depend on the dominant measure. We take μ itself as the dominant measure. This being so, note that if $v \in T_\xi(G/H)$ then $v = X(\xi)$ for some $X \in \mathfrak{g}$ so that $d\varphi_\xi(x)$ is given by evaluation of $U_\infty(X)$ (or rather an extension of it) on $\varphi(\xi)$. Therefore, existence of Fisher information is equivalent to $U_\infty(X)(\varphi(\xi)) \in L^2(\mu)$ as in Steele (1986).

4. The ℓ-condition and the proof of c)

In order to prove c) we make use of the ℓ-condition introduced in Le Cam (1970), which for families of the form $\{g\mu\}_{g\in G}$ turns out to be equivalent to differentiability. We take the set up of Theorem 3.1 and denote by $\mathfrak{g}$ the Lie algebra of the group G. In $\mathfrak{g}$ let $|\cdot|$ be any norm. In this context the ℓ-condition reads

$$(\ell) : \limsup_{A\to 0} \frac{\|\psi(ge^{tA}) - \psi(g)\|_2}{|A|} < \infty \; ; \; g \in G \, , \; A \in \mathfrak{g} \tag{4.1}$$

In principle this condition should be checked at every $g \in G$. However due to the unitarity of the representation U, if it is satisfied at some $g \in G$ then it is fulfilled everywhere. In fact, $\psi(g) = U(g)(1)$ and $\psi(ge^A) = U(g)U(e^A)(1)$ so $\|\psi(ge^A) - \psi(g)\|_2 = \|U(g)(\psi(e^A) - 1)\|_2 = \|\psi(e^A) - 1\|_2$ because U is unitary. Hence the ℓ-condition at some g is equivalent to the ℓ-condition at the identity of G.

The connection between differentiability and the ℓ-condition is provided by a result of Le Cam (1970) which says that if the ℓ-condition is satisfied on a measurable set S then with respect to the Lesbegue measure of the parameter space, ψ is almost always differentiable on S. From this result we see that if our model satisfies the ℓ-condition at just one point of G then it is differentiable everywhere. In fact, as was already remarked in the proof of b), to have differentiability of ψ everywhere it is enough that ψ is differentiable at some point. Conversely, it is clear that if ψ is differentiable at $g \in G$ then the ℓ-condition is satisfied at g. Thus we get the equivalence of the ℓ-condition and differentiability for models of the type $\{g\mu\}_{g\in G}$.

This being so we can prove c) by showing that if $\prod_n g_n\mu \sim \prod_n \mu$ for every sequence $(g_n)_{n\geq 1}$ with $g_n\, H \to H$ and $(g_n)_{n\geq 1} \in \ell^2_H$ then ψ satisfies the ℓ-condition. Before proceeding, let us note that due to the local equivalence of Riemannian distances, if $(A_n)_{n\geq 1}$ is a sequence in the Lie algebra g with $\sum_{n\geq 1}|A_n|^2 < \infty$ then $e^{A_n} \in \ell^2_H$. For the proof of c) it is possible to handle only sequences of the form e^{A_n}.

We need now a lemma proved in Shepp (1965) Lemma 4.

Lemma 4.1: Let $T : \mathbb{R}^d \to \mathbb{R}$ be a positive function satisfying $\limsup_{a\to 0} T(a) = \infty$. Then there exists a sequence $(a_n)_{n\geq 1} \subset \mathbb{R}^d$ with $\sum_{n\geq 1}|a_n|^2 < \infty$ and such that $\sum_{n\geq 1}|a_n|^2 T(a_n) = \infty$.

To prove c) in Theorem 3.1, suppose that the ℓ-condition for ψ is not satisfied (at the identity of G) and take $T : \mathrm{g} \to \mathbb{R}$ in the lemma above to be

$$T(A) = \frac{\|\psi(e^A) - 1\|_2^2}{|A|^2}.$$

Then since the ℓ-condition is not satisfied, there exists a sequence $(A_n)_{n\geq 1} \subset \mathrm{g}$ such that $\sum_{n\geq 1}|A_n|^2 = \infty$ and $\sum_{n\geq 1}|a_n|^2 T(a_n) = \sum_{n\geq 1}\|\psi(\ell^{A_n}) - 1\|_2^2 = \infty$. By Kakutani's theorem this implies that $\prod_n e^{A_n}\mu$ and $\prod_n \mu$ are singular contradicting the hypothesis.

5. Proof of a)

We now turn to the proof of a). By Kakutani's theorem we need to show that for sequences $g_n \in G$ with $g_n H \to H$ if $\sum_{n\geq 1} d(g_n H, H)^2 = \infty$ then $\sum_{n\geq 1}\|\psi(g_n) - 1\|_2^2 = \sum_{n\geq 1}\|U(g_n)(1) - 1\|_2^2 = \infty$. We know from b) that this happens in the differentiable case, so we try to regularize ψ (or φ) by a linear operator $T : L^2(\mu) \to L^2(\mu)$ which satisfies

(i) T is bounded

(ii) The mapping $g \to T(U(g)(1))$ is smooth (C^∞)

(iii) $T(U(g)(1)) = T(1)$ iff $U(g)1 = 1$, i.e., iff $g \in H$, or

(iii') T is one-one.

Suppose such a T exists. By ii) and iii) the mapping $\theta : gH \in G/H \to T(\psi(g))$ is one-one and smooth. Therefore by a known fact about differentiable mappings, there exists an open subset $U \subset G/H$ such that when restricted to U, θ

becomes an immersion. Fix $gH \in U$ and take a sequence g_n with $g_n H \to H$ and $\sum_{n\geq 1} d(g_n H, H)^2 = \infty$. Then $g\, g_n H \to gH$ and by the local equivalence of Riemannian distances we have that $\sum_{n\geq 1} d(g\, g_n H, gH)^2 = \infty$. Hence from the differentiable case of Section 2, we have that

$$\sum_{n\geq 1} \|TU(gg_n)(1) - TU(g)(1)\|_2^2 = \infty.$$

Now by the continuity of T as required in i),

$$\|TU(gg_n)(1) - TU(g)(1)\|_2 \leq \|T\|\, \|U(gg_n)(1) - U(g)(1)\|_2$$

so that $\sum_{n\geq 1} \|U(gg_n)(1) - U(g)(1)\|_2^2 = \infty$ which by the unitarity of the representation is equivalent to $\sum_{n\geq 1} \|U(g_n)(1) - 1\|_2^2 = \infty$ and this proves a).

A way of getting an operator T satisfying i), ii) and iii) is by representing a Brownian motion on G as convolution operators in $L^2(\mu)$. Actually the whole construction to follow is not specific to the representation U on $L^2(\mu)$ but works for any continuous unitary representation of G on a Hilbert space. Here continuity of the representation is essential in order that the theory becomes applicable. This is the reason why the continuity of φ is required.

For the rest of this section, U denotes a continuous unitary representation of G on the Hilbert space $(E, \langle\ ,\ \rangle)$.

Recall that the representation U lifts to the algebra (under convolution) of probability measures on the Borel sets of G by putting, for a probability ν and $v \in E$,

$$U(\nu)v = \int_G U(g)v\ \nu(dg). \tag{5.1}$$

The integral here is Bochner integral. Its existence is guaranteed by the fact that U is unitary so $\|U(g)v\| = \|v\|$. From this we also have that $\|U(\nu)v\| \leq \|v\|$ so $U(\nu)$ is bounded with operator norm $\|U(\nu)\| \leq 1$. Thus if we take $T = U(\nu)$, some ν, condition i) is automatically satisfied.

Smoothness (condition ii)) can also be obtained with operators of the form $U(\nu)$. In fact,

Lemma 5.1: Let π be a left invariant Haar are measure on G and suppose that ν is a probability with smooth density $f(g) = \frac{d\nu}{d\pi}(g)$. Then for all $v \in E$, the mapping $g \in G \to U(\nu) \circ U(g)v$ is C^∞.

Proof: Define $\tilde{v} : G \to E$ by $\tilde{v}(g) = U(g)v$. Since $||\tilde{v}(g)|| \leq ||v||$ $\tilde{v}$ is bounded so it is convolvable with any probability in G. Let $\check{\nu}$ be the image of ν under the mapping $g \in G \to g^{-1} \in G$. Then $\check{\nu}$ also has a smooth density and

$$\begin{aligned} \check{\nu} * \tilde{v}(g) &= \int_G \tilde{v}(h^{-1}g) \, \check{\nu}\,(dh) \\ &= \int_G U(h^{-1})U(g)v \, \check{\nu}\,(dh) \\ &= U(\nu) \circ U(g)v. \end{aligned}$$

Since convolution with smooth functions is smooth, we conclude that $g \to U(\nu) \circ U(g)v$ is smooth (see Warner (1972) for convolutions of vector valued functions).

In order to get ν with smooth density and such that $U(\nu)$ is an injective operator on the representation space E, we consider a Brownian motion on G.

Let $X_1, \ldots, X_d$ be a basis for the Lie algebra g of G. Viewing the elements of g as right invariant vector fields on G, consider the (Stratonovich) stochastic differential equation

$$dg_t = X_1(g) \circ dW_t^1 + \cdots + X_d(g) \circ dW_t^d \tag{5.2}$$

where $W_1, \ldots, W_d$ are independent Brownian motions.

A solution of (5.2) with initial condition $g_0 = 1$ is called a Brownian motion on G. It is known (c.f. Kunita (1984) Th. 5.1) that such a solution exists and is defined on the whole ray $[0, \infty)$. Take $t > 0$ denote by P the d-dimensional Wiener measure and let $\nu_t = P \circ g_t^{-1}$ be the probability law of the G-valued random variable g_t (ν_t becomes the transition probability of the Markov process associated to solutions of (5.2)). The infinitesimal generator of (5.2) is the second order differential operator $L = \frac{1}{2}\sum_{i=1}^{d} X_i^2$. Since this is an elliptic operator, each ν_t has a smooth density with respect to Lebesgue measure in G. This follows e.g. from Malliavin calculus (see for instance Watanabe (1984)). Therefore ν_t has also a smooth density w.r.t. any left invariant Haar measure π, so we are in the conditions of the above lemma.

We go now towards the proof that $U(\nu_t)$, $t > 0$, is a one-one operator in the Hilbert space E.

Due to the right invariance of the vector fields which are the coefficients of (5.2), it follows that a solution with initial condition g is given by $g_t g$ with g_t a solution starting at the identity. From this and the Markov property it follows that if ν_t is as above then $\nu_{t+s} = \nu_t * \nu_s = \nu_s * \nu_t$, $t, s \geq 0$.

Applying the representation U to ν_t we get thus a semi-group $t \to U(\nu_t)$ of bounded operators in E. Since $||U(\nu_t)|| \leq 1$, this is in fact a contraction semi-group. The infinitesimal generator of $U(\nu_t)$ is of course $U(L)$. Precisely, let $v \in E$

be a C^{∞} - vector for U. Then $g \to U(g)v$ is C^{∞} so Itô's formula applies. It gives

$$\begin{aligned} U(g_t)v &= v + \frac{1}{2}\sum_{j=1}^{d}\int_0^t U_\infty(X_j^2)(U(g_s)v)ds \\ &= v + \int_0^t U_\infty(L)(U(g_s)v)ds \end{aligned} \tag{5.3}$$

where g_t is the solution starting at the identity and U_∞ stands for the representation of right invariant differential operators on G on the C^{∞} vectors of E. From the definition of ν_t, it follows that for $v \in E$, $U(\nu_t)v = E[U(g_t)v]$ where the expectation is taken with respect to the Wiener measure P. Therefore, (5.3) gives

$$U(\nu_t)v = v + \int_0^t U_\infty(L)(U(\nu_s)v)ds$$

from where it is readily seen that the infinitesimal generator of $U(\nu_t)$ restricted to the C^{∞} - vectors is exectly $U_\infty(L)$. Hence this is a closable operator and its closure is the infinitesimal generator of the semi-group $U(\nu_t)$.

Now we use a result by Nelson and Stinespring (see Warner (1972) Th. 4.4.4.3) which assure that for elliptic operators of the form L, the closure of $U_\infty(L)$ is self - adjoint. This result and the following lemma on self - adjoint semi - groups guarantees that $U(\nu_t)$ is one-one.

Lemma 5.2: Let $T_t, t \geq 0$ be a contracting semi - group on a Hilbert space E with self - adjoint infinitesimal generator A. Then $\forall v \in E, t > 0, < T_t v, v > \geq 0$ with equality only if $v = 0$. In particular T_t is injective.

proof: By the spectral theorem we can suppose that E is the L^2 - space of some measure space (Ω, m) and that there exists a measurable $h : \Omega \to \mathbb{R}$ such that

$$dom(A) = \{f : \int_\Omega (1 + h(x)^2)\ |f(x)|^2\ m\ (dx) < \infty\}$$

and that

$$(Af)(x) = h(x)f(x), \quad f \in dom(A)$$

(see Davies (1980)). Since T_t is contracting, A is negative semi - definite and hence $h(x) \leq 0$ m - a.s.. This is enough to assure that if we set

$$(S_t f)(x) = e^{th(x)} f(x)$$

then S_t is a contraction semi - group with A as infinitesimal generator. By the uniqueness of a semi - group given the generator, $T_t = S_t$. For $f \in L^2(m)$, we have

$$< T_t f, f > = \int_\Omega e^{th(x)}\ (f(x))^2\ m\ (dx)\ \geq 0$$

with equality iff $f(x) = 0$ a.s.. The lemma is thus proved.

With this lemma we conclude the construction of an operator satisfying i), ii) and iii') and thus the proof of a) in Theorem 3.1.

6. Unbounded sequences

Theorem 3.1 covers only bounded sequences in G. It is not difficult however to give examples of quasi - invariant probabilities for which there are plenty of sequences $g_n \in G$ which are unbounded (in a sense to be precised below) and such that $\prod_n g_n\mu \sim \prod_n \mu$ (see Example 7.3).

We present here a result which assures in many situations that every unbounded sequence g_n separates $\prod_n g_n\mu$ and $\prod_n \mu$. In such situations the condition that $g_n H \to H$ in Theorem 3.1 can be raised. The result we give here is inspired on the notion of recurrence, more specifically on Poincaré's recurrence theorem. It reads.

Theorem 6.1: Take the previous setting with G acting on X and μ a quasi - invariant probability on X. Suppose $(g_n)_{n\geq 1} \subset G$ is a sequence with $\prod_n g_n\mu \sim \prod_n \mu$. Then for every measurable A with $\mu(A) > 0$ there exists an integer $i > 0$ and $x \in A$ such that $g_n x$ returns infinetely often to $g_i(A)$.

proof: First of all recall that we have also $\prod_n g_n^{-1}\mu \sim \prod_n \mu$ so by Kakutani's theorem $\sum_{m\geq 1} d_H(g_n^{-1}\mu, \mu)^2 < \infty$ where $d_H(g_n^{-1}\mu, \mu) = ||\psi(g_n^{-1}) - 1||_2$ is the Hellinger distance. It holds then that $d_H(g_n^{-1}\mu, \mu) \to 0$ as $n \to \infty$. However, $\{g\mu\}$ is a dominated model having a probability for basic measure. In these circunstances it is known that the Hellinger distance dominates to the variational distance

$$\rho(g\mu, \mu) = \ sup \ |g\mu(B) - \mu(B)|$$

with the supremium taken over the measurable sets B (c.f LeCam (1970)). We have thus that for every measurable B, $lim_{n\to\infty}|\mu(g_n B) - \mu(B)| = 0$.

Now, let A be as in the statement. Take $\varepsilon > 0$ with $\varepsilon < \mu(A)$ and an integer $H > 0$ such that $|\mu(g_n A) - \mu(A)| < \varepsilon$ if $n \geq H$. It is required to prove that for some integer $i > O$, there is an infinite number of n's such that $g_n^{-1} g_i A \cap A \neq \phi$, i.e., $g_i A \cap g_n A \neq \phi$.

Suppose to the contrary that for every $i \geq H$ the set $\{j > i : g_i A \cap g_j A = \phi\}$ is finite. Then we can get a sequence $i_k \to \infty$ such that the intersection between any two of the sets $g_{i_k} A$ is void. Since $\mu(g_{i_k} A) \geq \mu(A) - \varepsilon > 0$ this contradicts the fact that μ is probability.

In order to illustrate a way of applying this theorem to our original problem we include the following corollary. In it we use the following terminology: *A sequence*

y_n in a topological space Y is said to converge to ∞ it this happens in the one - point compactification of Y, i. e., if for any compact $K \subset Y, y_n \notin K$ if $n \geq H$ some $H > 0$. The sequence is unbounded if some subsequence converges to ∞. It is bounded otherwise.

Corollary 6.2: Suppose that μ is positive on open sets of X. Suppose also that for every sequence $g_n \in G$ with $g_n \rightarrow \infty$, $g_n x \in X$ is unbounded. Then $g_n \in E(\mu)$ only if g_n is bounded. In this case Theorem 3.1 applies.

proof: Since X is assumed to be locally compact, there is a compact K with $\mu(K) > 0$. Since $g_n x$ is unbounded, passing to a subsequence it can be assumed that $g_n x \rightarrow \infty$. But this contradict the fact that $g_n x$ returns infinetely often to the compact $g_i(K)$.

A particular instance of this corollary is when $X = G$ where G acts by left translations.

Typically Theorem 6.1 applies when there is "no recurrence" of sequences $g_n \in G$ with $g_n \rightarrow \infty$. We do not formulate any general statement about this but give latter some examples in which such a situation occurs.

7. Examples

7.1 Take $G = \mathbb{R}^d = X$ with the action given by translation: $(x, y) \in G \times X \rightarrow x + y \in X$. The case $d = 1$ is treated in the pioneering work by Shepp (1965). Theorem 3.1 above specializes to Theorem 1 of Shepp (1965) except from the fact that it is restricted to bounded sequences. However Corollary 6.2 applies so this restriction can in fact be raised. Note that the action of G on X is transitive so by Proposition 3.4, the continuity assumption in Theorem 3.1 a) can be taken for sure.

7.2 Let G be the group of rigid motions on $\mathbb{R}^d$ $X = \mathbb{R}^d$ and the action of G on X the canonical one. This is the situation covered by Steele (1986). Our Theorem 3.1 a) gives a slight extension of Theorem 1 in Steele (1986) (our formulation allows basic measure which are invariant by subgroups). Also b) and c) of Theorem 3.1 specilizes verbatim to Theorems 2 and 3 of Steele (1986) respectively. We note that the restriction in Theorem 1 of Steele (1986) that the sequence converge to the identity is not necessary. In fact, G and its action on $X = \mathbb{R}^d$ are easily seen to be under the conditions of Corollary 6.2.

7.3 This is an example (more properly a family of examples) showing a way of constructing sequences $g_n \rightarrow \infty$ in G and belonging to $E(\mu)$. Let K be a compact group and G an one - parameter "irrational flow" on K, i. e., a one parameter subgroup whose closure has dimension bigger then one. For instance K could be the two -

torus $T^2 = \mathbb{R}^2/Z^2$ and G the subgroup $\{(t, \alpha t) mod Z^2 : t \in \mathbb{R}, \alpha$ a fixed irrational$\}$. Make K act by left translation itself and restrict this action to G. We take for μ a measure of the type $d\mu = c\ f\ d\ \lambda$ where $d\lambda$ is Haar measure on $K, f = 1 + \varphi$ with φ a smooth positive function with support contained in some small mighborhood of the identity and c is a constant of normalization. With μ taken this way only the identity in K fixes μ. Theorem 3.1 applies to the family $\{k\mu\}_{k\in K}$ and since there are sequences $g_m \in G$ with $g_m \to \infty$ (in the topology of G) and $g_m \to 1$ (in the topology of K), it is clear that there are also sequences $g_n \subset G$ with $\prod_n g_n\mu \sim \prod_n \mu$ such that $g_n \to \infty$ in the intrinsic topology of G.

7.4. This example illustrates an aplication of Theorem 6.1 in a situation not covered by Corollary 6.2. Take $G = Gl^+(d, \mathbb{R})$ the group of invertible $d \times d$ matrices with positive determiniant, $X = \mathbb{R}^d$ and the linear action of G on X. Let μ be the normal distribuition $N(0, 1)$. Then $\{g\mu\}_{g\in G}$ becomes the family of zero mean normal distribuitions and H in our previous notation is the compact group $S0(d, \mathbb{R})$ of orthogonal matrices. Of course, Theorem 3.1 applies in its full generality.

In order to handle unbounded sequences, embed G in the space of $d \times d$ matrices with its $\mathbb{R}^{d^2}$ - canonical topology. Then it is easy to see that if $g_n \to \infty$ in G then either i) there is a subsequence $g_{n_k} \to \infty$ in $\mathbb{R}^{d^2}$ or ii) there is a subsequence g_{n_k} converging in $\mathbb{R}^{d^2}$ to a singular matrix. As to the first case an argument similar to the proof of Corollary 6.2 shows that $\prod_n g_n\mu \perp \prod_n \mu$. For the second case appeal directly to the "recurrence" approach of Theorem 6.1 to show that $\prod_n g_n\mu \perp \prod_n \mu$. For this, suppose that g_n is a sequence in $Gl^+(d, \mathbb{R})$ converging to the singular matrix P. Put $V = imP$. Clearly $k = dimV \leq d-1$. If we check that $\cup_n g_n^{-1}(V)$ is not dense we are done. In fact, in this case there is a compact $K \subset \mathbb{R}^d$ with non void interior such that $g_n K \cap V = \phi$, every n. Since $g_n \to P, g_n x \to Px \in V$, all $x \in \mathbb{R}^d$, so it is impossible for $g_n x$, $x \in K$, to return infinetly often to $g_i K$, any $i > 0$. Thus the condition of Theorem 6.1 is violated and it happens that $(g_n)_{n\geq 1} \notin E(\mu)$.

To see that $\cup_n g_n^{-1}(V)$ is not dense, work on the Grassmannian of k - planes in $\mathbb{R}^d$. Passing to a subsequence if necessary we can assume that $g_n^{-1}(V)$ converges in the Grassmanniam. This is enough to ensure that the set $(\cup_n g_n^{-1}(V)) \cap S^{d-1} \subset \mathbb{R}^d$ is compact thus showing that $\cup_n g_n^{-1}(V)$ is not dense $\mathbb{R}^d$.

We stress that this method of dealing with unbounded sequences does not depend on the specific μ but only on the way G acts on X.

7.5 Let $G = Gl^+(d, \mathbb{R})$ and $X = G/SO(d, \mathbb{R})$ the space of positive definite dxd matrices. The action of G on X is given by $g(s) = gsg^t$, $s \in X$ (with t meaning transposition). It is clear that if $g_n \to \infty$ in G then $g_n s g_n^t$ is unbounded in X, so Corollary 6.2 applies and Theorem 3.1 works for bounded or unbounded sequences in G. Here the probability μ can be taken to be any quasi - invariant measure.

A specific μ could be the Wishart distribution $W(1, d, n)$ with n degress of freedom and having the identity as scale matrix. The family $\{g\mu\}_{g\in G}$ becomes the family Wishart distributions $W(\sum, p, n) = g\mu$ with $\sum = gg^t$.

7.6 The above example is in fact specific to the following class: Let G be a semi-simple or reductive Lie group and K maximal compact subgroup. Take $X = G/K$ and μ any quasi-invariant probability on X. By taking, e. g., a Cartan decomposition of G one checks easily that the action of G on X is under the conditions of Corollary 6.2, so Theorem 3.1 applies for any sequence $g_n \in G$. Note that because of Proposition 3.4, the assumption in Theorem 3.1 a) do not need to be checked.

Another example covered by this class of examples is the hyperboloid model (c. f. Barndorff-Nielsen (1967)). For this model G is the semi-simple Lie group $SO(1, d-1)$ and $K = SO(d-1, \mathbb{R})$ and the symmetric space G/K is the unit hyperboloid $H^{d-1} = \{(x_0, \ldots, x_{d-1}) \in \mathbb{R}^d = x_0^2 - (x_1^2 + \ldots + x_{d-1}^2) = 1\}$.

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